Frege and the Idea of Formal Language

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1. Introduction

I am supposed to talk about the relation between foundational studies and the philosophy of language.[1] After having wondered what could be the most important link between these two disciplines, I decided to devote my talk to the emergence of the idea of formal language.

It is well known that this idea has exerted great influence on the philosophy of language in this century. Take the concept of the logical form of a sentence: from early days of analytic philosophy until today, this concept has played a central role in the philosophy of language. There is no doubt that this concept of logical form could not arise without the concrete example of formal languages, in particular, a formal language for the standard logic. Or, we can take as another example a solution to the problem of creativity of language: the problem was to explain how the finite beings like us could produce and understand a possibly infinite number of sentences, and the widely accepted answer in the contemporary philosophy of language is to appeal to the recursive constructions allowed in a language. Again it is obvious that this answer comes from setting up formal language as a model of language in general.

Of course, it is also true that there have been many philosophers who argue, sometimes vehemently, against taking formal language as embodying the essence of our linguistic activity. Some emphasized an informal character of our linguistic exchange, some urged us to attend to the open-ended variety of linguistic activities we are engaged in, and some tried to pursuade us that the search for the essence of language is fundamentally misguided and consequently in vain.

However, today I will say no more about the merits and demerits of formal language as a conceptual tool in the philosophy of language. The subject I want to take up here is rather how the concept of formal language emerged from the considerations on the nature of mathematics in the beginning years of foundational studies.

It is often remarked that both the foundational studies in mathematics and the modern philosophy of language had their starts in Frege's great works. But this supposed common origin of the two disciplines should make us pause. How come

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such a fortunate coincidence? Or, is it not a coincidence that the two disciplines had their starts in the same person? It is not really obvious that the philosophy of mathematics and that of language should have much in common. What has mathematics to do with language? One might be tempted to say something similar to the following: of course, a language is indispensable, if we want to communicate the results of our mathematical activity; still, it is just a means for communication, not an essential element of our mathematical activity; there is no reason to expect something vital to mathematics will be revealed by the analysis of the language in which the results of mathematical activity are presented.

Nevertheless, I believe linguistic consideration plays an important part in understanding our mathematical activity. In particular, it gives us an essential first step towards understanding the nature of justification in mathematics. In the following, I wish to show how the idea of formal language has originated in Frege's efforts to gain such understanding.

2. "A formula language, modeled upon that of arithmetic..."

The full title of Frege's now famous booklet is, according to one English translation of it,[2] *Begriffsschrift, a formula language, modeled upon that of arithmetic, for pure thought*. Please note the reference to "a formula language of arithmetic" in this title. This means that at the very beginning of the foundational studies of this era, there was already the idea of approaching mathematics from a linguistic point of view. To investigate historically and document fully how such a point of view had originated and developed must be a vast undertaking. What I can offer here is only a speculation as to why a sort of linguistic approach to mathematics should seem natural and what lights it could bring to our understanding of the nature of mathematical justification.

Through the centuries, the use of the so-called mathematical symbols has been of paramount importance to mathematics. And, the rapid development of algebra since the Renaissance had made them even more important. It is customary to treat the entire class of these mathematical symbols as a special vocabulary added to everyday language. In fact, when we are doing informal mathematics, that is, when we are not working in a formal system, we recognize an intuitive distinction between mathematical symbols and words. This might suggest a view that mathematical symbols are not really a part of a language.[3]

[2] in Jean van Heijenoort (ed.), *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*. 1967, Harvard University Press.

[3] This view might get some support from the fact that in some languages such as Japanese mathematical symbols come from an entirely different set of characters. Also it should be noted that usually there are no need to translate mathematical

But at the same time, we can see that, among mathematical symbols, there are some that function like names, and some that function like verb phrases. The former class of symbols, in turn, can be divided into those like "0" and "π" that denote some definite mathematical object, and those like "+" and "cos" that denote a function or operation. The mathematical symbols which correspond to the verb phrases are mainly those which express some relation between mathematical objects. The typical examples are "$>$" (the relation of greater than) and "$\cong$" (the congruence relation between figures). If there are any mathematical symbols which have no obvious counterparts in everyday language, they are the so-called variables like "x" for real numbers and "$\triangle ABC$" for figures. If we don't make further distinctions among various kinds of function names or various sorts of variables, most of the mathematical symbols we come across at an elmentary level may belong to one or the other of these three categories, namely, the names for objects or functions, the relational expressions, and the variables. This observation lends a support to a claim that mathematical symbols form a sort of language and that they have a discernible grammatical structure just as an ordinary language has.

If it is justified to see a mathematical symbol as a word just like a word we encounter in our everyday language, then it will be equally justified to see a mathematical formula, which consists wholly of mathematical symbols, as a sentence like an ordinary sentence in our language.

Still, an ordinary mathematical prose is not just a sequence of mathematical formulas, nor consists wholly of mathematical symbols. In it, there are also ordinary-looking words like "real number", "vector", or "prime", and perfectly ordinary words like "if", "not", or "every". If we say that mathematical symbols are words, and mathematical formulas are sentences, then the next bold step would be to claim that those ordinary and ordinary-looking words that occur in a mathematical prose are just another sort of symbols and the sentences formed from these words are just another sort of formulas. Further, if we want to emphasize the equally symbolic nature of these words, it might be a good idea to replace them by some artificial symbols that look similar to mathematical symbols. Thus we can reach a wholly symbolized autonomous language for mathematics.

3. The demand for a "gapless" proof

Though it may be an interesting conceptual discovery that mathematical

symbols from one language to another. Even in the case of proper names, which some philosophers have argued are not a part of any language, some translation is necessary. (A possible objection : though a written symbol need not be translated, i.e. replaced by another symbol, the mathematical symbol used in the language as spoken has to be "translated"!)

symbols together with a portion of common language form a language-like structure, this idea of "symbolizing" mathematics seems to be no more than a curious proposal for writing mathematics in a cipher-like new notation. In the hands of Frege, however, this became not just a proposal of a new notation, but a call for an adoption of a new kind of language, which was based on a consideration of the structure and mechanism of language in general.[4]

How did it come about? I believe the answer to this question belongs to a piece of common wisdom now. It's because Frege's chief interest in analyzing the language used in mathematics was to represent a mathematical proof in a rigorous way.

However, it is very important to make clear what "rigor" should mean here. Frequently Frege speaks of the need of a "gapless" proof for a complete justification of a mathematical statement. But, what makes one proof full of gaps and another without a gap? What is the criterion for a proof to be a gapless proof? If the soundness of a deduction is all that matters, it seems enough that at each step the conclusion follows validly from the premises. To this, one might object that it is never enough for the premises to entail the conclusion, because what is at issue here is not just the soundness of the argument but the justification it lends to the conclusion, and for the justificatory purpose the important thing is that each step of the deduction should be *recognizable* to be valid.

This is perfectly just. Let us imagine that we are presented with a proof of some mathematical statement; what makes us feel that there is a gap in the proof, if we do? Perhaps there are some spots in the proof where we are not sure what justifies the transition from one statement to another. Or we might say in these spots we miss some general rule that justifies the transition, because there is no explicit invocation of such a rule nor we can guess at once what it should be. Then,

[4] What differences are there between a notation and a language? This question seems to be a pressing one because "*Begriffsschrift*" is sometimes translated as "conceptual *notation*". However, we have also a more literal translation in "concept-script" or "ideography".
Here I can make only a few tentative remarks. Firstly, although there is now a strong tendency in the formal studies to assimilate any notational differences to the differences of languages, there are some evident cases of the notational differences that do not amount to the difference of languages; such is the case with many variants of the language of classical propositional logic. Secondly, it seems a necessary condition for two linguistic forms being a notational variant of each other that they have the same expressive power. Does Frege's system—to which he refers as a *Formelsprache*, a formula-language—have the same expressive power as the language used in an ordinary mathematical prose? I think the answer is negative. For one thing, in a mathematical prose there are various sorts of remarks that are no part of any proof (the remarks about history of the subject, the remarks about possible applications, and so on); even a proof itself usually contains some remarks about proof strategy and related matters.

one necessary condition for a proof to be gapless would be that it is made clear for each step of the proof what general rule is appealed to. Of course, these general rules must be recognizable to be valid, if the proof should be both sound and justificatory.

Now there are two possibilities here: either we think it is enough that for each transition in the deduction we can supply a justification in the form of an explicit statement of a general rule, or we require that there must exist a common fund of the general rules which we can specify all at once and from which the justification of any move in the proof should come. In the former case, theoretically at least, there are no limits to the range of justificatory means available to us; we could add a new principle for mathematical reasoning as we go along, if it would be recognized to be a valid one. In the latter, we have a closed set of justificatory means; even though it is possible to have an infinite number of the general rules that could be derived from the rules in the original set, every justificatory move should be reducible to the basic principles specified once and for all.[5]

However, apart from these two possibilities, there is another path well worth pursuing, namely, to see to what extent we can go with some preferred set of the principles for reasoning in various parts of mathematics. This sort of attempts do not presuppose that there is a single set of general principles for reasoning in the entire mathematics. Still, we have to specify at the outset what principles are allowed in the proofs of the relevant part of mathematics. If some specific set of reasoning principles are shown to be sufficient to the proofs in a branch of mathematics, that will teach us something about the nature of that part of mathematics.

You may realize this is just the aim of various axiomatizations of mathematical theories from Euclid on. And, I think, this is also the path Frege took with regard to "arithmetic", which meant in his case the theory of numbers in general, including the real numbers as well as the natural numbers. But, he went further than anyone before him in that he was not content with identifying only those principles of reasoning that are specifically mathematical. One of his aims was to determine the nature of arithmetical reasoning, and his well-known answers are that there is no intrinsically arithmetic reasoning, and that the so-called arithmetical reasoning is nothing but logical reasoning. In trying to establish this, Frege became the first logician that succeeded in explicitly formulating the basic principles of pure logical reasoning adequate for mathematical proof.

Then, we can see there could be another motive to require of a proof to be "gapless"; even if we are certain of the validity of the transition from one statement to another, it might be possible that we are not sure about the nature of the

[5] As will be mentioned later, it is now customary to distinguish mathematical axioms from logical inference rules. Here and in the following, "the principles of mathematical reasoning" are intended to comprise both of them.

transition because we don't know whether the principle used there could be reduced to some more basic principles of another nature. In particular, if we want to show that a specific kind of reasoning is sufficient for a certain branch of mathematics, it is crucial to determine whether each transition is reducible to the specified range of the rules for reasoning. In fact, when Frege speaks of "gapless proofs", what is at issue is almost always to determine whether the transition is purely logical or involves something extra-logical—which Frege called "something intuitive (Anschauliches)".[6] As a typical example, let me cite a passage from *The Foundations of Arithmetic* :

> ... the mathematician rests content if every transition to a fresh judgement is self-evidently correct, without enquiring into the nature of this self-evidence, whether it is logical or intuitive. (§ 90 — Austin translation)[7]

So, the collection of the basic principles of logical reasoning should be specified once and for all, and it must be shown that any logical transition (and, in the case of Frege, any transition in arithmetical proofs) is reducible to these basic principles. In what manner should this specification of the fundamental principles of logic be done ? Frege demanded that this should be done in such a manner as to exclude every conceivable misinterpretation or misapplication ; he demanded that these principles be specified in reference to only the outer forms of the symbols we use in the representation of a proof.

4. The birth of a formal language

Now, at last, I can bring together the two threads I think we can detect in Frege's approach to the foundations of mathematics : one is a sort of linguistic point of view which sees the mathematical symbolism, in particular, the algebraic symbolism, as an autonoumous language, and the other is the demand for a mathematical proof to be "gapless".

As we saw, in Frege's case the latter demand was intimately connected with his "logicist" program for the theory of numbers in general. Accordingly, he tried to reduce any step in an arithmetical proof to a number of steps that can be sanctioned by the basic principles of logic alone. In his formulation of them, these principles

[6] In Frege's work, there is another motivation to reduce arithmetical reasoning to logical one : he wanted to secure the most reliable foundation for arithmetic, and believed that "[t]he most reliable way of carrying out a proof...is to follow pure logic" (Preface to *Begriffsschrift*, 1st paragraph. J.v.Heijenoort (ed.), *Op. cit.* p. 5).

[7] Among other places, see in particular the first paragraph of the Preface to *Begriffsschrift*. Also see "Über die wissenschaftliche Berechting einer Begriffsschrift" (1882).

were divided into two classes, namely, logical axioms and logical inference rules.

I think it is advisable to make two changes here in order to get a clearer perspective on Frege's achievement. Firstly, we need not sympathize with Fregean logicism ; if we want to investigate the nature of mathematical reasoning, there are no need to limit ourselves to purely logical principles of reasoning. It goes without saying that logical principles play an essential part in any proof in mathematics, and perhaps it may be rush to say that arithmetic cannot be reduced to logic after all ; but, a more liberal attitude has a clear advantage. Secondly, if we want to have intrinsically mathematical principles of reasoning as well as logical ones, it is best to formulate the latter as inference rules. Logic has been traditionally concerned with inference, and, as many authors have pointed out,[8] it was unfortunate that the pioneers of modern logic presented their systems as systems of logical truths codified in the form of logical axioms. In the case of intrinsically mathematical principles of reasoning, it doesn't matter whether we formulate them as axioms or inference rules ; the principle of mathematical induction can be formulated as a rule of inference as well as an axiom (or an axiom scheme), and there seems to be no reason to prefer one to the other.

Then, what we have to do in the first place for an analysis of the proofs in a certain branch of mathematics is to identify the basic principles of reasoning in it. They consist of basic logical principles formulated as inference rules, and basic mathematical principles specified as axioms or rules. Next, every relevant proof has to be reformulated in such a way that we can identify at once what principles are used at each step in the proof. Now, what guarantees the immediate identification of the principles involved ? It is not enough that there is an explicit reference to the principles involved ; for, even then, we may have to rely on another piece of inference in order to ascertain that the principles indicated are actually employed at that point of the proof ; this is a familiar phenomenon in the case of informal proofs, where sometimes the principles indicated are not employed in the originally stated form but in some logically equivalent form.

How can we stop this sort of regress in the identification of the principles involved at a step of a proof ? I think Frege's way of representing a proof in his "formula language" can be seen as an attempt to stop the regress at the level of linguistic forms actually used in a representation of a proof. If from linguistic forms alone it is possible to determine whether a given principle is employed or not, then what is needed is only a kind of syntactic ability, the ability to discern a pattern in linguistic forms. Thus, at the beginning of Part II of *Begriffsschrift* Frege says, immediately after the exposition of his "formula language" :

[8] For example, M.Dummett, *Frege : Philosophy of Language.* 1973, Duckworth. pp. 432f.

> We have already introduced a number of fundamental principles of thought in the first chapter in order to transform them into rules for the use of our signs.[9]

However, anything that deserves the name of a rule must have general applicability, and a rule for the use of signs is no exception. In the case of the latter, general applicability is made possible by a schematic character of the rule or some extra rule for substitution. Take *modus ponens* as an example; we usually state it as a schematic rule

$$\frac{\begin{array}{l} A \\ \text{If } A \text{ then } B \end{array}}{B}$$

To determine whether a given step is an application of this rule, we have to know at least the following: (1) that there is a sentence having the form of a conditional, (2) that among the premises of the step there is a sentence that is identical with the antecedent of the conditional, (3) that the conclusion of the step is the same as the consequent of the conditional. It is by no means a mechanical matter to establish these facts, if we are operating within everyday language; for, in it, there are many ways to express a conditional, and in some languages a sentence must undergo a certain modification if it is to be embededded within a complex sentence.

This may mean that some regimentation of everyday language is inevitable if we want to transform the basic principles of reasoning into rules for the use of signs as Frege says. But a better course is to invent a new language in which the recognition of an application of a rule can be immediate. What properties should such a language have? As we can see from the simple example above, a schematic rule presupposes that there is a certain range of expressions that can be put in for a schematic letter. Or we can say with each schematic letter a certain grammatical category is associated; in the example above, if we want to be explicit, we should say "for arbitrary sentences A and B" and there should be no room for doubt as to which expression counts as a sentence. Moreover, we must be able to recover the grammatical components relevant to the application of the principles of reasoning; in the example above again, we must be able to determine uniquely the antecedent and consequent of a conditional sentence. In general, the language should have the principles of grammatical constructions by means of which we can recognize the structure of a sentence relevant to the application of the principles of reasoning.

In Frege's *Begriffsschrift*, we can find such an idea of formal language, the exact nature of which was to be finally clarified through the works of Gödel, Church, Kleene and Turing in the 1930's. The modern logic started by Frege is sometimes

[9] *Begriffsschrift* §13. (J.van Heijenoort (ed.) *Op. cit.* p. 28.)

called by the name "symbolic logic", but the use of symbols, or schematic letters, is not the defining characteristic of it; the use of them is almost inevitable if logic is to be a systematic study of inferences. What distinguishes the modern logic from the earlier logic lies rather in the fact that the former studies the inferences in everyday language through their counterparts in formal language. And I think this has been a fruitful method in studying the inferences in mathematics, too.