Localizability and elementary particles

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Abstract. The well-definedness of particles of any kind depends on the limits, approximations,
or other conditions that may or may not be involved, for example, whether there are interactions
and whether ostensibly related energy is localizable. In particular, their theoretical status
differs between its non-relativistic and relativistic versions: One can properly define interacting
elementary particles in single-system non-relativistic quantum mechanics, at least in the case of
non-zero mass systems; by contrast, one is severely challenged to define even these properly in
the relativistic quantum field theories that now underlie the study of particle physics. Here, the
impact of localizability on this status is reviewed in relation to the work of Paul Busch on positive
operator valued measures that significantly probes the relevance of quantum unsharpness to it.

1. Introduction

In quantum theory, the definability of objects such as elementary particles when required to
be localizable depends on the version of the theory under consideration and the interactions,
limits, and other conditions imposed on the system via which they would be instantiated; cf.
e.g. [1, Chapter 1]. Combining special relativity and quantum theory, as is needed for the
study of elementary particles, requires that a field theory, relativistic quantum field theory
(RQFT) be considered; cf. e.g. [2, Chapter 2]. Straightforward attempts within RQFT to
describe particles—something typically done via energy-momentum eigenstates—that might
subsequently be localized fail when interactions are allowed. This is so because adequate creation
and annihilation operators needed for their accounting via corresponding number operators
are definable only when no interaction can take place. Strictly speaking, this last condition
is never satisfied in the absence of approximation for common interactions such as those of
electromagnetism; cf. e.g. [3] and references therein. From a more operationalist perspective,
there are also specific difficulties for the provision of the mathematical entities needed to define
spatial localization consistently with relativistic causality requirements; it would appear that
the operation of localization is impossible even though experiments are standardly made via
what is assumed to be the local detection of particles. Yet, it might still be thought these
latter difficulties could be resolved by appealing to unsharp localization operators because,
before the work of Paul Busch, it was *sharp localization* that had been studied at the formal
level in relativistic quantum theory. With localization defined via quantum observables, Busch
considered whether these observables might be simply *unsharp*, so that the apparent issues with
creation, annihilation, and number operators might eventually be finessed.

Whether unsharpness indeed provides a solution to this quandary was considered by Busch
and coworkers specifically in the mathematical context of positive operator valued measures

(POVMs), and significant results were obtained: (i) a generalization of Schlieder's theorem to show that the local commutativity of localization observables implies that they are, at best, *unsharp in a strong sense* and (ii) that, in important special cases, two (discrete) unsharp (as well as sharp) observables commute, for any given state, if and only if the statistics of a measurement of one is unaffected by a nonselective Lüders measurement of the other [4]. One is thus naturally led to the reconsideration of the assumptions that have been made in defining localization. Busch subsequently found that the local commutativity assumption is a necessary consequence of Einstein causality for local unsharp measurements. Accordingly, he indicated that it could be the specific requirement that localization operators *themselves be local* that is responsible for the difficulties encountered in regard to localization [4].

Here, a review of the above and related results, as well as their interplay is provided and further aspects of unsharp measurement in quantum theory that relate to the operators playing a role in localization are briefly discussed. It is emphasized that, in agreement with the position taken by Busch, the assumption that localization itself be local could be given up in order for quantum theory to accord with special relativity and that, in a relativistic physics with elementary particles, theories or models of measurement in the space-time context should include the quantum aspects of measurement devices to better illuminate the physics involved.

2. The Schlieder theorem

In their analysis of localization in relativistic quantum theory, Busch *et al.* considered the following Hilbert-space of states and group representation associated with what is sometimes called a localization system [4] which has been the standard setting for the subject.

(1) (State space) A complex Hilbert space $\mathcal{H}$, with rays representing the pure states

under the conditions

(2) (Spectral condition) A strongly continuous unitary representation (CUR) $a \mapsto U(a)$ in $\mathcal{H}$ of the Minkowski space (M) translation group;

(3) (Hamiltonian-boundedness condition) The generator $H(a)$ (Hamiltonian) is bounded below for future directed, timelike unit vectors a,

the latter being also sometimes called the energy-boundedness condition. In this setting, they examined the Schlieder theorem [5], with the goal of generalizing it, in the valuable form given it by Malament [6] which considers a spatial localization observable in relation to what he called the *localization-event structure*.

(i) (Localization event structure) With a foliation of Minkowski space M selected via a family $\mathcal{S}$ of parallel spacelike hyperplanes S, each S is equipped with a family $\mathcal{F}(S)$ of subsets (spatial sets), including a covering family of bounded subsets, such that $\mathcal{F}(S + a)$ is the translation by a of the sets from $\mathcal{F}(S)$, and there is a map $\Delta \mapsto E_\Delta$ from $\mathcal{F}(S)$ to effects of $\mathcal{H}$ for each S.

The Malament theorem assumes satisfaction of the following explicit conditions for localization and locality in an inertial frame of reference:

(ii) (Translational covariance) For all $a \in M$,

$$U(a)E_\Delta U(a)^* = E_{\Delta+a}.$$

(iii) (Localizability) For each $S \in \mathcal{S}$ and $\Delta_1, \Delta_2 \in \mathcal{F}(S)$,

$$\text{if } \Delta_1 \cap \Delta_2 = \emptyset \quad \text{then} \quad E_{\Delta_1}E_{\Delta_2} = E_{\Delta_2}E_{\Delta_1} = 0.$$

(iv) (Local Commutativity) For $S_1, S_2 \in \mathcal{S}$, $\Delta_1 \in \mathcal{F}(S_1), \Delta_2 \in \mathcal{F}(S_2)$,

$$\text{if } \Delta_1, \Delta_2 \text{ are spacelike separated, then } \mathrm{E}_{\Delta_1}\mathrm{E}_{\Delta_2} = \mathrm{E}_{\Delta_2}\mathrm{E}_{\Delta_1}.$$

The Schlieder theorem in Malament's form was then written [4]

> If $(\mathcal{H}, a \mapsto U(a), \Delta \mapsto P_\Delta)$, where the P_Δ are projections (special cases of E_Δ), satisfies (ii)–(iv), then $P_\Delta = 0$ for all bounded spatial sets Δ.

The relevant implication of this theorem which, again, regards only projection-valued measures (i.e. the sharp case) is that *there can be no physical adequate notion of localizability* in the above context, with the implication that there is no adequate standard notion of localization of particles in relativistic quantum theory; cf. [7]. The Schlieder theorem excludes local number operators from Wightman's quantum field theory [8] or algebraic quantum field theories; cf. [9] and references therein.

3. Generalizing the Schlieder theorem to the case of unsharp observables

To find out whether some alternative approach to achieving a well-defined sense of localization in relativistic quantum theory may be possible in light of this result, Busch looked to the intrinsically *unsharp* observables defined via POVMs rather than only the special case of projection-valued measures. He addressed the question of whether a theorem analogous to Malament's form of Schlieder's theorem holds for unsharp localization observables by considering *non-projective* effects E_Δ in addition to projections P_Δ, that is, with the localization event structure such that the maps $\Delta \mapsto E_\Delta$ are POVMs defined on spatial Borel sets. Busch found that to be the case, proving the following.

> **Theorem** (Busch). If the structure $(\mathcal{H}, a \mapsto U(a), \Delta \mapsto E_\Delta)$ satisfies conditions (ii)–(iv), then $E_\Delta = 0$ for all (bounded) spatial sets Δ.

He then noted that the physical basis for the algebraic condition imposed in the assumptions *changes* in the move from projective measures to positive operator valued measures, in that the localization, when imposed via $E_{\Delta_1}E_{\Delta_2} = 0$ for disjoint spatial sets, no longer achieves the *physical* goal of imposing the condition "If the system is in Δ_1, it certainly is not in Δ_2 whenever these sets are disjoint" [4]. Therefore, he introduced, instead of (iii), the condition

(iii′) For all states $\varphi \in \mathcal{H}, \|\varphi\| = 1, \Delta_1, \Delta_2 \in \mathcal{F}(S)$,

$$\text{if } \Delta_1 \cap \Delta_2 = \emptyset \quad \text{then} \quad \langle\varphi|E_{\Delta_1}\varphi\rangle = 1 \implies \langle\varphi|E_{\Delta_2}\varphi\rangle = 0 \,,$$

which is a direct consequence of the assumption that $\Delta \mapsto E_\Delta$ is a (not necessarily normalized) POVM, tantamount to $P_{\Delta_1}^{(1)} \leq P_{\Delta_2}^{(0)}$, where $P_\Delta^{(1)}$, $P_\Delta^{(0)}$ denote the spectral projections of E_Δ associated with the eigenvalues 1 and 0, respectively, which is equivalent to (iii) in the case of projections, but better captures what is required for localization via effects, in that (iii′) implies only $P_{\Delta_1}^{(1)} P_{\Delta_2}^{(1)} = 0$; $E_{\Delta_1}E_{\Delta_2} \neq 0$ in general.

Assuming the alternative algebraic condition (iii′), which is a strong causality assumption that entails localizability when one considers $\Delta_2 = \Delta_1 + ta$ with t a time (cf. [9, Sec. 2]), Busch proved the following.

> **Theorem** (Busch). If the structure $(\mathcal{H}, a \mapsto U(a), \Delta \mapsto E_\Delta)$ satisfies conditions (ii), (iii′) and (iv), then $P_\Delta^{(1)} = 0$ for all (bounded) spatial sets Δ .

He took this result to indicate that any localization observable satisfying the conditions (ii), (iii′), and (iv) would necessarily be *strongly unsharp*, in that its effects E_Δ do not have 1 as an eigenvalue (cf. [4]), and posed the associated question of "whether among the strongly unsharp, covariant localization observables there exist any that satisfy local commutativity" (i.e. iv), pointing out that "an indication to the negative is provided by a recent theorem stating that *spacetime* localization observables cannot belong to any quasilocal algebra" [10].[1]

4. The strongly unsharp

After Busch proved the above theorems, Halvorson and Clifton provided an additional one on the basis of which they argued that even an appeal to strongly unsharp localization does not mitigate the force of the problems with localization indicated by the theorems of Schlieder and Malament, namely, that under the same assumptions of Busch's generalization, but with the addition of the condition

(NAV) (No absolute velocity) Let a be a spacelike translation of M. Then there is a pair (b, c) of timelike translations of M such that $a = b - c$,

the following holds [9].

> **Theorem** (Halvorson–Clifton).   Suppose that the unsharp localization system $(\mathcal{H}, a \mapsto U(a), \Delta \mapsto E_\Delta)$ satisfies the conditions (ii),(iii′), (iv), and NAV. Then $E_\Delta = 0$ for all Δ.

This result was seen by its authors as further evidence in support of the need for a field-based ontology in that it shows more strongly that a relativistic quantum mechanics of individual localizable systems, such as particles, cannot be constructed along recognizable lines.

To strengthen the argument against a particle ontology within quantum field theory, Halvorson and Clifton produced another result even more squarely focused on particles by addressing the issue of *number operators* rather than position operators, that is, operators the eigenvalues of which would be considered the number of particles within a given region of spacetime. That is, they consider a number structure $(\mathcal{H}, a \mapsto U(a), \Delta \mapsto N_\Delta)$, where the E_Δ, that is, a system of local operators over spacetime M, which is a generalization of the event-localization structure where the operator eigenvalues is not restricted to 0 and 1, and not discrete as would be intuitive, but also even non-discrete non-negative, to reach what they identify as the *minimal requirement* for a sensible notion of localizable particles [9, Sec. 6].

In addition to the assumptions of Busch's theorem, the Halvorson–Clifton theorem makes two additional ones: The first is (particle-)number conservation, namely that if $\{\Delta_n\}$ where n is a natural number is a disjoint covering of Σ, then the sum $\Sigma_n N_{\Delta_n}$ converges to a densely defined, self-adjoint operator N on the Hilbert state space $\mathcal{H}$ and $U(a)NU^*(a) = N$ for any timelike a so that the number operator is well defined (a condition satisfied in non-interacting QFT); the second is the additivity of the number operator, i.e. if Δ and Δ' are disjoint subsets of a single spacetime hyperplane then $N_\Delta + N_{\Delta'} = N_{\Delta+\Delta'}$. The conclusion of this additional theorem of Halvorson and Clifton is that, under with all these assumptions $N_\Delta = 0$ for all Δ, that is, in every state there will be no particles in any local region.

5. Relation to causality

In quantum theory, two forms of causality may be relevant: weak Einstein causality, the requirement that any changes of observables' *expectation values* take place with at most light speed, and strong Einstein causality, the requirement that individual property values change with at most light speed; cf. [4]. Strong causality implies localizability and is natural for relativistic

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[1] Beneduci and Schroeck Jr have made steps toward answering this by introducing a different space localization observable for the photon [11].

theories. However, Malament's original theorem does not require it, but assumes only the weak form; cf. [9]. Busch demonstrated that such *weak* Einstein causality is logically equivalent to the local commutativity condition [12]; as a consequence of Lüders theorem [13], two (discrete) observables represented by self-adjoint operators commute if and only if, for any state, the statistics of measurement of one are unaffected by nonselective *Lüders measurements* of the other, which is extendable to observables having non-discrete spectra [4]. One then notes that unsharp phase-space observables can be jointly measurable without commuting with each other. Thus, even weak Einstein causality appears open to violation in the face of the generalized Lüders theorem (applying to POVMs beyond the projectors) demonstrated by Busch and Singh [14], in two cases that Busch argued are the only ones of genuine physical significance [4]. However, the above results are obtained with the assumption that arbitrary measurement accuracy can be obtained as in classical physics.

Denying arbitrary accuracy provides a possible way out of an apparent difficulty for Einstein causality in the context of POVMs, in that it is operationally suspect, at least in quantum mechanics: Busch argued that localizing a quantum system entirely within a sharp, finite boundary can only be done using an infinite potential well or, in the context of quantum field theory, with enough energy to make significant particle-pair creation possible, leading to further complications which clearly take one beyond simple, single-particle analyses [4].

In addition, as Busch also pointed out, the concept of localization involves globality in that the collection of all bounded spatial subsets of S and *translational covariance* itself are global in character; cf. [15]. But even if localization observables allowing sharply localized states aren't required to be local, so as not to menace weak causality, one might still be concerned that there could be an instantaneous spread of wave functions which might be thought support superluminal signaling. Hegerfeldt's theorem is understood to show that superluminal spreading of wave functions occurs in any quantum theory in which there are states of localized systems. To understand this, one can consider, in particular, the following conditions—in addition to translation covariance and a Hamiltonian bounded from below—imposed by Hegerfeldt in the following forms [9, 16]:

(NIWS) If $\Delta \subseteq \Delta'$ and the boundaries of Δ and Δ' have finite distance, then there is an $\epsilon > 0$ such that $P_\Delta \leq P_{\Delta'+t}$ whenever $0 \leq t < \epsilon$;

(Monotonicity) If $\{\Delta_n\}$, where n is a natural number, is a downward nested family of subsets of Σ such that $\cap_n \Delta_n = \Delta$, then $\wedge_n P_{\Delta_n} = P_\Delta$.

Here each operator of the form P_Δ is a projection onto the corresponding spatial set, Δ, with the interpretation that it represents the proposition that the particle in question is certainly localized in Δ. The conclusion of Hegerfeldt is that a particle localized at some initial time in any finite region Δ must remain so localized; the assumptions of monotonicity, translational covariance, Hamiltonian boundedness, and NIWS can be satisfied only for entirely trivial dynamics; otherwise, instantaneous wave function spreading will occur [9].

The natural states in which to consider quantum systems would seem therefore to be unlocalized ones, such as those of momentum. Nonetheless, as Busch further argued, an instantaneous wave function spread would not in itself underwrite superluminal signaling: Any signal sender could not *control* the particle in question by releasing the particle or not to communicate a bit of information as this would required an infinite amount of energy, as it could only be accomplished via an infinite potential well (or the equivalent) and, if the particle is free, it cannot be under sufficient control to enable any quasi-signal to be received at space-like separation, even though it had been instantaneously localized. Thus, the notion of local measurement and local observable algebra would be subject to reformulation under some new, truly operationally significant notion of causality [4].

The results considered above bring into question the significance of the quantum-field theoretical postulates of local commutativity and strong and weak causality, the last being introduced via a local commutativity requirement [4]. The situation thus clarified by the work of Busch *et al.* has motivated recent analyses of Barat and Kimball [18] and Terno *et al.* [17, 19] focusing on energy density, which is directly related to photodetection, as the quantity best indicating the presence of any putative particle. Terno *et al.* argued that causality is not endangered by the requirements of localization, concluding that "energy density cannot be 'localized' enough to violate causality".

Terno *et al.* constructed an operator density POVM element with the probability of detecting a particle in a volume Δ assumed proportional to the integral of the energy density within it. The probability of finding a particle in the volume, given that a detection has occurred anywhere at all, was taken to be

$$\Pi(t, \mathbf{x}) = H^{-1/2} \hat{T}_{00}(t, \mathbf{x}) H^{-1/2}, \tag{1}$$

the first moment of which is proportional to the Born-Infeld position operator; cf. [20]. Their conclusion was then reached based on previous results of Barat and Kimball [18] showing that physical states of a real scalar field described by configuration-space wave-functions $\psi(t, \mathbf{x})$ are such that the requirement that the probability of a particle be outside a sphere of radius R be bounded by

$$\text{Prob}_{\notin R} < C^2 \exp(-2\gamma R), \tag{2}$$

C being a constant and $\gamma > m$ cannot be satisfied for the one-particle state $|\Psi\rangle$ having positive normal-ordered Hamiltonian energy density $T_{00}(t, \mathbf{x})$ for the Hamiltonian H [18]. Their POVM element is allowed to be non-local, as is evidenced by the presence of the Hamiltonian in its definition. In taking this operational approach, inconsistency with the causality requirements of quantum field theory appears to be avoided.

6. Conclusion

The work on localizability of Busch and others for elementary particle physics in a well-rounded world picture shows that the *theory of measurement* in relativistic quantum mechanics is in need of greater study. The approach that has been followed so far is one where the constitution of the instruments used for carrying out the operation of localization—the possible significance of which was also noted by Busch [4]—has not come under sufficiently explicit consideration. If the discussion is carried out fully in the realm of particle physics, as would serve the consistency of application of relativistic physical theory, the measuring instruments taken to localize particles should be considered themselves to be constituted by fields or elementary particles, something which thus far has not been carefully done. Thus, detailed theoretical descriptions at the *atomic-* and *subatomic-levels* of the instruments that would be involved in the operation of localization appear necessary for progress in exact treatments of the phenomena of particle physics. It could also serve to enrich notions such as Heisenberg's actualization of quantum potentiality, cf. [21], by providing greater physical detail to the measurement operations in RQFT, which goes beyond the quantum mechanics of systems fixed particle number in relation to which the former was first introduced. It is, therefore, suggested that this be done in future investigations of localizability in quantum theory.

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