Recent Work in Relevant Logic

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1 Introduction

Relevant logics are a group of logics which attempt to block irrelevant conclusions being drawn from a set of premises. The following inferences are all valid in classical logic, where A and B are any sentences whatsoever:

- from A, one may infer $B \to A$, $B \to B$ and $B \vee \neg B$;
- from $\neg A$, one may infer $A \to B$; and
- from $A \wedge \neg A$, one may infer B.

But if A and B are utterly irrelevant to one another, many feel reluctant to call these inferences acceptable. Similarly for the validity of the corresponding material implications, often called 'paradoxes' of material implication. Relevant logic can be seen as the attempt to avoid these 'paradoxes'.

Relevant logic has a long history. Key early works include Anderson and Belnap 1962; 1963; 1975, and many important results appear in Routley et al. 1982. Those looking for a short introduction to relevant logics might look at Mares 2012 or Priest 2008. For a more detailed but still accessible introduction, there's Dunn and Restall 2002; Mares 2004b; Priest 2008 and Read 1988.

The aim of this article is to survey some of the most important work in the field in the past ten years, in a way that I hope will be of interest to a philosophical audience. Much of this recent work has been of a formal nature. I will try to outline these technical developments, and convey something of their importance, with the minimum of technical jargon. A good deal of this recent technical work concerns how quantifiers should work in relevant logic. This is the topic of §2. §3 describes other advances in the recent technical literature. In §4, I discuss several recent attempts to give a philosophical interpretation of the most prominent semantics for relevant logic (the Routley-Meyer ternary relation semantics), and highlight some problems. These three sections may be read independently of one another. Those unfamiliar with the ternary relation semantics might like to read the introduction to §4, or consult Mares 2012, before going further.

2 Quantification

In this section, I'll discuss recent work on semantics for quantified relevant logics. I'll focus on three problems in the area: giving a semantics on which quantified relevant logic is complete (§2.1), restricted quantification (§2.2), and propositional quantification (§2.3).

2.1 Completeness

A long-standing problem, raised by Fine (1989), is that standard quantified relevant logics (such as **RQ**, the quantified version of **R**) are incomplete over the standard Tarskian semantics. Mares and Goldblatt (2006) address the issue directly, by providing a semantics on which **RQ** is complete. This is one of the most important recent developments in relevant logic.

The key notion in Mares and Goldblatt's semantics is that of a *proposition*: a proposition is a set of points, but not all sets of points count as propositions. Which sets of points count as propositions is primitive to frames (although it is subject to restrictions). Sentences are interpreted as functions from variable assignments to propositions, which Mares and Goldblatt call *propositional functions*. They define operations $-$, $\cap$ and $\Rightarrow$ on sets of points (and derivatively on propositional functions), which are used to interpret negation, conjunction and implication, respectively. Models are defined to ensure that, when they include propositional functions P and Q, they also include $-P$, $P \cap Q$ and $P \Rightarrow Q$.

So far, these models correspond to Routley-Meyer models of propositional relevant logic. (In particular, $-$ is defined in terms of a $*$ operator and $\Rightarrow$ in terms of a ternary relation R.) The surprise comes when we get to $\forall_n$, the propositional function corresponding to universal quantification. Mares and Goldblatt define an operation $\sqcap$ on sets of sets of points, so that $\sqcap X$ is the greatest lower bound of X in the subset lattice on propositions. The definition ensures that, if $\bigcap X$ is a proposition, then $\sqcap X = \bigcap X$; but in general, $\sqcap X$ may be a proposition even if $\bigcap X$ is not. The propositional function $\forall_n P$ is then defined by taking this greatest lower bound of all the propositions got from assigning an individual to variable x_n in the propositional function P. In symbols, where f is a variable assignment, I the set of individuals and $f[i/n]$ is the variable assignment differing from f only in assigning individual i to variable x_n:

$$(\forall_n P)f \;=\; \bigsqcap_{i \in I} P(f[i/n])$$

Valuations V and variable assignments f assign sentences A to sets of worlds $|A|_{Vf}$. Mares and Goldblatt show that, for any sentence A, $|A|_V$ is a propositional function and $|A|_{Vf}$ is a proposition. Interestingly, if the set of propositions *Prop* contains all hereditary sets of points, or either *Prop* or I is finite, then universally quantified sentences have the standard, Tarskian truth-conditions and, as a result, the logic is incomplete. But otherwise, the quantified logic **QR** is complete over this semantics. However, **QR** is in some respects weaker than one would like: $\neg\forall x(A \vee B) \vee A \vee \forall x B$ (with x not free in A) is not derivable in **QR**. Anderson and Belnap's quantified logic **RQ** is got from **QR** by adding the principle of *existential confinement*:

$$\forall x(A \vee B) \to (A \vee \forall x B)$$

where x is not free in A. For **RQ** to be complete, two further restrictions need to be added to the semantics. First, for all points x, y, if $x \sqsubseteq y$ then $x = y$; and, if X is a proposition, then so is its boolean complement (the set of all points not in X).

Mares and Goldblatt then show that **RQ** is complete over this restricted version of the semantics.

2.2 Restricted Quantification

Beall et al. (2006) develop an account of *restricted quantification* in a relevant logic setting. They consider generalised quantifiers, of the form *all Fs are Gs*, *some Fs are Gs* and *no Fs are Gs*. They note that, if *all Fs are Gs* receives the usual quantificational treatment in terms of a material conditional, then the inference from *all Fs are Gs* and Fa to Fb will be invalid in relevant logics. But this inference seems perfectly valid, so how should a relevant logician analyse *all Fs are Gs*? Part of the problem the authors face is that it would be counter-productive to render the inference from *all Fs are Gs* and Fa to Fb valid only at the cost of re-introducing *explosion* ($A, \neg A \vdash B$) or *contraction* ($(A \to (A \to B)) \vdash A \to B$). So doing would trivialise the resulting logic in the presence of naïve set theory or the Liar or Curry paradoxes (the very cases in which weak relevant logics are supposed to triumph).

The obvious suggestion is to use a standard relevant conditional in place of the material conditional in analysing *all Fs are Gs*. But, as Beall et al. note, this will have the bizarre consequence of invalidating the inference from *everything is a G* to *all Fs are Gs*. Instead, their approach is to introduce a stronger relevant conditional, '$\mapsto$', and analyse *all Fs are Gs* as $\forall x(Fx \mapsto Gx)$. '$\mapsto$' receives the usual relevant logic truth-condition, but in terms of a ternary relation R', restricted so that $R'xyz$ only if everything true at x is also true at z. (Formally, this idea is implemented by using a truth-preserving ordering $\sqsubseteq$ on worlds, and setting $R'xyz$ iff $Rxyz$ and $x \sqsubseteq z$, where R is the usual ternary relation.) The falsity-condition for '$\mapsto$' is much closer to the classical condition, however: $A \mapsto B$ is false at x just in case A is true and B false at x. This conditional trivially validates $A \vdash B \mapsto A$, and so validates the inference from *everything is G* to *all Fs are Gs*.

Beall et al. (2006) then analyse *no Fs are Gs* as *all F are non-Gs*, i.e. $\forall x(Fx \mapsto \neg Gx)$, rather than as the negation of *some Fs are Gs*. This move validates the inference from *nothing is G* to *no Fs are Gs* and, in the presence of excluded middle, validates *either no Fs are Gs, or some Fs are Gs*.

2.3 Propositional Quantification

From the early days of relevant logic, Anderson and Belnap showed an interest in *propositional quantification*. Propositional quantifiers bind variables p, q, thought of as ranging over propositions. A number of concepts important to relevant logic can be defined using propositional quantification. An *enthymematic* or 'suppressed premise' conditional $A \supset B$ can be defined by saying that there's some true proposition whose conjunction with A implies B:

$$\exists p(p \wedge (A \wedge p \to B))$$

One can also define *falsum* as $\forall p p$, and hence can define a form of negation $\neg A$ as $A \supset \forall p p$. Interestingly, when the underlying relevant logic is **E**, the theorems in

$\supset$, $\neg$, $\wedge$ and $\vee$ coincide with propositional intuitionistic logic (Anderson and Belnap 1961).

Routley and Meyer (1973) introduced a semantics for the propositional quantifiers which takes propositions to be *hereditary sets* of points, upwards-closed with respect to the partial order $\sqsubseteq$: if $x \in P$ and $x \sqsubseteq y$, then $y \in P$. As is usual in possible-worlds accounts of propositions, entailment between propositions is just set-theoretic inclusion: P entails Q iff $P \subseteq Q$. Quantification is then essentially second-order: $\forall p$ quantifies over all subsets of the domain. On this basis, Routley and Meyer conjectured that **RP** (**R** plus propositional quantification) was incomplete on their semantics. Kremer (1993) later showed that they were right.

So the question arises, is there an alternative semantics (based on Routley-Meyer relational models) for which **RP** is complete? Goldblatt and Kane (2010) shows that there is. The key idea in this semantics is that not all hereditary sets of points should count as propositions. Each model comes with a fixed set *Prop* of propositions (as in Mares and Goldblatt 2006, §2.1 above), which are both hereditary and closed under the operations used to interpret the connectives. The propositional quantifiers $\forall p$ and $\exists p$ are then interpreted as the greatest lower bound and the least upper bound, respectively, in the subset lattice on *Prop* (as in §2.1). When *Prop* does not include all hereditary sets, this need not be the intersection of those propositions, for that intersection need not be an element of *Prop*. So the proposition expressed by $\forall p A$ will in general be stronger than the conjunction of all propositions resulting from substituting propositional constants for the variable p in A. This is what we should expect: $\forall p A$ should entail all its instances, but needn't be entailed by their (infinite) conjunction.

Using a semantics along these lines, Goldblatt and Kane give completeness results for a range of logics, including **RP**. They also show that **EP** (as well as **RP**) is incomplete on these semantics when *Prop* includes all hereditary sets. In other words, the restriction placed on *Prop* is essential to obtaining a complete system.

3 Other Formal Work

3.1 Decidability results

One of the most important formal results of the last few years in relevant logic is the discovery that the implication fragment of *ticket entailment*, $\mathbf{T}_\rightarrow$, is decidable. This was independently discovered by Bimbó and Dunn (2013) and Padovani (2012). $\mathbf{T}_\rightarrow$ has four axiom schemes:

(B) $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$

(B′) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

(I) $A \rightarrow A$

(W) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

(a.k.a. prefixing, suffixing, identity and contraction) and uses *modus ponens* as its only rule. Ticket entailment **T** and its implication fragment $\mathbf{T}_{\rightarrow}$ were introduced by Anderson and Belnap (1975), who also raised (but did not solve) the question of their decidability. It has been known for some time now that $\mathbf{R}_{\rightarrow}$ and $\mathbf{E}_{\rightarrow}$, the implication fragments of the relevant logics **R** and **E**, are decidable (Kripke 1959), whereas neither **R** nor **E** are themselves decidable (Urquhart 1984). Urquhart also showed that **T** is not decidable (1984), but the decidability of $\mathbf{T}_{\rightarrow}$ proved much more difficult to establish.

The problem is important, in part because it is equivalent to an important type inhabitation problem in combinatory logic. Such logics consist of atomic combinators, each with a fixed type. Then the question arises: given some arbitrary type τ, is there a way of combining the available atomic combinators to produce a term of type τ? This is the type inhabitation problem for a given basis of atomic combinators (see, e.g., Trigg et al. 1994). If we take the combinators B, B′, I and W as our basis (whose types correspond to the axioms listed above), then the type inhabitation problem for this basis is equivalent to the decidability of $\mathbf{T}_{\rightarrow}$.

Bimbó and Dunn (2013) and Padovani (2012) established that $\mathbf{T}_{\rightarrow}$ is decidable using different techniques. Bimbó and Dunn (2013) proceed by reducing the decidability problem for $\mathbf{T}_{\rightarrow}$ to the decidability problem for $\mathbf{R}_{\rightarrow}$. Padovani (2012), by contrast, exploits the equivalence of the $\mathbf{T}_{\rightarrow}$ decidability problem with the type inhabitation problem for hereditary right-maximal terms of the λ-calculus (Broda et al. 2004). Broda et al. had previously solved a restricted version of the latter problem (2004); Padovani established the result in full generality.

3.2 Consistency Results

One application of (weak) relevant logics is to naïve set theory, a theory of sets containing the comprehension axiom

$$\exists x \forall y (y \in x \leftrightarrow Ay)$$

As is well known, this axiom trivialises classical set theories. But when the underlying logic is a weak relevant logic, triviality can be avoided. There is a question of just how weak this underlying logic needs to be to avoid triviality. The logic cannot contain $A \vee \neg A$, $(A \wedge (A \rightarrow B)) \rightarrow B$, $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ or $(A \rightarrow \neg A) \rightarrow \neg A$. But how strong can it be, before triviality occurs? Brady (1971; 1983; 2006) has investigated this question in depth. Using 3-valued semantic machinery, he has shown that various weak relevant logics, including $\mathbf{TW^dQ}$ and $\mathbf{TN^dQ}$, do not trivialise naïve set theory. These results are important, but the way they are achieved is highly complex. It is not clear how to extend Brady's methods to other logics, for example.

More recently, Brady (2013) shows how to use *metavaluations* instead of model-theoretic valuations to give consistency proofs for naïve set theory. Metavaluations are a way of capturing theoremhood, so that $vA = 1$ iff $\vdash A$ in the logic in question. Brady makes use of Slaney's (1984) second metavaluation v^*, which captures the non-theoremhood of negations: $v^*A = 1$ iff $\nvdash \neg A$. The aim of a consistency proof is then to show that $vA = 1$ only if $v^*A = 1$.

Using this technique, Brady (2013) shows that a range of logics are able to support a consistent naïve set theory. This includes logics stronger than those previously shown to be consistent with naïve set theory. In particular, Brady's metavaluation method shows that some logics containing $A \to (B \to A)$ are consistent with naïve set theory (which his previous model-theoretic method was unable to show). Perhaps most interestingly of all, Brady shows that the logic $\mathbf{TJ^d}$ plus $A \to ((A \to (B \to C)) \to (B \to C))$ supports consistent naïve set theory. This logic is in some ways close to **R**: indeed, replacing $B \to C$ with B in the previous axiom gives **R**. (In other ways it is much weaker than **R**, since it does not contain $A \vee \neg A$, $(A \wedge (A \to B)) \to B$, $(A \to (A \to B)) \to (A \to B)$ or $(A \to \neg A) \to \neg A$.)

3.3 Heyting Implication

As already noted in §2.2, there are affinities between the Routley-Meyer semantics and the Kripke semantics for intuitionistic logic. This is especially so when we place a truth-preserving ordering $\sqsubseteq$ on points in a Routley-Meyer model, for then Routley-Meyer models contain all the semantic machinery needed for interpreting the intuitionistic connectives. So the question arises: what would happen if we were to extend a relevant logic with intuitionistic connectives? Would we then be able to prove sentences (stated without using the new intuitionistic connectives) which we couldn't prove before?

Of particular interest in this regard is *Heyting implication*, $A \supset B$, which can be defined proof-theoretically as follows:

$$A \vdash B \supset C \qquad \text{iff} \qquad A \wedge B \vdash C \tag{1}$$

When models include a truth-preserving partial order $\sqsubseteq$ on points, $\supset$ can be interpreted in the way familiar from Kripke semantics for intuitionistic logic:

$$x \vDash A \supset B \text{ iff, for all points } y,\ x \sqsubseteq y \text{ and } y \vDash A \text{ only if } y \vDash B \tag{2}$$

Restall (1998) showed that the addition of Heyting implication (via equation 1) to the propositional relevant logic **R** is a *conservative* addition. If a $\supset$-free sentence A is not a theorem of **R** before the addition of $\supset$ to the language, then A will not become a theorem once $\supset$ is added. The argument here is wonderfully elegant. If $\supset$-free A is not a theorem of **R**, then it is not satisfied by some Routley-Meyer model. Since we can interpret $\supset$ in that model using equation 2, all one needs is to show that the axioms and rules of **R** (in the language with $\supset$) remain sound on such models. That comes down to showing that $\sqsubseteq$ is truth-preserving: if $x \vDash A$ and $x \sqsubseteq y$ then $y \vDash A$. The interesting new case to consider is when A is $B \supset C$ which, given equation 2, follows from the transitivity of $\sqsubseteq$. When we consider a first-order logic such as **RQ**, however, this strategy won't go through so easily. This is because the soundness of the quantifier axioms do not depend only on $\sqsubseteq$ preserving truth.

Goldblatt (2009) shows that the addition of $\supset$ is conservative for a large class of quantified relevant logics with an identity predicate $\approx$, including $\mathbf{RQ}^{\approx}$. But it remains an open question whether **RQ**, without an identity predicate, can be conservatively extended by Heyting arithmetic.

3.4 Four-Valued Semantics

Several approaches, going back to Dunn 1976, use a four-valued approach to negation in relevant logic. (This is not the only approach: perhaps the more popular is to use the 'Routley star' operator on points. I won't discuss this approach here.) On four-valued approaches, a valuation v makes independent assignments of the values *true* and *false* to primitives (so that each may receive just one, both or neither of these values). Connectives are then given truth-conditions and falsity-conditions. Dunn's clauses for negation are the usual ones from paraconsistent logic:

$$v(\neg A) = \textit{true} \text{ iff } v(A) = \textit{false}$$
$$v(\neg A) = \textit{false} \text{ iff } v(A) = \textit{true}$$

There is a problem, however, in the treatment of relevant implication. The truth-condition for $\rightarrow$ is standardly given using the ternary relation R. But what should the falsity-condition for $\rightarrow$ be? Routley (1984) uses a second ternary relation R', which greatly complicates the meta-logical reasoning. Restall (1995a) gives a falsity-clause for $\rightarrow$ using a single ternary relation R, but his semantics is not adequate for strong relevant logics such as **R**. The problem, in short, is getting $\rightarrow$ and $\neg$ to interact as they should. (In **R**, this includes validating the contraposition and reductio axioms.)

The problem seems to boil down to this. It is hard to ensure that, if A and B are true at exactly the same points, then $\neg A$ and $\neg B$ are also true at exactly the same points. Mares (2004a) notices that this can be dealt with automatically by working with propositions qua sets of points. This is the general approach of *neighbourhood semantics* (Scott 1970). On this approach, equivalent propositions are identical. If A and B are true at exactly the same points, they correspond to the same proposition. Negation is thought of as a semantic operator on propositions: the proposition expressed by $\neg A$ is the result of applying that semantic operator to whatever proposition is expressed by A. So, if A and B express the same proposition, then so must $\neg A$ and $\neg B$. This will guarantee that $\neg A$ and $\neg B$ are true at exactly the same points whenever A and B are. Mares (2004a) thus combines neighbourhood semantics with a single ternary relation R for interpreting implications. The approach is four-valued in the sense that the proposition expressed by $\neg A$ is not always the boolean compliment of the proposition expressed by A. Exactly one, both or neither of a proposition and its negation may be true at a point.

Semantic operations $\Rightarrow$, $\circ$ and ^{F} are then defined on propositions P and Q:

- $P \Rightarrow Q$ is the set of worlds x such that, for any worlds y and z such that $Rxyz$, if $y \in P$ then $z \in Q$.
- $P \circ Q$ is the set of worlds x such that there are worlds y and z such that $Rxyz$, $y \in P$ and $z \in Q$.
- ^{F} is an operator such that $P^{F^F} = P$, $(P \cap Q)^F = (P^F \cup Q^F)$, and $(P \Rightarrow Q)^F = (P \circ Q^F)$.

These operators are used to interpret implication, fusion (i.e., intentional conjunction) and negation, respectively. (Conjunction and disjunction are interpreted via union and intersection, respectively.)

This approach provides semantics on which **R** is sound and complete. And, by altering the properties of the ternary relation R, it can also provide semantics on which the contraction-free **RW** is sound and complete.

3.5 Natural-Deduction-Based Semantics

Brady (2010) develops a form of semantics which is heavily based on the natural deduction proof theory for the logics in question (which in Brady's case, are weak relevant logics such as **DW** and **DJ**). In relevant natural deduction systems, sentences are written with subscripts, or *indices*, which indicate which assumptions were used to derive that sentence. An assumption must be used in deriving some sentence for it to count as one of the premises proving that conclusion. Similarly, in using *conditional proof* to derive $A \to B$, one must genuinely use the assumption that A in the sub-derivation of B. B's indices must include the index on the assumption that A. For a sentence A to be a theorem, it must have a derivation ending in $A_\varnothing$, i.e., with no indices remaining on A.

Brady uses this idea to construct a semantics in which truth-values (1 and 0) are assigned to pairs of signed sentences and sets of indices. A signed sentence is an unsigned sentence prefixed with either a 'T' or an 'F'. Thus, a valuation v may set $v(\mathrm{T}A, x) = 1$, where x is a set of indices (natural numbers, in this case). The use of signed sentences allows for (what corresponds to) a four-valued semantics. Relative to a set of indices x, a sentence A may be just true, just false, both true and false (when $v(\mathrm{T}A, x) = v(\mathrm{F}A, x) = 1$) or neither true nor false (when $v(\mathrm{T}A, x) = v(\mathrm{F}A, x) = 0$). The clause for true implications $\mathrm{T}A \to B$, for example, is as follows:

$$v(\mathrm{T}A \to B, x) = 1 \text{ iff } v(\mathrm{T}A, n) = 1 \text{ only if } v(\mathrm{T}B, x \cup \{n\}) = 1$$

In other words, $A \to B$ is true relative to an index x just in case, on the assumption (given index n) that A is true, it follows that B is true relative to the combined index $x \cup \{n\}$. This semantic clause corresponds very closely to *conditional proof*, the introduction rule for $\to$.

This approach calls out for a philosophical interpretation: what are proof indices doing in the semantics? Mares (2010) provides one such interpretation, on which

> A hypothesis $A_{\{k\}}$ in a derivation in the natural deduction system is interpreted as supposing that there is a world in which there is a situation s_k that contains the information that A. (Mares 2010, 211)

On this view, the natural deduction system is interpreted as a system for reasoning explicitly about information holding in situations. This brings the natural deduction system for relevant logic much closer to the standard ternary relation semantics.

3.6 Disjunctive Syllogism

Disjunctive syllogism, the inference $A \vee B, \neg A \vdash B$, is not derivable in the relevant logics **E** or **R** (and hence not in weaker relevant logics either). Neither is $(A \vee B) \wedge \neg A \to B$ a theorem of **R** or **E**. (If it were, one could prove the irrelevant $A \wedge \neg A \to B$, which is precisely what relevant logicians want to avoid.) Nevertheless, disjunctive syllogism is *admissible* in some relevant logics. This means that the rule is theorem-preserving: if both $A \vee B$ and $\neg A$ are theorems, then B is a theorem too. We write this rule (which I'll call DS) as follows:

$$\frac{\vdash A \vee B \qquad \vdash \neg A}{\vdash B}$$

(This rule has the same form as *generalisation* is modal logic: the inference from $\vdash A$ to $\vdash \Box A$.)

Robles and Méndez (2010; 2011) consider weaker relevant logics for which admissibility of disjunctive syllogism is an open question, and consider what happens when DS is added as a primitive rule. These logics include **B** with the reductio rule (which they call $\mathbf{B}_r$), and **TWR** (ticket entailment without contraction but with the reductio axiom). For each logic, they construct a Routley-Meyer ternary semantics on which the logic is sound and complete.

This does not tell us whether disjunctive syllogism is admissible in those base logics ($\mathbf{B}_r$ and **TWR**). Nor is the fact that those logics (with DS added) have a Routley-Meyer ternary relation semantics particularly interesting in itself. The ternary-relation approach to semantics is extremely powerful and versatile: many logics (including classical, intuitionistic and various modal logics) can be given ternary relation semantics. What would be interesting is to discover just how weak a relevant logic needs to be before disjunctive syllogism becomes inadmissible. That's an interesting question due to a deep relationship between DS and the cut rule. In Genzen's classical sequent calculus LK, the cut rule is formulated as:

$$\frac{\Gamma_1 \vdash A, B \qquad \Gamma_2, A \vdash \Delta}{\Gamma_1, \Gamma_2 \vdash B, \Delta}$$

A special case of this is:

$$\frac{\vdash A, B \qquad A \vdash}{\vdash B}$$

which, given the classical sequent rules for $\vee$ and $\neg$, becomes DS. This will be an interesting avenue for researchers to explore in the future.

4 Interpreting the Semantics

One of the most important developments in relevant logic was the Routley-Meyer semantics (Routley and Meyer 1972a;b; 1973) for $\to$. This semantics gives pride of place to a ternary relation R on points (just as Kripke semantics uses a binary relation). These points might be thought of as possible worlds, situations or 'set-ups'. The key semantic clause for $\rightarrow$ is:

$$x \vDash A \rightarrow B \text{ iff, for all } y, z\text{: } y \vDash A \text{ only if } z \vDash B \tag{3}$$

This form of semantics turns out to be incredibly powerful and adaptable. By imposing a number of conditions on R, one can obtain semantics for a range of logics (including, but not limited to, many relevant logics) of $\rightarrow$. That gives a strong pragmatic justification for the ternary semantics. But many feel it needs a philosophical justification as well. This would be some account of what the points are, and what it means for them to stand in relation R to one another. The paradigm here is the possible worlds interpretation of Kripke semantics for modal logic, on which the points are possible worlds and (binary) R is a relation of relative possibility: Rwu says that world u is possible, from the standpoint of w.

In this section, I'll discuss several attempts to give a similar interpretation of the ternary relation R of the Routley-Meyer semantics.

4.1 Information-Based Interpretations

A popular way to interpret the ternary relation R is in terms of information. One can view the points x, y, z as pieces of information, with R saying that z contains the combination of the information in x and y. This idea goes back to Urquhart's semilattice semantics (1972), on which $x \vDash A \rightarrow B$ iff, for any y, $y \vDash A$ only if $x \sqcup y \vDash B$. Here, $x \sqcup y$ is thought of as the combination of the information in x and y. (We can capture Urquhart's idea in terms of the ternary relation by setting $Rxyz$ when $x \sqcup y = z$. However, it turns out to be better to think of z as containing, but perhaps not being identical to, the combined information in x and y, so that $Rxyz$ iff $x \sqcup y \sqsubseteq z$.) Alternatively, one can think of $Rxyz$ as saying that x is an information channel between information sites y and z (Restall 1995b). On this reading, $A \rightarrow B$ holds at x when x connects A-sites only to B-sites.

More recently, Mares (2004b; 2009; 2010) offers an alternative information-theoretic interpretation of R in terms of *situated inference*. He says

> a situation x can be said to contain the information that $A \rightarrow B$ if on the hypothesis that there is an y in the same world that contains A, we can derive that there is a situation z in the same world in which B. ... [This theory] is about making inferences from the perspective of situations about the situations in a world. (Mares 2010, 211)

This analysis interprets R in terms of what can be derived from what, so that $Rxyz$ says 'all the information that we can derive really using the information in both x and y is all contained in z' (Mares 2010, 212). Mares thinks of this in terms of *situated inference*, facilitated by 'informational links' (Mares 2004b). An informational link is a 'perfectly reliable connection, such as a law of nature or a convention' (Mares 2009, 426). A sufficient condition for $Rxyz$, for example, is that a law of nature of x's world relates y to z.

These various ways of thinking about R in terms of information have been popular, and they allow for interplay between relevant logic and other theoretical frameworks for reasoning about information, such as *situation semantics* (Barwise and Perry 1983). There is a worry with any such interpretation, however. Information (as the term is used by Mares, at least) has to be cognitively accessible, for 'what counts as a situation depends on the discriminatory capacities of human beings' (Mares 2009, 350). So, if situation x carries the information that A, then it should be possible for someone in x to get the information that A, and hence to come to know that A. But this idea is in tension with excluded middle, $A \vee \neg A$, which is valid in strong relevant logics (including **E** and **R**). There's no reason to think that there is a situation in which, for every A, either the information that A or the information that $\neg A$ is available.

One can reply (correctly) that the semantics for **R** or **E** doesn't require every point (situation) to support excluded middle. In fact, to avoid the paradoxes of material implication, it is vital that some points are incomplete (i.e., neither A nor $\neg A$ holds there). Points are usually divided into 'normal' and 'non-normal', with validity determined by the normal points only. The objection is that (for logics like **R** and **E**) this semantics requires normal points to support excluded middle and yet, on the current interpretation, there can be no such points. (The objection does not threaten weaker relevant logics in which excluded middle is not a theorem.) In §4.3 below I'll consider an alternative interpretation of R which avoids this worry.

4.2 Conditionality Interpretations

Beall et al. (2012) take a different approach to interpretations of R in the ternary semantics. They argue that whichever way we think of conditionality in general gives us a suitable interpretation of R. They consider three general ways of thinking about conditionality:

(i) as the exclusion of counterexamples;

(ii) as an operator or function; and

(iii) as the kind of notion supported by conditional logic.

I'll consider options (i) and (ii) only here.

Suppose, along with reading (i), we think of conditionality as the ruling out of certain situations: $A \to B$ says that there are no A-situations which are not also B-situations. That's a very classical way of thinking about '$\to$', where a counterexample to $A \to B$ is any situation where A is true but B is false. One can also think of a modal strict conditional $\Box(A \to B)$ in these terms: y is a counterexample to $\Box(A \to B)$ at x when y is accessible from x and A but not B is true at y.

The difficulty with running this interpretation in the case of the ternary relation semantics is that the points of evaluation of antecedent and consequent may differ. To check whether $A \to B$ holds at x, we need to check for A at y and B at z

whenever $Rxyz$. A counterexample to $A \to B$ at x, therefore, depends on what goes on at some pair of points y and z. In just this way, Beall et al. (2012) propose to treat 'split points' $\langle yz \rangle$ as potential counterexamples. Truth and falsity at a split point $\langle yz \rangle$ are fixed by truth at y and falsity at z, respectively. $Rxyz$ then says that $\langle yz \rangle$ is accessible (or possible relative to) x. So, just as in the modal strict conditional case, $\langle yz \rangle$ is a counterexample to $A \to B$ at x when $\langle yz \rangle$ is accessible from x, A is true at $\langle yz \rangle$ but B is false there. We then get the standard Routley-Meyer clause (3) for $\to$.

If this approach is to provide a philosophical interpretation of R (as opposed to a useful bit of formal machinery), then the notion of a split point must be well understood. Notice that $\langle yz \rangle$ cannot in general be thought of as the pair of points y, z, for the pair y, z is identical to the pair z, y. But not so for split points, in which the order of the points is essential (as the definitions of truth and falsity at a split point make clear). For the same reason, we can't think of $\langle yz \rangle$ as the (mereological) composition of y and z. We might think of $\langle yz \rangle$ as some sort of list or sequence of y and then z. But what is a list or sequence of two situations, and in what sense are sentences true or false relative to such lists or sequences? In general, one needs a story on which y and z, *taken in that order*, constitute a counterexample to $A \to B$, which does not assume that they constitute a counterexample when taken the other way around.

Now let's turn to Beall et al.'s second option for interpreting R. This involves thinking of the conditional $A \to B$ in terms of an operator or function, taking us from A to B. Intuitionists might think of this in terms of a function from a proof of A to a proof of B, for example. More generally, suppose it makes sense to *apply* one situation y to another x, as if x were a function and y an argument. And suppose the 'result' of this application, xy, is in some sense contained in some other situation, z. Let's notate this $xy \sqsubseteq z$. If this makes sense, then we can set $Rxyz$ when $xy \sqsubseteq z$. This gives us an understanding of R in terms of functional application and containment relations between situations. It's easy to make sense of these notions when the 'situations' are proofs, programs, sets of evidence, or other syntactic constructions. This provides good reason to think that intuitionists and other constructivists can make sense of the ternary relation in this way.

This way of interpreting R resurrects the worry from §4.1, however. Suppose x and y are the kinds of entries which can be applied functionally to one another, such as proofs or sets of evidence. What justification do we then have for thinking that there's some such x at which, for every A, either A or $\neg A$ holds? If there are no such points, then excluded middle cannot be valid and we will be unable to give semantics along these lines for strong relevant logics such as **R**. I'll now turn to an interpretation of R which avoids this worry.

4.3 The Truthmaker Interpretation

An important feature of the points x, y, z in the ternary semantics is that they may be *partial*: it may be that neither A nor $\neg A$ holds at some point x. Such points have a natural interpretation in epistemic terms, as information states, evidence or proofs. But, I've claimed, such interpretations lead to problems in justifying excluded middle. This suggests that a non-epistemic interpretation of partial states would be preferable, at least when considering strong relevant logics.

Restall (1996) suggests one such reading: the points are *truthmakers* (facts, states of affairs, or whatever else does truthmaking work). Restall briefly describes a truthmaker semantics which gives the first-degree fragment (i.e., without embedded conditionals) of the logic **RM**. (van Fraassen (1969) had already spotted that a facts-based approach can give semantics for Anderson and Belnap's first-degree entailment.) Jago (2012) develops the idea further, using a truthmaking semantics for $\to$ based on Urquhart's semilattice semantics (Urquhart 1972).

This approach can overcome the excluded middle worry I raised in §4.1 for epistemic interpretations of points. A truthmaker for A will typically not be a truthmaker for B or for $\neg B$, unless there is some close relationship between A and B. So many truthmakers satisfy the partiality requirement. Yet plausibly, there are 'big truthmakers', such as whole possible worlds, which do make each instance $A \vee \neg A$ true. These can serve as the 'normal' (validity-determining) points in the semantics.

This approach has not been investigated in much detail, but it promises a philosophical interpretation of relevant logic in terms of familiar truth-like notions. It also suggests that principles of relevant logic are pertinent to the metaphysical debate over truthmaking. (The metaphysical debate between Rodriguez-Pereyra (2009) and Jago (2009) over truthmaking would become a debate over the properties of $\sqcup$ in the relevant logic semantics, for example.) As Restall says, the approach is of interest 'to all those who seek to understand contemporary work on relevant logic, and for those who wish to form a robust theory of truthmaking' (Restall 1996, 339).

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References

Anderson, A. and Belnap, N. (1962). Tautological entailments, *Philosophical Studies* 13(1): 9–24.

Anderson, A. and Belnap, N. (1963). First degree entailments, *Mathematische Annalen* 149(4): 302–319.

Anderson, A. R. and Belnap, N. D. (1961). Enthymemes, *The journal of philosophy* 58(23): 713–723.

Anderson, A. R. and Belnap, N. D. (1975). *Entailment—the logic of relevance and necessity*, Princeton University Press, Princeton, NJ.

Barwise, J. and Perry, J. (1983). *Situations and Attitudes*, Bradford Books, MIT press.

Beall, J., Brady, R., Dunn, J. M., Hazen, A., Mares, E., Meyer, R. K., Priest, G., Restall, G., Ripley, D., Slaney, J. et al. (2012). On the ternary relation and conditionality, *Journal of philosophical logic* 41(3): 595–612.

Beall, J., Brady, R. T., Hazen, A. P., Priest, G. and Restall, G. (2006). Relevant restricted quantification, *Journal of Philosophical Logic* 35(6): 587–598.

Bimbó, K. and Dunn, J. M. (2013). On the decidability of implicational ticket entailment, *Journal of Symbolic Logic* 78(1): 214–236.

Brady, R. (2006). *Universal Logic 2006*, CSLI Publications, Stanford, CA.

Brady, R. T. (1971). The consistency of the axioms of abstraction and extensionality in a three-valued logic., *Notre Dame Journal of Formal Logic* 12(4): 447–453.

Brady, R. T. (1983). The simple consistency of a set theory based on the logic CSQ, *Notre Dame Journal of Formal Logic Notre-Dame, Ind.* 24(4): 431–449.

Brady, R. T. (2010). Free semantics, *Journal of philosophical logic* 39(5): 511–529.

Brady, R. T. (2013). The simple consistency of naive set theory using metavaluations, *Journal of Philosophical Logic* pp. 1–21.

Broda, S., Damas, L., Finger, M. and Silva e Silva, P. (2004). The decidability of a fragment of BB′IW-logic, *Theoretical computer science* 318(3): 373–408.

Dunn, J. (1976). Intuitive semantics for first-degree entailments and coupled trees, *Philosophical studies* 29(3): 149–168.

Dunn, J. and Restall, G. (2002). Relevance logic, in D. Gabbay and F. Guenthner (eds), *Handbook of Philosophical Logic*, second edn, Vol. 6, Kluwer Academic, Dordrecht, pp. 1–136.

Fine, K. (1989). Incompleteness of quantified relevance logics, in J. Norman and R. Sylvan (eds), *Directions in relevant logic*, Kluwer, Dordrecht.

Goldblatt, R. (2009). Conservativity of heyting implication over relevant quantification, *Review of Symbolic Logic* 2: 310–341.

Goldblatt, R. and Kane, M. (2010). An admissible semantics for propositionally quantified relevant logics, *Journal of philosophical logic* 39(1): 73–100.

Jago, M. (2009). The conjunction and disjunction theses, *Mind* 118: 411–415.

Jago, M. (2012). Truthmaking logics. Unpublished draft.

Kremer, P. (1993). Quantifying over propositions in relevance logic: nonaxiomatisability of primary interpretations of $\forall p$ and $\exists p$, *The Journal of symbolic logic* 58(1): 334–349.

Kripke, S. A. (1959). The problem of entailment, *Journal of Symbolic Logic* 24(4): 324.

Mares, E. (2004a). 'Four-valued' semantics for the relevant logic **R**, *Journal of Philosophical Logic* 33(3): 327–341.

Mares, E. (2004b). *Relevant Logic: A Philosophical Interpretation*, Cambridge University Press.

Mares, E. (2010). The nature of information: a relevant approach, *Synthese* 175: 111–132.

Mares, E. (2012). Relevance logic, in E. N. Zalta (ed.), *The Stanford Encyclopedia of Philosophy*, summer 2012 edn.

Mares, E. D. (2009). General information in relevant logic, *Synthese* 167(2): 343–362.

Mares, E. D. and Goldblatt, R. (2006). An alternative semantics for quantified relevant logic, *The Journal of Symbolic Logic* 71: 163–187.

Padovani, V. (2012). Ticket entailment is decidable, *Mathematical Structures in Computer Science* 1(1): 1–40.

Priest, G. (2008). *An introduction to non-classical logic: from if to is*, Cambridge University Press.

Read, S. (1988). *Relevant logic*, Blackwell Oxford.

Restall, G. (1995a). Four-valued semantics for relevant logics (and some of their rivals), *Journal of Philosophical Logic* 24(2): 139–160.

Restall, G. (1995b). Information flow and relevant logics, in J. Seligman and D. Westerståhl (eds), *Logic, language and computation: The 1994 Moraga Proceedings*, CSLI Press, pp. 463–477.

Restall, G. (1996). Truthmakers, entailment and necessity, *Australasian Journal of Philosophy* 74(2): 331–40.

Restall, G. (1998). Displaying and deciding substructural logics 1: Logics with contraposition, *Journal of Philosophical Logic* 27(2): 179–216.

Robles, G. and Méndez, J. M. (2010). A Routley-Meyer type semantics for relevant logics including Br plus the disjunctive syllogism, *Journal of philosophical logic* 39(2): 139–158.

Robles, G. and Méndez, J. M. (2011). A Routley-Meyer semantics for relevant logics including TWR plus the disjunctive syllogism, *Logic Journal of IGPL* 19(1): 18–32.

Rodriguez-Pereyra, G. (2009). The disjunction and conjunction theses, *Mind* 118: 427–443.

Routley, R. (1984). The american plan completed: alternative classical-style semantics, without stars, for relevant and paraconsistent logics, *Studia Logica* 43(1-2): 131–158.

Routley, R. and Meyer, R. (1972a). The semantics of entailment II, *Journal of Philosophical Logic* 1: 53–73.

Routley, R. and Meyer, R. (1972b). The semantics of entailment III, *Journal of philosophical logic* 1: 192–208.

Routley, R. and Meyer, R. (1973). The semantics of entailment I, in H. Leblanc (ed.), *Truth, Syntax, and Semantics*, North-Holland, pp. 194–243.

Routley, R., Plumwood, V. and Meyer, R. K. (1982). *Relevant logics and their rivals*, Ridgeview Publishing Company.

Scott, D. (1970). Advice on modal logic, *Philosophical problems in logic*, Springer, pp. 143–173.

Slaney, J. K. (1984). A metacompleteness theorem for contraction-free relevant logics, *Studia Logica* 43(1-2): 159–168.

Trigg, P., Hindley, J. R. and Bunder, M. W. (1994). Combinatory abstraction using B, B′ and friends, *Theoretical computer science* 135(2): 405–422.

Urquhart, A. (1972). Semantics for relevant logics, *Journal of Symbolic Logic* 37(1): 159–169.

Urquhart, A. (1984). The undecidability of entailment and relevant implication, *Journal of Symbolic Logic* pp. 1059–1073.

van Fraassen, B. (1969). Facts and tautological entailments, *Journal of Philosophy* 66(15): 477–487.