Abstract

Throughout science and mathematics, expert inquirers often reformulate existing problem-solving procedures and theories. But what value is there to reformulating, particularly when one already knows how to solve a given problem? Is reformulating merely instrumentally valuable for other practical or epistemic aims, or does it constitute a distinctive kind of epistemic achievement? I argue that by changing what we need to know to solve a problem, significant reformulations constitute a kind of intellectual value. Whereas some reformulations are trivial notational variants, a significant reformulation provides an inferentially distinct problem-solving plan. I situate my preferred position as a middle ground between deflationary and metaphysically-substantial alternatives.[1]

Contents

Throughout science, mathematics, and engineering, we often have multiple, compatible methods for solving problems. For each theory that scientists advance, they typically develop multiple ways of expressing or formulating its physical content. Often, the motivations for reformulating are practical: scientists wish to solve problems more quickly, simply, or elegantly. Sometimes, the aim is explicitly to clarify conceptual foundations, often by applying new mathematical techniques. Either way, the results of reformulating are a significant aspect of scientific progress. Reformulations often change how we understand the world, spawning new areas of research that probe the properties and scope of the reformulated theory. Similar remarks apply to reformulations in mathematics. These often lead to new proofs of old theorems, sometimes stemming from unexpected connections between seemingly disparate mathematical domains.[2]

[1] This paper reformulates a chapter of my dissertation titled "Between Instrumentalism and Fundamentalism about Reformulations." I thank the same people who helped me with that chapter, along with two anonymous referees for this journal.

[2] For discussions of the value of alternative proofs in mathematics, see John W. Dawson Jr, *Why Prove it Again? Alternative Proofs in Mathematical Practice* (Cham: Birkhäuser, 2015); and Rebecca Lea Morris, "The Values of Mathematical Proofs," in *Handbook of the History and Philosophy of Mathematical Practice*, ed. Bharath Sriraman

Insofar as we look to the sciences to motivate norms of inquiry, we should consider norms governing when to reformulate. Under what conditions is it wise to reformulate an existing problem-solving procedure or theory?[3] Schematically, consequentialism provides a straightforward answer to this decision-theoretic question: it is wise to reformulate whenever the expected utility of reformulating outweighs the expected utility of sticking with known problem-solving strategies, taking into account epistemic, practical, and moral factors.[4] But this consequentialist answer is only a schema. Determining when it is wise to reformulate still requires assessing and weighing various values involved. Here, I focus on an axiological question rather than a decision-theoretic one: what is the nature of the value(s) that reformulations provide? Answering this axiological question is prior to determining when it is wise to reformulate.

Theory reformulation is a kind of theory change, wherein we change how a *single* theory is formulated or expressed. This differs from cases of theory change that involve changing from one theory to another. Theory change of this latter kind provides an obvious sort of epistemic value: a new theory might solve problems that we couldn't solve before, providing knowledge that we could not have obtained otherwise. For instance, quantum mechanics answers questions that classical mechanics simply does not resolve. There is nothing particularly puzzling about the epistemic value of such changes.

A similar appraisal applies to competing theories or rival formulations of the same theory, such as rival 'interpretations' of quantum mechanics (where many view these as rival theories in their own right).[5] Of two rival formulations, one might provide answers that are closer to the truth. By positing rival ontologies, competing formulations straightforwardly lead to different ways of understanding the world. It is no surprise then that philosophy of science and the foundations of mathematics have historically focused on competing formulations of a given subject matter.

But what should we make of cases where two formulations are *compatible*, in the sense that we are not forced to choose between them? In such cases, we can simultaneously accept and use either formulation. Compatible formulations provide neither competing ontologies nor competing descriptions or predictions. Instead, they provide logically consistent problem-solving procedures for a shared class of problems. Physics supplies a wellspring of examples. Within classical mechanics, there are no less than five ways of formulating a large variety of problems, including the Newtonian, Hamiltonian, Lagrangian, Hamilton–Jacobi, and Routhian formulations of classical mechanics. These formulations differ in their mathematical strategies for solving the equations of motion for classical systems, and—within their shared domain of applicability—they describe the same physical states of affairs. Similarly, nonrelativistic quantum mechanics

(Springer International, 2021), 1–32.

[3] For norms of inquiry in connection to problem-solving, see Christopher Hookway, "Fallibilism and the Aim of Inquiry," *Proceedings of the Aristotelian Society, Supplementary Volumes* 81 (2007): 1–22 and Jane Friedman, "The Epistemic and the Zetetic," *Philosophical Review* 129, no. 4 (2020): 501–36.

[4] Thanks to Michele Odisseas Impagnatiello for suggesting a decision-theoretic viewpoint, which prompted the contrast with axiology below.

[5] The same could be said for different 'pictures' of quantum mechanics—such as the Schrödinger, Heisenberg, or interaction pictures—provided that one interprets these pictures as positing rival ontologies (e.g. concerning whether it is wavefunctions or operators that evolve in time). However, many physicists and philosophers interpret these pictures of quantum mechanics as being not only empirically but also physically equivalent (Lawrence Sklar, "Saving the Noumena," *Philosophical Topics* 13, no. 1 (1982): at p. 90; David Bohm, *Quantum Theory*, Dover (Englewood Cliffs: Prentice-Hall, 1951), at p. 380). Section 8.1 returns to this example.

can be formulated in a variety of distinct mathematical garb, including wave mechanics, matrix mechanics, density operators, and path integrals.[6] In what follows, my chief aim is to assess the value of compatible approaches to solving the same problem, abbreviating these simply as *reformulations*. Although I focus on cases of mathematical and empirical inquiry, the account I develop applies to problem-solving in general.[7]

1 A Spectrum of Responses

The value of compatible reformulations is puzzling for at least the following reason. For a given problem, no particular formulation is necessary for providing a solution. Any compatible formulation would suffice. In this way, each compatible formulation seems to render the others dispensable for the purposes of problem-solving. Nevertheless, many reformulations seem to constitute a particular kind of intellectual progress, deepening our understanding.

To characterize the value of reformulating, I will consider a spectrum of philosophical positions. We can visualize these positions as lying along a continuum from maximally deflationary to maximally inflationary, i.e. involving substantial metaphysical commitments. My goal is to defend a position I call *conceptualism*, occupying a middle ground between these extremes.[8] Conceptualism illuminates a particular kind of non-practical, epistemic value that compatible formulations provide, what we might call *intellectual value*. This comprises aspects of what many would call the 'purely epistemic,' although my aim is not to adjudicate the bounds of the epistemic.[9]

The seemingly most deflationary position denies that compatible reformulations provide any kind of value beyond mere convenience. According to what I will call *conventionalism*, this is all there is to say about reformulations. Reformulations provide convenient footholds for forging ahead, facilitating the solution of problems we could solve with other methods if only we were willing to sacrifice the time and energy.[10] Conventionalism holds that there is nothing deep or epistemically significant about reformu-

[6] Daniel F. Styer et al., "Nine Formulations of Quantum Mechanics," *American Journal of Physics* 70, no. 3 (2002): 288–97

[7] Typically, to solve an empirical problem, it is necessary to make measurements. Different measurement techniques or devices can provide compatible approaches to measuring the same physical quantity. Although they fall within the scope of compatible formulations, I do not focus on such cases here. I thank a referee for raising this question.

[8] Conceptualism emphasizes the role that concepts play in theory reformulation and understanding, including what Manders calls "expressive means": Kenneth Manders, "The Euclidean Diagram (1995)," in *The Philosophy of Mathematical Practice*, ed. Paolo Mancosu (Oxford: Oxford University Press, 2008), 80–133; and Kenneth Manders, "Expressive Means and Mathematical Understanding" (2012). These include the mathematical, linguistic, diagrammatic, and notational resources we use to express theories. For a helpful account of conceptual resources, see Alejandro Pérez Carballo, "Structuring Logical Space," *Philosophy and Phenomenological Research* 92, no. 2 (2016): 466ff.

[9] For discussion of the purely epistemic, see Ernest Sosa, *Judgment and Agency* (Oxford: Oxford University Press, 2015), at pp. 45, 172. Cohen considers various contested conceptions of the epistemic in Stewart Cohen, "Theorizing about the Epistemic," *Inquiry* 59, nos. 7-8 (2016): 839–57.

[10] A deflationary attitude is attractive for at least some notational choices, particularly what I'll describe as trivial notational variants. North describes—but does not fully endorse—this attitude in her discussion of coordinate systems: "Coordinate systems are labeling devices, tools that we impose...Since many such descriptive tools can be used, we tend to choose one for reasons of convenience" (Jill North, *Physics, Structure, and Reality* (Oxford: Oxford University Press, 2021), at p. 22).

lations. They merely amount to a different choice of convention, no different in kind than a change in notation. Seen through this lens, the seeming intellectual triumphs of wholesale theoretical reformulations are simply one notational change after another, convenience piled atop convenience. Conventionalism holds that reformulations differ only in degree—rather than kind—from trivial notational changes.[11]

Conventionalism belongs to a family of views that I will call *instrumentalism*. What these views have in common is reducing the value of reformulations entirely to their instrumental value for accomplishing other epistemic or practical aims.[12] On the practical side, such aims include solving problems quickly, easily, or with fewer computational resources, along with manipulative control over target systems. On the epistemic side, such aims include prediction, discovery, descriptive adequacy, and increasing our credence or confidence in solutions to problems (such as via corroboration or by reducing the risk of error). I leave open whether some practical aims are epistemic as well. Goldman, for instance, classifies problem-solving speed or efficiency as epistemic.[13] Regardless, I contend that we can isolate a non-practical dimension of the epistemic, namely *the intellectual*.

Instrumentalism contends that although reformulations are one method for achieving these practical and epistemic goods, reformulations are not constitutive of these goods. In this way, reformulations remain dispensable at least in principle. In Section 4, I argue that various versions of instrumentalism fail to respect a key aspect of scientific and mathematical practice, namely an intuitive distinction between trivial notational variants and non-trivial reformulations, where the latter are intellectually significant.

At the other extreme of our continuum lies *fundamentalism*. It proposes a metaphysical picture similar to David Lewis's. Lewis posits that some properties belong to an elite set of *perfectly natural properties*, with physics aiming to provide a partial inventory of these.[14] Ted Sider speaks instead of a theory's conceptual structure, which must match the structure of reality in order for the theory to be "fully successful."[15] Sider's framework suggests that two formulations of a theory can state the same truths about the world while nonetheless disagreeing about which concepts are more fundamental, i.e. more joint-carving. According to Sider, successfully describing fundamental structure has epistemic value. These pictures motivate a metaphysically-committal account of the value of reformulations. Insofar as reformulating is sometimes constitutive of writing a theory in more joint-carving terms, fundamentalists can interpret some reformulations as non-instrumentally valuable.

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[11] Some deflationary positions about the nature of scientific representation might be seen as inspiring or motivating conventionalism. See for instance Jonathan Cohen and Craig Callender, "There is no Special Problem about Scientific Representation," *Theoria* 55 (2006): 67–85.

[12] See Korsgaard's distinction between instrumental value vs. final value (which need not be intrinsic): Christine M. Korsgaard, "Two Distinctions in Goodness," *Philosophical Review* 92, no. 2 (1983): 169–95. Focusing on theories or models that misrepresent reality, Le Bihan distinguishes their heuristic and intrinsic epistemic value (Soazig Le Bihan, "Enlightening Falsehoods: A Modal View of Scientific Understanding," in *Explaining Understanding*, ed. Stephen R. Grimm, Christoph Baumberger, and Ammon Sabine (New York: Routledge, 2017), pp. 111, 122). Some have developed instrumentalism to give a deflationary account of the value of scientific explanations (Bas C. van Fraassen, *The Scientific Image* (Oxford: Oxford University Press, 1980), at pp. 93–94; and Tania Lombrozo, "The Instrumental Value of Explanations," *Philosophy Compass* 6, no. 8 (2011): 539–51).

[13] Alvin I. Goldman, *Epistemology and Cognition* (Cambridge, MA: Harvard University Press, 1986), at p. 122

[14] David Kellogg Lewis, "New Work for a Theory of Universals," *Australasian Journal of Philosophy* 61, no. 4 (1983): pp. 357, 364

[15] Theodore Sider, *Writing the Book of the World* (Oxford: Oxford University Press, 2011), at p. vii

For those willing to endorse additional metaphysical commitments, fundamentalism offers a non-instrumentalist account of the value of reformulating. However, it comes at the cost of difficult problems of epistemic access. As I argue in Section 5, these epistemic access problems partly spoil the positive story that fundamentalism can tell. The account I develop occupies a middle ground between instrumentalism and fundamentalism. Section 6 dubs this third strategy *conceptualism*: it focuses on how reformulations improve our epistemic position with regards to solving problems. I will argue that reformulations have non-instrumental value simply in virtue of how they restructure problem-solving. Successful reformulations clarify what we need to know to solve problems, improving our understanding of the world. Like instrumentalism, my account does not require substantial ontological commitments. Like fundamentalism, it accommodates the intuition that many reformulations are more than just instrumentally valuable.

A rival middle ground position—*explanationism*—holds that reformulations can be valuable in virtue of providing alternative explanations. Due to the vast number of different accounts of scientific (and mathematical) explanation, explanationism provides a schema, to be filled in with a particular account of explanation. Different accounts of explanation give rise to different versions of explanationism, making it difficult to argue decisively against. Nonetheless, I develop a general problem that seemingly afflicts all versions of explanationism: reformulations manifest a number of differences that *prima facie* do not appear to be explanatory differences. Instead, these differences involve changes to the epistemic or inferential structure of problem-solving. They involve changes to how scientists and mathematicians go about structuring a search space through different inferential rules.[16] Hence, I believe that a general account of reformulating requires focusing on how formulations structure problem-solving. Answering explanatory why-questions is, after all, just one kind of problem inquirers face.

Of course, nothing prevents fundamentalists or explanationists from adopting my conceptualist analysis but wanting to add more. A fundamentalist might wish to append additional commitments to fundamental structure. An explanationist might wish to append additional commitments to explanatory differences. My view is not incompatible with either of these augmentation strategies. Instead, conceptualism stands opposed to either fundamentalism or explanationism *being the end of the story* regarding the value of reformulations. My goal is to show that on their own, various versions of instrumentalism, fundamentalism, and explanationism provide inadequate accounts of reformulation. These negative arguments motivate a need for conceptualism as a positive account of the value of reformulations.

To make headway on the axiological question, Section 2 provides two simple illustrations of compatible formulations. Already in these cases, we see a range of values that compatible reformulations might manifest. Section 3 uses these examples to motivate three desiderata that any satisfying account of reformulations must meet. Subsequent sections argue that of the various accounts considered, only my preferred position—

[16] In the context of mathematical theories, Pérez Carballo argues that "to accept a mathematical theory is to adopt a particular way of structuring logical space" (Pérez Carballo, "Structuring Logical Space," *op. cit.*, p. 488). Similarly, Le Bihan's modal account of scientific understanding holds that understanding of phenomena is "afforded when one knows how to navigate the possibility space for these phenomena" (Le Bihan, "Enlightening Falsehoods: A Modal View of Scientific Understanding," *op. cit.*, p. 131). See also Armond Duwell, "Understanding Quantum Phenomena and Quantum Theories," *Studies in History and Philosophy of Modern Physics* 72 (2020): 278–91.

conceptualism—meets these three desiderata.

2 Two Simple Illustrations

Scientific reformulations are often rich and complex, involving advanced concepts from mathematics or sundry sciences. While inherently interesting, such examples require a wealth of background knowledge to assess. Fortunately, some simple examples illustrate characteristic features that arise.

Consider the following problem, discussed in the cognitive science literature on problem-solving and expertise.[17] Two trains—located at stations 50 miles apart—head toward each other at 25 miles per hour. While they are moving, a bird flies back and forth between them at 100 miles per hour. The problem is to figure out how many miles the bird travels before the trains meet. One *hard approach* to this problem involves calculating the distance the bird flies on each round-trip between the two trains. Stipulating that the bird always takes the shortest distance between the trains, one can determine the overall distance by summing a geometric series, with a term for each leg of the journey. Alternatively, an *easy approach* to solving this problem involves simply determining how long the bird is in flight. This equals the amount of time it takes for the trains to reach each other, namely, one hour. Hence, the easy approach entails immediately that the bird travels 100 miles as it flies between the trains.[18]

As a second example, consider an application of Gauss's law in electromagnetism. We are handed a ball containing static point charges of total charge Q. Our task is to quantify the strength of the electric field coming out of the ball. In other words, we need to determine the *electric flux* Φ_E, defined as the integral of the electric field **E** over the surface.[19] Naïvely, it would seem that to calculate the flux we need to know the electric field vector at each point passing through the surface. And to determine these electric field vectors, it would seem that we need to know the exact distribution of charges within the ball. Incredibly, Gauss's law shows us that we in fact do not need to know anything about either the charge distribution or the electric field to determine the flux. Instead, the electric flux simply equals the total amount of charge contained within our surface divided by a constant ϵ_0, known as the vacuum permittivity. Hence, knowledge of ϵ_0 and the total charge Q suffices for knowing the flux.[20]

In both cases, we have two compatible ways of solving the same problem. The pro-

[17] Goldman, *Epistemology and Cognition, op. cit.*, p. 132

[18] The *mutilated checkerboard problem* provides a similar example: after removing the squares from two opposite corners of a checkerboard, can the remaining squares be tiled with 31 dominoes? For discussion and additional examples, see Merim Bilalić et al., "When the Solution is on the Doorstep: Better Solving Performance, but Diminished Aha! Experience for Chess Experts on the Mutilated Checkerboard Problem," *Cognitive Science* 43, no. 8 (2019).

[19] More precisely, the electric flux Φ_E through a closed surface S is the surface integral of the component of the electric field normal to the surface, i.e. we integrate the scalar product of the electric field vector **E** with the differential of the normal vector to the surface **da**: $\Phi_E \equiv \oiint_S \mathbf{E} \cdot \mathbf{da}$. Thanks to Gordon Belot for suggesting this example.

[20] For systems with appropriate symmetry, Gauss's law supplies another simple compatible reformulation. In such cases, we can calculate the electric field itself purely algebraically, eliminating the need for integration. In contrast, a non-symmetry-based approach would apply Coulomb's law and a superposition principle for electric fields, integrating for the electric field.

cedures do not disagree about the way the world is. They provide the same answer to the problem and ultimately for the same physical reasons, albeit differently organized. Our axiological question is the following: what value is there to having more than one approach to solving the same problem? What do we gain by reformulating a problem-solving procedure or theory?

Instrumentalism contends that reformulating is not valuable for its own sake but merely as a means to other practical or epistemic ends. In each of our two illustrations, both formulations solve the same problem, so locally we do not have any non-practical epistemic differences (at least at first glance). Each compatible formulation provides epistemic goods such as (approximate) truth, prediction, or knowledge. Remaining differences between the formulations seem to be practical ones, such as differences in computational simplicity, efficiency, and convenience.

For instance, it is easier and faster to solve the bird–train problem by figuring out how long the bird is in flight than by calculating a geometric series. Likewise, it is easier and faster to apply Gauss's law to determine the electric flux than to painstakingly apply Coulomb's law. The easier methods may in turn decrease the risk of making a calculational mistake, but this is an epistemic difference in-practice, rather than in-principle. Later, I will consider whether global differences in problem-solving fruitfulness allow instrumentalism to draw epistemically significant differences between formulations. Perhaps one formulation generalizes to a wider range of phenomena, leading to increased instrumental value. For reasons considered in Section 3, I will argue that differences in fruitfulness still miss important epistemic differences between the approaches.

By contrast, on Lewis's fundamentalist framework, a formulation does better the closer it comes to a canonical language that carves nature at its joints. A concept carves nature *perfectly* at its joints only if it is fundamental, but joint-carving is not an all or nothing affair. Instead, different concepts within the special sciences can be more or less joint-carving.[21] Sider enriches this picture by arguing that differences in joint-carving generate differences in the epistemic value of formulations. Given two languages for describing the world, if one of them carves nature better at the joints, then it has epistemic value that the other one lacks. Sider illustrates this in the context of the predicates green and grue, claiming that "it's *better* to think and speak in joint-carving terms. We ought not to speak the 'grue' language, nor think the thoughts expressed by its simple sentences."[22]

In the case of the bird and the trains, it is plausible that neither formulation is more joint-carving than the other. The geometric series approach keeps track of the causal details of the bird's trajectory, while the easy approach shows that we do not need this information to solve the problem. Yet, neither approach is obviously more fundamental. In cases like this, a fundamentalist might agree with an instrumentalist that this reformulation has no more than instrumental value.[23]

The Gauss's law case is more interesting. As one of Maxwell's laws of electrodynamics, Gauss's law plausibly is more fundamental than Coulomb's law. Gauss's law is related to conservation principles, which themselves have a close connection with laws

[21] Lewis, "New Work for a Theory of Universals," *op. cit.*, p. 347

[22] Sider, *Writing the Book of the World*, *op. cit.*, p. 61

[23] The conceptualist view I defend in Section 6 accommodates the intuition that even in this simple case, there are intellectually significant differences between the two approaches to the bird–train problem.

of nature and fundamental symmetries.[24] Additionally, Gauss's law applies to not only static but also moving charges, and it is therefore more general than Coulomb's law. A fundamentalist might view this difference in fruitfulness as evidence that the Gauss's law approach gets closer to fundamental joints in nature.[25]

On the view I defend in Section 6, we can grant both that reformulations have instrumental value and that they could—for all we know—have epistemic value coming from tracking fundamental structure. What matters is that we can be sure of one source of their non-instrumental epistemic value: reformulations clarify what we need to know to solve problems.[26] By changing our epistemic situation, reformulations accrue epistemic value independently of any further metaphysical role they might play. In short, a significant reformulation leads to a different way of understanding the world. This is in contrast to trivial or insignificant reformulations, considered in the next section.

3 Three Desiderata

I will argue that any satisfying account of reformulations must meet three desiderata. First, it must distinguish trivial notational variants from significant reformulations. Whereas some reformulations are merely matters of arbitrary, conventional choices, others appear to be epistemically significant. Second, a successful account must make sense of local differences between reformulations that arise when solving the same class of problems. Although reformulations often lead to differences in solving wider classes of problems, appealing only to global differences does not address important local differences. Indeed, it is plausible that such local differences ground global differences in fruitfulness. Finally, the criteria that an account employs ought to be epistemically accessible. An account is less satisfying insofar as it appeals to features of the world that might readily elude us. Here, I independently motivate these three desiderata. Sections 4 and 5 argue that both instrumentalism and fundamentalism fall short of meeting them.

Not all reformulations are epistemically significant. Some amount to nothing more than trivial notational variants. These include simple notational substitutions for typographical preference, the use of a right-handed rather than a left-handed coordinate system, conventions for summation, etc. I take it as a datum of scientific and mathematical practice that these trivial notational variants are epistemically insignificant. At the very least, they are much less epistemically valuable than paradigmatic cases of reformulation,

[24] See Franco Strocchi, *An Introduction to Non-Perturbative Foundations of Quantum Field Theory* (Oxford: Oxford University Press, 2013), Ch. 7.

[25] As Tappenden notes, defenders of joint-carving may take differences in fruitfulness or fertility as evidence that one formulation is more fundamental than another (although Tappenden himself does not endorse a metaphysically robust notion of 'fundamentality') (Jamie Tappenden, "Mathematical Concepts: Fruitfulness and Naturalness," in *The Philosophy of Mathematical Practice*, ed. Paolo Mancosu (Oxford: Oxford University Press, 2008), 276–301). I will remain neutral on whether fruitfulness plays this evidential role, at least when it comes to metaphysically robust notions of 'fundamental.' For an argument that fertility is not a fundamental virtue, see Daniel Nolan, "Is Fertility Virtuous in its Own Right?," *British Journal for the Philosophy of Science* 50, no. 2 (1999): 265–82.

[26] Roughly, I intend 'knowledge' in the sense of non-luckily acquired, justified true belief. This includes true beliefs regarding how to apply a false—but perhaps approximately true—theory, such as knowing how to apply the Ptolemaic model of our solar system, Newtonian mechanics, or nonrelativistic quantum mechanics (each of which is strictly speaking false). My account is compatible with more fine-grained views concerning the nature of knowledge or knowledge attributions.

including the two simple cases presented in Section 2. A successful account of compatible reformulations must provide principled grounds for distinguishing trivial notational variants from significant reformulations, affording greater epistemic value to the latter. This requirement supplies the *first desideratum.* To satisfy it, an account must avoid both (i) over-generating cases of significant reformulations (e.g. by classifying *all* reformulations as epistemically significant) and (ii) under-generating such cases (e.g. by classifying all reformulations as trivial notational variants).

To meet the first desideratum, an account must provide a principled distinction between clear cases of trivial vs. significant reformulations. This does not require providing necessary and sufficient conditions, since there may be vague cases that do not fall neatly into either category. Instead, it suffices to justify the datum that there is an epistemically significant difference between trivial vs. significant reformulations, with the latter being objectively more epistemically valuable (at least in clear cases). This distinction is objective in the sense that its truth does not depend on how agents feel or what they believe about it. Regarding the meaning of "epistemically significant" or "epistemic differences," there are many candidates, given the contested nature of the word 'epistemic.' Different accounts may specify different meanings for these terms. I describe my preferred account in Section 6, which focuses on non-practical dimensions of answering questions, i.e. solving problems.

The second desideratum constrains what we can appeal to when meeting the first. In clear cases, we can distinguish trivial from significant reformulations at the local level of individual problems or problem-types. This is another apparent datum of mathematical and empirical inquiry that any satisfying account must save. Given two compatible reformulations, there is a class of problems that they both solve. Within this shared domain of problems, significant reformulations display an epistemic difference, while trivial reformulations do not. Since these epistemic differences arise locally, we should account for them through local aspects of the formulations. It should not be necessary to consider global differences in fruitfulness or problem-solving scope. Unless shown otherwise, we should assume that these global differences arise from differences at the local level of solving individual problems. The second desideratum embodies these demands: a satisfying account of reformulations must provide local criteria for distinguishing trivial vs. non-trivial reformulations. Section 4 shows how the first two desiderata pose a serious problem for instrumentalism.

Besides the need to locally distinguish trivial from significant reformulations, a *third desideratum* presents itself: the criteria of significance should be epistemically accessible. To the extent that there are manifest epistemic differences between trivial and non-trivial reformulations, the criteria we use to explicate these differences should be manifest as well. Our account of reformulation should not be hostage to the lucky success of risky inferences. An account with epistemically inaccessible criteria may have the resources to address the first two desiderata, but it would be difficult to determine when the criteria are met. Accounts of reformulation that rely on risky inferences will face problems of underdetermination, leading to skeptical scenarios. The more difficult it is to determine whether the criteria are satisfied, the more severe these skeptical scenarios will be. In science, these worries about underdetermination are well motivated: there are numerous historical examples of scientists making needlessly risky inferences that were shown to be unfounded.[27] This is not an idle philosopher's skepticism. There are principled, practice-based reasons for seeking to avoid risky inferences whenever possible.

An additional reason favors the third desideratum. Appraising compatible formulations is a challenge facing philosophers of many different temperaments, from constructive empiricists to those willing to posit Aristotelian essences. Ideally, an account of reformulation should have a widely-acceptable minimal core. This core should be as minimal in its ontological commitments or posits as possible. Nothing precludes those with additional metaphysical commitments from embellishing this account further, but it is harder to deconstruct a more metaphysically committal account into a version acceptable for the a-metaphysical. Section 5 shows how this third desideratum severely limits the appeal of fundamentalism, at least as the core of an account of reformulations.

4 Problems facing Instrumentalism

Recall that instrumentalism assesses reformulations based entirely on their instrumental value for various epistemic or practical aims. This suggests the following *instrumentalist criterion* for distinguishing trivial from non-trivial reformulations: a reformulation is significant if and only if it leads to an instrumentally valuable difference. Since most instrumentally valuable differences between formulations are epistemically accessible, instrumentalism easily satisfies the third desideratum. The challenge for instrumentalism is to satisfy the first desideratum without running afoul of the second. In other words, instrumentalism must distinguish between intuitive cases of trivial and non-trivial reformulations without either (i) over-generating cases of significant reformulations or (ii) appealing solely to global differences in problem-solving scope or fruitfulness. I will argue that various instrumentally valuable differences each violate at least one of these conditions.

First, consider practical differences in convenience, such as problem-solving speed or ease of solution. Although significant reformulations often differ along these dimensions, so do paradigmatic cases of trivial notational variants. For instance, we find it extremely difficult to read mirror images of words.[28] Similarly, scientists sometimes develop strong psychological preferences for certain notational conventions. For instance, physicists working in particle physics phenomenology tend to use a different space-time metric convention than those working in general relativity or string theory. The former tend to use a mostly minus $(1, -1, -1, -1)$ metric while the latter use a mostly plus $(-1, 1, 1, 1)$ metric. There are many compelling practical reasons to prefer one convention over the other, based on the kinds of problems that most commonly arise in either domain. Burgess and Moore note that, although just a choice of convention, "some physicists approach this issue with almost religious conviction."[29] An instrumentalism focused on these kinds of practical differences would diagnose these two metric conventions as

[27] Examples include Newton's inference from absolute acceleration to the existence of absolute velocity, late 18th-century inferences to the existence of caloric as the carrier of heat, and 19th century inferences to the existence of an aether for the propagation of light as an electromagnetic wave.

[28] For an illustration, see Wittgenstein's remark 151 in the following: Ludwig Wittgenstein, *Philosophy of Psychology: A Fragment*, 4th ed., trans. G.E.M. Anscombe, P.M.S. Hacker, and Joachim Schulte (Wiley-Blackwell, 2009 [1949]), p. 209. Framing-effects from the presentation of statistics in terms of decimals or ratios provide a further example. See Daniel Kahneman, *Thinking, Fast and Slow* (Macmillan, 2011).

[29] Cliff Burgess and Guy Moore, *The Standard Model: A Primer* (Cambridge: Cambridge University Press, 2006), p. 518

significant reformulations. Such verdicts would vastly over-generate the class of significant reformulations, thereby running afoul of the first desideratum. Even if these practical differences between trivial notational variants are sometimes important, they still appear to be *different in kind* from the intellectual differences that conceptualism highlights.[30]

A structurally similar objection applies to versions of instrumentalism that focus on in-practice epistemic differences such as reducing the risk of error, or increasing one's degree of confidence in a solution. For instance, most people are less likely to make a calculational error using the easy approach to the bird–train problem than the geometric series approach. Yet, we also see reductions in the risk of error when using a trivial notational variant that we are more comfortable or familiar with. Hence, this criterion does not distinguish trivial from significant reformulations. Similarly, we gain increased confidence in a solution whenever we solve a problem anew, whether using a trivial or a significant reformulation. This is no different than how double checking an answer can increase our confidence in it.[31]

Turning to differences in prediction, control, or descriptive adequacy, we see that these differences do not arise locally. By definition, two compatible formulations both solve a shared class of problems. Hence, they locally provide the same predictions, are equally approximately true, and provide the same degree of manipulative control. It is therefore difficult to see how there could be local differences along these dimensions.

To meet the first desideratum, instrumentalism seemingly must appeal to global differences, such as differences in fruitfulness. When we broaden our scope to consider how reformulations differentially generalize in different contexts, sometimes certain formulations succeed where others fail. For instance, the easy solution to the bird–train problem applies even to a bird executing exquisite loop-de-loops between the trains. In contrast, the geometric series solution requires that the bird fly in straight lines (otherwise, we would require further information about the bird's trajectory). Similarly, the Gauss's law approach applies to moving charges, while the Coulomb's law approach requires that the charges are static. In each case, one formulation is more fruitful than the other, applying to a strictly wider range of problems.

No doubt, differences in fruitfulness are epistemically valuable. They constitute differences in the predictions we can make and the phenomena we can save. However, they are not differences that arise at the local level of shared problem-solving. We should expect that these global differences are symptoms of underlying local differences in problem-solving. Appealing solely to global differences relinquishes the goal of identifying local differences that are *prima facie* significant. It would be more satisfying if we could accommodate global differences in terms of local, epistemically significant differences between formulations. We should give up on the second desideratum only if other promising accounts fail to meet it as well. For this reason, instrumentalism is not enough to account for the significance of reformulations. Instrumental differences are part of a larger story, but they are not the whole story.

[30] Moreover, what counts as computationally simpler or more convenient is often a matter of taste or pedagogical training. Ideally, we would satisfy the first desideratum by giving an objective distinction between clear cases of trivial and significant reformulations.

[31] As Davidson notes, "it is often worthwhile to increase our confidence in our beliefs, by collecting further evidence or checking our calculations" (Donald Davidson, "Truth Rehabilitated," in *Truth, Language, and History* (Oxford: Clarendon Press, 2005), pp. 6–7).

5 Problems facing Fundamentalism

According to many scientific realists, science aims at the truth. Fundamentalism proposes a further aim for empirical inquiry: an ideal scientific theory must describe the world in a fundamental language.[32] Two descriptions of a subject matter can both be true, while one of them is more fundamental. Lewis contends that physics aims at providing an inventory of natural properties. According to Lewis, "the businesss [sic] of physics is not just to discover laws and causal explanations. In putting forward as comprehensive theories that recognize only a limited range of natural properties, physics proposes inventories of the natural properties instantiated in our world."[33] Likewise, Sider argues that describing the world in joint-carving terms leads to greater epistemic value than merely having a true theory:[34]

> The goal of inquiry is not merely to believe truly (or to know). Achieving the goal of inquiry requires that one's belief state reflect the world, which in addition to lack of error requires one to think of the world *in its terms*, to carve the world at its joints. Wielders of non-joint-carving concepts are worse inquirers.[35]

Although neither Lewis nor Sider are explicitly concerned with compatible formulations, their commitments to fundamental structure suggest a *fundamentalist criterion* for assessing reformulations: a reformulation is epistemically valuable whenever it leads to a more joint-carving formulation. Using this criterion, fundamentalism straightforwardly meets the first two desiderata from Section 3. It proposes an objective epistemic difference between trivial notational variants and significant reformulations. Whereas trivial notational variants are equally joint-carving, significant reformulations exhibit a difference in fundamentality: namely, one formulation is more joint-carving than the other. Furthermore, the fundamentalist criterion of significance is local: these differences in fundamentality arise at the level of individual problem-solving. Fundamentalism thereby satisfies the second desideratum as well. Although evidence for differences in joint-carving might come from global considerations of fruitfulness, the differences themselves arise locally (if they arise at all).

The main problems facing fundamentalism arise from its substantial ontological commitments. Many empiricists and scientific anti-realists (and even some realists) disavow commitments to perfectly natural properties or fundamental structure. Relying on these commitments precludes fundamentalism from providing a minimal account of the value of reformulations. In response, a fundamentalist might be inclined to say: so much the worse for the metaphysically-averse. But there are independently compelling reasons to

[32] Similarly for ideal logical and mathematical theories, regarding relevant putative features of reality.

[33] Lewis, "New Work for a Theory of Universals," *op. cit.*, p. 364

[34] Dasgupta has challenged Sider's contention that more fundamental descriptions necessarily have greater epistemic value (Shamik Dasgupta, "Realism and the Absence of Value," *Philosophical Review* 127, no. 3 (2018): 279–322). He argues that fundamentalists must *explain* where this epistemic value comes from, but that no explanation is forthcoming. However, fundamentalists can simply note that knowledge of fundamental structure may be *primitively* valuable, in which case there would be nothing further to explain. In other words, the correct explanation—and hence the best explanation—would be that such knowledge is primitively valuable. Reality may not accord with the notion of 'explanation' that Dasgupta presupposes (for instance, reality might violate versions of the principle of sufficient reason).

[35] Sider, *Writing the Book of the World*, *op. cit.*, p. 61

be wary of appeals to fundamental structure. One reason comes from fundamentalism itself: Occam's razor. If we can provide a positive account of reformulations with fewer metaphysical commitments, then this account will be simpler. Fundamentalists would then, by their own lights, have reasons to take such an account seriously.[36] This provides one reason in favor of the conceptualist account I provide in Section 6.

More substantial metaphysical commitments typically involve posits that are less epistemically accessible. It is difficult to know if and when theory formulations track perfectly natural properties. Beyond appeals to intuition, fundamentalists can rely on theoretical virtues as evidence for greater fundamentality. Whether and when to believe that differences in virtues—such as simplicity or fruitfulness—track fundamentality seems ultimately decided by appeals to philosophical intuition. Some scientific realists and fundamentalists may be sufficiently optimistic about these aspects of philosophical methodology. For them, these epistemic access problems may not be substantially more troubling than our access to physical unobservables posited by mainstream scientific theories. Nevertheless, an account of reformulations is epistemically more secure to the extent that it does not rely on controversial methodological commitments.[37] Ideally, we should seek an ontologically minimal account of reformulations that even empiricists can adopt. More metaphysically-committed philosophers then remain at liberty to invoke additional ontological commitments when assessing reformulations.

Epistemic access problems also lead to problems of underdetermination. Consider an epistemically possible world where neither the Gauss's law nor the Coulomb's law formulation of the electric flux problem is more fundamental than the other (see Section 2). This world is empirically indistinguishable from the one fundamentalists might take us to be in, where the Gauss's law formulation is putatively more fundamental. In either world, our physical theories make exactly the same predictions about both observables and unobservables. Yet, the two worlds disagree about whether the Gauss's law formulation is more fundamental, and hence about whether the two formulations are trivial notational variants. In a world where the two formulations are equally fundamental, the fundamentalist criterion of significance classifies the two formulations as trivial notational variants (at least *qua* fundamental structure). In the other, this criterion says that the formulations are significantly different. But since both worlds would be empirically indistinguishable, it is difficult to know which one we are in.[38]

As a result, fundamentalism makes the significance of compatible formulations hostage to empirically inaccessible facts about fundamental structure. Even worse, these inaccessible facts do not have any bearing on how the formulations appear to us. Metaphysical facts about differences in joint-carving do not change how we solve problems or understand the world using our theories. Fundamentalism meets the first two desiderata *in*

[36] Thanks to Dave Baker for stressing this point.

[37] Cohen and Callender argue that perfectly natural properties face additional problems of epistemic access beyond the usual skeptical challenges to knowledge of physical unobservables: Jonathan Cohen and Craig Callender, "A Better Best System Account of Lawhood," *Philosophical Studies* 145 (2009): p. 13. Woodward provides further reasons against flatly invoking natural properties: James Woodward, "The Problem of Variable Choice," *Synthese* 193 (2016): p. 1056.

[38] van Fraassen develops a similar underdetermination problem for mathematical Platonism through his parable of the lands of Oz vs. Id: Bas C. van Fraassen, "Platonism's Pyrrhic Victory," in *The Logical Enterprise*, ed. Alan Ross Anderson, Ruth Barcan Marcus, and R. M. Martin (New Haven: Yale, 1975), 39–50. Cohen and Callender, "A Better Best System Account of Lawhood," *op. cit.* provide this kind of argument against the epistemic accessibility of perfectly natural properties.

principle. But in virtue of failing the third desideratum, fundamentalism makes it difficult to know when significant differences arise. To avoid underdetermination problems, we should strive for an account of compatible formulations that does not depend on relatively inaccessible facts, e.g. about fundamental structure. Even if we are mistaken or woefully ignorant about the world's fundamental structure, we should be able to satisfactorily interpret significant epistemic differences between reformulations.

6 Conceptualism

With myriad problems facing instrumentalism and fundamentalism, I now develop an account of reformulations that avoids these problems. Because it focuses primarily on how different concepts restructure problem-solving, I call my account *conceptualism*. Conceptualism accounts for the significance of reformulations in terms of how they structure problem-solving, based on the inference rules deployed. Section 6.1 provides examples of these inferential differences. They constitute differences in the inferential structure of a problem-solving procedure. Section 6.2 shows how conceptualism easily satisfies all three desiderata, providing a positive, local, and ontologically minimal account of reformulations. Finally, Section 6.3 considers in more detail the notion of sameness or equivalence of inferential structure.

6.1 Inferential structure

Consider the toy example from Section 2 involving a bird flying between two trains. The hard procedure requires knowing the distance the bird travels on each leg between the trains (or alternatively, the time spent on each leg). The easy procedure shows that we do not need to know the bird's detailed trajectory: it suffices to know the speed at which the bird flies and the amount of time spent flying. Similar considerations apply to calculating the electric flux emanating from a charged body using Gauss's law. This law shows that we do not need to know the distribution of charges within the charged object or the electric field at each point on the surface. Instead, it suffices to know the total amount of charge the object contains.

More complicated cases of reformulation also display differences in what we need to know to solve problems. In the quantum mechanics of atoms and molecules, we can often use symmetry arguments to solve problems without needing to know many details about a system's dynamics. In contrast, elementary methods that eschew appeals to symmetries require more detailed information to solve the same problems.[39] The Lagrangian formulation of classical mechanics illustrates a similar moral. It tells us that to calculate the equations of motion for a classical system, we do not need to know the constraint forces acting on the system. In contrast, the Newtonian formulation requires knowledge of these constraint forces. My conceptualist account focuses on differences in what we need to know to solve problems, arguing that such differences in inferential structure underpin significant reformulations.

[39] For cases of symmetry-based reformulation in quantum mechanics and quantum chemistry, see my discussions in Josh Hunt, "Interpreting the Wigner–Eckart Theorem," *Studies in History and Philosophy of Science* 87 (2021): 28–43 and Josh Hunt, "Epistemic Dependence and Understanding: Reformulating Through Symmetry," *British Journal for the Philosophy of Science* 74, no. 4 (2023): 941–74.

Each compatible formulation constitutes a plan for problem-solving, comprising a set of inferential steps. Each inference step takes us from an input set to an output, ultimately resulting in a solution. Two problem-solving plans are *inferentially equivalent* just in case what you need to know to carry out one plan is the same as what you need to know to carry out the other plan. For instance, what an English speaker needs to know to carry out the easy approach to the bird–train problem is the same as what a French speaker needs to know, even though the English speaker uses English and the French speaker uses French. The propositions that these two speakers need to know are the same, even though they are voiced in different languages, using different sentences. Similarly, two Turing machines are inferentially equivalent provided that they carry out an identical algorithm, even if their individual command lines are written in a different order or using different symbols. Section 6.3 further defends these equivalence claims.[40]

To speak more precisely about inferential structure, it is convenient to introduce a three-part relation between (i) input information, (ii) an inference rule, and (iii) output information. Call these *inferential relations*. An inferential relation is satisfied provided that applying the inference rule to the input yields the output. Each inferential relation fixes what one needs to know to apply it, namely (i) the input information and (ii) the inference rule.[41] Two problem-solving plans are inferentially equivalent if and only if they share the same inferential relations. Hence, a sufficient condition for being inferentially inequivalent is having different inferential relations. This amounts to a difference in what one needs to know to solve problems using the two formulations.[42]

Compatible formulations involving Arabic vs. Roman numerals provide a simple illustration of inferentially inequivalent plans. Imagine a lecture hall with 21 rows of 16 seats each. Our task is to determine how many people it can seat. Arabic numerals allow us to multiply 16 by 21 using a standard algorithm from grade school. This algorithm takes advantage of Arabic numeral's positional notation to modularize the problem into a series of single-digit multiplication and addition sub-problems, such as calculating six times two. To use this multiplication algorithm in full generality, one needs to know (i) 100 single-digit multiplication facts (i.e. the times table up to 9) and (ii) how to add Arabic numerals.

Now, imagine reformulating this multiplication problem using Roman numerals, i.e. calculating XVI times XXI. Since Roman numerals are a sign-value system rather than a positional one, our familiar algorithm does not work.[43] We must rely instead on an

[40] For an account of mathematical proofs based on Michael Bratman's planning theory of intention, see Yacin Hamami and Rebecca Lea Morris, "Plans and Planning in Mathematical Proofs," *The Review of Symbolic Logic* 14, no. 4 (2021): 1030–65.

[41] I do not intend here to take a stance on whether knowledge-how reduces to knowledge-that. If one denies the reduction, then they can read "knowledge of the inference rule" as meaning "knowing how to use the inference rule." The concept of inferential relations makes more precise what I have elsewhere referred to as "epistemic dependence relations" (EDRs) (Hunt, "Interpreting the Wigner–Eckart Theorem," *op. cit.*; Josh Hunt, "Understanding and Equivalent Reformulations," *Philosophy of Science* 88, no. 5 (2021): 810–23; Hunt, "Epistemic Dependence and Understanding," *op. cit.*).

[42] My proposal for 'inferential equivalence' is naturally gradated: two problem-solving plans are *more or less* inferentially equivalent to the extent that they share the same inferential relations. In this way, some problem-solving plans are inferentially more similar than others.

[43] In an additive system, the string represents the sum of its individual numerals. For simplicity, I do not consider subtractive notation such as "IV" for four, representing this instead as "IIII." Everything I say below could be adapted to this case. See Michael Detlefsen et al., "Computation with Roman Numerals," *Archive for History of Exact Sciences* 15, no. 2 (1976): 141–48.

inferentially inequivalent problem-solving plan. Rather than using an addition table, we instead use seven simplification rules such as replacing "IIIII" by "V". We also use a multiplication table of 49 separate multiplication facts (such as L times L equals MMD), which must be augmented for factors above one million.[44] When it comes to figuring out that 16 times 21 equals 336, these two formulations display different inferential structures, characterized by differences in what one needs to know to solve the problem. They thereby amount to inferentially different plans for problem-solving.

In contrast, merely interchanging the symbol '5' everywhere with the symbol 'V' does not result in an inferentially different plan. With this symbol substitution, we could use either problem-solving plan described above. Hence, inferential (in)equivalence concerns not only the notation we use but also *how* we use that notation, i.e. the problem-solving plans supported by that notation or other relevant concepts.

As this simple example already illustrates, how a formulation *presents* input information, i.e. its mode of presentation, can undergird the value of reformulating. Different modes of presentation can make available different inference rules. For instance, the standard multiplication algorithm is only applicable to information presented in a positional system. Presenting numbers in a sign-value system, such as Arabic numerals, makes available different inference rules. Aronowitz and Lombrozo develop an analogous point in the context of cognition, describing how mental simulation can alter what information is *accessible*. In a process they call "representational extraction," mental simulation provides "a change in accessibility conditions, and hence in how existing information can be deployed."[45] This is similar to how significant reformulations can make available inference rules that lead to distinct problem-solving plans.

6.2 Satisfying the three desiderata

Conceptualism proposes a straightforward criterion for assessing the significance of reformulations: a reformulation is significant (i.e. non-trivial) when it results in an inferentially inequivalent problem-solving plan. As argued in the preceding section, a sufficient condition for inferential inequivalence is that two formulations differ in what one needs to know to apply them. I now show that this criterion satisfies all three desiderata from Section 3. It provides a principled distinction between intuitive cases of trivial and nontrivial reformulations that is both local and epistemically accessible.

I leave open whether meeting this criterion is also *necessary* for a reformulation to

[44] Details of this algorithm, which relies on the distributive law, can be found in Dirk Schlimm and Hansjörg Neth, "Modeling Ancient and Modern Arithmetic Practices: Addition and Multiplication with Arabic and Roman Numerals," in *30th annual conference of the Cognitive Science Society* (Red Hook, NY: Curran, 2008), p. 2100. The problem of determining which of two numbers is greater also involves different inferential relations in these two formalisms; see Mark Colyvan, *An Introduction to the Philosophy of Mathematics* (Cambridge: Cambridge University Press, 2012), pp. 133–34.

[45] Sara Aronowitz and Tania Lombrozo, "Learning through Simulation," *Philosophers' Imprint* 20, no. 1 (2020): p. 15. Similarly, Lombrozo describes how the process of "learning by thinking"—such as through thought experiments—can "create a representation with novel affordances, one that's newly available to processes of explicit reasoning and argumentation, no matter that in some sense the relevant knowledge was there all along" Tania Lombrozo, "'Learning by Thinking' in Science and in Everyday Life," in *The Scientific Imagination*, ed. Arnon Levy and Peter Godfrey-Smith (Oxford: Oxford University Press, 2020), p. 239. Both Lombrozo (*ibid.*, p. 237) and Aronowitz and Lombrozo focus on psychological aspects of simulation and inference—involving "different mental representations" and "different mental processes" (Aronowitz and Lombrozo, "Learning through Simulation," *op. cit.*, p. 15). Here, I focus on propositional reasoning throughout, rather than distinct mental processes.

be significant in the relevant sense (a sense that sets aside practical dimensions such as problem-solving speed or reducing the practical risk of error). Conceivably, there might be two formulations that differ at the level of joint-carving but not at the level of what one needs to know to solve problems.[46] If joint-carving differences do have objective epistemic value, then this would provide a separate sufficient condition for significance. However, as Section 5 argues, this metaphysical criterion is not epistemically accessible.

The first desideratum demands a principled distinction between trivial notational variants and significant reformulations. Unlike significant reformulations, two problem-solving plans that are trivial notational variants fail the above criterion: they have identical inferential structure, deploying the same inferential relations for solving problems.[47] Symbol substitution provides the simplest case: substituting every instance of a symbol α with a previously unused, arbitrary symbol β does not alter a formulation's inferential relations. Likewise, even though many scientists prefer to work in a right-handed coordinate system, working in a left-handed coordinate system preserves the same inferences. In relativistic theories, the arbitrary choice between a mostly positive or a mostly minus metric convention does not lead to inferential differences. Hence, these two conventional choices are trivial notational variants. This is the case despite the fact that many physicists have a personal—and sometimes subfield-wide—preference for one convention over the other. After discussing the other two desiderata, I return in Section 6.3 to the question of whether there is any sense in which trivial notational variants exhibit different inferential relations.

The conceptualist criterion also satisfies the second desideratum, which demands that we *locally* distinguish trivial from significant reformulations. Inferential differences arise at the local level of solving individual problems. We can assess whether two compatible formulations are inferentially distinct by considering their shared class of problems. We need not appeal to differences in their fruitfulness or scope. Differences in fruitfulness are no doubt also epistemically significant, but conceptualism shows how they arise from local differences in inferential relations. It is in virtue of restructuring our solution procedures that some formulations become more fruitful than others for certain classes of problems. Differences in fruitfulness are not a reason for significance; they are a symptom.

Finally, conceptualism satisfies the third desideratum by proposing a criterion that is epistemically accessible. Differences in inferential relations are not empirically underdetermined. We learn about them simply by analyzing how various formulations support problem-solving. For instance, when we discover a new way to solve a problem, we learn that we don't need to know certain facts that an earlier formulation requires.

In virtue of our relatively easy access to inferential structure, conceptualism avoids the underdetermination problems that afflict fundamentalism. Even in a world where we are radically wrong about which formulation is more fundamental, we can identify inferential differences. In contrast, fundamentalism relies on differences in fruitfulness or other super-empirical virtues as *evidence* for metaphysical differences. This involves an inductively risky inference to the existence of underlying differences in fundamental

[46] For instance, in first-order logic, perhaps the existential quantifier is more fundamental than the universal quantifier, even though they are inter-definable using negation. See Sider, *Writing the Book of the World*, *op. cit.*, pp. 217–18 for discussion.

[47] Framed in terms of syntactic symmetries, the inferential structures or plans that two trivial notational variants provide are *invariant* under the reformulation.

structure. Conceptualism provides a method for appraising reformulations that avoids these risky inferences. Even anti-realists about physical unobservables can recognize inferential differences between formulations.

Moreover, like fundamentalism, conceptualism shows how significant reformulations have final epistemic value, as opposed to being merely instrumentally valuable for other epistemic or practical aims. First, as Section 6.1 argues, intellectually significant reformulations constitute a clarification of inferential structure. Second, improved understanding of inferential structure provides improved understanding of the theory or problem-solving strategy, since part of what it is to understand a theory is to know what is necessary or sufficient for applying the theory to solve problems.[48] Clearly then, significant reformulations improve our understanding of a theory. Finally, to the extent that a theory or problem-solving procedure is about worldly phenomena, an improved understanding of a theory leads to improved understanding of the relevant phenomena. For instance, someone who improves their understanding of classical mechanics thereby comes to better understand classical phenomena such as pendula. In this way, significant reformulations improve our understanding of the world, constituting a kind of scientific progress.[49] Regarding logic or mathematics, the formulation or problem-solving approach may itself be 'the phenomena,' e.g. in the event that nothing in reality stands behind these domains of inquiry. Significant reformulations in logic or math would thereby improve our understanding of these disciplines directly, constituting logical or mathematical progress.[50]

Scientific realists, fundamentalists, and others may hanker for a metaphysically deeper account of inferential structure. They may seek to ground inferential differences in explanatory differences or differences in fundamental structure. For instance, perhaps some inferential differences correspond to differences in what information is explanatorily relevant to the problem-solution. According to this explanationist proposal, information that *we do not need* to solve a problem is explanatorily irrelevant. In the case of the bird and the trains, we do not need to know the detailed trajectory that the bird takes. Some may therefore be inclined to say that such details are explanatorily irrelevant. However, one difficulty with this inference is that it takes us from considering the inferential structure of a formulation to considering more contentious explanatory relations in the world. Philosophers who endorse causal-mechanical accounts of explanation may have a different intuition. From a causal-mechanical standpoint, the bird's detailed trajectory explains the distance it travels. It remains explanatorily relevant, despite the fact that we do not need to know it in order to solve certain problems. Conceptualism shows that we can positively assess reformulations without resolving these kinds of explanatory disputes. Section 7 considers explanationism in more detail.

Nothing prevents philosophers with a more optimistic view of theoretical virtues from making further inferences about physical or metaphysical facts that ground infer-

[48] For instance, de Regt equates understanding a theory with "being able to use the theory" (Henk de Regt, *Understanding Scientific Understanding* (Oxford: Oxford University Press, 2017), p. 23).

[49] By considering compatible reformulations of the same explanation, Hunt, "Understanding and Equivalent Reformulations," *op. cit.*, p. 819 strengthens this argument to encompass differences in understanding why phenomena occur (a.k.a. 'explanatory understanding'). I argue that such cases can provide differences in understanding-why while nevertheless representing the same explanation (and hence being explanatorily on a par).

[50] See Pérez Carballo, "Structuring Logical Space," *op. cit.* for a non-factualist approach to mathematics. Here, I intend to remain neutral on metaphysical questions of mathematical or logical ontology. If mathematical Platonism is true, then the case of mathematical progress parallels the case of scientific progress.

ential structure. They are welcome to do so if so inclined. Nevertheless, these additional commitments preclude fundamentalism and non-deflationary versions of explanationism from providing a metaphysically minimal account of reformulations, based on epistemically accessible resources. If instrumentalism could meet the first two desiderata, it would already provide a minimal account. But as it stands, instrumentalism is inadequate. At the other extreme, fundamentalism commits us to more than is necessary. Conceptualism, I have argued, is just right.

6.3 Sameness of inferential structure

My argument in Section 6.2 assumes that paradigmatic cases of trivial notational variants have the same inferential structure, i.e. they are *inferentially equivalent*. To characterize inferential equivalence, one must accommodate the following difficulty: even obvious cases of trivial notational variants require knowing slightly different things—in some sense of "different"—simply because they involve different notation. For instance, to solve a problem using a left-handed coordinate system, one needs to understand the relevant convention. This left-handed convention is *ipso facto* different than that of a right-handed convention. To take a linguistic analogy, knowing what "dogs bark" means requires knowing some English, while knowing what the synonymous expression "die Hunde bellen" means requires knowing some German.

Understanding any sentence requires understanding a notation. Yet the notation does not thereby become part of the content of the sentence. Whatever inferential differences exist between trivial notational variants, they wholly concern the notation rather than what we use the notation to represent or accomplish. These reflections motivate an *aboutness criterion* for inferential equivalence: two formulations are inferentially equivalent just in case any differences in what an agent needs to know to apply the formulations are merely *about the notation*, rather than about either the content of the formulations or what the formulations allow us to achieve. Notation is a vehicle for communicating content, not the content itself.[51]

Complementing this appeal to aboutness, we can characterize inferential equivalence using an account of synonymy of meaning. Different accounts of meaning may yield different accounts of inferential equivalence. For my purposes, it suffices to illustrate one such account, showing that it is possible to provide a principled distinction between trivial notational variants and significant reformulations. Since my account relies on an equivalence between problem-solving plans, it is natural to use Gibbard's account of meaning, based on a similar notion of planning.[52]

Following Gibbard, we can understand the synonymy of "dogs bark" and "die Hunde bellen" as follows. Both sentences voice the same thought, which we can denote either as DOGS BARK or DIE HUNDE BELLEN. To believe the sentences are synonymous (in a given situation) is simply to plan to use "dogs bark" if I am an English speaker in those situations that I would plan to use "die Hunde bellen" if I were a German speaker, and vice versa. Hence, synonymy of meaning amounts to equivalence of plans. Although

[51] See North, *Physics, Structure, and Reality*, *op. cit.*, pp. 20, 32 for a related discussion of conventions or arbitrary choices in terms of aboutness. To make this criterion more precise, one could apply Yablo's account of aboutness: Stephen Yablo, *Aboutness* (Princeton: Princeton University Press, 2014).

[52] Allan Gibbard, *Meaning and Normativity* (Oxford: Oxford University Press, 2012)

the English and German speaker know different languages, once we abstract away these linguistic differences, they know the same thing, namely the thought that DOGS BARK.

Applying Gibbard's account of synonymy to inferential structure supports the conceptualist criterion I have been defending. Consider a problem that one can solve using either a left-handed or a right-handed coordinate convention. In the left-handed case, I appeal to an inferential relation (IR) expressed in the left-handed convention, denoted 'IR$_{left}$.' In the right-handed case, I appeal to an IR expressed in the right-handed convention, denoted 'IR$_{right}$.' IR$_{left}$ and IR$_{right}$ voice the same inferential relation provided that when working in a left-handed convention, I plan to use IR$_{left}$ in the same situations as I would plan to use IR$_{right}$ if I were working in a right-handed convention. Hence, although I technically need to know something different to work with IR$_{left}$ rather than with IR$_{right}$ (and vice versa), these two inferential relations are synonymous. The difference in what I need to know is wholly about my notation, rather than a genuine inferential difference in problem-solving plan.

Unlike trivial notational variants, significant reformulations provide different plans for solving problems. Ultimately, this is borne out as a difference in the inferential relations that they exploit or make available. For example, in the bird–train problem, someone using the hard formulation needs to determine the distance the bird travels on its first segment, second segment, etc. (or, alternatively, the time spent on each segment). They then need to know how to sum the distance on these segments, relying on an inference rule for summing an infinite geometric series. An agent following the easy formulation does not need to determine this information, nor rely on this inference rule. This is a genuine difference in the inferential structure of these problem-solving plans.

7 Problems with Explanationism

I have argued that conceptualism meets the three desiderata laid out in Section 3. It provides a middle ground between instrumentalism and fundamentalism about reformulations. Of course, other intermediate positions might meet these three desiderata as well. *Prima facie*, one approach that seems attractive involves tracking putative differences in explanation. Perhaps two compatible formulations are significantly different just in case they exhibit explanatory differences, such as by providing different explanations or suggesting different factors as being explanatorily (ir)relevant. I will call this schematic proposal *explanationism*. It satisfies the first desideratum by holding that trivial notational variants are explanatorily on a par, whereas significant reformulations manifest explanatory differences. Provided that these explanatory differences are local and epistemically accessible, explanationism will meet the second and third desiderata as well. In this section, I argue that conceptualism has important advantages over explanationism. In particular, conceptualism characterizes the epistemic differences between reformulations without taking a stand on the contentious topics of scientific and mathematical explanation.

Whether or not two compatible formulations have an explanatory difference depends on the nature of explanation. Different accounts of explanation give diametrically opposed verdicts on the simple examples that we have considered. Hempel's deductive–nomological account would characterize both the easy and hard approaches to the bird–train problem as equally explanatory: both appeal to the same law-like statement (distance as a function of rate and time), the same initial conditions, and provide equally rigorous derivations of the explanandum.[53] Hence, a Hempelian explanationist would have to view these as trivial notational variants. On a causal–mechanical account of explanation, the hard approach to the bird–train problem might be viewed as more explanatory, since it explicitly tracks additional causal details. Likewise for the Coulomb's law approach to calculating electric flux, since this approach explicitly calculates the electric field from each individual charge. A unificationist account of explanation suggests the opposite verdict: by eliminating reference to these additional causal details, the simple approach to the bird–train problem and the Gauss's law approach both apply to a wider range of phenomena.[54] Despite combining aspects of causal and unificationist approaches, Strevens' kairetic account of explanation would agree with the unificationist verdict here. Roughly, the kairetic account characterizes information that can be abstracted from a causal model—while still saving the phenomena—as being explanatorily irrelevant.[55]

These different verdicts illustrate the important role that philosophical assumptions play in assessing what information counts as explanatorily relevant. In contrast, we can recognize that one formulation does not require information that another requires, without presupposing further philosophical claims about explanation. Hence, we can discern more securely that reformulations display inferential differences than explanatory differences. Moving from a recognition of these inferential differences to claims about explanatory relevance requires further philosophical principles.

For instance, when we find out that knowledge of the distance traveled on each leg of the bird's journey is unnecessary for solving the bird–train problem, we might be tempted to infer that this information is explanatorily irrelevant. Doing so requires endorsing a philosophical principle like the following: contextually-unnecessary but causally efficacious information is explanatorily irrelevant. Proponents of causal–mechanical pictures of explanation may have different philosophical intuitions about whether this contextually-unnecessary information is explanatorily irrelevant. They might instead argue that tracking this information provides a deeper explanation, even if this deeper explanation is unnecessary for many purposes. My point here is a simple one: settling this sort of philosophical dispute is downstream from characterizing central epistemic differences between compatible formulations. We can account for many of the epistemic and methodological advantages of reformulations without settling these further questions about explanation or explanatory relevance.

Most accounts of explanation agree on at least one thing: explanations provide answers to why-questions.[56] Explanatory information describes the reasons why an event occurred or a fact is true. This aspect of explanation provides a second argument for

[53] Carl G. Hempel, "Aspects of Scientific Explanation," in *Aspects of Scientific Explanation, and other Essays in the Philosophy of Science* (New York: Free Press, 1965), 331–496

[54] For this traditional dialectic between causal-mechanical vs. unificationist accounts of explanation, see Wesley C. Salmon, "Scientific Explanation: Causation and Unification," in *Causality and Explanation* (Oxford: Oxford University Press, 1998), 68–78.

[55] Michael Strevens, *Depth: An Account of Scientific Explanation* (Cambridge, MA: Harvard University Press, 2008), p. 97. Hunt, "Epistemic Dependence and Understanding," *op. cit.* considers accounts of scientific explanation in more detail.

[56] This includes both irrealist frameworks such as van Fraassen's pragmatic account and realist approaches such as Skow's causal–grounding account of reasons why. See van Fraassen, *The Scientific Image*, *op. cit.* and Bradford Skow, *Reasons Why* (Oxford: Oxford University Press, 2016).

viewing explanatory differences as logically downstream from the epistemic differences that concern conceptualism. Logically, why-questions form a proper subset of a larger category of scientific and mathematical questions. Not all problems take the form of requests for explanatory information or reasons why, as Toulmin noted decades ago.[57] Hence, not all problem-solving procedures provide explanations, even if they succeed at providing solutions. Questions about whether a solution procedure is explanatory typically go beyond whether it provides the correct solution. We see this, for instance, in the case of mathematics: a rigorous proof of a mathematical theorem may not count as explanatory. For instance, Lange argues that proofs by mathematical induction often fail to be explanatory.[58]

In privileging conceptualism over explanationism, I do not deny that philosophical questions about explanation and explanatory relevance are important. My point is merely that various versions of explanationism could agree with my conceptualist analysis of reformulations, while disagreeing about the nature of explanation. Conceptualism is thereby better suited to provide a minimal core for an account of reformulations. Having adopted this minimal core, one can then defend further philosophical principles about explanation and explanatory relevance.[59] In this way, my complaint against explanationism is similar to my complaint against fundamentalism: to assess important epistemic differences between reformulations, explanationism has to presuppose more than is necessary.

8 Conclusion

Conceptualism holds that reformulations can possess non-instrumental value simply by clarifying what we need to know to solve problems. Significant reformulations provide inferentially different plans for solving the same problem or class of problems. This is in contrast to trivial notational variants, which do not alter what we need to know to solve problems. I have shown how conceptualism provides a middle ground between instrumentalism and fundamentalism, preserving the positive features of those accounts while avoiding their drawbacks.

Cast in terms of an aim of inquiry, conceptualism suggests that inquirers ideally ought to clarify what they need to know to solve the problems that interest them. Doing so provides a kind of non-practical epistemic value, which we might call *intellectual value*. Intellectual value is importantly different from the kinds of practical value that can arise from good notational choices. For instance, I might have a strong preference for a right-handed coordinate convention, or for reading words that run left to right rather than right to left. I might be considerably faster or more reliable with one notation than the other. Such practical differences may strike some as being genuinely epistemic, an issue that I have left open here. Regardless, they are importantly different from the kinds of non-practical epistemic differences that significant reformulations provide. Such dif-

[57] Stephen Toulmin, *Foresight and Understanding: An Enquiry into the Aims of Science* (Bloomington: Indiana University Press, 1961), p. 21

[58] Marc Lange, "Why Proofs by Mathematical Induction are Generally not Explanatory," *Analysis* 69, no. 2 (2009): 203–11

[59] Elsewhere, I use philosophical disagreements about explanatory relevance to motivate an expressivist account of explanation: Josh Hunt, "Expressivism about Explanatory Relevance," *Philosophical Studies*, 2022,

ferences in problem-solving plans exist independently of anyone's preferences, comfort-level, speed, or risk of error. They exist even for ideal computers.

Of course, if one computer yields a solution in five minutes while another takes two weeks, that is practically important for belief-formation. However, if the two computers implement the same problem-solving plan (e.g. program), then there is no intellectually significant difference between them. Conceptualism neatly captures this distinction between practical and non-practical dimensions of epistemic value. By figuring out what we need to know to solve problems, we enhance our understanding of reality.

One might worry that this improved understanding requires that compatible reformulations track underlying differences in explanation or fundamentality. If so, then the value of reformulating would collapse into either explanationist or fundamentalist value. To rebut this objection, it suffices to note that this kind of intellectual improvement need not involve greater knowledge of either fundamental structure or explanatorily (ir)relevant factors, as I have argued in Sections 5 and 7. An intellectually significant reformulation may be equally fundamental as a prior formulation, and it may provide the same explanation.[60] Hence, the kind of epistemic value that conceptualism focuses on floats free from the sources of value highlighted by explanationism and fundamentalism. The intellectual value of reformulating does not require downstream or additional benefits such as greater fruitfulness, more accurate explanations, or more fundamental descriptions.

8.1 A Final Illustration: 'Pictures' of Quantum Mechanics

One final example provides an illuminating summary of Section 1's spectrum of responses to the value of reformulating. In nonrelativistic quantum mechanics, a family of reformulations arises that are distinct from both (i) competing interpretations (such as Bohmian Mechanics vs. Many Worlds) and (ii) compatible formulations such as wave mechanics vs. path integrals. Known as different 'pictures' or representations of quantum mechanics, this plethora of more fine-grained formulations arises even just in the context of wave mechanics. They include the Schrödinger, Heisenberg, and interaction pictures.[61]

Like frames of reference in classical mechanics, these pictures differ in how they encode time-dependency of physical observables, whether it be through (i) the wavefunction, (ii) operators, or (iii) some combination of these two. In the Schrödinger picture, the quantum wavefunction evolves with time, while operators corresponding to observables remain constant. The Heisenberg picture swaps the time-dependency of wavefunctions and operators, treating the quantum wavefunction as constant, while operators evolve in time.[62] Construed literally, these pictures seem to evoke rival ontologies, as North highlights.[63]

Yet as noted at the outset, the value of reformulating is not particularly puzzling in the

[60] In providing the same explanation, compatible reformulations can still furnish different understandings of why the explanandum phenomenon occurred, as I argue in Hunt, "Understanding and Equivalent Reformulations," *op. cit.*

[61] I thank a referee for prompting me to illustrate my position using this example.

[62] In both pictures, a further choice of basis functions must also be made, such as the position or momentum basis for configuration space in the Schrödinger picture. These basis choices constitute further reformulations where intellectually significant differences can arise.

[63] North, *Physics, Structure, and Reality*, *op. cit.*, pp. 13, 219–20

case of incompatible formulations or rival theories. If we interpret these pictures as rival formulations of quantum mechanics, then clearly they are not trivial notational variants. Some pictures could be more approximately true than others, with perhaps one being closest to the truth and hence of greatest epistemic value. However, physics practice standardly treats these approaches as *compatible* formulations of quantum mechanics: one can freely choose which picture they wish to use in problem-solving, sometimes even changing pictures midstream. On the assumption that these pictures *are* compatible, it becomes interesting to ask whether they constitute significant reformulations, as opposed to being trivial notational variants.[64]

An instrumentalist about compatible formulations would argue that the value of these different pictures is merely instrumental for our practical and other epistemic ends. Casual remarks by physicists lend support to this instrumentalist appraisal, such as Bohm's remark that "Whether we use a Schrödinger or Heisenberg representation depends entirely on which is more convenient in the problem with which we are dealing."[65] Similarly, many introduce the interaction picture of quantum mechanics as particularly convenient for dealing with time-dependent perturbation theory.[66] It is uncontentious that these pictures are all at least empirically equivalent. No matter which picture one chooses to calculate in, one obtains the same empirical predictions, including probabilities for measurement outcomes and expectation values for observables.[67] Against instrumentalism's deflationary construal of these reformulations, fundamentalism, explanationism, and conceptualism all ask whether these different pictures of quantum mechanics might provide a kind of non-instrumental epistemic value.

For the fundamentalist, these pictures constitute significant reformulations provided they differ in how well they carve reality at the joints. One picture would have to represent or mirror the fundamental structure of the quantum world more closely than another. Perhaps a principled reason could be found for privileging one picture over others. However, one challenge to this approach is the existence of an infinite number of distinct representations, all related by unitary transformations. The mathematical and empirical equivalence of these representations suggests that metaphysically privileging one of them could be arbitrary. This situation illustrates the epistemic access worries developed in Section 5: fundamentalism makes the intellectual significance of these formulations hostage to epistemically inaccessible facts.

A structurally similar worry arises for explanationism. For the explanationist, different pictures of quantum mechanics qualify as significant reformulations only if they evince explanatory differences. Yet insofar as many are inclined to interpret these formulations as not only empirically but also physically equivalent, it is difficult to see what such explanatory differences could consist in. If they are physically equivalent, then they

[64] Additionally, anyone who favors interpreting these pictures as rival or incompatible formulations would need to tell an error theory for why physicists seem uninterested in choosing between them. In physics practice, one is not seen as embroiling themselves in contradiction or inconsistency if they use one picture in a particular context, while using a different picture in another.

[65] Bohm, *Quantum Theory*, *op. cit.*, p. 380

[66] Ramamurti Shankar, *Principles of Quantum Mechanics*, 2nd ed. (Springer, 1994), p. 486

[67] Brian C Hall, *Quantum Theory for Mathematicians*, vol. 267 (Springer, 2013), p. 79. This empirical equivalence follows from the pictures being *unitarily equivalent*, a kind of mathematical equivalence codified in the Stone–von Neumann theorem. For details, see Laura Ruetsche, *Interpreting Quantum Theories* (Oxford: Oxford University Press, 2011), Ch. 2.3.

provide the same reasons why for any physical event and hence the same explanation.[68] If there are underlying explanatory differences between these pictures, identifying them will plausibly involve additional philosophical commitments.

My aim is not to preclude either a fundamentalist or explanation-centric interpretation of reformulations, but merely to show that neither is needed to characterize intellectually significant differences in such cases. Various pictures may be equally fundamental and equally explanatory, while still providing different ways of understanding quantum mechanics. Conceptualism can account for intuitively significant differences between these pictures, without embroiling itself in controversial metaphysical or explanatory commitments. Following Section 6.1, different pictures of quantum mechanics constitute significant reformulations provided that they deploy different inferential structures for problem-solving. As I now detail, such differences are numerous.

To use a particular picture to calculate an expectation value of an observable, one needs to know the relevant equation that describes time evolution. In the Schrödinger picture, since the wavefunction evolves in time, one must calculate this evolution using the Schrödinger equation. In the Heisenberg picture, since operators evolve in time, one must calculate this evolution using a different equation, aptly called the Heisenberg equation.[69] Determining the time-evolution of the relevant quantity requires different calculations in the two pictures, constituting a difference in inferential structure.

As instrumentalism emphasizes, various pictures differ in convenience for certain problems. Conceptualism accommodates the intuition that there is more to these pictures than *mere* differences in calculational convenience. Many of these differences in convenience arise from intellectually significant differences. For instance, a particular picture might illuminate properties or relations that other pictures obscure. Perhaps most strikingly, the Heisenberg picture makes manifest a structural analogy between classical and quantum mechanics, known as Ehrenfest's theorem.[70] In the Heisenberg picture, the time derivative of quantum observables mirrors the time derivative structure of the corresponding classical observables. Similarly, in the context of time-dependent perturbation theory, the interaction picture makes manifest how the interacting part of a Hamiltonian accounts for the time evolution of states and operators.[71] This inferential difference underwrites the practical advantages the interaction picture affords for solving problems in this domain.

[68] For instance, Sklar notes that "Hardly anyone would deny that the Schrödinger and Heisenberg 'representations' are, indeed, representations of one and the same theory" (Sklar, "Saving the Noumena," *op. cit.*, p. 90). Similarly, Bohm notes that "as is always the case in a unitary transformation, we simply describe the same phenomena in a different language" (Bohm, *Quantum Theory*, *op. cit.*, p. 380).

[69] Where H is the Hamiltonian operator and $\psi(t)$ is the wavefunction in the Schrödinger picture, the time-dependent Schrödinger equation is $i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$. In contrast, where $A(t)$ is an operator in the Heisenberg picture, the Heisenberg equation of motion is $i\hbar\frac{d}{dt}A(t) = [A(t), H]$ (J. J. Sakurai and Jim Napolitano, *Modern Quantum Mechanics*, 2nd ed. (Boston: Addison-Wesley Press, 2011), p. 83). Here, '$[A(t), H]$' denotes the commutator between $A(t)$ and the Hamiltonian, structurally analogous to the Poisson bracket in classical mechanics.

[70] Ehrenfest's theorem relates the time evolution of an operator's expectation value to the expectation value of a commutator between that operator and the Hamiltonian: $\frac{d}{dt}\langle A\rangle = \langle [A, H]\rangle/(i\hbar) + \left\langle \frac{\partial A}{\partial t}\right\rangle$. For a time-independent operator, the $\frac{\partial A}{\partial t}$ term vanishes. In the Heisenberg picture, Ehrenfest's theorem follows almost immediately from the Heisenberg equation of motion, whereas its derivation from the Schrödinger equation is more involved.

[71] Shankar, *Principles of Quantum Mechanics*, *op. cit.*, pp. 490, 486

Here, I have focused on non-relativistic quantum mechanics. Yet, as is often the case with compatible reformulations, additional differences arise when we widen the domain of application. In the context of relativistic quantum mechanics, the Heisenberg picture makes manifest the Lorentz invariance of quantum fields, in virtue of treating space and time coordinates on equal footing. In contrast, the Schrödinger picture treats these coordinates differently, thereby obscuring Lorentz invariance.[72] Partly for this reason, many calculations in quantum field theory are more convenient using the Heisenberg picture. Again, while some may be inclined to take these differences as evidence for metaphysically privileging one picture over others, conceptualism shows that this metaphysical privileging is not necessary. We can account for intuitively significant differences between the pictures based simply on differences in the plans they provide for problem-solving. In this case as in others, conceptualism identifies when and how a compatible reformulation enhances our understanding.

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[72] In the Heisenberg picture, operators are functions of spacetime while wavefunction states are independent of spacetime, thereby treating space and time on a par. In contrast, these coordinates perform different roles in the Schrödinger picture, with wavefunctions depending only on time and operators depending only on spatial coordinates. See Steven Weinberg, *The Quantum Theory of Fields*, vol. 1 (Cambridge University Press, 1995), p. 109.

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