



# Type-2 Fuzzy Sets and Newton's Fuzzy Potential in an Algorithm of Classification Objects of a Conceptual Space

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## Abstract

This paper deals with Gärdenfors' theory of conceptual spaces. Let  $\mathcal{S}$  be a conceptual space consisting of 2-type fuzzy sets equipped with several kinds of metrics. Let a finite set of prototypes  $P_1, \dots, P_n \in \mathcal{S}$  be given. Our main result is the construction of a classification algorithm. That is, given an element  $\tilde{A} \in \mathcal{S}$ , our algorithm classifies it into the conceptual field determined by one of the given prototypes  $\tilde{P}_i$ . The construction of our algorithm uses some physical analogies and the Newton potential plays a significant role here. Importantly, the resulting conceptual fields are not convex in the Euclidean sense, which we believe is a reasonable departure from the assumptions of Gärdenfors' original definition of the conceptual space. A partitioning algorithm of the space  $\mathcal{S}$  is also considered in the paper. In the application section, we test our classification algorithm on real data and obtain very satisfactory results. Moreover, the example we consider is another argument against requiring convexity of conceptual fields.

**Keywords** Type-2 fuzzy sets · Fuzzy sets · Fuzzy set valued function · Conceptual spaces · Prototypes · Fuzzy Newton's potential

**Mathematics Subject Classification** 68T30 · 68T37 · 68W05

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# 1 Introduction

## 1.1 General Overview of the Paper

In his pioneering book (Gärdenfors, 2000) Gärdenfors introduced the theory of some geometric structures which he called the conceptual spaces and showed their application to knowledge representation. More formal definition of the conceptual space is given in Sect. 2 (see also Rickar 2006; Rickard et al. 2007 for more information). A conceptual space is a multidimensional (usually metric) space  $\mathcal{S}$  such that its every dimension (or a finite set of dimensions) is understood as some measurable (or observable), numeric (or symbolic) information/property and is called a *quality*. Pairwise disjunct regions in  $\mathcal{S}$  are called *concepts* (or *conceptual fields*). This notion originate from the prototype theory of Kamp and Partee (1995). Following (Kamp & Partee, 1995) and linguists' language the most representative subset of a concept is called its *prototype*. Typically, in most of the literature the prototype is considered as a point but here we relax this constraint because it is more natural in our setup. The motivation for this decision will become clear later.

Since the beginning of Gärdenfors theory one of the most controversial constraint in his definition concerns convexity of the conceptual fields. One side of this dispute, with Gärdenfors at the helm, proclaimed that if two objects  $O_1, O_2$  belong to the same concept  $C$  then all objects located between them (in the geometric sense) should also have the same properties and therefore be inside  $C$ . The others like (Bechberger & Kühnberger, 2017; Hernández-Conde, 2017) reject this claim. Here we present solution that fit somewhere in between. We recommend reading recent paper (Ströbner, 2022), which considers several versions of convexity in the context of conceptual spaces.

In this paper we consider a conceptual space  $\mathcal{S}$  whose elements are tuples constructed from a point  $x$  in some (metric) space  $X$  and a fuzzy number  $x^* \in \mathcal{F}_1([0, 1])$  (the set of type-1 fuzzy sets with their supports in  $[0, 1]$ ). Collection of such elements (where  $x$  are distinct) can and will be considered as type-2 fuzzy sets (which were defined by Zadeh in 1975 in Zadeh (1975a, b, c)).

Specifically, this paper considers the problem of classification of a given type-2 fuzzy set  $A$  into the conceptual field defined by one of the given prototypes  $\tilde{P}_i$ ,  $1 \leq i \leq n$ . Since every point of  $\mathcal{S}$  is a fuzzy set it is more natural for a prototype to be also a fuzzy set and not just a crisp number.

Typical procedure of assigning an object to some concept is based on the Voronoi tessellation method (Okabe et al., 2011). In this paper in order to divide the space  $\mathcal{S}$  into partitions we use fuzzy Newton's potential operator, type-2 and type-1 fuzzy sets. The concepts obtained by our algorithm are convex but only in a more general geodesic sense. Basically in our setting the conceptual space is equipped with additional fuzzy information - the degree of membership. It can be treated as a density of the mass spread across some region in the space. Then it seems natural for computing the shortest path between two objects to use method that takes into account gravitation of the space. Specifically, the metric space most commonly used in applications is the Euclidean space. But Euclidean space is a space equipped with geometry related to

the zero-curvature metric. It is an example of one of the simplest Riemannian space. In this space a curve connecting two points and having the shortest length (so called *geodesic curve*) is a segment (a fragment of a straight line). The convexity becomes a *geodesic convexity* in Riemannian spaces (see do Carmo 1992). For a set in a non-Euclidean space to be geodesically convex is far from being just convex in Euclidean sense.

The novelty of this work is that although our conceptual space can be equipped with different metrics, in the algorithm classifying the elements of conceptual space into the concept fields we do not use metrics directly. Instead, we use a function of the metric - namely generalized Newton's potential (or more appropriately - fuzzy Newton's potential). In this way, our partition of the conceptual space into conceptual fields is significantly different from the partitioning obtained by means of the classical Voronoi diagram method.

Considering such a space and finding analog of fuzzy Newton's potential defined in Urban and Grzelińska (2017) for our structure which, in turn, allowed us to construct a classification algorithm is the main achievement of this work. Our approach (cf. Urban and Grzelińska 2017) involves a fundamental departure from the convexity of dimensions and/or conceptual fields. This departure is also shared by Hernández-Conde (2017) (see Gärdenfors' answer to Hernández-Conde Gärdenfors (2019)).

In a broad sense, the problem of classifying a given object into appropriate domain is closely related to the problem of data classification (Gordon, 1999). Recently, the theory and application of data classification has attracted the interest of many scientists from different areas, who are working on related problems, e.g. statistics, economics, sociology, psychology, linguistics, and AI (see e.g. Aggarwal 2015; Dougherty 2013; Suthaharan 2016).

Our algorithm, as we will show, can also be used to obtain a partitioning of a given space into concept fields for the prototypes consisting of a finite set of type-2 fuzzy sets. But as in classical approaches problem of unclassified boundaries remains unresolved (cf. Douven et al. 2013; Dietz 2003.)

## 1.2 Why We Gave Up Strict Euclidean Convexity?

We say that a subset  $X \subseteq \mathbb{R}^D$  is *convex* if for all  $x_1, x_2 \in X$  its convex combination of the form  $\alpha x_1 + (1 - \alpha)x_2$  is in  $X$  for all  $\alpha \in [0, 1]$ . In our setup we are considering sets from  $\mathcal{S} = \mathbb{R}^D \times \mathcal{F}_1([0, 1])$ . One can think about the second component of the Cartesian product above as a space of errors measurement or (un)certainity degree expressed by the fuzzy number. Sometimes taking a linear combination of fuzzy numbers can make sense but as we show in Sect. 5.2 more natural is to treat fuzziness as a "weight" or "density". As we will see, then the Euclidean convexity is lost.

## 1.3 Why do We Use Type-2 Fuzzy Sets as Prototypes?

In Urban and Grzelińska (2017)  $D$ -dimensional type-1 fuzzy prototypes were considered and using the theory of gravitation partitioning algorithm was constructed. Such a generalization has many applications. Consider the following example from cognitive

science (cf. Jäger 2009). Not all people see colors in the same way. For this reason, the prototype of a given color for one person will not necessarily be the prototype for another person. Moreover, everybody has a tendency to view the prototype of the color, say red, not as a one object described by three parameters (wavelength, saturation, and hue), but rather by a spectrum of red colors. Why? Those color dimensions are continuous which means that for any value one can select another point within arbitrarily small  $\varepsilon$ -range. In reality human perception is not that precise and some values can be indistinguishable. So the prototype of the color red is simply a subset of the 3-dimensional conceptual space  $\mathbb{R}^3$ . This example can be very easily considered using a Voronoi diagram. However, it was observed in Urban and Grzelińska (2017) that even this generalized model is still inadequate. Although someone may see the color red as a spectrum, he or she is also able to show that some red colors from the prototype set are 'redder' than the others. For example we could consider some environmental factor influencing on observation of redness. Taking this into account a completely different method which solves the case described above by using  $D$ -dimensional type-1 fuzzy sets was proposed in Urban and Grzelińska (2017). The *characterizing function* evaluated at a given point shows to what extent this point represents the concept.

In this work we go a step further and we use prototypes being  $D$ -dimensional type-2 fuzzy sets.

We continue with the example of colors.

Suppose we have a given prototype of the color red. That is, we are given a certain region  $R$  in  $\mathbb{R}^3$ . Now assume that perfect red has weight 1. Let us choose any point (color)  $r$  from  $R$ . In our model we introduce an additional fuzzy effect by saying that the weight of  $r$  is not a number but the interval contained in  $[0, 1]$ . In other words we are given a mapping  $w$  defined on  $R$  which assigns to each point  $r \in R$  an interval  $[a_r, b_r] \subset [0, 1]$ . The interval  $[a_r, b_r]$  can be thought of as a fuzzy interval number. Hence we have to deal with type-2 fuzzy sets.

## 1.4 Structure of the Paper

The paper is organized as follows. In Sect. 2 we recall the basic concepts and definitions related to conceptual spaces. In this section we also review some literature about conceptual space theory. Definition of Voronoi diagrams is given.

Fuzzy sets of type-1 and 2 are considered in Sect. 3. In particular arithmetic operations on type-1 fuzzy sets are introduced via  $\alpha$ -cuts.

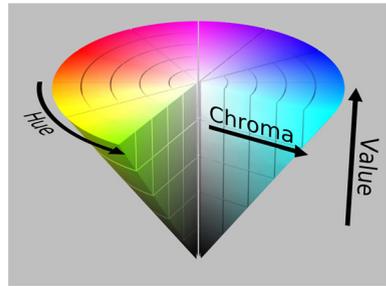
We recall the methods of ranking the fuzzy sets of type-1 according to the review paper by Brunelli and Mezei (2013).

Finally, we recall a few methods of integration of fuzzy set valued functions.

In Sect. 4 we define our conceptual space  $\mathcal{S}$  which is the space of fuzzy sets of type-2 in  $\mathbb{R}^D$  and define some metrics in  $\mathcal{S}$ .

Finally, in Sect. 5 we define our classification algorithm, introduce the concept of Newton's fuzzy potential, as well as provide motivation for its definition and adequacy to the problem under consideration. In Sect. 5.2 we present a realization of our algorithm.

**Fig. 1** Representation of color mapped on HSV space



In Sect. 6 we apply our classification algorithm to the real data containing recordings of birdsongs.

Finally, Sect. 7 contains summary and conclusions.

## 2 Conceptual Spaces

Since Gärdenfors introduced in Gärdenfors (2000) conceptual space (cf. also Gärdenfors 1988; Gärdenfors 1996 and his monograph Gärdenfors 2017) to the cognitive sciences, the generalization of conceptual space as well as the reformulating of the original structure has appeared in literature; see e.g. (Rickard, 2006) and (Rickard et al., 2007). What we understand as a conceptual space is given in the following definition.

**Definition 1** A conceptual space  $\mathcal{S}$  is a subspace of some metric space  $(X, d)$ , together with family of prototypes  $\mathcal{P} = \{P_1, \dots, P_k\}$ , where every  $P_i \subset \mathcal{S} \subset X$ .

The space  $\mathcal{S}$  is divided on  $k$  partitions called *concepts*. An element  $P_i$  which belongs to  $i$ -th concept and is its “most representative example” is called a *prototype*. Usually  $P_i$  are singleton sets and such partitions are usually obtained using Voronoi diagrams (see Okabe et al. 2011). That is the concept  $C_i$  corresponding to the prototype  $P_i$  is the set

$$C_i = \{x \in \mathcal{S} : d(x, P_i) < d(x, P_j) \text{ for all } j \neq i\},$$

where  $d(x, A) = \inf\{d(x, a) : a \in A\}$ .

**Remark 2.1** Notice that Voronoi tessellation gives a convex partitioning as a result.

Let us continue with a classical HSV color space example. Imagine a space  $[0, \pi) \times [0, 1] \times [0, 1]$  where dimensions are hue, saturation and brightness accordingly and prototypes are  $\{(0, 1, 1), (\frac{\pi}{3}, 1, 1), (\frac{2\pi}{3}, 1, 1)\}$ . They are the most typical red, green and blue. Now for a given point  $x$  for some partitioning we classify it as for example red or green depending on in which concept  $x$  lies (see Fig. 1<sup>1</sup>).

<sup>1</sup> [https://commons.wikimedia.org/wiki/File:HSV\\_color\\_solid\\_cone\\_chroma\\_gray.png](https://commons.wikimedia.org/wiki/File:HSV_color_solid_cone_chroma_gray.png)

## 2.1 Conceptual Spaces: Some Relate Works

The notion of conceptual space—an idea originally introduced to cognitive studies and philosophy by Gärdenfors (1988, 1996) has found myriad applications in various fields of science over the years. Even as seemingly exotic as music theory (Gärdenfors, 1988; Forth et al., 2010) or application to the study of the concept of metaphor (Gärdenfors, 1996).

For a reformulation of the notion of conceptual space see the works of Rickard (2006), Rickard et al. (2007) and Bechberger and Kühnberger (2017).

Interested reader in the applications of conceptual spaces in various fields of science can find a lot of information in the proceedings of the conference devoted to the applications of conceptual spaces (Kaipainen et al., 2019; Zenker & Gärdenfors, 2015) and in the references contained therein.

The problem under current consideration comes from the area of cognitivism. Cognitivism is an interdisciplinary science whose main aim is to analyze and model the brain activity of human beings (see e.g. Kriegeskorte and Kievit 2013 for some application of conceptual spaces in this area), as well as their senses. Cognitivism also constitutes a basis for other sciences. One example of this is cognitive linguistics, which is relevant to the problems discussed in this paper, see e.g. (Gärdenfors, 2011).

The most common in applications is  $\mathbb{R}^D$  with the  $\ell^2$ -metric (i.e. Euclidean one) and the partitioning is obtained by the use of the Voronoi diagram. This creates the problem of points lying on the edges of the respective conceptual fields. Attempts to solve this problem can be found in the works of Dietz (2003) and Douven et al. (2013), where the authors introduced a new type of Voronoi diagrams (with “thinner” and “fuzzy” boundaries of polygons). Abandoning the request for convexity we found in Urban and Mróz (2019) a quite natural conceptual space, in which we do not encounter such a problem.

Conceptual space representing knowledge is a very attractive tool in applications. On the other hand, we should notice that the whole theory of conceptual spaces built by Gärdenfors is also an extremely elegant theory. However, one of the weak points of the notion of conceptual space is the difficulty to find an easy and automated procedure that would be able to reproduce reality on its geometric representation.

Recently there has been a lot of work devoted to this problem. There have been published works presenting how conceptual spaces are created in a knowledge-driven manner, relying on prior knowledge how to form concepts. To mention a few papers that treat this type of problems we write (Alshaikh et al., 2019; Bouraoui & Schockaert, 2018; Banaee et al., 2018; Jameel et al., 2017; Jameel & Schockaert, 2016, 2018).

## 3 Type-1 and Type-2 Fuzzy Sets

There are multiple definitions of fuzzy sets (See Viertl 2010; Viertl and Hareter, 2005). For our convenience we use one describing them by their *characterizing function*. It is necessary to emphasize what operations we mean when we say “integrate fuzzy function” or “compare fuzzy numbers” since a lot of such procedures are available. One can think it is a weakness but we claim the opposite. Since many definitions are

available we can seek throughout them and choose the one that suits our purposes the best. It is worth to mention that our algorithm is parameterized by those methods and therefore very flexible.

**Definition 2** An  $n$ -dimensional fuzzy set  $A^*$  in  $X \subseteq \mathbb{R}^n$  is a function  $A^* : X \rightarrow \mathbb{R}$  satisfying:

- (1)  $A^* : X \rightarrow [0, 1]$ ,
- (2)  $\exists!x \in X$  such that  $A^*(x) = 1$ ,
- (3) for every  $\alpha \in (0, 1]$   $\alpha$ -cut  $A^*_\alpha$  of the fuzzy set  $A^*$ ,

$$A^*_\alpha = \{x \in X : A^* \geq \alpha\}$$

is a non-empty, compact, and convex set. We also assume that the following set

$$A^*_0 := \overline{\{x \in X : A^*(x) > 0\}} = \text{supp } A^*$$

has the same properties.

**Remark 3.1** For  $n = 1$  fuzzy sets defined in this way are often called fuzzy numbers. So for arbitrarily  $n$  we can think of them as a fuzzy points.

The set of all  $n$ -dimensional fuzzy sets with  $X \subseteq \mathbb{R}^n$  is denoted by  $\mathcal{F}_1(X)$ .

**Remark 3.2** Note that in the definition of a  $n$ -dimensional fuzzy set the set  $X$  contained in  $\mathbb{R}^n$  can be replaced by any set contained in any metric space or even in any topological space. However, in our work we will not consider such generalizations.

**Definition 3** Let  $X \subseteq \mathbb{R}^n$  be an arbitrary set. Type-2 fuzzy subset of  $X$  denoted by  $\tilde{A}$  is a map of the form

$$\tilde{A} : X \rightarrow \mathcal{F}_1([0, 1])$$

The support of the function  $\tilde{A}^2$ ,

$$\text{supp } \tilde{A} = \overline{\{x \in X : \tilde{A}(x) \neq 0\}}$$

is called the *base* of the type-2 fuzzy set  $\tilde{A}$ . The base of  $\tilde{A}$  is a subset of  $\mathbb{R}^n$  for which the map  $\tilde{A}$  is non-trivial (See also Mendel and Bob John, 2002).

In other words, the type-2 fuzzy set is a map from the set  $X$  into the family of fuzzy numbers having their supports contained in the interval  $[0, 1]$ . The family of all type-2 fuzzy sets in  $X$  contained in  $\mathbb{R}^n$  is denoted by  $\mathcal{F}_2(X)$ . It is worth seeing that  $\mathcal{F}_2(X) \subset X \times \mathcal{F}_1([0, 1])$ .

<sup>2</sup> To be more precise by  $\neq 0$  we mean that it is  $\neq \mathbf{F}(0)$  where  $\mathbf{F}(0)$  is natural embedding of number into fuzzy numbers defined in(4)

### 3.1 Operation on Fuzzy Sets

We start with the Minkowski arithmetic performed on subsets of  $\mathbb{R}^n$ .

**Definition 4** Let  $A, B \subset \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ . Then

$$\begin{aligned} A + B &:= \{a + b : a \in A, b \in B\}, \\ A \cdot B &:= \{ab : a \in A, b \in B\}, \\ \lambda \cdot A &:= \{\lambda a : a \in A\}. \end{aligned}$$

The Minkowski arithmetic of sets immediately induces an arithmetic of  $n$ -dimensional vectors from  $\mathcal{F}_1(\mathbb{R}^n)$  via their  $\alpha$ -cuts:

**Definition 5** The sum  $x^* \oplus y^*$  and multiplication  $x^* \odot y^*$  of two  $n$ -dimensional fuzzy sets  $x^*$  and  $y^*$  are defined via their  $\alpha$ -cuts as follows

$$\begin{aligned} (x^* \oplus y^*)_\alpha &= x_\alpha^* + y_\alpha^*, \\ (x^* \odot y^*)_\alpha &= x_\alpha^* \cdot y_\alpha^*. \end{aligned}$$

Similarly, the multiplication of a fuzzy set  $x^*$  by a crisp real number  $\lambda$ ,  $\lambda \odot x^*$  is defined by the equation

$$(\lambda \odot x^*)_\alpha = \lambda \cdot x_\alpha^*.$$

### 3.2 Ranking Methods for Fuzzy Numbers

Comparison of 1-dimensional fuzzy numbers will play an vital role in the description of our algorithm. Here we present only a few methods in order to illustrate our partitioning algorithm. More examples and details on comparing methods can be found for example in Brunelli and Mezei (2013).

There are basically two approaches to the problem. Fuzzy and crisp. We will consider the second method since everything can be applied to the first method without any problem (see Brunelli and Mezei 2013, p. 629). The crisp method of ranking fuzzy numbers  $\mathcal{F}_1(\mathbb{R})$  use an appropriate map which sends element of  $\mathcal{F}_1(\mathbb{R})$  into the real line  $\mathbb{R}$ . Thus we are given a map  $M : \mathcal{F}_1(\mathbb{R}) \rightarrow \mathbb{R}$  and we say that a fuzzy number  $x^*$  is not greater than  $y^*$  (we denote this by  $x^* \preceq y^*$ ) if  $M(x^*) \leq M(y^*)$ .

In the fuzzy method fuzzy binary relation is used. That is we are given function  $M : \mathcal{F}_1(\mathbb{R}) \times \mathcal{F}_1(\mathbb{R}) \rightarrow [0, 1]$ . If  $A^*, B^* \in \mathcal{F}_1(\mathbb{R})$  then  $M(A^*, B^*) \in [0, 1]$  is interpreted as the degree to which  $A^*$  is greater than  $B^*$ . Fuzzy numbers are ranked (with respect to  $M$ ) in the following way

$$M(A^*, B^*) \geq M(B^*, A^*) \text{ then } A^* \succ_M B^*.$$

Here we list only two examples of crisp methods. For ranking based on fuzzy binary relation see the mentioned paper (Brunelli & Mezei, 2013).

- Adamo’s method (Adamo, 1980). One simply evaluates the fuzzy number based on the rightmost point of the  $\alpha$ -cut  $[\underline{a}_\alpha, \bar{a}_\alpha]$  for a given  $\alpha$ :

$$AD_\alpha(A^*) = \bar{a}_\alpha.$$

- The *center of gravity* of a fuzzy number was introduced in Østergaard (1976) as

$$CoG(A^*) = \int_{-\infty}^{+\infty} \frac{x A^*(x)}{A^*(x)} dx.$$

### 3.3 Integrals of Fuzzy Set Valued Function

Since we want to define our fuzzy Newton potential as an integral from a fuzzy set valued function, we are interested in methods of integrating of such functions. It turns out that there is a lot of definitions of integrals of fuzzy set valued functions.

In this section we recall several types of integrals of the functions taking values in the set of type-1 fuzzy sets.

Let  $\tilde{f}$  be a fuzzy set valued function, i.e.  $\tilde{f} : \mathbb{R}^n \rightarrow \mathcal{F}_1(\mathbb{R})$ .

We start with the most natural definition given in Viertl (2010).

For  $0 < \alpha \leq 1$  and  $x \in \mathbb{R}^n$ , consider  $\alpha$ -cuts of  $\tilde{f}(x)$ . Since by our assumption  $\alpha$ -cuts are convex and compact thus  $\alpha$ -cut is an interval. Therefore we can write, for every  $\alpha \in (0, 1]$ ,

$$(\tilde{f}(x))_\alpha = [\underline{f}_\alpha(x), \bar{f}_\alpha(x)].$$

The real valued functions  $\underline{f}_\alpha(\cdot)$  and  $\bar{f}_\alpha(\cdot)$  are called  $\alpha$ -level functions.

**Definition 6** Let  $\tilde{f}$  be a type-1 fuzzy valued function defined on the measure space  $(X, \mathcal{B}, \mu)$ , where  $\mathcal{B}$  is a  $\sigma$ -field of Borel subsets of  $X$ , and  $\mu$  is a measure on  $\mathcal{B}$ . If all  $\alpha$ -level functions  $\underline{f}_\alpha(\cdot)$  and  $\bar{f}_\alpha(\cdot)$  are integrable with finite integrals  $\int_X \underline{f}_\alpha(x) d\mu(x)$  and  $\int_X \bar{f}_\alpha(x) d\mu(x)$ , then the generalized integral  $\int_X \tilde{f}(x) d\mu(x)$  is the fuzzy number  $J^*$  whose  $\alpha$ -cuts  $J_\alpha^*$  are defined by

$$J_\alpha^* = [\underline{J}_\alpha, \bar{J}_\alpha] := \left[ \int_X \underline{f}_\alpha(x) d\mu(x), \int_X \bar{f}_\alpha(x) d\mu(x) \right], \quad \alpha \in (0, 1].$$

By the representation lemma (Viertl 2010, Lemma 2.1) the characterizing function of the fuzzy integral  $J^*$  is, for every  $x \in X$ , given by

$$J^*(x) = \sup \left\{ \alpha \mathbf{1}_{[\underline{J}_\alpha, \bar{J}_\alpha]}(x) : \alpha \in [0, 1] \right\}.$$

The resulting fuzzy integral is denoted by

$$\int_X \tilde{f}(x) d\mu(x) = J^*.$$

There are many other integrals that one can use (Jang et al., 2004; Bongiorno et al., 2012). Just as an example, let's take the Henstock-Kurzweil integral (HK) (Yeong, 2011), which is a generalization of the Riemann integral. The generalization of the integral (HK) to fuzzy functions (fHK) is given in Bongiorno et al. (2012). In Bongiorno et al. (2012), functions with a 1-dimensional domains of the form  $[a, b] \subset R$  are considered.

To make the definition of (fHK) for functions with multi-dimensional domains is possible with the help of the monograph (Yeong, 2011; see also Hai and Gong, 2003).

At this point we omit both the definition of the integral (HK) and leave out the details leading to the object (fHK) because in the examples that we are going to present the integral turns into a sum and the whole theory of the integral is not needed.

## 4 Metrics on $\mathcal{S}$

A number of metrics can be introduced in  $\mathcal{S}$ . We will show couple of natural metrics in this space. Since any object  $\tilde{A} \in \mathcal{F}_2(X)$  can be thought of as a curried<sup>3</sup> function of two arguments (one from  $X$  and another from  $[0, 1]$ ) first what comes to mind is some kind of supremum metric. But since  $X$ -parts of domains of two objects which (in our case) going to be disjoint (5.1) taking supremum on both arguments would not take  $X$ -part into consideration. Hence another approach is needed but it is also very natural. One can simply take a linear combination of metric on  $X$  and metric on  $\mathcal{F}_1([0, 1])$ .

$$d_{(w_1, w_2)} = w_1 d_X + w_2 d_{\mathcal{F}_1}, \text{ where } w_1 + w_2 = 1.$$

It is easy to check that  $d_{(w_1, w_2)}$  is a metric when  $d_X$  and  $d_{\mathcal{F}_1}$  are metrics.

For the  $X$  part of a space the most natural is to take some standard  $\ell^p$ -metric  $d_{\ell^p}$  for some  $p \geq 1$ . More interesting is a fuzzy part where one could use for example supremum metric  $d_\infty$  but a better solution is to construct a special metric using Hausdorff metric and  $\alpha$ -cuts.<sup>4</sup>

Now, let  $A^*, B^* \in \mathcal{F}_1([0, 1])$ , we define

$$d_H^1(A^*, B^*) = \int_0^1 d_H(A_\alpha^*, B_\alpha^*) d\alpha,$$

$$d_H^2(A^*, B^*) = \sup_{\alpha \in [0, 1]} d_H(A_\alpha^*, B_\alpha^*).$$

It is easy to check that  $(\mathcal{F}_1([0, 1]), d_H^1(X))$  and  $(\mathcal{F}_1([0, 1]), d_H^2(X))$  are metric spaces.

In a sense our space is just an Euclidean space where every its point is equipped with fuzzy number.

<sup>3</sup> Currying is a transformation of a function that takes multiple arguments into higher-order function with single argument.

See: <https://en.wikipedia.org/wiki/Currying>

<sup>4</sup> A brief introduction to properties of those metrics can be found in "Appendix".

We use the following notation. If  $\tilde{x} \in \mathcal{S}$ , then (by Definition 3)  $\tilde{x} = (x, x^*)$ , where  $x \in \mathbb{R}^n$  and  $x^* : [0, 1] \rightarrow [0, 1]$ .

So to obtain metric  $d_{(w_1, w_2)}$  on that space we simply substitute some  $\ell^p$ -metric for  $d_X$  and one from  $\{d_\infty, d_H^1, d_H^2\}$  for  $d_{\mathcal{F}_1}$  and apply the to appropriate parts of arguments. E.g.

$$d_{(\frac{1}{2}, \frac{1}{2})}(\tilde{x}, \tilde{y}) = \frac{1}{2}d_{\ell^2}(x, y) + \frac{1}{2}d_H^1(x^*, y^*).$$

In our opinion Hausdorff derivative metrics  $d_H^1$  and  $d_H^2$  are “better” in a way.

Consider natural embedding  $\mathbf{F}$  of usual, real numbers into  $\mathcal{F}_1(\mathbb{R})$  defined as follow<sup>5</sup>

$$\begin{aligned} \mathbf{F} : \mathbb{R} &\rightarrow \mathcal{F}_1(\mathbb{R}), \\ \mathbf{F}(x) &= \mathbf{1}_{\{x\}}. \end{aligned}$$

Now let  $x^*, y^* \in \mathbf{F}(\mathbb{R})$  then

$$d_\infty(x^*, y^*) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

So  $d_\infty$  restricted to  $\mathbf{F}(\mathbb{R})$  acts like a discrete metric. It may be undesired since more natural metric on real line is simply distance between numbers  $|x - y|$ .

Now by substituting  $d(x, y) := |x - y|$  in definition (7) and taking  $x^*, y^* \in \mathbf{F}(\mathbb{R})$  we get:

$$\begin{aligned} d_H(x_\alpha^*, y_\alpha^*) &= \max\{\delta(x_\alpha^*, y_\alpha^*), \delta(y_\alpha^*, x_\alpha^*)\} \\ &= \delta(x_\alpha^*, y_\alpha^*) = \sup\{\delta(x, y_\alpha^*) \mid x \in x_\alpha^*\} \\ &= \delta(x, y_\alpha^*) = \inf\{d(x, y) \mid y \in y_\alpha^*\} \\ &= d(x, y) = |x - y|. \end{aligned}$$

It is therefore true for  $\alpha \in [0, 1]$  that

$$d_H^1 \upharpoonright_{\mathbf{F}(\mathbb{R})} (x^*, y^*) = d_H^2 \upharpoonright_{\mathbf{F}(\mathbb{R})} (x^*, y^*) = |x - y|.$$

Moreover we can also achieve discrete metric between numbers using  $d_H^i$  as in  $d_\infty$  case, simply by taking  $d$  to be discrete metric. (Proof is similar).

### 5 Classification Algorithm

In this section we present two algorithms one for classification of a given point in the space to the specific conceptual field and a second one for partitioning the space. But before we need to see how fuzzy Newton’s potential operator looks like in our setting.

<sup>5</sup>  $\mathbf{1}_A(x) = 1$  if  $x \in A$  and  $\mathbf{1}_A(x) = 0$  if  $x \notin A$ .

### 5.1 Type-2 Fuzzy Newton’s Potential

Let  $\mathcal{P} = \{\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_k\}$  consists of  $n$  type-2 fuzzy sets with the corresponding base sets  $P_i \subset \mathbb{R}^n$ . A fuzzy prototype is each type-2 fuzzy set  $\tilde{P}_i$  contained in  $\mathcal{P}$ .

For every  $\tilde{x} = (x, x^*) \in \mathcal{S}$ , whose base  $x$  does not intersect with any base  $P_i$ ,  $i = 1, \dots, k$  i.e.

$$\{x\} \cap \bigcup_{i=1}^k P_i = \emptyset,$$

we can compute fuzzy Newton’s potentials with the analogy of the situation where elements  $\tilde{P}_i$  were of type-1 (this case was considered in Urban and Grzelińska (2017)). The  $i$ -th potential has the following form

$$\Psi_i(\tilde{x}) = \begin{cases} -f_{P_i} \ln(d(\tilde{x}, \tilde{y})) \odot y^* dy & \text{if } n = 2, \\ -f_{P_i} d(\tilde{x}, \tilde{y})^{2-d} \odot y^* dy & \text{for } n \geq 3. \end{cases} \tag{5.1}$$

In analogy with the classical case we can think that the fuzzy numbers  $\Psi_i(\tilde{x})$  indicates the energy of the potential field at point  $x$ , which is generated by the mass concentrated in the domain  $P_i$  with the fuzzy density  $\tilde{P}_i$ .

### 5.2 Algorithm

The algorithm finds best concept domain for every element  $\tilde{x} = (x, x^*) \in \mathcal{S}$  such that  $\{x\} \cap \bigcup_{i=1}^k P_i = \emptyset$  and works as follows.

---

#### Algorithm $\tilde{x} \in \mathcal{S}$ concept domain assignment algorithm

---

**Require:**  $\tilde{x} = (x, x^*) \in \mathcal{S}$   
**Require:**  $\mathcal{P} = \{\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_k\}$  ▷ Set of prototypes  
**Require:**  $f : P(\mathcal{S}) \times (\mathbb{R}^n \rightarrow \mathcal{F}_1([0, 1])) \rightarrow \mathcal{F}_1([0, 1])$  ▷ Fuzzy integral  
**Require:**  $\preceq \subset \mathcal{F}_1([0, 1]) \times \mathcal{F}_1([0, 1])$  ▷ Fuzzy number ranking  
**Ensure:**  $\{x\} \cap \bigcup_{i=1}^k P_i = \emptyset$   
1: **for**  $\tilde{P}_i \in \mathcal{P}$  **do**  
2:      $P_i \leftarrow \text{supp } \tilde{P}_i$   
3:     **if**  $n = 2$  **then**  
4:          $\Psi_i \leftarrow -f_{P_i} \ln(d(\tilde{x}, \tilde{y})) \odot y^* dy$   
5:     **else**  
6:          $\Psi_i \leftarrow -f_{P_i} d(\tilde{x}, \tilde{y})^{2-d} \odot y^* dy$   
7:     **end if**  
8: **end for**  
9:  $j \leftarrow \text{index of } \max_{\preceq} \{\Psi_i \mid 1 \leq i \leq k\}$   
10: **return**  $\tilde{P}_j$  ▷ Return prototype of assigned concept domain

---

**Remark 5.1** In the step 9, if there are several indices to which correspond the maximum integrals, the element  $\tilde{x}$  cannot be classified<sup>6</sup>. Elements on the edges of  $P_i$  cannot be qualified to any concept domain.

Now, when classification is defined one cannot easily partition the space  $\mathcal{S}$ . Both components (strict and fuzzy) can be infinite. So simple checks for all elements cannot be always done. But we can easily construct approximate partition. Just select some representative points and run classification for them. For strict  $X$  subspace one can divide it to finite number of sets. Then construct the set  $\mathcal{X}$  by choosing arbitrary representative form each of them. Fuzzy part is a functional space therefore partitioning it can be tricky. Therefore we could select representatives as a Dirac's deltas  $\{\mathbf{F}(x_0), \dots, \mathbf{F}(x_n)\}$  for some  $x_0, \dots, x_n \in [0, 1]$ . The procedure looks like this. .

---

### Algorithm $\mathcal{S}$ partitioning

---

**Require:**  $\mathcal{P} = \{\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_k\}$  ▷ Set of prototypes  
**Require:**  $f : P(\mathcal{S}) \times (\mathbb{R}^n \rightarrow \mathcal{F}_1([0, 1])) \rightarrow \mathcal{F}_1([0, 1])$  ▷ Fuzzy integral  
**Require:**  $\preceq \subset \mathcal{F}_1([0, 1]) \times \mathcal{F}_1([0, 1])$  ▷ Fuzzy number ranking  
1: **for**  $\tilde{x} = (x, x^*) \in \mathcal{X} \times \{\mathbf{F}(x_0), \dots, \mathbf{F}(x_n)\}$  **do**  
2:   **if**  $\{x\} \cap \bigcup_{i=1}^k P_i = \emptyset$  **then**  
3:     Assign domain to  $\tilde{x}$   
4:   **end if**  
5: **end for**

---

However, this algorithm has some deficiencies. This algorithm is undefined for some points in the  $\mathcal{S}$ -space. We have two cases.

- (1) Procedure does not assign any domain to any point from prototype supports. We have to assume that they are pairwise disjoint. Then every point in base belongs to prototype domain. If it not the case we could simply compare their fuzzy values at the intersection points with  $\preceq$  relation. But in case of equality we remain undefined.
- (2) Similar problem occurs on the boundary of two domains when "attraction" of more than one prototype is equally strong.

## 6 Example

In this section we present motivating, real data example. It should clarify why we use fuzzy numbers and why relaxation on convexity condition is needed. Dataset we are going to use comes from Kaggle<sup>7</sup> and contains recordings of birdsongs from different bird species that can be found in Britain. From it we going to use recordings themselves as well as information about location where recording was made and name of recorded bird. The idea is as follows.

Chose some set of birds. For each species select one recording to be a prototype and one as a test sample. Prototypes embedded in some metric space gives fuzzy

<sup>6</sup> This type of situation also occurs in the case of partition obtained using the Voronoi diagram method.

<sup>7</sup> <https://www.kaggle.com/rtatman/british-birdsong-dataset>.

conceptual space. For each test sample song run a classification algorithm and check whether it matches proper prototype.

Lets think about how we can represent those data in our abstract, fuzzy setup.

We want to construct conceptual space  $\mathcal{S}$  that is a subspace of  $R^D \times \mathcal{F}_1([0, 1])$ .

First part is easy. Since we have access to geographic location lets take  $S^2 \subset \mathbb{R}^3$  which is a sphere representing a globe. Obviously we set the points in  $S^2$  to be longitude and latitude. So now we know that desired conceptual space is  $\mathcal{S} = S^2 \times \mathcal{F}_1([0, 1])$ . Only downside of choosing such a space is a difficulty of having nice visual representation of fuzzy part. We could only draw points on a globe but to justify our setup we selected very small coefficient for Euclidean part so such picture would not give us much information.

Next we need second component, namely a fuzzy number. For that we choose scaled Fourier transform of the recording. Frequency spectrum has some really nice properties like:

- It doesn't depend on the length of recording.
- It isolates the dominant frequency so we can expect that same species should have pretty similar spectra.
- For real-world signal it is never a single frequency and it resembles a fuzzy set.

For our example we have chosen seven prototypes and seven matching birdsong. Following table presents file ids chosen for prototype and test sample respectively.

Bird species	Prototype	Test sample
Eurasian Treecreeper	146,753	70,129
Corn Bunting	125,777	123,167
Common Wood Pigeon	94,614	130,993
European Nightjar	27,039	67,229
Common Reed Bunting	102,048	102,803
European Greenfinch	99,155	124,013
Rook	143,002	27,145

But before we could run an algorithm we needed to adjust the data by doing some preprocessing.

First of all the files contain a lot of noise so some denoising procedure where needed. It was done by hand in Audacity<sup>8</sup> but details are beyond scope of this paper. So for example after this step *Eurasian Treecreeper* song form file looked like in Figs. 2, 3 and 4.

Secondly one of conditions from Definition 2 is that all  $\alpha$ -cuts have to be convex. In other words our spectrum have to be concave. We ce achieve that by losing some points and connecting local maxima and the result can be seen on Sect. 6.

Lastly concaved frequency spectra had to be normalized. Notice that in our setup  $\mathcal{F}_1([0, 1])$  is a set of bounded functions of type  $[0, 1] \rightarrow [0, 1]$ . So for domain we have to have set  $[0, 1]$ . We normalize it by interval  $[0\text{Hz}, 18,000\text{Hz}]$  which we accept as a

<sup>8</sup> [www.audacityteam.org](http://www.audacityteam.org).

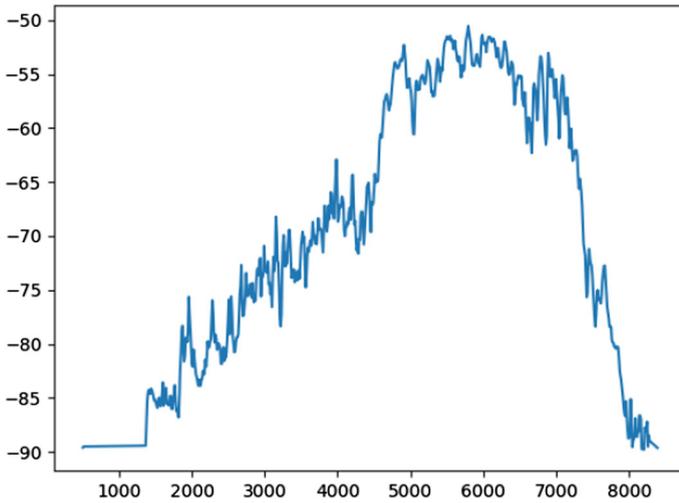


Fig. 2 *xc146753.flac* denoised spectrum

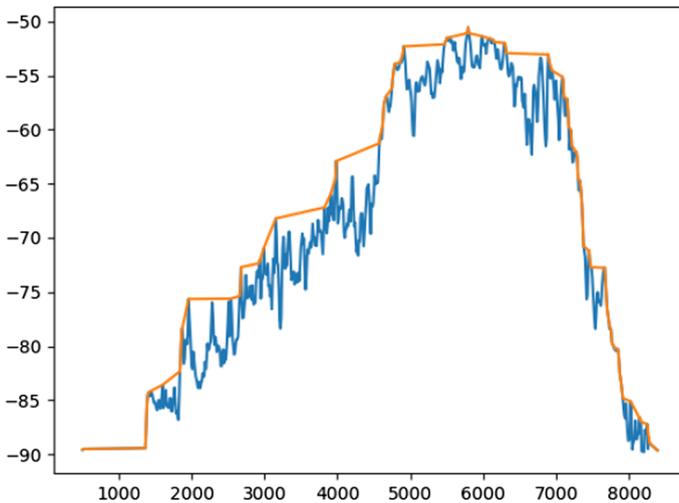


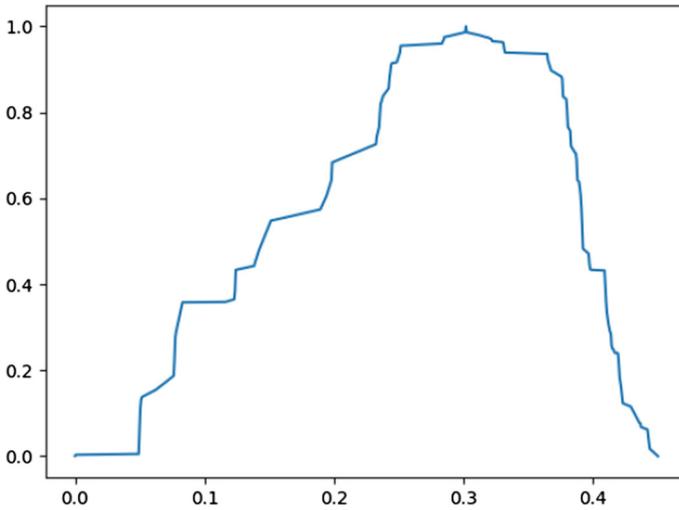
Fig. 3 Concaved spectrum

band audible to the average human. Vertical axis were normalized for every spectrum separately (by taking global maximum as 1). Since one of the conditions of being fuzzy number is to have exactly one global maximum equals to 1. See Sect. 6.

At the end we got the fuzzy number. So the set of prototypes consists of tuples  $((x_i, y_i), f_i)$  where  $x_i, y_i, f_i$  are longitude, latitude and concaved normalized spectrum of the  $i$ -th birdsong.

Now for selected prototypes and some other songs of a species matching one of the prototype we can run an classification algorithm with the following parameters:

- Center of gravity ranking method.



**Fig. 4** Concaved and normalized spectrum

- Fuzzy integral defined in this paper.
- Metric  $d_{(0.01,0.99)}((x, x^*), (y, y^*)) = 0.01 \cdot d_{\ell^2}(x, y) + 0.99 \cdot d_H^1(x^*, y^*)$

As one can see we put only small weight to the Euclidean distance since birds are very mobile creatures and it had a lot of sense to do so. It is easy to see that the obtained concepts are not Euclidean convex.

In this classification example six out of seven birds were classified correctly to its corresponding prototype. The result really depends on how well the actual birdsong were isolated from a noisy sample. But even when the first (denoising) step was done a bit loosely or wasn't done at all, the success rate was greater than 75%.

This example should also justify why having “multiple points” prototype is useful. We can have two or more birdsong in one species like one for fighting another for mating. All of them belong to same prototype but can have entirely different spectra.

## 7 Summary and Conclusions

In this paper we generalized notion of the Gärdenfors conceptual space by two ideas. To begin with, we added fuzziness into the values contained inside conceptual space. Then this notion allowed us to treat those values as ‘blurred’ and consequently consider them as with some density added. Density spread across some space gives us a notion of a mass. In such a context the shortest paths between points are not necessarily a strait line cause ‘heavier’ object attracts stronger and it may cause curving the path. We believe that this simple construct can have a significant impact on theory of conceptual spaces. And we hope that this paper could raise again the issue whether the concepts need to be convex. Our position is clear—convexity is necessary but in its more general, geodesic sense (see do Carmo 1992.)

The classification algorithm in the  $\mathcal{S}$  space does a bit of sophistication at first glance. Thus, it becomes even more interesting and unexpected to some extent that the a priori sophisticated and complicated algorithm is applicable to the most ordinary possible problem of classifying bird species based on their songbirds.

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**Code Availability** The computations were performed using the Python programming language.

## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

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## Appendix

### $\ell^p$ Metrics

Let  $X \subseteq \mathbb{R}^n$ . The most natural approach is to consider some metric from the class of metrics defined for any parameter  $p \geq 1$  as:

$$d_{\ell^p} : X \times X \rightarrow \mathbb{R},$$

$$d_{\ell^p}(x, y) = \left( \sum_{i=1}^n d_i(x_i, y_i)^p \right)^{1/p}.$$

It is obvious that this class contains Euclidean ( $p = 2$ ), maximum ( $p = \infty$ ) and taxi metrics ( $p = 1$ ).

### Supremum Metric, $d_\infty$

The definition of the supremum metric in space  $\mathcal{F}_1([0, 1])$  is a simple consequence of the fact that it is the space of bounded functions from  $[0, 1]$  to  $[0, 1]$  what follows from definition (2).

Let  $B(X, Y)$  be the set of all bounded functions from a set  $X$  into the metric space  $Y$  equipped with the supremum metric  $d_\infty$ ,

$$d_\infty : B(X, Y) \times B(X, Y) \rightarrow \mathbb{R},$$

$$d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|.$$

One can easily check that  $(B(X, Y), d_\infty)$  is a metric space. Clearly,  $\mathcal{F}_1([0, 1])$  is the space of bounded functions, then with the usual supremum metric  $\mathcal{F}_1([0, 1])$  is a metric space.

### Hausdorff Metrics

Let  $(Y, d)$  be a metric space. Consider the family  $\mathcal{C}(Y)$  of non-empty compact subsets of  $Y$ , i.e.

$$\mathcal{C}(Y) = \{A \subseteq Y \mid A \neq \emptyset \text{ and } A \text{ is compact}\}.$$

We introduce metric in the space  $\mathcal{C}(Y)$ . For  $x \in Y$  and  $A, B \in \mathcal{C}(Y)$  define

$$\delta(x, A) = \inf\{d(x, y) \mid y \in A\},$$

$$\delta(A, B) = \sup\{\delta(x, B) \mid x \in A\}.$$

The *Hausdorff metric* is the following function

$$d_H : \mathcal{C}(Y) \times \mathcal{C}(Y) \rightarrow \mathbb{R},$$

$$d_H(A, B) = \max\{\delta(A, B), \delta(B, A)\}$$

The next is the very well known theorem which says that  $d_H$  is in fact a metric in the space  $\mathcal{C}(Y)$ .

**Theorem 7.1** (Kigami 2001, Proposition 1.1.5) *The Hausdorff space  $(\mathcal{C}(Y), d_H)$  is a metric space. Moreover, if  $(Y, d)$  is a complete metric space then  $(\mathcal{C}(Y), d_H)$  is a complete metric space.*

#### **Proof** sketch

We only prove that  $d_H$  is a metric. For completeness of the Hausdorff space  $(\mathcal{C}(Y), d_H)$  see e.g. more advanced textbooks on topology or fractals; e.g. (Kigami 2001, Proposition 1.1.5).

Clearly,  $d_H$  is symmetric and  $d_H(A, B) = 0$  if and only if  $A = B$ .

In order to prove the triangle inequality it is enough to write, for any  $a \in A, c \in C$ ,

$$\begin{aligned} \delta(a, B) &= \inf_{b \in B} d(a, b) \leq \inf_{b \in B} (d(a, c) + d(c, b)) \\ &= d(a, c) + \delta(c, B) \leq d(a, c) + \delta(C, B). \end{aligned}$$

Minimizing over  $c$  gives  $\delta(a, B) \leq \delta(a, C) + \delta(C, B)$ . Maximizing over  $a$  gives  $\delta(a, B) \leq \delta(A, C) + \delta(C, B)$ . As a consequence  $\delta(A, B) \leq \delta(A, C) + \delta(C, B)$ . Interchanging the role of  $A$  and  $C$  in the above calculation shows that  $\delta(B, A) \leq \delta(B, C) + \delta(C, A)$  and the triangle inequality follows.  $\square$

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