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Some highs and lows of hylomorphism: on a paradox about property abstraction

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Abstract We defend hylomorphism against Maegan Fairchild's purported proof of its inconsistency. We provide a deduction of a contradiction from *SH+*, which is the combination of "simple hylomorphism" and an innocuous premise. We show that the deduction, reminiscent of Russell's Paradox, is proof-theoretically valid in classical higher-order logic and invokes an impredicatively defined property. We provide a proof that *SH+* is nevertheless consistent in a free higher-order logic. It is shown that the unrestricted comprehension principle of property abstraction on which the purported proof of inconsistency relies is analogous to naïve unrestricted set-theoretic comprehension. We conclude that logic imposes a restriction on property comprehension, a restriction that is satisfied by the ramified theory of types. By extension, our observations constitute defenses of theories that are structurally similar to *SH+*, such as the theory of singular propositions, against similar purported disproofs.

Keywords Appendix B Paradox · Free logic · Higher-order logic · Hylomorphism · Impredicative definition · Property abstraction · Property comprehension · Ramified type theory · Russell-Myhill Paradox · Russell's Paradox

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1 A purported proof that simple hylomorphism is inconsistent

In a recent article Maegan Fairchild (2017) argues by means of what she considers a Russellian paradox that a particular metaphysical theory, *simple hylomorphism* (*SH*), is inconsistent. *SH* is what Fairchild deems a "minimal version" (p. 34) of contemporary hylomorphism. Advocates of contemporary hylomorphism include Fine (1982), Sosa (1999), Johnston (2006), Koslicki (2008) and Rea (2011). We here defend *SH* against Fairchild's purported disproof. We do not endorse *SH*. Indeed we find its central concept, *embodiment*, excessively unclear. However, *SH* is structurally sufficiently like other metaphysical theories according to which objects are composed (in part) of properties or concepts—such as the theory of property-involving or concept-involving propositions (which we do endorse) and certain theories of facts—that we believe our defense is of general interest quite independently of hylomorphism. Fairchild uses the "paradox" to argue that *SH* is inconsistent. (The argument, insofar as the theory from which it proceeds is not a deliverance of naïve intuition, is not a paradox or antinomy in the usual philosophical sense. However because some theories of property-involving composite objects do have the approval of naïve intuition, analogous arguments directed at those theories qualify as paradoxes in this sense.) We instead draw the very different conclusions that *SH*, which we show is consistent, is inconsistent with unrestricted property comprehension and that logic imposes a restriction.

SH posits for any individual *x* and any property F of *x*, an individual "*x qua* F"—symbolized '*x*/F'—that bears a special relation to F that is not instantiation. This special relation between the posited individual *x*/F and the property F is called 'embodiment'. Fairchild appears to adopt the following definition (p. 34 and p. 35*n*10):

$$y \textit{ embodies } \mathrm{F} =_{df} \exists x(y = x/\mathrm{F}).$$

The slash '/' (and likewise the word '*qua*') may be regarded as a term for the *qua*-function, which assigns to an individual *x* (the "base") and a property F (the "form") the corresponding *qua*-object, *x*/F, if there is one and is undefined otherwise.[1] Hylomorphists, although not of a single mind regarding the exact nature of the *qua*-function, are guided by a common thought. As Fairchild explains (pp. 33–34) the idea, it is roughly that a statue *s*, for example, is a *qua*-object *m*/F, where *m* is the matter that constitutes *s* and F is the property of being statue-shaped; furthermore *s* (that is, *m qua* statue-shaped)[2] embodies being statue-shaped insofar as *m*'s being statue-shaped "explains" *s*'s having the modal properties it has—for

[1] Fine (1982) is explicit that a *qua*-object exists only when the base has the form (p. 100). Fairchild does not specify what, if anything, ⌜α/Π⌝ is supposed to mean when α designates *x*, Π designates F, and *x* lacks F. Either way, the *qua*-function is undefined when *x* lacks F.

[2] The word '*qua*', which putatively designates a particular function from individuals and properties to *qua*-objects, is as we use it, an "indirect" (*ungerade*) operator in that the occurrence of 'Michelle's husband' in 'Barack *qua* Michelle's husband' designates not its default (or "customary") designatum, Barack, but its indirect designatum, the property of being Michelle's husband. The slash designates the very same function as '*qua*', but it is not an indirect operator. (In Fairchild's alternate use '*qua*', like '/', is also not indirect. Where we say 'Barack *qua* Michelle's husband' and '*m qua* statue-shaped', she would

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example, perhaps, *s*'s being essentially a statue (that is, being necessarily a statue if existent). For Fairchild's purposes as well as for our present purposes, the exact natures of the *qua*-object, the *qua*-function, and the embodiment relation are largely irrelevant.

Unless otherwise indicated we adopt logic of second order (or by an alternative count, logic of third order) with lambda-abstraction, under a Russellian intensional interpretation whereby the monadic-predicate variables 'F', 'G', etc. range over properties of the individuals over which the singular-term ("individual") variables '*x*', '*y*', and '*z*' range, rather than classes (or characteristic functions), and monadic-predicate constants designate such properties.[3] We assume that an expression consisting of the slash flanked by its argument expressions is a singular term. We use the definite descriptions operator '$\imath$' to form a designator of the thing that uniquely answers to the description. (These anti-Russellian choices with respect to terms formed by means of the slash and definite descriptions ease exposition, but do not materially affect our arguments.) A free logic is employed in connection with non-designating singular terms—saliently non-designating terms formed by means of the slash.[4]

Footnote 2 continued
say, deviating from Standard English-*cum*-Latin usage, 'Barack *qua* being Michelle's husband' and '*m qua* being statue-shaped').

[3] The symbol 'λ' is a variable-binding operator that forms a compound functional expression from an open expression. Strictly speaking, the lambda-abstract $\ulcorner\lambda\alpha[\xi_\alpha]\urcorner$ designates the function that assigns to any value of α the designatum of ξ_α under the assignment of that value to α. In the special case of an open formula ϕ_α, $\ulcorner\lambda\alpha[\phi_\alpha]\urcorner$ is a compound monadic predicate. For the purposes of this paper, we follow common practice in taking a monadic predicate to designate a property (namely, the one that corresponds to the relevant propositional function that is, strictly speaking, designated). In English, this corresponds to the formation of the compound monadic predicate $\ulcorner$is a thing such that $\phi_{it}\urcorner$ from the open formula ϕ_{it} in which the pronoun 'it' functions as a free variable, for example, the formation of 'is a thing such that it is snub-nosed and it is wise' (in short, 'is snub-nosed and wise') from 'it is snub-nosed and it is wise'. Such lambda-abstracts are governed by inference rules of *lambda-conversion*. In particular, *lambda-expansion* (property abstraction) permits the inference from the formula ϕ_β to $\ulcorner\lambda\alpha[\phi_\alpha]\beta\urcorner$, where ϕ_β is the result of uniformly substituting free occurrences of the singular term β for the free occurrences of the singular-term variable α in ϕ_α. In English, this corresponds to the inference from 'Socrates is snub-nosed and Socrates is wise' to 'Socrates is snub-nosed and wise'. Fairchild appears to treat the properties over which the monadic-predicate variables range as intensions, that is, as functions from possible worlds to extensions (p. 35*n*10). We do not here object to treating predicates as designating intensions. Given this understanding of the entities over which the monadic-predicate variables range, a lambda-abstract $\ulcorner\lambda\alpha[\phi_\alpha]\urcorner$ is definable by means of the second-order definite description $\ulcorner \imath\, \Pi\Box\forall\alpha(\Pi\alpha \leftrightarrow \phi_\alpha)\urcorner$ where Π is a monadic-predicate variable. Compare Church (1974a, p. 30). Alternatively, the definite-description operator '$\imath$' is definable in terms of lambda-abstraction together with an appropriate higher-level function. (So that, for example, $\ulcorner \imath\, \alpha\phi_\alpha)\urcorner$ is definable by means of $\ulcorner I\lambda\alpha[\phi_\alpha]\urcorner$ where 'I' is the function that assigns to any property the only individual that has that property if such exists, and is undefined otherwise.) Lambda-abstraction underlies all variable binding and is therefore more basic than definite-description formation.

[4] First-order free-logical UI ($\forall$-Elim) licenses the inference from $\ulcorner\forall\alpha\phi_\alpha\urcorner$ and the supplementary premise $\ulcorner\exists\gamma(\gamma = \beta)\urcorner$ to ϕ_β, where the variable γ does not occur free in the singular term β and ϕ_β is the result of uniformly substituting free occurrences of β for the free occurrences of the variable α in ϕ_α. First-order free-logical EG ($\exists$-Intro) licenses the inference from ϕ_β and the same supplementary premise $\ulcorner\exists\gamma(\gamma = \beta)\urcorner$ to $\ulcorner\exists\alpha\phi_\alpha\urcorner$. First-order free-logic also involves corresponding modifications of $\forall$-Intro and $\exists$-Elim. As we use the term here, *free second-order logic* analogously modifies the classical logic of the quantifiers to

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Fairchild's argument that *SH* is inconsistent derives a contradiction in a manner familiar from Russell's Paradox for sets (and similar paradoxes such as the Russell-Myhill Paradox for propositions). The derivation of the contradiction uses *SH* and the claim that there is a *qua*-object *q* and a property that *q* embodies and also lacks. It also relies on a comprehension schema for property abstraction.

The theory *SH* is given by two axioms:

Existence $\forall x \forall \mathrm{F}(\mathrm{F}x \rightarrow x/\mathrm{F} \text{ exists})$
Uniqueness $\forall x \forall y \forall \mathrm{F} \forall \mathrm{G}(x/\mathrm{F} = y/\mathrm{G} \leftrightarrow x = y \ \& \ \mathrm{F} =_2 \mathrm{G})$.[5]

The predicate 'exists' may be taken as defined by '$(\lambda x)[\exists y(x = y)]$'. The equality sign '$=_2$' is of the type of a binary relation both between properties of individuals (that is, between objects over which 'F' and 'G' range) and between propositions. The equality sign '=' is of the type of a binary relation between individuals whereas '$=_2$' is of higher type.

Fairchild invokes the following modal property-abstraction comprehension schema, which she takes to be a "liberal conception of properties" (pp. 35 *passim*):

MPC $\exists \mathrm{F} \Box \forall x(\mathrm{F}x \leftrightarrow \phi_x)$,

where ϕ_x is any formula in which 'F' does not occur (free). Fairchild cites the following instance:

$MPC_{\mathbf{S}}$ $\exists \mathrm{F} \Box \forall x[\mathrm{F}x \leftrightarrow \exists \mathrm{G}(x \text{ embodies G} \ \& \ {\sim}\mathrm{G}x)]$

The very purported property of being a *qua*-object such that there is a property it embodies and also lacks is one putative witness for (one truth-making instance of) this instance of *MPC*. We shall let '**S**' abbreviate '$\lambda y[\exists \mathrm{G}(y \text{ embodies G} \ \& \ {\sim}\mathrm{G}y)]$' throughout. If one understands by 'property' a modal intension (that is, a function from possible worlds to extensions), as Fairchild appears to do, **S** is *the* witness for $MPC_{\mathbf{S}}$.[6] Even on a finer-grained understanding of 'property', **S** is the *principal*

Footnote 4 continued
take account of monadic predicates that do not designate any element of the universe over which the monadic-predicate variables range.

[5] Reflection on *SH* reveals that the example by which Fairchild explains the guiding idea of hylomorphism is oversimplified. Let *m* be the marble that constitutes Michelangelo's *David* and let F be its specific shape. Even if *m qua* statue-shaped has F as David does, it, unlike *David*, presumably could have had a very different shape. So it is a mistake to identify *David* with *m qu*a statue-shaped. Compare Fine (1982), which identifies *Goliath* with *Goliath*-matter *qua Goliath*-shaped and not with *Goliath*-matter *qua* statue-shaped. Even the specific shape (or suitable variations thereon) is probably not enough. Even if *m*/F is so-shaped, it could have come to have F entirely by natural forces instead of by design. Arguably, *David* is essentially a sculpted statue not sculpted primarily by anyone other than Michelangelo. If so, then, not only is it incorrect to identify *David* with *m qua* statue-shaped, it is even incorrect to identify *David* with *m*/F. Compare Fairchild (2017, p. 34*n*5).

[6] It is by no means clear that hylomorphists are prepared to regard the form of a statue as a modal intension. On a more standard understanding of a property, and correspondingly of lambda-abstraction, there are infinitely many witnesses for an instance of *MPC*, all of which determine the same intension. For example, being a *qua*-object that both embodies a property it lacks and is such that the number one is odd is another witness for $MPC_{\mathbf{S}}$ distinct from **S**.

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witness. (Our discussion of **S** applies *mutatis mutandis* to any witness.) Thus if *MPC* is correct, then the following is true:

$C_1 \quad \forall x[\mathbf{S}x \leftrightarrow \exists \mathrm{G}(x \text{ embodies G} \ \& \ \sim \mathrm{G}x)]$

Although Fairchild cites *MPC* in support of C_1, her argument does not actually use the full force of *MPC*. Instead, it uses a non-modal schematic consequence of *MPC*, which suffices for the derivation of C_1:

$PC \quad \exists \mathrm{F}\forall x(\mathrm{F}x \leftrightarrow \phi_x)$

In particular, the argument utilizes the following instance:

$PC_{\mathbf{S}} \quad \exists \mathrm{F}\forall x[\mathrm{F}x \leftrightarrow \exists \mathrm{G}(x \text{ embodies G} \ \& \ \sim \mathrm{G}x)]$

C_1 is also provable using classical lambda-conversion instead of *PC*. (See note 3.) Assuming for the time being that **S** is a genuine property we shall also say that a *qua*-object is a *stone-caster* iff it has **S**. We borrow this terminology from Salmón (forthcoming), who uses 'stone-caster' as a term for a singular proposition that predicates a property that the proposition itself lacks. The term 'stone-caster' is less apt for *qua*-objects than it is for singular propositions of the sort relevant to the Russell-Myhill Paradox; however we think it is salutary to bear in mind a striking similarity between *qua*-objects and true singular propositions (or singular facts, especially in contrast to concrete objects like statues).[7]

Fairchild provides forceful considerations in favor of the claim that (the *SH* proponent should hold that) there is a *qua*-object *q* and a property that *q* embodies and also lacks (p. 36*n*13).[8] The *qua*-object *q* is then a stone-caster. Thus by *Existence*, *q qua* stone-caster (that is, *q*/**S**) exists. Instantiating the universal quantifier in C_1 we obtain the following:

$C_2 \quad \mathbf{S}q/\mathbf{S} \leftrightarrow \exists \mathrm{G}[q/\mathbf{S} \text{ embodies G} \ \& \ \sim \mathrm{G}q/\mathbf{S}]$

The *qua*-object *q qua* stone-caster is either itself a stone-caster, or else it is not. Which is it? Suppose *q qua* stone-caster is a stone-caster, that is, suppose **S***q*/**S**. By *Uniqueness*, *q qua* stone-caster embodies only one property, **S**. Thus, since (by left-to-right C_2) it embodies a property it lacks, *q qua* stone-caster lacks **S**. Therefore *q*

[7] The term 'stone-caster' derives from John 8:7: "Let s/he who is without sin cast the first stone." We shall later question the legitimacy of the definition of the term 'stone-caster' as a term for a special kind of *qua*-object. This will have the effect of also questioning much of the discussion to follow that incorporates 'stone-caster' as potentially meaningless. "My propositions serve as elucidations in the following way: anyone who understands me eventually recognizes them as nonsensical, when he has used them—as steps—to climb up beyond them. (He must, so to speak, throw away the ladder after he has climbed up it.) He must transcend these propositions, and then he will see the world aright" (Wittgenstein [1922] 2018, 6.54).

[8] Fairchild gives an intricate argument for the claim. We offer (without hereby endorsing) instead a simple alternative: if there is any non-*qua*-object *m*, then there is a property (not being a *qua*-object) that *m qua* non-*qua*-object (that is, *m*/not being a *qua*-object) embodies and also lacks. Where our alternative relies on there being an instantiated property of not being a *qua*-object (in conjunction with *Existence*), her argument relies on the plausible assumptions that there are at least two individuals, *a* and *b*, and that there are at least three properties, one had only by *a*, one had only by *b*, and one had only by *a* and *b* (in conjunction with *Existence* and *Uniqueness*). Both arguments are compelling. Fairchild's argument will be more appealing than our alternative to an *SH* proponent who, like Sosa (1999), thinks that it is mere dogma to suppose that there are non-*qua*-objects (p. 141).

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qua stone-caster is not a stone-caster, that is, (1) $\sim\mathbf{S}q/\mathbf{S}$. Being a non-stone-caster, *q qua* stone-caster has every property it embodies (by right-to-left C_2). Thus, since *q qua* stone-caster embodies **S**, it has **S**. Therefore *q qua* stone-caster is a stone-caster, that is, (2) $\mathbf{S}q/\mathbf{S}$. (1) and (2) are a contradiction.[9] Fairchild concludes that *SH* is inconsistent.

Fairchild's conclusion is overstated. A more fitting conclusion faithful to the spirit of her position is that *SH* is inconsistent with the exceedingly plausible assumption that if *SH* is true, then there is a *qua*-object *q* and a property that *q* embodies and also lacks. (See note 8.) Given the plausibility of this conditional, we may regard its consequent as a minor supplement to *SH*. Let *SH+* be *SH* conjoined with this supplement. We may then charitably understand Fairchild as purporting to prove that *SH+* is inconsistent.

Fairchild's derivation demonstrates that *SH+* is inconsistent with $PC_{\mathbf{S}}$, and hence with *PC*. That is, instead of proving that $\vdash\ \sim SH$, Fairchild's argument proves that $\vdash PC_{\mathbf{S}} \rightarrow\ \sim SH+$. Fairchild's argument relies on *PC* as a product of a liberal theory of properties, but to serve her objective of proving *SH+* inconsistent, *PC* must be not merely a product of a theory of properties; it must be a valid schema of logic. And in fact, *PC* is sometimes taken as an axiom schema of second-order logic. Alternatively, *PC* is provable in classical second-order logic using a rule of substitution for monadic-predicate variables or using classical lambda-conversion. One way or another, *PC* is in fact a valid schema of classical second-order logic (with a free logic for terms formed by means of the slash).[10] Given her objective, instead of citing *MPC* as a substantive theory of properties Fairchild would have done better to cite *PC* as a valid schema of classical second-order logic. So understood, her argument (as modified) does indeed constitute a valid *reductio ad absurdum* disproof (that is, proof of the negation) of *SH+* in classical second-order logic.

2 A proof that simple hylomorphism is consistent

Classical second order logic is committed to $PC_{\mathbf{S}}$ and thus to some property like **S** (given embodiment). Classical second-order logic with lambda-abstraction embraces **S** itself. Either way, *SH+* is excluded. Notoriously, however, classical logic is an artificial idealization. For example, '$\exists x[x = f(a)]$' is classically valid,

[9] Our rendering of Fairchild's argument corrects reasoning and other errors in both her informal prose (pp. 35–36) and her "more rigorous" presentation (pp. 35–36*n*10, 14, and 15).

[10] Compare Church (1974a), pp. 29–30. The following is a (schematic) proof of *PC* in classical second-order logic with lambda-abstraction.

1.	$\forall x(\phi_x \leftrightarrow \phi_x)$	logic (ϕ_x is any formula without free 'F'.)
2.	$\forall x[\ \lambda y[\phi_y](x) \leftrightarrow \phi_x\ ]$	1, lambda-expansion, logic
3.	$\exists \mathrm{F}\forall x(\mathrm{F}x \leftrightarrow \phi_x)$	2, 2^{nd}-order existential generalization on $\ulcorner\lambda y[\phi_y]\urcorner$

See note 3. A similar schematic argument may be given for *MPC* in classical second-order logic with lambda-abstraction and modality. This argument assumes as an additional premise that the lambda-abstract is a rigid designator.

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although if '*a*' designates France and '*f*' is a symbol for the partial *king-of* function, which assigns to any kingdom its ruling monarch, the classically valid sentence is indisputably untrue. Arguably it is altogether false. Classical logic artificially disallows symbols for partial functions like the *qua*-function. It is precisely for this reason that *SH* employs free logic and includes *Existence* as a substantive axiom. Free first-order logic is more realistic than classical first-order logic, hence more widely applicable. Just as the presence of non-designating slash-terms requires a free logic with respect to such terms, the presence of lambda-abstraction recommends that a free second-order logic be adopted with respect to the generated predicates. Lambda-abstracts are sufficiently like definite descriptions that the logic should take account of the possibility that they are improper. (See notes 3 and 4.) Thus the logic that underlies *SH* is a free second-order logic.

Despite being inconsistent in classical second-order logic, *SH+* is like '$\sim\exists x[x = f(a)]$' in that it is obviously consistent in some more appropriate sense: consistent in *real* logic. There is even something very much like a model-theoretic proof that *SH+* is consistent. (The slight differences between a model-theoretic proof and our proof will be explained below.) We reinterpret both singular-term variables and monadic-predicate variables as ranging over the pure sets. Accordingly, we reinterpret 'λ' as an operator for set abstraction. On this reinterpretation predicates designate their semantic extensions. We interpret a monadic-predication formula $\ulcorner \Pi\alpha \urcorner$ where α is a singular term and Π is a monadic predicate in the normal way, so that it is true iff the designatum of α is an element of the semantic extension of Π. Further, we reinterpret the slash as a symbol for a restricted ordered-pair operation. Let us say that an ordered pair $< x, y >$ is an *∈-pair* iff $x \in y$. If the designatum of α is an element of the extension of Π, then $\ulcorner \alpha/\Pi \urcorner$ designates the ∈-pair < the designatum of α, the designatum of Π > ; otherwise $\ulcorner \alpha/\Pi \urcorner$ is undefined. (The modal-logical operator '□' receives its standard interpretation as an operator for metaphysical necessity.) We call this *the ∈-Pair Interpretation*. Both *Existence* and *Uniqueness*, so interpreted, are deliverances of any standard set theory such as ZF, so that *SH* is true on the ∈-Pair Interpretation. Further, the sentence '$\exists x\exists$G[x embodies G & $\sim$Gx]' is also true on the ∈-Pair Interpretation. For instance, let *r* be the ∈-pair of the empty set ∅ and its singleton {∅}. The ∈-pair *r* exists and it is not an element of its own second element. It follows that contrary to Fairchild, *SH+* is consistent (if ZF set theory is true).

The truth of *SH+* on the ∈-Pair Interpretation and the provability in classical logic of *PC* are reconciled by recognizing that the logic underlying *SH* is a free second-order logic rather than classical second-order logic. This is to be expected, since PC_S, which is inconsistent with *SH+*, is true in every classical second-order model. (See note 10.) The ∈-Pair Interpretation is in effect a free second-order model of *SH+*. It is not a model in the set-theoretic sense. Where classical models provide consistency proofs relative to an underlying set theory, our interpretation provides a consistency proof relative to an underlying class theory. As with the intended interpretation of ZF the universe of the ∈-Pair Interpretation is (standardly) a proper class rather than a set. More germane to the present inquiry, the ∈-Pair Interpretation does not conform to the classical model-theoretic requirement that every predicate have a semantic extension, even if only the empty set. On the ∈-Pair

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Interpretation some predicates fail to designate anything in the universe over which the monadic-predicate variables 'F' and 'G' range. On this interpretation if the extension of Π would otherwise be a proper class, then Π does not designate and has no semantic extension.[11] For example, on the standard view that the class of all sets is a proper class, '$\lambda x[x = x]$' fails to designate on this interpretation. The logic underlying *SH* is a free second-order logic in which *PC* fails. Further, classical lambda-expansion is invalid in such a free logic, and is replaced by a free-logical version that requires $\ulcorner\exists\Pi(\Pi =_2 \lambda\alpha[\phi_\alpha])\urcorner$ (where Π does not occur free in ϕ_α) as a supplementary premise. This is similar to free-logical existential generalization, which licenses the inference from ϕ_β and $\ulcorner\exists\gamma(\gamma = \beta)\urcorner$ (where γ does not occur free in β) to$\ulcorner\exists\alpha\phi_\alpha\urcorner$. (See note 4).

What then goes wrong with Fairchild's *reductio ad absurdum* argument against *SH*+? The ∈-Pair Interpretation casts light on this question. Fairchild's argument requires that PC_S be a truth of the logic underlying *SH*. On the ∈-Pair Interpretation PC_S asserts the existence of a set, *s*, whose elements are exactly the ∈-pairs that are not elements of their own second elements. The resemblance to the putative set in Russell's Paradox is too striking to miss. In the *∈-Pair Paradox* we consider the counterpart of *q qua* stone-caster, $< r, s >$. Is it an element of *s*? By familiar reasoning, if it is, then it is not; and vice versa. Thus, although *r* exists, there can be no such ∈-pair as $<r, s>$. If PC_S is taken as a premise rather than a theorem, the resulting derivation is valid in both classical and free second-order logic. Moreover, the derivation provides for (what is in effect) a proof that there is no such set as *s*, since if there were such a set as *s*, then *r* would be an element of it (since $< \varnothing, \{\varnothing\} > \notin \{\varnothing\}$), and thus there would be such an ∈-pair as $< r, s >$. Since *SH*+ is true on the ∈-Pair Interpretation and PC_S is inconsistent in both classical and free second-order logic with *SH*+, PC_S is false on the ∈-Pair Interpretation. Indeed, on the ∈-Pair Interpretation, *PC* is simply naïve set comprehension.

The ∈-Pair Interpretation thus demonstrates (*1*) that *SH*+ is consistent; (*2*) that the naïve comprehension schema *PC* is not a valid schema of free second-order logic; and therefore (*3*) it cannot legitimately be relied upon to prove *SH*+ inconsistent. The ∈-Pair Interpretation is an ordered-pair interpretation that is like the intended interpretation of *SH*+ in treating $\ulcorner\alpha/\Pi\urcorner$ as undefined when $\ulcorner\Pi\alpha\urcorner$ is false. Sosa (1999) exploits an analogous similarity in letting an instantiation-pair of an individual and one of its properties be a philosophical avatar for a *qua*-object. However the ∈-Pair Interpretation is not isomorphic with the intended interpretation

[11] That is, the ∈–Pair Interpretation may be regarded as being obtained in two stages. First, the singular-term variables are taken as ranging over the pure sets, while the monadic-predicate variables are taken as ranging over classes of pure sets and 'λ' is interpreted as an operator for class abstraction. The slash is interpreted as a symbol for the ordered-pair operation. Already at this stage the slash stands for a partial function, undefined when its second argument is a proper class, since proper classes are not elements of ordered pairs. The interpretation obtained at the first stage satisfies both *PC* and *Uniqueness*, but *Existence* fails for any instance in which 'F' is assigned a proper class. At the second stage the slash is restricted to ∈-pairs and (more important) the proper classes of the first stage are excised, so that the monadic-predicate variables now range over "small classes" of pure sets and any lambda-abstract that designates a proper class at the first stage is stripped of its designatum and therewith of its semantic extension.

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of *SH+*, under which monadic predicates are terms for properties (whether conceived of as modal intensions or more standardly) rather than sets (or characteristic functions). Whereas on the ∈-Pair Interpretation '$\lambda x[x = x]$' fails to designate, on the intended interpretation it designates a (universal) property.

3 Reinterpretation as an argument that simple hylomorphism is false

Fairchild's reliance on *PC* as a product of a liberal theory of properties rather than as a truth of logic suggests the more temperate position that *SH+*, though consistent, is false. Indeed one may regard its inconsistency with *PC* as a good reason to reject *SH+*. But consideration of what we call *the Paradox of Property Singletons* argues otherwise. We shall let '**R**' abbreviate '$\lambda y[\exists G(y = \{G\} \ \& \ \sim Gy)]$', which designates the property of being the singleton of some property the singleton itself lacks.[12] If the reader will forgive an understatement, the formation of a singleton is at least as well understood as the formation of a *qua*-object from an individual and one of its properties. The following instance of *PC* is thus no less legitimate than $PC_{\mathbf{S}}$ is:

$PC_{\mathbf{R}}$ $\quad \exists F \forall x[Fx \leftrightarrow \exists G(x = \{G\} \ \& \ \sim Gx)]$

Assuming this instance of *PC*, **R** is the principal witness for it. (See note 6 and surrounding text.) If singletons of properties are to be countenanced as individuals, then they are evidently subject to the following minimal condition:

*Existence** $\quad \forall F[\{F\}$ exists].[13]

If instantiating $PC_{\mathbf{S}}$ to '*q*/**S**' is legitimate, then instantiating $PC_{\mathbf{R}}$ to '{**R**}' is as well. Yet the result,

C_3 $\quad \mathbf{R}\{\mathbf{R}\} \leftrightarrow \exists G[\{\mathbf{R}\} = \{G\} \ \& \ \sim G\{\mathbf{R}\}]$,

is straightforwardly inconsistent with *Existence**, yielding (1*) $\sim \mathbf{R}\{\mathbf{R}\}$ and (2*) $\mathbf{R}\{\mathbf{R}\}$. *Existence** is a trivial set-theoretic consequence (via *Separation*) of the set-theoretic truism (via *Pairing*) that every property (assuming properties can be *ur*-elements) is an element of some set or other. Rejection of *Existence** in order to block the contradiction would be excessively ad hoc: If there is a property **R**, then it has a singleton. Faced with the present choice between *Existence** and $PC_{\mathbf{R}}$, the

[12] The notation '{G}' is a first-order singular term for a set. It may be taken as defined by '$\iota x \forall F(F \in_2 x \leftrightarrow F =_2 G)$', where '$\in_2$' is a dyadic predicate for a relation between a property of individuals and an individual. As we use it, 'ι' validates the schema $\ulcorner \Pi(\ldots \iota \alpha \phi_\alpha \ldots) \leftrightarrow \exists \beta[\forall \alpha(\phi_\alpha \leftrightarrow \alpha = \beta) \ \& \ \Pi(\ldots \beta \ldots)\urcorner$ where α and β are distinct individual variables and Π is a simple predicate.

[13] The analog of *Uniqueness*, namely

*Uniqueness** $\quad \forall F \forall G[\{F\} = \{G\} \leftrightarrow F =_2 G]$,

is a logical consequence of *Existence**. (Refer to note 12.) Fairchild is sympathetic to overcoming her challenge to *SH* by rejecting *Uniqueness* while retaining *Existence* (pp. 38–39). There is no analogous option for the Paradox of Property Singletons.

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recommended solution is clearly to reject $PC_{\mathbf{R}}$.[14] This discredits reliance on *PC* even merely to falsify *SH*+ (rather than to prove it inconsistent).

The situation with the more temperate *reductio ad absurdum* argument against *SH*+ is thus seen to be very similar to the one described by Saul Kripke in his classic essay "A Puzzle about Belief":

> Someone wishes to give a *reductio ad absurdum* argument against a hypothesis in topology. He does succeed in refuting this hypothesis, but his derivation of an absurdity from the hypothesis makes essential use of the unrestricted comprehension schema in set theory, which he regards as self-evident. (In particular, the class of all classes not members of themselves plays a key role in his argument.) Once we know that the unrestricted comprehension schema and the Russell class lead to contradiction by themselves, it is clear that it was an error to blame the earlier contradiction on the topological hypothesis. (Kripke 1979, pp. 253–254)

Analogously, Fairchild (on the more temperate interpretation) wishes to give a *reductio* argument against *SH*+. Her derivation of an absurdity makes essential use of an unrestricted comprehension schema *PC* in property theory, which she regards as self-evident. Once we know that *PC* leads to contradiction when combined with *Existence**, it is clear that it was an error to blame the earlier contradiction on *SH*+.

4 Avoiding impredicativity

Due consideration of the parallelism between Fairchild's argument and the Paradox of Property Singletons points to the obvious culprit, which is common to both and which is the only plausible suspect in the latter case: the "property generator" *PC*.[15] Evidently, $PC_{\mathbf{R}}$ is an untrue instance of *PC*. This observation points to the obvious move for the *SH* proponent to make: embrace that $PC_{\mathbf{S}}$ is simply another untrue instance of *PC*. If **S** does not exist, then neither does *q qua* stone-caster (since there is no **S** for it, or for any *qua*-object, to embody). Conversely, the postulation of **S** leads to *q qua* stone-caster, which in turn leads to contradiction. The *SH* proponent should reject the entire pernicious package: *q qua* stone-caster, **S**, and *PC*. The *SH* proponent who rejects **S** can still admit that there is a property that *q* embodies and also lacks. It must not be inferred, however, that *q* thereby has **S**; there is no **S** for *q* to have.

The situation is familiar from Russell's Paradox. One who rejects the Russell set can still admit that the set of lemons is not an element of itself. It must not be inferred, however, that the set of lemons is thereby an element of the Russell set; there is no Russell set for the set of lemons to be an element of. Just as there is no set corresponding to the "condition" (open formula) '$x \notin x$' even though some

[14] Salmón (unpublished) provides a fuller treatment of the Paradox of Property Singletons.

[15] By saying that *PC* "generates" properties we mean merely that it posits for any open formula a corresponding property.

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individuals (like the set of lemons) satisfy the open formula, so too the proponent of *SH* should hold that there is no property corresponding to '∃G[*x* embodies G & ∼G*x*]' even though some individuals (like *q*) satisfy it. Whereas often no harm results from conflating an open formula and its corresponding predicate, such conflation is always risky. Fairchild's explicit conflation of the open formula '∃G[*x* embodies G & ∼G*x*]' and its corresponding predicate 'λ*x*∃G[*x* embodies G & ∼G*x*]' (p. 35*n*10) may contribute to her treating *PC* as self-evident. Perhaps too calling an open formula a "condition" encourages an insufficiently reflective attitude toward *PC*.

There is an independent and philosophically respectable rationale for replacing *PC* with a more discriminating variant. As Fairchild recognizes, **S** is *impredicatively defined*. That is, **S** is introduced ("defined") by property abstraction from an interpreted open formula (a "condition") that quantifies over a totality that purportedly includes **S** itself.[16] Exactly analogously, the putative property **R** in the Property-Singletons Paradox is also impredicatively defined. Impredicative definition smacks of circular definition. For over a century, since Henri Poincaré (1906) put forward his Vicious-Circle Principle, many (and not only mathematical constructivists) have looked upon impredicative definition with profound suspicion. Again, rejection of **S**—this time due to its impredicativity—and with it *PC*, is the conventional move for the *SH* proponent to make.[17]

The ramified theory of types (with axioms of reducibility) is a logical apparatus that was designed for theorizing, in a manner that preempts paradoxes of impredicativity, about such things as properties that quantify over properties (propositional functions that are abstracted by quantifying over propositional functions), functions that are abstracted by quantifying over functions, and propositions that quantify over propositions. Russell (1903) proposed without endorsing a ramified-type-theoretic resolution of the Russell-Myhill Paradox. Whitehead and Russell ([1927] 1963) endorse ramified type theory (with axioms of reducibility) as resolving paradoxes of impredicativity.[18] This apparatus is thus

[16] This notion of impredicativity is a special case of, but stricter than, the broader notion, largely based on Henri Poincaré's vicious-circle principle (1906), to wit, that of introducing a particular element of a class by quantifying over the elements of that class. Although it is not impredicatively defined in the stricter sense, in this broader sense the putative set involved in Russell's Paradox (the set of all and only those sets that are not elements of themselves) is said to be "impredicatively defined." However, as F. P. Ramsey in effect pointed out (1925, p. 204), so also is the idea of fixing reference by a superlative definite description, for example, 'the shortest spy', 'the first child to be born in the 22nd Century', etc. (We thank C. Anthony Anderson for supplying this reference.) The stricter sense of 'impredicative', which is likely what is usually meant in the relevant literature, is uniformly adhered to throughout the present essay.

The notion of *definition by abstraction* involved in the stricter notion of impredicativity is to be sharply distinguished from the distinct notion in the neo-logicism literature that bears the same label.

[17] According to this response, the lambda-abstract abbreviated by '**S**' does not designate on the intended interpretation of *SH*, consequently '∀F' in *Existence* cannot be legitimately instantiated to it. The free second-order logic employed here requires the supplementary premise '$\exists$F(F $=_2$ $\lambda y[\phi_y]$)' for lambda-expansion, thus rendering the inference at line 2 in the derivation in note 10 fallacious. The missing supplementary premise is itself a property-comprehension schema.

[18] See Church (1976), Russell (1903, Appendix B), Russell (1908), and Whitehead and Russell (1927) 1963, *12, pp. 161–167. Whitehead and Russell's axioms of reducibility entail that every level *n* property

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tailor-made for theorizing about *qua*-objects whose form is impredicatively defined and it would be natural for the hylomorphist to utilize it. Ramified type theory replaces impredicative definition with stratification of properties and propositions. For example, the abstraction of a property F of individuals through generalizing over a plurality of properties including F is replaced with abstraction of a level $n + 1$ property F^{n+1} through generalizing over properties of level n, $n \geq 1$. Given that the property of being blue in color is level 1, the property of being the same color as the sky is level 2. The property of having some level 2 property or other in common with Henry is level 3. In this way, ramified type theory rejects the unrestricted property-comprehension schema *PC*, which generates impredicatively defined properties and therewith certifies **S** as a legitimate property. A compelling solution to Fairchild's Russellian "paradox" is to reject $PC_{\mathbf{S}}$ and to supplant it with property comprehension in a suitable ramified type theory.[19]

One ramified property-comprehension schema is the following:

$PC^n \quad \exists F^n \forall x(F^n x \leftrightarrow \phi^n)$

where ϕ^n is any formula of level n, $n \geq 1$.

For present purposes the level of a formula may be taken to be that of the highest-level simple predicate or operator occurring free within it, or one level higher than the highest-level simple predicate or operator occurring bound within it, whichever is higher. PC^n authorizes abstraction of a property F^n of level n that invokes quantification over properties G^m of level m but requires that $m < n$.

Ramified type theory requires analogous stratifications of *Existence* and *Uniqueness*. All these modifications amount to treating each of *PC*, *Existence*, and *Uniqueness* as "typically ambiguous" (Whitehead and Russell [1927] 1963, *65, pp. 415–416), that is, as a schema for which it is to be taken that its instances at each level are asserted. The property-comprehension schema *PC* sanctions the postulation of **S** as an un-leveled property of *qua*-objects. By contrast, PC^n sanctions the postulation of a property $\mathbf{S}^{n+1}$, for each level $n \geq 1$, which is the level $n + 1$ property of being a *qua*-object that embodies an n-level property it lacks. On this conception un-leveled **S** is not a legitimate property (and likewise for un-leveled **R**). The modifications to *Existence* and *Uniqueness* (and *Existence**) retain the spirit of those principles, whereas the modification to *PC* defeats its point by effectively precluding such properties as **S** (as well as **R**).

Fairchild surveys a range of responses to her argument that are supposed to parallel resolutions that have been offered to Russell's Paradox. She fails to mention Russell's resolution to Russellian paradoxes that invoke impredicativity. Ramified type theory (with axioms of reducibility), though controversial, is in fact the

Footnote 18 continued
for $n \geq 1$ is co-extensive with a level 1 property. It is sometimes said—following a highly misleading (at best) remark of Quine (1967, p. 152)—that the axioms of reducibility restore the paradoxes of impredicativity. The claim is incorrect, however, as Russell had already noted in 1908 (last paragraph of section V). See Church (1974b, p. 356).

[19] Where ϕ_x invokes quantification over properties of individuals the proof of *PC* in note 10 thereby invokes the basic form of impredicative definition. By contrast, impredicative *PC* is not a theorem of ramified type theory. In a suitable formulation of ramified type theory it is not even well-formed.

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industry-standard resolution of such paradoxes. The thought that there are stone-casters of various levels although there is no encompassing property of being a stone-caster (of some level or other) need not be unpalatable to the *SH* proponent. Indeed, it is entirely to be expected that a careful *SH* proponent will countenance stone-casters^{n+1} for each level $n \geq 1$ and reject un-leveled **S**, and therewith *PC*, as illegitimate.[20]

5 Reaction and overreaction

Fairchild in effect briefly considers resolving the inconsistency by rejecting the claim that to every open formula there corresponds a genuine property. She offers the following cryptic reply.

> But weakening our (otherwise consistent) theory of properties in light of the problems [*sic*][21] for simple hylomorphism would surely be an overreaction. For those who already take issue with the liberal conception of properties, the minimal theory [*SH*] may be safe. Intuitively, however, this argument doesn't show us that the property involved [**S**] is somehow pathological. Instead, it seems to cast suspicion on the object [*q qua* stone-caster] that embodies the property. Something has gone wrong in the theory of embodiments [*SH*], not in the theory of properties on which it relies. (pp. 36–37)

It is unclear what is it for a theory to be "otherwise consistent". On the most charitable (while still plausible) interpretation, Fairchild's thought is that although the liberal theory of properties is not consistent with *SH*+, it is consistent *simpliciter*. But the same is true *mutatis mutandis* of *SH*+, as Fairchild more or less concedes when she admits that the minimal theory "may be safe".[22] Fairchild nonetheless clearly thinks that *PC* has an exalted status, so that rejecting it would be a misguided "overreaction". She regards it as intuitively obvious that whereas the contradiction reveals a problem with the (purported) *qua*-object, *q*-*qua*-stone-caster (*q*/**S**), it does not reveal any problem with the (purported) property of being a stone-caster (**S**).

We agree with Fairchild that intuitively the contradiction indicates that there is no such *qua*-object as *q*/**S**. A resolution must explain why. Given that *q* exists, the *qua*-object theorist has two choices: either **S** cannot be the form of a *qua*-object (in spite of the fact that **S** exists and *q* has it), or there is no such property as **S**. *Existence* precludes the first option. Note that the (putative) ordered pair < *q*,

[20] The proposal for restricting *Existence* that Fairchild discusses on pp. 37–38 bears a superficial similarity to the present suggestion but is radically different.

[21] The use of the plural "problems" is misleading, since it suggests that Fairchild provides considerations against *SH* in addition to her Russellian challenge.

[22] We take the force of Fairchild's "may be" to be to leave open the possibility that *SH* + is "unsafe" for reasons unrelated to her argument. The fact of this concession renders the interpretation of Fairchild's argument that we offer in §3 exceedingly plausible in spite of its poor fit with her tagline that *SH* is inconsistent.

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S > , which for Sosa (1999) serves as a philosophical avatar for a *qua*-object if *q* has **S**, exists if both *q* and **S** exist. Moreover the purported property **S** is intuitively dubious. Compare our discussion of the Paradox of Property Singletons and our argument that there is no such set as *s* in connection with the ∈-Pair Paradox.

While the contradiction casts doubt on **S**, it does nothing, as we have already observed, to discredit **S**'s defining "condition", which is the interpreted open formula '∃G(*x* embodies G & ∼G*x*)'. It would indeed be an overreaction to reject the open formula (in contrast to the lambda-abstract that is the open formula's corresponding predicate) as defective. As long as 'embodies' is provided with a clear sense, the open formula genuinely expresses a singular existential proposition under an assignment of a value to '*x*'. It does not follow, however, that there is a corresponding property.

Fairchild's contention that the contradiction does not intuitively discredit **S** amounts to an unsupported and summary dismissal of a venerable tradition that sees impredicativity as intuitively pathological. It is incumbent on one who uses **S** to discredit a metaphysical theory of property-involving composite objects to address the classical Russellian stance regarding impredicativity. Even if Russell's ban on impredicativity is ultimately to be rejected, logic itself evidently calls for *some* theory that imposes a restriction on property comprehension. While we do not endorse *SH*, dismissing it on a ground that would also reject any metaphysical theory that liberally countenances property-involving composite objects—such as set theory with properties as *ur*-elements and standard theories of propositions—is dubious. A permutation of the first sentence of the remarks quoted at the beginning of this section has significantly greater claim to being true: But weakening (the consistent) simple hylomorphism in light of the problem for the liberal theory of properties would surely be an overreaction.

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