METAPHYSICS AND CONVENTION IN DIMENSIONAL ANALYSIS, 1914–1917

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This article recovers an important, century-old debate regarding the methodological and metaphysical foundations of dimensional analysis. Consideration of Richard Tolman's failed attempt to install the principle of similitude—the relativity of size—as the founding principle of dimensional analysis both clarifies the method of dimensional analysis and articulates two metaphysical positions regarding quantity dimensions. Tolman's position is quantity dimension fundamentalism. This is a commitment to dimensional realism and a set of fundamental dimensions that ground all further dimensions. The opposing position, developed primarily by Percy Bridgman, is quantity dimension conventionalism. Conventionalism is an antirealism regarding dimensional structure that holds our nonrepresentational dimensional systems to have basic quantity dimensions fixed only by convention. This metaphysical dispute was left somewhat unsettled. It is shown here that both of these positions face serious problems: fundamentalists are committed to surplus dimensional structure; conventionalists cannot account for empirical constraints on our dimensional systems or the empirical success of dimensional analysis. I show that an alternative position saves what is right in both: quantity dimension functionalism.

1. Introduction

In this article, I study a dispute about the methodological foundations of dimensional analysis to clarify its metaphysical foundations. Consideration of the debate started by the failed attempt of Richard Tolman to install the principle of

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similitude—the relativity of size—as the founding principle of dimensional analysis both clarifies the method (and limits) of dimensional analysis and articulates two metaphysical positions regarding quantity dimensions. One view, which I call "fundamentalism," holds that there is objective dimensional structure and that there is a set of objectively basic (i.e., fundamental) quantity dimensions. Another view, conventionalism, holds that dimensional systems do not represent any objective dimensional structure and that basic quantity dimensions are determined by convention. Objections to both positions presented in the historical debate are found to have (limited) validity, and a third, alternative position, functionalism, is introduced. For the functionalist, the objective aspect of dimensional structure is modal structure. Quantity dimension functionalism allows for a synthesis of two methodological conceptions of dimensional analysis that, prima facie, are in tension: that dimensional analysis is a logical method and that dimensional analysis provides explanations.

The historical discussion will be restricted to the debate prior to Percy Williams Bridgman's landmark *Dimensional Analysis* (1922) and will focus primarily on an exchange between Bridgman and Tolman.[1] Other significant contributors to the debate, Edgar Buckingham and Tatiana Ehrenfest-Afanassjewa, cannot be given their full due here.

In what remains of this introduction, I introduce dimensional analysis as a method for problem-solving in physics, clarify its role as a logical method, and clarify an all-important and not often made distinction between unit systems and dimensional systems. This introduction provides all the necessary background for the rest of the article.

1.1. Dimensional Analysis in Action

Dimensional analysis is well known to even beginning students in physics, although explicit instruction in the method is far from universal. Dimensional analysis finds use in (often heuristic) arguments in fundamental physics and in technical engineering applications alike. Let's consider an example of dimensional analysis in action.

possible. This work further benefited from comments from audiences at HOPOS and &HPS meetings as well as comments from Susan Sterrett, Caspar Jacobs, and Daniel Mitchell: thank you. This work was supported by a grant from the Center for History of Physics at the American Institute of Physics.

1. In this way, it differs from the brief but more comprehensive account of the debates regarding dimensional analysis by Walter (1990). Her account is more comprehensive in that it covers the debates before and after *Dimensional Analysis*, but it is more myopic in its focus on Bridgman—rightly so, as Walter's book is a biography of Bridgman.

Say we are tasked with deriving the equation for the period of oscillation, t, of an arbitrary pendulum. We assume that the system can be adequately described in terms of the following quantities: the mass of the pendulum, m, the length of the pendulum, l, and the constant acceleration of gravity, g.[2] Next, we assume that these quantities are all reducible to mechanical dimensions:

$$\begin{aligned} [t] &= \mathrm{T} \\ [m] &= \mathrm{M} \\ [l] &= \mathrm{L} \\ [g] &= \mathrm{LT}^{-2}. \end{aligned}$$

The square brackets are a function from quantities to their dimensions, here given in terms of the basic mechanical dimensions, mass, length, and time (unitalicized capital letters denote dimensions).[3]

The problem is to find the form the of the function f such that $t = f(m, l, g)$, and so $[t] = f([m], [l], [g])$. This is the principle of dimensional homogeneity:[4]

> Principle of Dimensional Homogeneity. Every representationally adequate physical equation is dimensionally homogeneous, and an equation is dimensional homogeneous if the quantity terms on each side have the same dimension.[5]

We assume that this function f takes the form of a monomial $km^{\alpha}l^{\beta}g^{\gamma}$, with numerical scale factor k.[6] From this assumption and the principle of dimensional homogeneity, it follows that there is a set of linear equations to be solved for

2. This condition of "adequate description" is often called "completeness" (e.g., Buckingham 1914). That phrasing gives the wrong idea. Dimensional analysis requires only that all of the relevant quantities are considered; many quantities that are also descriptive of the system (indeed, there is an infinity of them) are excluded because of irrelevance, redundancy, and so forth.

3. Following Maxwell (2002), there is a slightly different convention in which, for example, $[L]$ represents the length dimension rather than L, and so forth.

4. This principle was first made explicit by Fourier in his *Théorie Analytique de la Chaleur*: "It must now be remarked that every undetermined magnitude or constant has one dimension proper to itself, and that the terms of one and the same equation could not be compared, if they had not the same exponent of dimension" (1878, 128). For more on the geometrical roots of dimensional analysis, see De Clark (2017) and Roche (1998).

5. Each of these terms is a monomial of quantity variables (or constants) and dimensionless scale factors; addition and subtraction distinguish terms. This captures the intuition that it makes no sense, for example, to add a length to a mass, to subtract a force from a velocity, and so forth.

6. This is due to Bridgman's (1931) lemma. For discussion, see Berberan-Santos and Pogliani (1999) and Jalloh (forthcoming).

the exponents of the relevant quantities such that the monomial has the dimensions of t. The following equations are to be solved:

$$\mathrm{M}: 1\alpha + 0\beta + 0\gamma = 0$$

$$\mathrm{L}: 0\alpha + 1\beta + 1\gamma = 0$$

$$\mathrm{T}: 0\alpha + 0\beta - 2\gamma = 1.$$

The Greek variables stand for the exponents of the variables in the monomial, and each coefficient is the exponent of the indicated basic quantity dimension had by the corresponding quantity m, l, or g. By inspection, $\alpha = 0$. With two equations and two variables (β and γ), we find the solution to be $\beta = 1/2$ and $\gamma = -1/2$, so $t = k\sqrt{l/g}$, where k is some undetermined dimensionless constant.[7] QED

1.2. Dimensional Analysis as Logic

Dimensional analysis was commonly thought of as a logical method by those who developed its foundations (see Gibbings 1982). I attempted to make the logical character of dimensional analysis evident in the previous demonstration by distinguishing assumptions that draw on our prior physical knowledge and the workings of dimensional analysis itself. In discussing his foundational paper on dimensional analysis (Buckingham 1914), Buckingham wrote:

> Some three or four years ago, having occasion to occupy myself with practical hydro- and aerodynamics, I at once found that I needed to know more about the method in order to use it with confidence for my own purposes. . . .
>
> I had therefore, as it were, to write an elementary textbook on the subject for my own education. My object has been to reduce the method to a mere algebraic routine of general applicability, making it clear that Physics came in only at the start in deciding what variables should be considered, and that the rest was a necessary consequence of the physical knowledge used at the beginning; thus distinguishing sharply between what was *assumed*, either hypothetically or from observation, and what was mere logic and therefore certain.[8]

7. Such derivations can be done more systematically by way of the Π-theorem, a fundamental result of dimensional analysis (see discussion and references in sec. 2.2).

8. Edgar Buckingham to Lord Rayleigh (John William Strutt), November 15, 1915, MP 2017-2296, 33, Niels Bohr Library and Archives, American Institute of Physics (College Park, MD; hereafter "AIP").

It is clear from this letter that Buckingham understood dimensional analysis as a logical method insofar as it was certain and so did not depend on any further empirical claims (i.e., a priori). Modeling dimensional analysis on deductive logic, we can say that it provides a form of valid argument (more abstractly, transformation rules): if such-and-such quantities have such-and-such dimensions, relative to a dimensional system (see sec. 1.3), then they are related by so-and-so functions.[9] In our extended post–logical empiricism hangover, such a distinction between logic and experience may seem hopeless and, worse, old-fashioned—we cannot accept Buckingham's conception of dimensional analysis.[10]

I would like to rehabilitate an idea of dimensional analysis as logic by abandoning Buckingham's *epistemic* conception of logic while accepting that it stands apart from ordinary physics in an important way. The relations between dimensional analysis and experiment are too complex to segregate dimensional analysis from empirical assumptions, but there is still a sense in which dimensional analysis stands above (or below) the ordinary practice of physics in a way similar to relative standing of logic and ordinary reasoning. For this rehabilitation, I will draw on Gil Sagi's (2021) recent defense of an exceptionalist conception of logic as a *methodological discipline*—this contrasts with the usual exceptionalist conceptions of logic on an epistemic basis (e.g., because it is, a priori) that is now so unfashionable after Quine (1951). In adding dimensional analysis to the roster of methodological disciplines, I am accepting the invitation left open by Sagi that "perhaps there are other methodological disciplines targeting scientific practice" (2021, 9741). I offer the claim that dimensional analysis is the methodological science peculiar to quantitative science, here narrowly considered as peculiar to quantitative *physical* science, and so can synonymously be understood as *the logic of quantities*.

What is a methodological discipline? We may do well to start with the characterization given by Sagi (2021):

9. That the generation of Π-terms and so functional relations can be computed completely and without arbitrariness is shown in Gibbings (2011). That does not mean that in ordinary practice there is not an art in determining "which" Π-terms (i.e., functional relations) are of interest for the relevant system.

10. In a later letter to Rayleigh on January 7, 1916 (MP 2017-2296, 33, AIP), Buckingham already expresses his feeling that his methodological strictures chafed against the zeitgeist: "It is evidently desirable that this subject should receive a clear exposition. Tolman does not, I imagine, care much for the distinctions between known facts, assumptions made for the sake of building up theories, and purely logical operations on these facts or assumptions. And it seems that many of the very clever rising generation of physicists have much the same feeling. I, on the other hand, regard these distinctions as very essential to clear thinking and sound progress" (p. 6).

> As a start, by a methodological discipline, I mean a discipline that produces tools, methods or a methodology for some practice. I take a method to be a systematic procedure or system of rules for carrying out a practice. There may be methods for very specific practices (measuring the distance between the earth and the moon, solving differential equations) or general methods advising a whole discipline (how to conduct a scientific experiment, how to prove a mathematical theorem). . . . A methodology, in general, is aimed at a higher level of scientific practice, as it concerns the production and selection of scientific theories. A methodology, I assume, may give rise to a method (for, e.g., theory choice) or consist of a compendium of methods (for reasoning in science). (9736)

A methodological discipline is defined relationally to what we may call a "client" discipline. The methodological discipline aids practitioners in aligning their scientific practice to the aims of their first-order client discipline. Put differently, the aims of a methodological discipline are to ensure that the products of some client discipline (e.g., theories or models) meet the internal aims of that client discipline (e.g., prediction, explanation). I am proposing that dimensional analysis has physics (broadly construed) as a client discipline—dimensional analysis provides principles and derivational techniques that allow physicists to check the validity of their quantitative equations and to efficiently derive new ones.[11]

What is the relation between a methodological discipline and a client discipline? An intriguing characterization of the relation between them given by Sagi involves an extension of the use-mention distinction: client disciplines use tools, methods, and concepts that are mentioned (e.g., criticized, constructed) by the corresponding methodological discipline. Although physics uses concepts of quantity, principles of homogeneity, and dimensional systems, it is left for dimensional analysis to discuss the nature of quantities, to justify and determine the consequences of dimensional homogeneity (e.g., the Π-theorem), and to elaborate and distinguish dimensional systems.[12] It is important that this exceptionalist, relational conception of methodological disciplines does not lapse into a sort of epistemic foundationalism as attacked by Quine (1951). We can capture both the special position of a methodological discipline and its revisability by distinguishing two phases of research:

11. A similar distinction between "framed" and "framing" inquiry has been articulated and defended by Henne (2023).

12. A closely related and analogous methodological discipline is metrology, which provides the (experimental) physicist with units of measurement, values for constants, rules for error propagation, and so forth. Metrology is an important case to consider as the divide between methodological discipline and client discipline(s) has become sociologically and institutionally regimented in a clarifying way.

Business as Usual. The methodological discipline constructs, describes, and regiments the techniques and concepts used by the client discipline. The rules set by the methodological discipline exert normative force on the practitioners of the client discipline; when there is a discrepancy, the principles set by the methodological principle take precedence.

Negotiation. Problems or developments in the client discipline lead to reconsideration of the principles of the methodological discipline and the relationship between the two—neither discipline takes normative priority to the other. This phase ends by the establishment of a new "business as usual" paradigm between client and methodological disciplines.

In the business as usual phase, the client-provider relation is as expected, and the methodological discipline provides tools and methods that hold normative force over the practices of the client discipline (they are *relatively* a priori in the sense of Friedman [2001])—an equation of physics found to violate dimensional homogeneity is an equation to be corrected (or at least used with great care in special circumstances). In the negotiation phase, usual business is disrupted, and internal pressures from the client discipline (e.g., empirical results, paradoxes) lead to adjustments in the methodological principle and even shifts in what aspects of the relevant scientific practice belong to which discipline. The historical episode to be considered here is usefully described in these terms: In the early twentieth century, pragmatic matters (above all, the development of airplanes; see Sterrett [2005]) led to a formalized business deal between the nascent methodological discipline of dimensional analysis and the physical sciences. While this deal quickly came to be "business as usual," Tolman attempted in 1914 to renegotiate the deal. Inspired by radical developments in the client discipline, physics, Tolman attempted to augment the foundations of the methodological discipline with a new relativity principle and thereby provide new constraints on the client discipline. Although Tolman's negotiation failed, it made explicit many implicit aspects of the initial deal between dimensional analysis and physics, some of which have yet to be fully clarified. In the next subsection, I clarify an important aspect of the usual deal. In the rest of the article, I raise and attempt to answer one question left to be negotiated: To what extent do features of our dimensional systems represent objective structure?

1.3. Dimensional Systems and Unit Systems

Dimensional analysis depends on some assumptions regarding physical quantities. They must form a complete dimensional system, meaning that the complete set of quantities are reducible to products of powers of fundamental units multiplied by a numerical scale factor (for proofs, see Bridgman [1931] and Berberan-Santos and Pogliani [1999]):

$$Q_i = k_i u_a^{\alpha} u_b^{\beta} u_c^{\gamma} \ldots$$

Q_i is some arbitrary quantity, k_i is some numerical factor, and u_x is some fundamental unit. The Greek exponents are known as dimensions, following Fourier (1878).[13] Each basic unit is assigned a basic dimension. In a mechanical dimensional system, for example,

$$\begin{aligned} m &= u_{\mathrm{M}} \\ l &= u_{\mathrm{L}} \\ t &= u_{\mathrm{T}} \end{aligned}$$

where l, m, and t are arbitrary mass, length, and time quantities set to be units by convention, (e.g., a kilogram, a meter, and a second). Each of these units has a basic dimension:

$$\begin{aligned} [m] &= \mathrm{M} \\ [l] &= \mathrm{L} \\ [t] &= \mathrm{T}. \end{aligned}$$

In abstraction from the actual units, we can use these basic dimensions to derive the dimensions of all other mechanical quantities.[14] Thus, dimensional systems, which are determined by the basic dimensions, are more coarse-grained than unit systems. For each dimensional system there is an infinite set of logically possible coherent unit systems that are all interconvertible and form what I call a "dimensional group."[15] For example, the dimensions of force, F, and the dimensions of velocity, V, are given so:

$$\begin{aligned} [F] &= \mathrm{MLT}^{-2} \\ [V] &= \mathrm{LT}^{-1}. \end{aligned}$$

13. This sometimes leads to expressions like "has exponent d in dimension X," which are equivalent to expressions like "has dimension X^d."

14. Italicized capital letters are variables for quantities; for the remainder of this section, I retain lowercase variables for units (excluding dimensionless constants k_i). Unitalicized capital letters represent dimensions.

15. There is some complexity in the nature of units that I am suppressing. The important thing is that dimensional groups consist of units defined by "similar scales" (Ellis 1964). The group structure of similar unit systems is not to be confused with the group structure of dimensional systems (which unit systems inherit; see de Boer 1995).

These dimensional formulae correspond to definitions of mechanical units:

$$
\begin{aligned}
f &= k_f m l t^{-2} \\
v &= k_v l t^{-1}.
\end{aligned}
$$

For a coherent system of mechanical units, $k_f = k_v = 1$.[16] We can distinguish basic quantities, which have dimensional exponent 1 in only one of the basic dimensions (and exponent 0 otherwise), and derived quantities, which have arbitrary dimension in any of the basic dimensions. Basic quantities are measured by fundamental units, and derived quantities are measured by defined units. The dimensions of the derived quantities encode formal relations between them and the basic quantities: these relations identify the transformation rules for derived quantities on changes in the fundamental units.

For any derived mechanical quantity Q, its defined unit, q, will be a monomial function of the fundamental units, just as described above:

$$q = m^{\alpha} l^{\beta} t^{\gamma}.$$

The Greek dimensional exponents determine how the defined unit changes with arbitrary scalar transformations of the fundamental units,

$$\frac{q'}{q} = \left(\frac{m'}{m}\right)^{\alpha} \cdot \left(\frac{l'}{l}\right)^{\beta} \cdot \left(\frac{t'}{t}\right)^{\gamma},$$

where the primed units are the new units. If we halve the fundamental time unit $2t' = t$ and leave the mass and length units unchanged, for example, the unit of force, f, will quadruple because $\gamma_f = -2$ and the velocity unit, v, will double because $\gamma_v = -1$:

$$
\begin{aligned}
\frac{f'}{f} &= \left(\frac{m'}{m}\right)^{1} \cdot \left(\frac{l'}{l}\right)^{1} \cdot \left(\frac{t'}{t}\right)^{-2} = \left(\frac{t}{2t}\right)^{-2} = 4 \\
\frac{v'}{v} &= \left(\frac{m'}{m}\right)^{0} \cdot \left(\frac{l'}{l}\right)^{1} \cdot \left(\frac{t'}{t}\right)^{-1} = \left(\frac{t}{2t}\right)^{-1} = 2.
\end{aligned}
$$

16. The usage of the terminology "complete" and "coherent" varies widely. I am also making a distinction between dimensional and unit systems that is not usually made, although see Abraham (1933). I reserve "complete" for dimensional systems with a reduction base, as I go on to describe. I reserve "coherent" for any unit system of a complete dimensional system such that the derivative quantities are defined with dimensionless scale factors $k_i = 1$. Complete equations, which are interpreted according to a complete dimensional system, are unit-invariant (in algebraic form) for any coherent unit system of that dimensional system. This captures the lessons of Grozier (2020), although he does not make the distinctions I make, as the mistakes he diagnoses could be avoided by the recognition of the distinction between dimensional systems and the more fine-grained unit systems.

The use and operation of these unit transformation rules and their duality with dimensional formulae are uncontroversial. Although many of the methods that dimensional analysis provides to physics are uncontroversial, controversy remains regarding the meaning of its subject matter, quantity dimensions and dimensional formulae.

One interpretation of dimensional analysis harks back to Buckingham's conception of dimensional analysis as a *formal* logic concerned with conventionally decided transformation rules on defined or stipulated "objects." In this view, dimensional formulae are understood to be formal rules for the use of units and numerical representations of quantities, which are purely conventional. In this reading, representations of dimensions like M are purely syntactic shorthand for change ratios such as m'/m. The basis of a dimensional system and the corresponding formulae for derived dimensions are therefore reducible to rules of translation between ultimately conventional unit systems that regiment our practice of assigning numbers to objects and systems.

There is a competing interpretation of dimensional analysis that holds quantity dimensions to be entities in their own right, irreducible to mere convention and formal rules. In this view, dimensional formulae not only represent unit transformation rules but also reveal the metaphysical character of quantities. Not only is a unit of force defined, but a quantity of force is constructed or constituted by the dimensions of mass, length, and time. It is as if the basic dimensions are the fundamental substances from which the more complex derivative quantity dimensions are composed. With this interpretation, there is a uniquely correct dimensional system that represents the objective dimensional structure of quantities: its basic dimensions are *fundamental* dimensions, and its dimensional formulae represent *grounding* relations between the fundamental and derivative dimensions.[17]

To further explicate and critically examine these two interpretations of dimensional analytic methods and objects, I set them against questions regarding the objectivity of the two main features of dimensional systems discussed: basic quantity dimensions and dimensional formulae.

1.4. Metaphysical Questions and Answers

A dimensional system is to be understood as a formal system that consists simply in a set of basic—that is, independent—quantity dimensions (a basis) and a

17. This controversy dates back to the development of the dimensional calculus by Maxwell and others (see Mitchell 2017) and continues to the present day, with Skow (2017) arguing against the interpretation of dimensional formulae as denoting constitution relations (but defending them as definitional relations).

rule that all other (derivative) dimensions are products of powers of the basic dimensions.[18] While dimensional formulae are in a sense extraneous to the system—all derivative dimensions already "exist," given a basis—in physics, we care about particular derivative quantities like pressure or volume, and so we might also distinguish dimensional systems by the dimensional formulae for the set of canonical physical dimensions. Let's distinguish two aspects of a dimensional system that suffice to identify it: a basis and a set of dimensional formulae for derivative dimensions.[19]

Metaphysical questions concern the relations between dimensional systems and dimensional structure, if there is any. Dimensional structure would be the ontic analog of a dimensional system—if there is objective dimensional structure, then there is a dimensional system that correctly represents this aspect of the world. This brings us to the first ontological question, the general question of realism.[20]

> Dimensional Realism. Is there objective dimensional structure that corresponds to a dimensional system?

Alternatively, this can be put as a question: is there an objectively correct dimensional system for the world? A subsidiary question further specifies some particular aspect of dimensional systems that may be objectively determined:

> Fundamental Basis. Is there a fundamental dimensional structure that corresponds to a dimensional basis?

Is it the case that the dimensions M, L, and T form a unique basis for mechanical dimensions (with $[F] = \mathrm{MLT}^{-2}$)? Or is there another set—for example, F, L, and T (with $[M] = \mathrm{FL}^{-1}\mathrm{T}^{2}$)—that would serve just as well?[21] The general ontological question can be understood as raising the question of whether or not our

18. This rule is Bridgman's lemma (see references in sec. 1.1). Independence can be understood thus: two quantity dimensions are independent if neither depends on the other—that is, no product of powers of the one appears in the dimensional formula for the other and vice versa. One might say, "Well, I can define the dimensions of mass to be $\mathrm{L}^{-1}\mathrm{ML}$, so mass is not independent of length." The response is that no exponents of like dimension in dimensional formulae are allowed to go unsummed (in this case, the two powers of length cancel out). A set of basic dimensions spans a dimensional system in just the same way that a set of basis vectors spans a vector space (see Corrsin [1951] and de Boer [1995]).

19. We can alternatively represent a system just by dimensional formulae—that some quantity dimensions have a single dimension of power 1 indicates that they are basic.

20. This dimensional structure is supposed to be "joint-carving" in the sense of Sider (2011).

21. An explication of "just as well" is given in sec. 3.1.

dimensional systems represent anything at all. The fundamental basis question further speciates forms of realism. If a dimensional realist believes there is a set of objective basic quantity dimensions, they are a fundamentalist; if not, they are a functionalist. A conventionalist rejects objective dimensional structure tout court and so automatically rejects objective fundamental dimensional structure corresponding to the basis of a dimensional system.[22] The relationships between these metaphysical positions and the answers they provide to the questions above are summarized in the flowchart (fig. 1).

As I show, both fundamentalism and conventionalism about quantity dimensions are articulated and defended in the years 1914–17. A third view, functionalism is presented here as a synthesis of the two, responsive to problems of both historical positions.[23]

Tolman (1917) provides the first full articulation of quantity dimension fundamentalism. Quantity dimension fundamentalism combines a dimensional realism with a commitment to a fundamentality principle: there are fundamental quantity dimensions that metaphysically ground the derivative quantity dimensions.

> FUNDAMENTALISM. There is only one correct dimensional system, and it represents the dimensional structure of the world. Dimensional formulae describe the natures of quantity dimensions.

Tolman's fundamentalism comes out of a debate concerning his proposed principle of similitude, which was to replace the principle of dimensional homogeneity as the foundation of dimensional analysis.

22. One might wonder if it may speciate forms of antirealism as well. One might think that if an antirealist holds that there is such an objective basis set, they are an operationalist. The operationalist cannot hold that this set is objectively basic in the metaphysical sense that we are concerned with here; it must be an epistemic fundamentality (the operationalist distinction is often between primary and secondary quantities; see Ellis 1968). For this reason, operationalism is not considered here, although this is closer to the later view of Bridgman (1931; see also, Gibbings 2011).

23. Dialectically, this division of the logical space is similar to that in Skow (2017). The analogy would be that Skow's positivist stands in for my conventionalist, his contructivist for my fundamentalist, and his definitional connectionist for my functionalist. There are some differences: Skow's definitional connectionist is also a fundamentalist as they are committed to nonrelativity, the position that there is an objectively determined basis for our dimensional system. That said, Skow's definitional connectionist comes closer to my functionalist due to an emphasis on the necessary connections between distinct quantity dimensions (Skow 2017, 194). An appreciation of the full force of conventionalist symmetries would lead Skow's definitional connectionist to drop the idea of unique real definitions of derivative dimensions, and so fundamental dimensional structure, yielding a functionalist account.

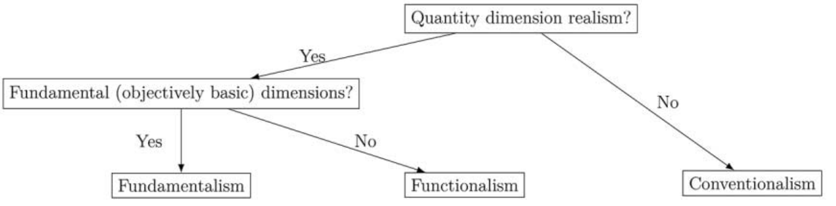

Figure 1. Flow chart through the logical space of quantity dimension metaphysics

> PRINCIPLE OF SIMILITUDE. "*The fundamental entities out of which the physical universe is constructed are of such a nature that from them a miniature universe could be constructed exactly similar in every respect to the present universe.*" (Tolman 1914, 244)[24]

Tolman conceptualized his principle of similitude as a relativity principle, the relativity of size (i.e., length scale). In the first instance, this principle is to be understood and was understood as a particular instance of quantity dimension fundamentalism. In this instance, Tolman held that there was only one fundamental mechanical dimension, length. With the adoption of certain laws as providing dimensional formulae that grounded mass, time, and other mechanical quantities in length, Tolman was able to recover the intuition behind his relativity of size principle: a universal scale transformation of lengths ought to be an empirical symmetry (e.g., a doubling of all the lengths overnight would not be empirically detectable). At first, Tolman defends his principle by giving up the metaphysical, fundamentalist reading of it. He ultimately recants and gives up the principle and defends a more tenable fundamentalist picture.

As it turns out, Tolman's principle of similitude is false, owing to its conflict with Newtonian gravity and the relevant confirming evidence thereof. This was pointed out almost immediately by Buckingham (1914) and amplified by Ehrenfest-Afanassjewa (1916b) and Bridgman (1916). Tolman himself thought a new theory of gravity was imminent.[25] The empirical disconfirmation of

24. A major warning is to be heeded here. In this article, "the principle of similitude" or "the method of similitude" refers to uses of Tolman's principle. More generally, "similarity methods" are just another term for using traditional dimensional analysis based on the principle of dimensional homogeneity and proportionality principles (see Sterrett 2017). At the risk of misunderstanding, I am sticking with the terminology used by those in the debate—although it is relatively clear that Buckingham (1914) intended to reclaim the terminology of similitude from Tolman. In the end, Buckingham won out.

25. The relationship between Tolman's principle and the emergence of novel theories of gravity, let alone questions about the nomological nature of the constants (see sec. 2.3), is much too large a topic to be dealt with here. I will note only that Nordström (1915) developed a version of his scalar gravitational theory (an early competitor to Einstein's general theory of relativity) that is consistent with Tolman's principle. The development and significance of such a theory is left for future work.

Tolman's principle does not undermine the interest of the methodological and metaphysical issues that were raised by the debate concerning his principle. The positions outlined in the debate and the arguments given for them have implications for the general study of dimensional systems.[26]

2. From Tolman's Principle of Similitude to Arguments against Fundamentalism

In this section, I discuss the debate surrounding Tolman's principle of similitude in three parts, roughly in historical order. Each subsection deals with a dialogue between Tolman and an interlocutor: Edgar Buckingham, Tatiana Ehrenfest-Afanassjewa, or Percy Bridgman. Each dialogue brings forward the metaphysical issues latent in the methodological debate, but special attention is paid to the dialogue with Bridgman, which leads to explicit metaphysical accounts of quantity dimensions.

A brief note on the scientific context for this debate is necessary. The concern with the foundations of dimensional analysis is connected to other radical changes in the foundations of physics in general.

2.1. Contextualizing Dimensional Analysis in the Wake of Relativity

This debate regarding the foundations of dimensional analysis was not about relativity, nor quantum mechanics.[27] That said, it is important for comprehending this debate to understand some of the fundamental questions that were raised by relativity, which caused Tolman to reconsider the very nature of physical quantities. Maila Walter situates the development of dimensional analysis as part of a broader reckoning with the radical consequences of relativity theory: "The dimensional analysis controversy revealed a generous amount of confusion about the meaning of relativity and measurement. . . . Einstein's abrogation of the traditional meaning of measurement has demonstrated that the relationship between mathematics and physical reality had to be reconsidered. The dispute over dimensions was just one manifestation of a general concern that would be stated with more precision and politicized by the logical positivists" (1990, 84). The following description of this broader context is based on Walter's more thorough accounting of the relevant foundational debates in the wake of relativity.[28]

26. A reader with pure metaphysical interest may skip to sec. 3.

27. For an initial look at the application of dimensional analysis to quantum mechanics, see Semay and Willemyns (2021). For a contemporary and systematic application of dimensional analysis to general relativity, see Porta Mana (2021).

28. One of the broader trends I will not discuss was the search for a natural and rationally determinable set of fundamental units (see Walter 1990).

The special theory of relativity was met with suspicion and disbelief when it was brought to the attention of American physicists—the promulgation and acceptance of the theory in America is due in no small part to the efforts of Gilbert Lewis and Richard Tolman in 1908.[29] Lewis and Tolman (1909), in American pragmatist fashion, describe the principle of relativity as grounded in the generalization of experimental facts (e.g., the Michelson-Morley experiment). The principle is accordingly understood as a constraint on what is measurable by Lewis and Tolman: "[Einstein] states as a law of nature that absolute uniform translatory motion can be neither measured nor detected" (712).

This is to say that only relative motion has "physical significance" or objectivity. This principle, combined with the postulate of the frame invariance of the speed of light, leads to the shocking consequences of relativity theory: time dilation and length contraction. Lewis and Tolman's grounding of relativity and its consequences in measurement lead them to an antirealist interpretation of such consequences:

> Let us emphasize once more, that these changes in the units of time and length, as well as the changes in the units of mass, force, and energy which we are about to discuss, possess in a certain sense a purely factitious significance; although, as we shall show, this is equally true of other universally accepted physical conceptions. We are only justified of speaking of a body in motion when we have in mind some definite though arbitrarily chosen point as a point of rest. The distortion of a moving body is not a physical change in the body itself, but is a scientific fiction. (1909, 717)[30]

Lewis and Tolman describe these phenomena as changes in units and "in a certain sense psychological." They claim that the acceptance of these distortions is the cost of retaining our fundamental conceptions of physics. The psychological unreality of these distortions owes to the fact that their occurrence appears to

29. They presented a paper, "Non-Newtonian Mechanics and the Principle of Relativity," at the Christmas meeting of the American Physical Society in 1908, as stated by Kevles (1995, 90). However, I can find no trace of an article in *Physical Review*, as he claims. The article (draft completed in May 1909) was published both in *Philosophical Magazine* and the *Proceedings of the American Academy of Arts and Sciences* the following year, with an inverted title: "The Principle of Relativity, and Non-Newtonian Mechanics." Here I cite the latter, American publication. A citation for the former can be found in Walter (1990). See also, Goldberg (1984, 1987) on the American response to relativity.

30. The special theory of relativity was seen as upending our fundamental concepts of physical quantities—when Lewis and Tolman refer to "units," they are conflating the functions of units as reference quantities (i.e., standards) and as numerical fixed points. The terminology of units versus quantities versus magnitudes was not standardized for decades.

depend on whether or not some observer considers herself at rest—a judgment lacking in objectivity due to the relativity principle.

The more proper evaluation of the situation is given in Lewis and Tolman's (1909) claim that absolute motion has no significance—dilation and contraction are artifacts of an arbitrarily chosen rest point, thereby retaining something of our "fundamental conceptions." This claim is a common feature of symmetry arguments and occurs in Tolman's argument for the principle of similitude, as well as recent debates on quantity symmetries.[31] In arguing for the existence of a symmetry transformation, and thereby the unreality of the supposed features of reality that vary under that symmetry, the basis for the symmetry argument seems to be undermined because there is no such feature to be transformed. In Einstein's case, this is absolute velocities. In Tolman's case, with the principle of similitude, it is absolute lengths. This is only a matter of charitable interpretation and convenience; any appearance of self-undermining can be removed by restating these relativity principles as statements about what objective structure there is. The theory of special relativity rejects any objective, frame-independent, velocity structure. Tolman's principle of similitude rejects any objective, absolute length magnitudes, which become dependent on a choice of comparative standard, analogous to how length quantity values are relative to a choice of unit standard.

2.2. Tolman versus Buckingham

The inciting event for the debate is Tolman's (1914) publication of "The Principle of Similitude," which puts forward a relativity principle—the relativity of size—as the founding principle of dimensional analysis.

> Relativity of Size. A global transformation of the length scale is both a dynamical and empirical symmetry—there is no objectively determined length scale.

I hope this is a useful updating of Tolman's principle in conformity with how we now generally understand the principle of relativity, as a symmetry principle. This gloss is good only insofar as it has the same consequences as Tolman's (1914, 244) own statement of the principle of similitude. Tolman exhibits the consequences of this principle by way of a thought experiment:

31. See Dasgupta (2013), Baker (2020), Wolff (2020), and Martens (2024) (and their citations) for more on the absolutism-comparativism debate in the metaphysics of quantity. The supposed mass doubling symmetry at the center of the debate is a direct analogue of Tolman's miniature universe transformation.

- Consider an observer O with a meter stick that measures the length of some extension, s, to be $l_s = 1m$.
- Now consider a counterpart world, a "miniature universe" in which there is a counterpart of the original observer, O', and both his "meterstick" and the extension s have been shrunk in length by a factor of x.
- Because both the length of the unit standard and the measured extension have changed by the same factor, the assigned value of the length will be invariant: $l_s' = 1m'$.
- The length quantity of the counterpart extension in the miniature universe of O', expressed in the units of O, will be $l_s' = x \cdot 1m$ or, more generally, $l' = xl$.

Given that this transformation equation, $l' = xl$, is expressed in a single system of units (it is true in either the units of O or O'), it must be understood as an equation of quantities—this accounts for Tolman's interpretation of the transformation to the miniature universe as a metaphysical transformation.[32] Accepting the speed of light postulate, their temporal measurements must also stand in the same relation: $t' = xt$. From assuming the invariance of other laws (e.g., Coulomb's law), Tolman derives a whole set of symmetry transformations (table 1).[33] From these results Tolman determined the functional form of several physical equations describing important physical phenomena: ideal gases, blackbody radiation, the electromagnetic field, and so on.

In the same year, Buckingham's landmark paper, "On Physically Similar Systems," presented the most influential proof of the Π-theorem. Buckingham argued that Tolman's principle was only a "particular case" of his result—this has some truth to it (see sec. 2.3). I will not here go through the full derivation of the theorem (see Gibbings [1982, 2011], Pobedrya and Georgievskii [2006], and Sterrett [2009, 2017, 2021]). Buckingham states the essential content of the theorem in terms of absolute units (equivalent to "coherent unit system," defined earlier). Using such an absolute/coherent system, the theorem shows that a duality exists between active and passive interpretations of changes of the

32. One might ask whether this equation necessarily involves expressions in both systems of units; however, that would confuse its role as a quantity equation with its role as a numerical equation. Both interpretations are available, but, in Tolman's argument, this must be an equation of quantities as the tranformation is ontic, not a mere formal translation (see de Courtenay 2015).

33. Table 1 is selectively adapted from Tolman (1914, 226). Note the invariant quantities and the corresponding theoretical commitments of Tolman's principle: the constancy of the speed of light, electromagnetic theory, and the laws of thermodynamics.

Table 1. Induced Transformations of Quantity Magnitudes under Similitude Transformations

Quantity Kind	Symmetry Transformation
Length	$l' = xl$
Time duration	$t' = xt$
Velocity	$v' = v$
Acceleration	$a' = x^{-1}a$
Mass	$m' = x^{-1}m$
Force	$f' = x^{-2}f$
Energy	$U' = x^{-1}U$
Energy density	$u' = x^{-4}u$
Electrical charge	$e' = e$
Entropy	$S' = S$
Temperature	$T' = x^{-1}T$

Note.—Adapted from Tolman (1914, 226).

fundamental units, corresponding to the distinction between transformations of formal and ontic dimensions (see sec. 2.3):[34]

> When absolute units are used, the validity of a complete physical equation is unaffected by changes in the fundamental units. Hence in changing from a system S to a similar system S' it is immaterial to the validity of the equation in question whether we do or do not retain our original fundamental units. If we alter the sizes of the fundamental units $[Q_1] \ldots [Q_k]$ in the same ratios as the kinds of quantity $Q_1 \ldots Q_k$ which they measure, the numerical value of any quantity of one of these kinds will be the same in both systems. And if we do not change the relations of the derived and fundamental units of our absolute system, every derived unit $[P]$ will change in the same ratio as every quantity P of that kind, so that the numerical value of every quantity in the system S will be equal to the numerical value of the corresponding quantity in the similar system S'. (Buckingham 1914, 354)[35]

34. This active-passive transformation duality can be made intuitive by considering the double interpretation of a fundamental unit in the case in which it is defined with respect to a material standard. A passive transformation corresponds to switch from a material meter-long standard for a length unit to a distinct, material, foot-long standard for a length unit. An active transformation corresponds to the (metaphysical) compression of a meter-long length standard to a length of 1 foot. The dual active-passive interpretation of the Π-theorem is dealt with in more detail by Jalloh (forthcoming).

35. Walter's discussion contains a claim that requires correction. Walter distinguishes similitude, "a simple way to investigate the manner in which a change of scale affects the properties of physical systems," from dimensional homogeneity, which requires that "the operation of addition and the relationship of

While Buckingham follows the Maxwellian tradition of discussing dimensional analysis in terms of invariance of "complete" equations under transformations of the fundamental units, we can understand his claim here as a generalization of Tolman's similitude principle—insofar as the principle of dimensional homogeneity is agnostic with respect to particular dimensional systems.[36] It is important to emphasize that the Π-theorem follows (almost) directly from the principle of dimensional homogeneity. Therefore, for all involved, the results of the Π-theorem—assuming an orthodox dimensional system—are results of the approach that I am calling "the principle of dimensional homogeneity" (where mass, length, and time are the basic mechanical quantity dimensions, etc.). There is a logical distinction between the principle and the principle plus a dimensional system, but the principle has no function independent of the adoption of a dimensional system (thus the "almost").

In a coherent unit system, the relations between basic and derived quantities are defined such that arbitrary changes in the magnitudes of the basic quantities induce changes in the derivative quantities such that representationally adequate and dimensionally homogeneous equations remain true. This is done without stipulating a particular invariance with respect to transformations of the length quantities. In brief, the theorem states that all physical equations are dimensionally homogeneous and so can be put in the form $A_1 + A_2 + \ldots + A_N = 0$, where each A term is a product of powers of the fundamental Q terms (the basic quantities of the dimensional system; e.g., masses, lengths, and times), and each term has the same dimension: $[A_i] = [A_j]$. Therefore, subtracting A_N and then dividing through by $-A_N$ yields an equation with dimensionless Π-terms:[37]

$$\Pi_1 + \Pi_2 + \ldots + \Pi_{N-1} = 1.$$

These dimensionless Π-terms will be invariant under any change of numerical value (passive transformation) or magnitude (active transformation) of the basic dimensions.[38] For example, let an arbitrary Π-term be the ratio between two masses $m_a = 2000$ g and $m_b = 1000$ g, so $\Pi_{a/b} = 2$. If we actively transform

equality are valid only for objects [i.e., quantities] of the same kind [i.e., dimension]" (1990, 86–87). The claim to be criticized is that "Buckingham, like everyone else" conflated these two bits of dimensional reasoning. This claim is false: Buckingham (1914) clearly distinguishes similitude and dimensional homogeneity as he uses the principle of dimensional homogeneity to provide a proof of the Π-theorem, which in turn defines a criterion for physical similarity. One follows from the other, but there is no indication that these are to be equated.

36. For more on the Maxwellian prehistory of this debate, see Mitchell (2017).

37. The dimensionless quantities and the theorem get their name from the fact that the dimensionless terms of the equation have the form of product-functions: $\Pi = \prod_i^N Q_i^{x_i}$.

38. This is the case where a basic dimension is understood as the set of all the quantities of that kind with an ordering that allows for the mapping by a choice of scale to a set of numbers (see Ellis 1964).

the dimension by doubling the masses, then $m_a = 4000$ g, $m_b = 2000$ g, then $\Pi_{a/b} = 2$. If we passively transform the dimension by a different choice of scale, a change to kilogram units, then $m_a = 2$ kg, $m_b = 1$ kg, then $\Pi_{a/b} = 2$.

Buckingham notes that Tolman's principle requires an assumption of speed, charge, and entropy as the invariants of its symmetries (table 1). For Buckingham, this is merely a specific realization of the general Π-theorem—that is, dimensional homogeneity. This specification is merely an unorthodox choice of dimensional system. Buckingham raises three objections to adopting this dimensional system and therefore Tolman's principle. First, it moves what are thought of as empirical laws from the client discipline of physics to the relatively a priori methodological discipline of dimensional analysis: "The unnecessary introduction of new postulates into physics is of doubtful advantage, and it seems to me decidedly better, from the physicist's standpoint, not to drag in either electrons or relativity when we can get on just as well without them" (Buckingham 1914, 356).[39]

Second, it makes this move unnecessarily. Buckingham goes on to show that the principle of dimensional homogeneity with the ordinary dimensional system can derive equations that Tolman credits the principle of similitude with. Third, Buckingham shows the essential inconsistency of Tolman's system and Newtonian gravity, due to variance of the gravitational constant across the supposed similitude transformation. Although Tolman himself derives the inconsistency of his approach with Newtonian gravity (see Tolman 1914, 254), Buckingham is the first to note this as a *problem* (see Buckingham 1914, 375). Ehrenfest-Afanassjewa is then the first to locate this discrepancy in the *gravitational constant* and makes much of this in her criticism of Tolman (see sec. 2.3).

Tolman (1915) responds to Buckingham and argues that the principle of similitude is superior to the principle of dimensional homogeneity on grounds of the latter's inability to constrain the functional form of equations with dimensional constants of unknown dimensions. These are cases in which dimensional homogeneity necessitates the introduction of dimensional constants: consider Stefan's law, $u = aT^4$. By the lights of the dimensional analyst, *in advance of the establishment of the dimensions of a*, the equation could have a different algebraic form—for example, $u = aT^3$.

In this case, the dimensional analyst is tasked with determining a function that relates the energy density of a blackbody, u, and its absolute temperature, T. Their respective dimensions, $ML^{-1}T^{-2}$ and Θ, are incommensurable, so the principle of dimensional homogeneity is of no help. Without either the dimensions of the mediating constant or the form of the function relating the two

39. Ehrenfest-Afanassjewa (1916b) makes this same complaint.

inhomogeneous quantities, the dimensional analyst armed only with the principle of dimensional homogeneity can make no derivations.

In contrast, the principle of similitude tells us that u must be numerically equivalent to its scale counterpart, u':

$$u = F(T) = u' = F(T') = x^4 F(x^{-1}T).$$

Referring to table 1, we see that u scales with x^4 and T with x^{-1}, so the solution for this equation requires taking temperature to the fourth power, and the equation is only fixed up to a scalar factor a, yielding Stefan's law:[40]

$$u = aT^4.$$

Now considerations of dimensional analysis nonarbitrarily yield the dimensions of the constant. As the dimensional analyst starts with neither the form of the equation nor the dimension of the constant, the principle of dimensional homogeneity is not determinative. If the dimensional analyst had the form of the law, the constraint of dimensional homogeneity would immediately yield the dimensions of the constant. If the dimensional analyst has the dimensions of the constant, the constraint of dimensional homogeneity would determine the functional, algebraic form of the equation. Tolman (1915) puts the relation of the two principles thus:

> Where dimensional constants enter, the principle of dimensional homogeneity is of no avail in predicting the form of a relation, since we cannot tell beforehand what the dimensions of the constant are going to be. For such problems we must have recourse to the principle of similitude. On the other hand, when dimensional constants do not enter into the relation, although we may apply either principle, the principle of similitude is usually the less powerful since it merely prescribes invariance when the different measurements are multiplied by powers of a single arbitrary multiplier x, while the principle of dimensional homogeneity prescribes the more drastic requirement of invariance when the multiplications are carried out with a different arbitrary multiplier for each fundamental property. (232)

Understanding Tolman's claim relies on distinguishing two ways in which a principle may be "stronger." The first way is that a principle may be *logically*

40. One way to think about the nature of the functional results yielded by either form of dimensional analysis is that the results give the family of curves that corresponds to the function but do not give the value of the coefficients. Those are found by experiment (see Gibbings [1974, 2011] on the relation of dimensional analysis to experiment).

stronger than another; in this case, the principle of similitude is the stronger principle as it provides more determinate derivations than the principle of dimensional homogeneity does, particularly in cases in which there is a dimensional constant of unknown dimension.[41] The second is that one principle may be more *robust* than another; in this case, the principle of dimensional homogeneity is a more robust principle as it is commutes with more supposed symmetry transformations—in particular, arbitrary transformations of the mass, length, and time that do not conform to the similitude transformations given in table 1. With some irony, the Euclidean standard is the guide to fundamentality that physicists adopt—the standard of logical strength. In contrast, for mathematicians, robustness seems to be the guide to fundamentality with respect to principles.[42] The question with respect to the operative, logical standard is one of efficiency: how much am I getting for what cost? On this standard, the principle of similitude would win out—if it weren't false.

2.3. Tolman versus Ehrenfest-Afanassjewa

An interpretative issue brings us back to the metaphysical considerations at hand. Tatiana Ehrenfest-Afanassjewa most clearly states an objection to Tolman's principle shared by the other respondents: the principle of similitude is merely an application of the principle of dimensional homogeneity to a special dimensional system, and if the assumption of this dimensional system is unfounded, the principle is specious.[43] She states in the first paragraph of her response:

> An accurate analysis shows that Tolman's considerations possess at least a close connection with the reduction to a definite hypothesis of the

41. Because the principle of dimensional homogeneity provides no constraint on the functional structure of such equations, one might like to say that these are cases in which the principle of dimensional homogeneity is inapplicable and that the principle of similitude enjoys a wider range of applicability.

42. The reason for the discrepancy between the standards of mathematics and physics comes from different standards of modality. From the physicist's perspective, many mathematical models are just that, and the robustness criterion is of no relevance when a class of possible worlds is fixed. Another way to put the debate between Tolman and his critics is that they disagree about the class of physically possible models (given the empirical data).

43. Walter's account of this historical debate is overly dismissive of Ehrenfest-Afanassjewa's contributions, especially her later, post-*Dimensional Analysis*, mathematical intervention (Ehrenfest-Afanassjewa 1926), which is described only as "extensive and confusing" (Walter 1990, 101). This dismissal is unfortunately mirrored in responses by Bridgman (1926) and Campbell (1926)—although Bridgman includes Ehrenfest-Afanassjewa (1926) in the list of important references that appeared in between editions of *Dimensional Analysis* (the list can be found in the preface to the revised edition.) A major reconsideration of her work in dimensional analysis is under development, but see also San Juan (1947), Palacios (1964), and Johnson (2018) for developments of her approach to dimensional analysis. For a more general reevaluation of her work in mathematics and physics, see Uffink et al. (2021).

> conviction of the homogeneity [unit invariance[44]] of all the equations of physics, a conviction which is commonly used without any foundation. This is not the intention of the author, as appears from his third paper on the same subject, yet he really does nothing else but construct a system of dimensions of his own (indeed one that in some respects deviates from the C.G.S. [centimeter-gram-second unit] system), and he examines all equations with a view to homogeneity *as regards this system of dimensions*. (Ehrenfest-Afanassjewa 1916b, 1)

Although Tolman (1916) rejects the presentation of his principle as determining a system of dimensions, he accepts the presentation of the relationship between the two principles: the principle of similitude involves a further empirical ansatz, which is to be settled by the investigations into the nature of gravity, and the principle is to be given methodological priority due to its usefulness. His disagreement with Ehrenfest-Afanassjewa can be clarified by way of a distinction made in sec. 1.3. When Ehrenfest-Afanassjewa states that Tolman is establishing a principle of homogeneity restricted to a special set of dimensions, she is referring to formal dimensions—dimensions considered only as change ratios for a group of unit systems. When Tolman claims that this is not the case, he is considering ontic dimensions—dimensions considered as descriptions of the natures of quantities through their dimensional formulae.

> Formal Dimensions. Dimensions encode the transformations of numerical representations of quantities due to changes in unit systems.

> Ontic Dimensions. Dimensions are properties of quantities in physical systems; they encode similarity relations that are invariant between scaled systems.[45]

We could just as well distinguish these as unit dimensions and quantity dimensions.[46] Formal dimensions are merely formal devices translating between unit

44. Homogeneity—that is, unit invariance—is sometimes treated as the fundamental principle of dimensional analysis in lieu of dimensional homogeneity. Authors vary on which is to be taken as axiomatic and which is to be derived, but the cases in which unit invariance and dimensional homogeneity come apart are so few and spurious as to be dismissed for our purposes (but see Bridgman 1931). I treat both approaches as the "dimensional homogeneity" approach. For more on the mathematical definition of homogeneity, see Ehrenfest-Afanassjewa (1926), San Juan (1947), and Palacios (1964).

45. This distinction is given by Johnson (2018, 105–12). A similar distinction between dimension-first and unit-first attempts to provide a mathematical model for the quantity calculus is noted by Raposo (2018). For the connection between similarity relations and ontic quantity dimensions, see Sterrett (2009).

46. This distinction became clearer in the 1930s (see Abraham 1933).

conventions. Ontic quantity dimensions, according to the fundamentalist at least, correspond to objective dimensional structure.

Ehrenfest-Afanassjewa argues that Tolman's similitude transformations should be understood only as formal transformations—that is, unit changes.[47] She places conditions on Tolman's ontic interpretation of these transformations as indicating actual changes in size (e.g., a miniature universe):

> 1. that a model universe in the sense defined above is possible,
> 2. that we possess all equations which are wanted for a full description of the whole universe,
> 3. that the latter condition is especially fulfilled by those equations which in the C.G.S. system serve to fix the dimensions of the different quantities. (Ehrenfest-Afanassjewa 1916b, 4)

To these conditions, she raises three objections. First, the unit transformation coefficients (or scale factors) for time, length, and mass (and so on) are fixed independently of any investigation into the possibility of such model universes. Second, the full description condition necessitates that the transformation coefficients (she also says, in quotes, the "dimensions") of the derived quantities are fixed by the similitude transformation in a way that unnecessarily minimizes the number of basic dimensions by disallowing the introduction of novel (nonmechanical) basic dimensions (reducing "the number of degrees of freedom of the transformation"). Third, there is no reason to think that the current fundamental dimensions are sufficient to capture all of nature ("which should give a *necessary* reduction of the degrees of freedom" in the dimensional system), and Tolman's reduced mechanical basis (consisting of just length) is insufficient to capture Newtonian gravity.[48]

47. "The transition from the numbers x_i to x'_i may also be thought of in another way: instead of imagining measurements to be made with the same units in two different worlds, we may conceive the measurements to be carried out applying two different sets of units to the same objects ('in the same world')" (Ehrenfest-Afanassjewa 1916b, 3).

48. Ehrenfest-Afanassjewa suggests a strategy for saving the ontic interpretation of the dimensional symmetries: the scaling of dimensional constants, so as to guarantee quantity symmetries (for contemporary arguments about this strategy, see Roberts [2016], Jacobs [2023], Martens [2024], and Jalloh [forthcoming]). The introduced constant can be understood two ways: either as some real quantity, such as a postulated constant of matter, or else "denote it as a product of special values of the active variables occurring in the equation" (Ehrenfest-Afanassjewa 1916b, 5). She develops this more thoroughly as the introduction of "formal variables" in Ehrenfest-Afanassjewa (1916a). The upshot is that such an extension of the "'physical' meaning of the constants" trivializes the possibility of active scale transformations and the invariance of equations under such transformations and so "ceases to afford a criterion for distinguishing between equations which are 'physically allowable' and arbitrary equations" (Ehrenfest-Afanassjewa 1916b, 6).

Tolman objects to Ehrenfest-Afanassjewa's characterization of his principle as determining another "system of dimensions" distinct from that corresponding to the then standard centimeter-gram-second unit system—at least insofar as dimensions are understood in the ontic sense.[49] Tolman gives an initial statement of the fundamentalist conception of an ontic system of dimensions:

> The dimensions of a quantity may be best regarded, I believe, as a shorthand statement of the definition of that kind of quantity in terms of certain fundamental kinds of quantity, and hence also as an expression of the essential physical nature of the quantity in question. If, for example, we define force as mass times acceleration, the dimensions of force will be $[mlt^{-2}]$ and this may be regarded as a shorthand recapitulation of the definition of force in terms of mass, length and time, and also as an expression of the essential physical nature of force.
>
> The reason, now, why certain physical equations have to be dimensionally homogeneous is because in the cases under consideration the physical nature of the quantities equated has to be the same. (Tolman 1916, 9)

Tolman argues that the second principle invoked, that the dimensions of a quantity express the essential nature of that quantity, grounds the principle of dimensional homogeneity. That an equation must have terms of equal exponent in each basic dimension on either side follows if equations are taken to describe not only numerical equalities but also "quantity identities." Here Tolman assimilates the definition of derived quantity dimensions and their metaphysical constitution. That the nature of physical quantities does not unproblematically follow from their dimensional formulae is discussed in the literature (e.g., Skow 2017; Johnson 2018)—Tolman's conflation of definition and constitution is a target of Bridgman's conventionalist critique.

The ontic interpretation of dimensional systems makes clear Tolman's reason for denying that the principle of similitude provides one. According to the principle of dimensional homogeneity force is defined and constituted by mass, length, and time, according to the formula $[f] = \mathrm{MLT}^{-2}$. Under the system of dimensions that would be given by the principle of similitude, force is a function only of length, $[f] = \mathrm{L}^{-2}$. If Tolman were committed to a system of dimensions given by the principle of similitude, he would say the principle's

49. This is the dimensional system for which CGS is a coherent unit system (see sec.1.3). In this respect, there is no difference between the CGS system and a meter-kilogram-second system.

attributes force the nature of an inverse area. For this reason, Tolman retreats to treating his principle as an empirical ansatz regarding the possibility of miniature, indistinguishable universes that is available for (dis)confirmation through the implied theory of gravity. This is a retreat from his original ontic interpretation of his similitude transformations.

2.4. Tolman versus Bridgman

Tolman's principle qua empirical ansatz is the target of Bridgman's critique: "If the exact form of the equations and their mode of application should turn out to be exactly identifiable with the corresponding manipulations of the theory of dimensions, then the principle of similitude must be judged not to be new. . . . I shall try to show in this note that such an identification is possible; that in so far as the principle of similitude is correct it gives no results not attainable by dimensional reasoning, and that in its universal form as stated above it cannot be correct" (1916, 424).[50] Bridgman's aim is to show that Tolman's principle of similitude is more determinative than the principle of dimensional homogeneity at the cost of reliability.

Bridgman diagnoses Tolman's apparent examples of the greater determinativity of the principle of similitude by drawing attention to a special feature of the dimensional constants involved. In particular, "The principle of similitude may be applied with correct results to all those cases in which the dimensional constants have such a special form that they are not changed in numerical magnitude by the restricted change of units allowed by the principle" (Bridgman 1916, 425)

The dimensions of Stefan's constant, a, are $\mathrm{ML}^{-1}\mathrm{T}^{-2}\Theta^{-4}$, so we can express a as $N_a m l^{-1} t^{-2} \theta^4$, where N_a is some dimensionless number and m, l, t, and θ are units of mass, length, time, and temperature, respectively. Now apply the principle of similitude:

$$a = a' = N_a x m' x l'^{-1} x^2 t'^{-2} x^{-4} \theta'^{-4} = N_a m' l'^{-1} t'^{-2} \theta'^{-4}.$$

The x factors cancel, and the numerical value of Stefan's constant is invariably N_a. That only some such constants are invariant under dimensional scale transformations is evident in Tolman's failure to capture Newtonian gravitation: $G = N_G \mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}$ scales with factor x^{-2}. The conclusion of Bridgman's argument is that the method of similitude requires an assumption regarding

50. "The universal form" is the statement that the materials that constitute the universe could be used to create an empirically indistinguishable universe that differed only in size.

the dimensionality of the relevant constant(s) just as the method of dimensional homogeneity does: a user of the principle of similitude must assume that the dimensional constants that figure in the fundamental equations are such that their dimensional transformation coefficients cancel out. This assumption bears out surprisingly often. In addition to *a*, Bridgman cites the gas constant, the velocity of light, and the constant of quantum action. Is there some metaphysical significance to this seeming conspiracy of the dimensional constants?

Bridgman answers in the negative—the apparent conspiracy can be explained by the dimensional structure of our conventionally defined unit systems. By limiting valid unit transformations to those that leave that some choice of constants invariant (e.g., *c* and *e* in Tolman's system), a number of consistent systems of dimensions can be defined. Bridgman amplifies Buckingham's observation that the number of independent basic dimensions or units can be determined by the number of unit-invariant quantity relations (i.e., laws) we choose to accept as axiomatic (i.e., relatively a priori, as indicated in sec. 1.2). Apparently, then, the number of basic quantity dimensions (and number of dimensional constants) is conventional. If force were to be set as an additional fundamental quantity, for example, there would be a new dimensional constant in Newton's second law; instead, we take the law, with this would-be constant set to unity, as a unit-invariant axiom. Bridgman argues that we accept dimensional *definitions* not owing to some metaphysical identity but because of the frequency of the corresponding experimental fact.

Bridgman provides a helpful demonstration of the conventionality involved. I will modify his convention of using the square brackets [*x*] to using curly brackets {*x*} to denote the unitless numerical value of *x* (in line with contemporary standards; see JCGM 2012). Bridgman provides a description of each of the constants of nature in terms of the fundamental units (five constants and five basic units):[51]

$$G = \{G\}m^{-1}l^{3}t^{-2} = \{G'\}m'^{-1}l'^{3}t'^{-2}$$

$$c = \{c\}lt^{-1} = \{c'\}l't'^{-1}$$

$$k = \{k\}ml^{2}t^{-2}\theta^{-1} = \{k'\}m'l'^{2}t'^{-2}\theta'^{-1}$$

$$h = \{h\}ml^{-2}t^{-1} = \{h'\}m'l'^{-2}t'^{-1}$$

$$E = \{E\}e^{-2}ml^{-3}t^{-2} = \{E'\}e'^{-2}m'l'^{-3}t^{-2}.$$

51. *G* is the gravitational constant; *c* is the light constant; *k* is the (Boltzmann) thermodynamic constant; *h* is the quantum constant; *E* is the (Coulomb) electric force constant. The following two sets of equations are adapted from Bridgman (1916, 429).

These equations can be used to determine the value of the constants under changes of fundamental units. Alternatively, they can be reformulated to determine the unit transformations that keep the values of the constants fixed:

$$
\begin{aligned}
l'^2 &= \frac{\{h\}}{\{h'\}}\left(\frac{\{c\}}{\{c'\}}\right)^{-3}\frac{\{G\}}{\{G'\}}\,l^2 \\
t'^2 &= \frac{\{h\}}{\{h'\}}\left(\frac{\{c\}}{\{c'\}}\right)^{-5}\frac{\{G\}}{\{G'\}}\,t^2 \\
m'^2 &= \frac{\{h\}}{\{h'\}}\frac{\{c\}}{\{c'\}}\left(\frac{\{G\}}{\{G'\}}\right)^{-1}m^2 \\
\theta'^2 &= \frac{\{h\}}{\{h'\}}\left(\frac{\{c\}}{\{c'\}}\right)^{5}\left(\frac{\{k\}}{\{k'\}}\right)^{-2}\frac{\{G\}}{\{G'\}}\,\theta^2 \\
e'^2 &= \frac{\{h\}}{\{h'\}}\frac{\{c\}}{\{c'\}}\left(\frac{\{E\}}{\{E'\}}\right)^{-1}e^2.
\end{aligned}
$$

Tolman's transformation equations can be derived by holding all constants fixed except for G; however, different transformation equations can be defined by varying other constants and holding G fixed. In each of these systems, some constant or other is the odd man out (i.e., is variant under similitude transformations). Generally speaking, if we wish to freely vary some number of the fundamental units (as Tolman does for length), we have to vary the same number of universal constants. The indeterminacy of which constants are varied due to the conventional choice of which fundamental unit (i.e., basic dimension) to ground our dimensional system in (i.e., a choice of alternative similitude principles) was taken by Bridgman to undermine Tolman's characterization of his principle as an empirical ansatz to guide the development of a novel theory of gravity. There is no more reason to hope for a new theory of gravity guided by this principle than a new theory of electricity. The constant or physical theory that "the" principle of similitude is in tension with is a matter of arbitrary choice. This arbitrariness—reducing time to length rather than reducing length to time—is unavoidable for Tolman in the absence of an ontic conception of his dimensional system. In other words, the choice of dimensional system associated with the principle of dimensional homogeneity is arbitrary, and a generalized principle of similitude does not yield unique empirical predictions—which is to be expected given Tolman's retreat to presenting the principle as only defining a formal system of dimensions (see sec. 2.3).

Tolman presents a full-fledged metaphysical account of "measurable quantities" in his final response regarding the principle of similitude. This account is in no way reactionary—it does not constitute an argument in favor of the principle of similitude—but rather serves a foundational purpose:

> The time is already ripe for a much more comprehensive and systematic treatment of the field of mathematical physics than has hitherto been attempted, and the completion of this task would make it possible to derive all the equations of mathematical physics from a few consistent and independent postulates, and to define all the quantities occurring in these equations in terms of a small number of indefinables. The purpose of this article is to discuss from a somewhat general point of view the nature of the quantities which occur in the equations of mathematical physics and to consider a set of indefinables for their definition. We shall thus hope to help in the preparation for that more complete systematization of mathematical physics which is undoubtedly coming. (Tolman 1917, 237)

Tolman aims to prepare the way for a generally axiomatic treatment of physics as a whole.[52]

Tolman reintroduces his metaphysical posit by way of discussing the relation that holds between fundamental and derived quantities, which is represented by dimensional formulae: "*The dimensional formula of a quantity may be regarded as a shorthand statement of the definition of that kind of quantity in terms of the kinds of quantity chosen as fundamental, and hence also as a partial statement of the 'physical nature' of the quantity in question*" (1917, 242).[53] Tolman holds that the apparent necessity of five fundamental quantity dimensions (three mechanical ones, one for electromagnetism, and another for thermodynamics) is due to there being "five fundamentally different kinds of 'thing'": space, time, matter, electricity, and entropy.

Beyond being sufficient to account for all known physical quantities, Tolman puts forth two further conditions on a set of fundamental quantity dimensions. The fundamental quantities must be extensive—this allows for extensive methods of measurement for all derived quantities, even those that are themselves intensive (consider the role of a thermometer in measuring the temperature).[54] The

52. Appropriate to the generality of his aims, Tolman takes Russell's (1903) distinction of magnitude and quantity as his starting point. Tolman's system, including his fundamental distinction of intensive and extensive quantities, cannot be dealt with here in full.

53. That dimension can at most be only a partial description of the nature of a quantity is here set aside (see Lodge [1888] and Mari [2009]).

54. "In case the derived quantity has intensive rather than extensive magnitude some more or less artificial correlation of the magnitude in question with quantities having extensive magnitude will then have to be used, as has been done in the case of our ordinary temperature scale" (Tolman 1917, 248).

set of fundamental quantity dimensions must also be such that they provide an optimal level of simplicity to the system of quantities.

With all this on the table, Tolman argues that Bridgman's conventionalism is due to a confusion of quantity dimension and unit dimension:

> The fact that it has become usual to pick out the units for derived quantities in the way indicated has sometimes led to an unfortunate confusion as to the real significance of dimensional formulae. Thus there has grown up the practice of speaking of the dimensions of a unit when what is really intended is the dimensions of the quantity involved. It certainly seems best, however, to use the dimensional formula of a quantity as a shorthand restatement of its definition in terms of the fundamental kinds of quantity. The dimensional formula is thus a symbol for the physical nature of the derived quantity and a recapitulation of the *necessary* relation between different kinds of quantity rather than the statement of a relation between units which we find convenient. (Tolman 1917, 249)

The dimensional relations between quantities are necessary, not conventional. This necessity distinguishes quantity dimensions from unit dimensions, or dimensional systems from unit systems (see sec. 1.3). Generally speaking, a dimensional system or a unit system can be used to fix the other by defining a coherent system of units. Nonstandard dimensional systems are often defined in this way by setting a constant equal to 1 and eliminating one kind of unit for another—for example, the spatialization of time units in relativity theory on the adoption of the light postulate. If one takes this to be a true *elimination* of the constant c, then one adopts a *dimensional* system in which time and length *quantities* are equivalent.[55] Tolman rejects any such conventionalism regarding the basic quantity dimensions. For him, the reduction of the time dimension to the space dimension would be the same as reducing pressure to volume on account of using them to form a two-dimensional graph—a well-founded correlation is insufficient for a dimensional reduction, let alone the reduction of a *fundamental* quantity dimension.[56] By distinguishing the necessary

55. Physicists often talk in this manner, but it is apparent that they usually take this to be a change only in unit systems and not in dimensional systems. The "suppressed" constants return when it is time for physical interpretation (but see Rücker 1888).

56. Although Tolman is a metaphysical realist about dimension, he thinks what we take to be the number of dimensions is a manner of empirical inquiry. The special sciences, following the example of thermodynamics, may introduce new kinds of measurable quantities (e.g., economics). The reduction of the number of dimensions seemed to him impossible, but not logically so.

dimensional relations of quantities from the conventional "dimensional" relations of units, Tolman takes himself to be reiterating what I am calling the "ontic-formal dimension distinction" he made in Tolman (1916). This confusion between the "dimensions of quantity" and "dimensions of unit" he claims may be "a contributory cause for a number of criticisms which have been made on the principle of similitude" (Tolman 1917, 251). That said, Tolman stops short of an explicit defense of his principle and, as far as I've seen, never defends or makes use of it again. As I argue in the next section, the points he makes against Bridgman's libertine conventionalism do point the way to a metaphysics of quantity dimensions, but one weaker than the quantity-dimension fundamentalism that he develops over the course the debate concerning his principle of similitude.

2.5. Verdicts

As mentioned, the failure of Tolman's principle of similitude was overdetermined. There is, however, much to learn about the foundations of dimensional analysis from the debate concerning its relation to the principle of dimensional homogeneity. Here are the results we may take from each of the criticisms discussed in the previous sections.

Buckingham correctly shows that the principle of dimensional homogeneity can generate a broad class of symmetry transformations, of which Tolman's "relativity of size" is only a special case corresponding to the adoption of an unorthodox dimensional system. Tolman is right to claim that the principle of similitude is the more determinative principle because it can be used to derive functional equations for systems with unknown dimensional constants, whereas the principle of dimensional homogeneity is useless.

Ehrenfest-Afanassjewa sharpens the criticism that Tolman's principle is merely setting up a peculiar dimensional system. She argues that Tolman's dimensional system is an allowable, qua formal system, but Tolman has not met the conditions needed to give it an ontic interpretation. In particular, the ontic interpretation of the dimensional system will require the variability of the magnitude of the gravitational constant across the similitude transformation—a transformation she takes to be nomologically impossible (see Jalloh [forthcoming] on "constant necessitism"). Tolman capitulates that his principle only works as setting up a formal system of units—although he thinks this may still constrain the form of future theories of gravity—and puts forward a fundamentalist metaphysics of dimensions, independent of the form of fundamentalism (length fundamentalism) apparently adopted in his initial 1914 paper.

Bridgman shows the apparent extra domain of determinativity to not be an argument in favor of the methodological priority of Tolman's principle of similitude, contrary to Tolman. The epistemic benefit of the principle is limited as it depends on an assumption about the dimensions of the relevant constant but not its exact dimensional formula; its dimensions must be such that it is invariant under the similitude transformation. Although this turns out to generally be the case (with the notable exception of G), Bridgman shows that given the number of constants and the number of basic dimensions, any principle of similitude based on the scaling of a single such basic dimension would lead to some constant or another being left out. The similitude transformations follow from this conventional choice and dimensional homogeneity, and Tolman's chosen unit system fails to be empirically adequate in the case of gravity. Tolman, systematizing his response to Ehrenfest-Afanassjewa, does not defend the principle of similitude but rather aims to clarify a confusion. Tolman distinguishes between ontic quantity dimensions and formal unit dimensions and claims that Bridgman's conventionalist argument depends on a confusion between the two. Whereas unit systems are indeed conventional, dimensional systems, constituted by dimensional formulae, are supposed to be representative of the intrinsic metaphysical nature of the quantities they describe. We cannot *choose* the basic quantity dimensions. Tolman's retreat to an understanding of the principle of similitude as merely showing the convenience and viability of a particular kind of unit system marks a complete rejection of the ontic interpretation of the principle of similitude, but it also marks the beginning of a debate regarding the metaphysics of quantity dimensions. For more on Tolman's and Bridgman's views on their return to this issue in the 1930s, see appendix S1.

3. Recovering Dimensional Realism: Arguments against Conventionalism

In this section I summarize the two metaphysical accounts of quantity dimensions that emerge from the early methodological debate and propose a synthesis that overcomes difficulties with both positions. As described in section 1.4, fundamentalism, the metaphysics of dimensions espoused by Tolman, and conventionalism, the antimetaphysics espoused by Bridgman, can be understood as opposite positions regarding two theses:

Dimensional Realism. There is objective dimensional structure that corresponds to a dimensional system.

Fundamental Basis. There is a fundamental dimensional structure that corresponds to a dimensional basis.

The fundamentalist accepts both theses, and the conventionalist rejects both theses. The conventionalist case against fundamental basis relies on the symmetry in defining equations: we can just as well take $f = ma$ to define the force dimension in terms of the dimensions of mass and acceleration as we can take it to define the mass dimension in terms of the dimensions of force and acceleration. The conventionalist case against dimensional realism therefore follows: If there exist a multiplicity of acceptable bases, then there is no unique dimensional system that represents objective dimensional structure. The conventionalist takes the existence of such symmetry transformations and the following lack of a unique dimensional system to provide evidence for the further antirealist claim that there is no objective dimensional structure. Such an argumentative strategy is familiar from the space-time literature: if some putatively objective structure varies under a transformation that is a symmetry of the laws (dynamical symmetry) and leads to an empirically indistinguishable system (empirical symmetry), then that structure is not in fact objective. For example, Leibniz famously argued against the existence of absolute space-time positions by showing that a universal translation of positions 5 miles to the west would be both a dynamical and empirical symmetry.[57]

I make the case here that there is a dimensional realism that can be recovered in light of the conventionalist symmetry argument. The conventionalist would be too rash if they were to take their symmetry argument to show that there is no dimensional structure whatsoever. Earlier I distinguished dimensional systems by their basis dimensions (see sec. 1.4); however, I now show that the objective dimensional structure that is represented by such dimensional systems is more coarse-grained. I do not attempt to give a new model of dimensional systems that is "reduced," so that there is nothing in a dimensional system that does not correspond to objective dimensional structure. Rather, I present a "sophisticated" account of dimensional systems such that equivalent dimensional systems related by an isomorphism (a change of basis) are taken to represent the same objective dimensional structure.[58] To recover some form of dimensional realism, some distinctions regarding the relations between dimensional systems and

57. For developments of such symmetry arguments, see Ismael and van Fraassen (2003) and Dasgupta (2016). I leave undetermined what is to be done with the "surplus structure," whether it is to be straightforwardly eliminated from our ontology or shown to be reducible to a fundamental, objective structure interacting with some subjective aspect. For a classic exposition of the analogous space-time debate, see Earman (1989).

58. On the difference between reduced and sophisticated theories, see Dewar (2019) and Martens and Read (2020). I will not make a stand here on whether a reduced theory is preferable to a sophisticated one or if a reduced one is possible in this case. It is just the case that a sophisticated theory of dimensional systems is readily available to me while a reduced one is not.

dimensional structure must be made. I divide each realist thesis into two subtheses, yielding four fundamentalist commitments:

DIMENSIONAL REPRESENTATION. Dimensional systems represent objective dimensional structure.

DIMENSIONAL UNIQUENESS. There is a uniquely correct dimensional system that represents the objective dimensional structure of the world.

FUNDAMENTAL BASIS SIZE. The size of the set of basic quantity dimensions is objectively determined.

FUNDAMENTAL BASIS IDENTITY. The individual identities of basic quantity dimensions are objectively determined.

Dimensional representation and dimensional uniqueness make up dimensional realism. This analysis is to be understood similarly to van Fraassen's (1989) analysis of scientific realism. Dimensional representation is a statement that dimensional systems are to be taken literally; they purport to represent something objective and so can be judged to do so more or less adequately. Dimensional uniqueness says that only one such dimensional system is ultimately correct. Similarly, fundamental basis size and fundamental basis identity make up fundamental basis. There are two possibly objective aspects of the fundamental dimensional structure. I argue that we can be realist about one aspect of the basis of dimensional systems (size) without being realist about the other (identity).

The conventionalist argument against fundamental basis is only partially successful: conventionalist transformations of the identities, but not the number of basic quantity dimensions, are consistent with the empirical success of dimensional analysis. A dimensional system for mechanics that treats force as a basic quantity (and mass as derived) is as empirically adequate as a dimensional system that treats instead mass as a basic quantity instead. However, although there appears to be no natural constraint on which quantity dimensions appear as basic, there is a natural lower limit on the number of quantity dimensions that can adequately represent a physical system. In fact, in Tolman's rebuttal to Bridgman's conventionalism, he puts forward the essential argument in favor of the objectivity of the number of basic quantity dimensions: the problem of insufficient bases. The problem is that the reduction of the number of basic quantity dimensions reduces the determinative power of the principle of dimensional homogeneity; therefore, it seems that the reduced dimensional system misrepresents some dimensional structure necessary to have a determinative dimensional analysis of physical systems. For example, Tolman (1917, 250) shows that the dimensional analytic derivation of the equation for the centripetal force,

$$f = k\frac{mv^2}{r},$$

becomes much more indeterminate when the dimensions of length and time are equated (reducing the basic mechanical dimensions to two by making velocity dimensionless):[59]

$$f = k\frac{mv^n}{r}.$$

This is evidence that a dimensional system that collapses the length and time dimensions lacks the representational capacity to adequately describe the centripetal force—Palacios (1964) calls such violations of this natural constraint the problem of insufficient bases. However, Tolman went too far in holding that this shows that the identities of the basic quantity dimensions are objectively determined by nature; in fact, it is the number of basic dimensions that is so determined.

When dimensional realism is taken as a package deal, the conventionalist attack on objective basis identity is enough to justify an antirealism about quantity dimensions. However, we can divide dimensional realism into dimensional uniqueness and dimensional representation. If conventionalist critique requires the rejection of fundamental basis identity, then dimensional uniqueness must also be rejected. Dimensional realism can be salvaged as the conjunct of just dimensional representation and fundamental basis size. The form and ramifications of this moderated dimensional realism is discussed in section 3.3, but first the case against a thoroughgoing conventionalism needs to be given. In the next sections, I give two arguments against an antirealist conventionalism: the first is the problem of insufficient bases, which is revealed by the Rayleigh-Riabouchinsky paradox; the second is the inability of the conventionalist to account for the explanatory nature of dimensional analysis altogether.

3.1. The Generalized Rayleigh-Riabouchinsky Paradox and the Problem of Insufficient Bases

In an early exposé of dimensional analysis, Rayleigh (1915) uses dimensional analysis to derive equations for a number of systems, including a case of heat

59. This is even worse when you consider that v^n could be folded into k, hiding any dependence on velocity.

transfer between a rigid rod and a stream of fluid (Boussinesq's problem). Riabouchinsky (1915) showed that by reducing the number of dimensions involved in describing the system from four to three—by eliminating the independent dimension of temperature via adoption of the mechanical theory of heat—dimensional analysis results in a less determinate result. This appears to be a paradox: *more* knowledge about the system, that temperature has equivalent dimension to energy, yields a *less* informative result. This surprising result shows that not all laws can be taken to give reductive dimensional formulae—on pain of inadequate representation. This means that the multiplicity of a dimensional system is not fully conventional but rather is restricted on one side by nature.

We can better understand this so-called paradox and the problem it raises by consideration of a simpler case. The Rayleigh-Riabouchinsky paradox can be generalized to an observation regarding the determinacy of dimensional systems in general. The case of dimensional reduction I wish to consider in fact appears in Buckingham's (1914, 372–75) response to Tolman: the reduction of the mechanical dimensional system's basis from three to two basic dimensions by using Newton's force laws to define a dimensional formula for mass in terms of length and time.[60] I show how such a dimensional reduction is done and then how it leads to lower specificity in the derivation of the period of a pendulum compared with the treatment in section 1.1.

To begin, we set Newton's two force laws equal to each other,

$$G\frac{mm'}{r^2} = m'a,$$

simplifying the expression we get:

$$G\frac{m}{r^2} = a.$$

Now we make G into a dimensionless number and, for convenience, assume we are working in a coherent set of units such that $G = 1$. The resulting equation,

$$m = ar^2,$$

will define the unit mass, with dimensions

$$[m] = [a][r^2] = \mathrm{LT^{-2}L^2} = \mathrm{L^3T^{-2}}.$$

60. The choice of which dimension is reduced to the other two is arbitrary, although this kinematic reduction recalls a "Laplacian" reduction of mass (see Martens 2018).

We make the same assumption regarding the quantities, which may be involved in modeling a simple pendulum, in our reduced kinematic dimensional system:

$$\begin{aligned}[t] &= \mathrm{T} \\ [m] &= \mathrm{L}^3\mathrm{T}^{-2} \\ [l] &= \mathrm{L} \\ [g] &= \mathrm{L}\mathrm{T}^{-2}.\end{aligned}$$

It is not clear from inspection that mass is irrelevant to the pendulum period. We have to be more systematic and apply the algorithmic method supplied from the Π-theorem; this will also provide us with the concepts needed to understand in full generality the problem of insufficient bases.

Importantly, the Π-theorem informs us that for any system, the number of quantities that describe the system, N, and number of basic dimensions that derive the dimensions of those quantities, B, determine the number of dimensionless Π-terms that are sufficient to describe the system: $N - B$. In this case, $N = 4$ and $B = 2$, so we should expect there to be two Π-terms sufficient to describe the pendulum. To determine the forms of the Π-terms, we must solve two sets of equations for the dimensional exponents of the component terms. Each set is composed of equations for each basic dimension. Two equations and four variables means that the exponents of two variables must be arbitrarily determined. We choose the simplest case for each Π-term: $\Pi_1 \propto t^1 l^0$ and $\Pi_2 \propto t^0 l^1$. The Π-terms will each have the form

$$\begin{aligned}\Pi_1 &= t m^{\alpha_1} g^{\beta_1} \\ \Pi_2 &= l m^{\alpha_2} g^{\beta_2}.\end{aligned}$$

Now we set up the two sets of linear equations to determine exponents of zero in each basic dimension for the two Π-terms:

$$\begin{aligned}\mathrm{T}:&\ -2\alpha_1 - 2\beta_1 + 1 = 0 \qquad && -2\alpha_2 - 2\beta_2 + 0 = 0 \\ \mathrm{L}:&\ \ 3\alpha_1 + 1\beta_1 + 0 = 0 && \ 3\alpha_2 + 1\beta_2 + 1 = 0.\end{aligned}$$

These equations yield the following solutions:

$$\begin{aligned}\alpha_1 &= -\frac{1}{2} \qquad & \beta_1 &= \frac{1}{2} \\ \alpha_2 &= -\frac{1}{4} & \beta_2 &= \frac{3}{4},\end{aligned}$$

so

$$\Pi_1 = tm^{-\frac{1}{2}}g^{\frac{1}{2}}$$
$$\Pi_2 = lm^{-\frac{1}{4}}g^{\frac{3}{4}}.$$

Solving for t, we get the equation $t = km^{1/2}g^{-1/2}\Psi(lm^{-1/4}g^{3/4})$, where Ψ is some power function. This compares unfavorably with the more specific equation derived in the full mechanical dimensional system, $t = k\sqrt{l/g}$. The move to a reduced basis generates spurious Π-terms. On the other hand, at some point, an increase in the number of basic dimensions will not reduce the number of Π-terms that describe a system. Palacios uses these conditions to provide criteria for insufficient and superabundant bases: "*If it happens that in augmenting in some way a given basis, the number of independent π monomials decreases, then the original basis was incomplete, whilst if the same system of such monomials is always obtained then, the original basis is complete and the augmented one is superabundant*" (1964, 67). I dub a dimensional system that is neither insufficient (or incomplete) nor superabundant with respect to a physical system to be a "well-tuned" dimensional system.

That a dimensional system can be more or less well tuned, that there is an objective standard (maximally efficient dimensional analysis) for how well a dimensional system describes physical systems, belies the conventionalist position. The conventionalist cannot account for the differences among an insufficient, a well-tuned, and a superabundant dimensional system, whereas the dimensional realist has an easy answer: the well-tuned system accurately represents the dimensional structure of the physical system in question, the insufficient system lacks certain representational capacities, and the superabundant system has unnecessary resources. Nature constrains the number of bases from below; a general Occamist norm constrains the number of bases from above.

3.2. Accounting for Dimensional Explanations

Recently, philosophers of science have given accounts of how dimensional analysis provides explanations and in doing so have attempted to eliminate any sense of paradox from the Rayleigh-Riabouchinsky phenomena discussed in the previous section. Lange (2009) has argued that dimensional analysis provides explanations of derived laws that screen off the fundamental laws. Dimensional analysis explains certain similarity features of systems that are independent of various aspects of their constitution (and so the sometimes distinct sets of fundamental laws that govern the phenomena in question).[61] I want to

61. Lange considers the dimensional similarities of waves in a fluid and standing waves in a string (Lange 2009, sec. 4.)

emphasize something about how Lange's account of dimensional explanations applies to the generalized Rayleigh-Riabouchinsky paradox. Dimensional explanations using dimensional systems with more basic quantity dimensions (particularly ones considered derivative in conventional systems) apply to a larger set of counterfactual cases. They apply to systems independently of the values and the dimensions of the (in this system, dimensional) constants that link the additional basic dimensions into the laws.[62] Bridgman (1931, 59) considers a case in which volume is treated as an additional basic mechanical dimension independent of length, which allows for the derived equation to apply even in non-Euclidean geometries where $v = l^3$ may not hold.[63] This would introduce a dimensional volume constant ω, where $v = \omega l^3$ and $[\omega] = \mathrm{VL}^{-3}$, which could have a nontrivial value (i.e., not 1). In the thermodynamic case, it could be that the value of Boltzmann's constant or the gas constant was different, such that a unit of temperature would not be equivalent, in either value or dimension, to a unit of energy, invalidating any mechanical reduction of thermodynamics. In both cases, the derivation that allows for the possibility of the variations in constants—that is, does not treat the relevant laws as a priori—is the more explanatorily powerful in the sense that it is more general.[64] Pexton (2014) gives a different although consistent account of how dimensional analysis explains: dimensional analysis provides models of systems that make apparent patterns of modal dependence (i.e., counterfactuals). On Pexton's modal-model theory of dimensional explanations, Rayleigh-Riabouchinsky phenomena can be accounted for by the fact that for some systems, such dimensional reductions (e.g., that of temperature to energy) are simply irrelevant. It is no surprise that irrelevant factors can introduce noise (in the form of extra degrees of freedom) that interfere with the power of an explanation given by the model. As seen with Lange's account, there is a tradeoff between abstraction and explanatory power.

The conventionalist makes both the general explanatory power and also the dependence of explanatory power on a choice of dimensional system mysterious. Surely if some choice of convention is better than another, not as a matter of

62. Lange (2009) holds that this is a counterlegal. This depends on the somewhat controversial but underappreciated thesis that the values of the constants are part of the laws (e.g., nomologically necessary). For reasons why this may not be the case, see Jacobs (2023) and Jalloh (forthcoming).

63. This fails to hold in a very mundane case: a liter of volume was defined (by the CGPM in 1901, until 1964) as the volume occupied by a kilogram of pure water in standard conditions rather than as a cubic decimeter, as it is currently. While the two definitions aim to define the same quantity, the correspondence is not exact, meaning that the former definition requires a constant to relate the volume and length unit, and the conceptual independence of volume from length in this defintion requires that this constant be dimensional (see Petley 1983, 137).

64. The explanation is powerful because it applies to more possible (or impossible) worlds; the derivation has greater modal robustness.

what is convenient to deal with but in its *explanatory capacities*, we ought to question whether dimensional systems are indeed a matter of convention after all. The dimensional realist has a nicer story to tell about the explanatory power of dimensional analysis: dimensions exist, and some dimensional systems better describe (some aspects of) their natures than others.[65] However, the conventionalist critique still has some bite. Generally, the Rayleigh-Riabouchinsky paradox shows only that the number of basic quantity dimensions, the degrees of freedom in the dimensional system, is constrained by nature. Both practice and mathematical theory give reason to believe that the basis of a dimensional system is not unique.[66] This conventionalist constraint on our metaphysics of quantity dimensions can be seen by considering the symmetric nature of defining equations: the relation between volume and length is equally well expressed by the formulae $V = L^3$ and $L = V^{1/3}$. What is needed is a metaphysics of dimensions that captures the objective structure of dimensional systems while leaving open for convention a choice of basis. Furthermore, this structure needs to be such that it provides a foundation for the representational and explanatory success of dimensional analysis. In the next section, I introduce such a metaphysics of quantity, a moderate realism: quantity dimension functionalism.

3.3. Functionalism: The Best of Both Worlds?

If the empirical adequacy of our dimensional systems is to serve as a guide to our metaphysics of dimensions, then it seems that we must give up fundamental basis identity and its corollary, dimensional uniqueness, because the empirical success of our quantitative representations of systems are insensitive to some changes in our dimensional formulae. Sensitivity to other changes in our dimensional formulae—namely, the problem of insufficient bases and the dependence of explanatory on dimensional systems—drive us to be committed to fundamental basis size and dimensional representation. As neither position articulated in the past century provides an adequate metaphysics of dimensions according to these criteria, both fundamentalism and conventionalism are to be rejected.

In their stead, I offer quantity dimension functionalism. Here I repeat the basic formulation of the view given in section 1.4:

65. An extended argument for dimensional realism from dimensional explanations has been provided by Jacobs (2024).

66. Mathematical models of dimensional systems are legion. Often, dimensional systems are modeled as vector spaces or groups. A full account of the metaphysics of dimensional systems in light of these models must be postponed. For mathematical models of quantity dimensions and some of their physical and metaphysical implications, see San Juan (1947), Corrsin (1951), Palacios (1964), Whitney (1968a, 1968b), de Boer (1995), Tao (2012), Johnson (2018), and Raposo (2018, 2019).

> FUNCTIONALISM. There is an objectively correct set of dimensional systems—each system describing the dimensional structure of the world equally well. While there is no unique basis for these dimensional systems, the number of quantity dimensions that are fundamental is objectively determined.

This position is indeed a form of structural or sophisticated realism in which quantity dimensions are without fundamental intrinsic natures or quiddities but rather have their natures as a matter of their relative positions in the (quotient) dimensional system.[67] Quantity dimension functionalism is most closely related to the sense of functionalism in the space-time literature, wherein "spacetime is as spacetime does" (Knox 2013, 2019; Lam and Wüthrich 2018). However, as a "role functionalism," rather than a "realizer functionalism," there is no commitment to underlying primitives that "realize" particular dimensional roles; functionalists can be thoroughgoing structuralists regarding quantity dimensions.[68] The relative positions in dimensional structure are the invariant objects *described by* different dimensional formulae given a choice of basis—no quantity dimension is *reducible* to any particular dimensional formula but only to an ensemble of them related by the (conventionalist) symmetry transformations of the dimensional system. Ultimately, these positions are to be understood as nomic roles.[69]

Although these dimensional nomic roles are not reducible to the dimensional formulae that describe them, we can learn about the structure of these nomic roles from considerations of the structure of dimensional formulae. As mentioned, dimensional formulae—and so, the dimensional dependence relations they describe—are symmetric; some quantity dimensions cannot be said to ground others, except relative to a basis; and quantity dimension symmetries (i.e., active dimension scale symmetries; for discussion, see Roberts [2016],

67. For simplicity's sake I will collapse the set of objectively correct dimensional systems into a single dimensional system. In group theory, this operation is called "quotienting," and the resultant quotient dimensional system can be understood as that invariant under all of the transformations among different dimensional bases. A similar structuralist defense of dimensional realism can be found in Jacobs (2024), although he does not take on the historical orientation that I have here.

68. On the distinction between role functionalism and realizer functionalism, see McLaughlin (2006). I invoke the distinction to defend my "functionalist" label from the complaint that it surreptiously commits me to quiddities, the realizers of dimensional roles. In fact, the existence of realizers or quiddities is of no importance. It is important only that the role is more fundamental than the realizer.

69. I cannot give a general treatment of nomic essentialism/structuralism here; for a survey, see Wang (2016). It is worth noting that dimensional structure seems to be an additional "high-order mathematical feature," which tells in favor of a nomological rather than a causal structuralist account of physical properties (see Berenstain 2016).

Martens [2021], and Jalloh [forthcoming]) will be tightly constrained, as they will involve the transformations of all the quantity dimensions with relevant mutual dependency relations.[70] Such quantity dimensions symmetries define a class of dynamical symmetries—dimensional analysis is used to determine similarity relations, transformations under which two systems can be used as (dynamic) models of each other (for details, see Sterrett [2009, 2017]). These dimensional dependence relations therefore play a double role of identifying the quantity dimensions relative to each other and of constraining the forms of the laws.[71]

In this functionalism view, we can say something about the nature of the dimensional structure of the world as a whole. Dimensional structure is an order of modal structure that is more coarse-grained than that of the nomological modal structure of the laws as traditionally conceived. If the laws are considered as strict equalities between quantities, then dimensional structure captures the more coarse-grained proportionality relations between quantities that hold with natural necessity. These dependence relations are central to the nature of the laws, although they undetermine their "strengths"—their relative strengths being captured by the relative values of their characteristic constants.[72] Much more can be said to explicate the version of nomic essentialism to which the quantity dimension functionalist is committed to and to defend it from various objections in the metaphysics literature (most notably, by Sider [2020]). My aim here is only to introduce the view, which is hopefully sufficiently motivated by consideration of the alternatives with which the history of dimensional analysis has furnished us.

4. Conclusion

This article has exposited an unduly neglected debate regarding the methodological and metaphysical foundations of dimensional analysis and has evaluated the merits of the two major positions: conventionalism and fundamentalism. Both positions are found lacking. Conventionalism regarding quantity dimensions fails to account for the explanatory success of dimensional analysis and

70. On the viability of symmetric dependency relations, see Barnes (2018).

71. One might quibble here with my "constraining" language. With Campbell (1924) and Palacios (1964), one may argue that the laws constrain dimensional analysis by defining the relations between dimensional quantities. I do not want to establish any sort of priority claim here regarding the structure of the physical dimensions or the forms of the laws; they are mutually constraining, and I claim only that one takes precedence over the other depending on the epistemic context.

72. This last point was made clear to me by Bryan Roberts; see also, Dahan (2020) on the idea of constants characterizing the laws.

representational constraints on dimensional systems. Fundamentalism fails to fit with the conventionality found in scientific practice and fails to give reason to privilege any basis over others for a dimensional system. I have set forth the basic outline of a functionalist account of quantity dimensions in which the empirical constraints on the number of basic quantity dimensions and the conventionality regarding which quantity dimensions are treated as basic are respected. The metaphysical residue that the functionalist is realist about are the symmetric, nomologically necessary dependency relations between quantity dimensions, which correspond to the dimensional forms of the laws and so encode metaphysically robust proportionality relations.

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