The Bridgman-Tolman-Warburton Correspondence on Dimensional Analysis, 1934

A Supplement to "Metaphysics and Convention in Dimensional Analysis, 1914-1917" in *HOPOS*

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Contents

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1 Editorial Introduction

This correspondence marks the return of physicists Richard C. Tolman[1] and Percy W. Bridgman[2] to the topic of dimensional analysis. In the preceding decades Tolman and Bridgman were at the center of debates concerning the methodological and metaphysical commitments of dimensional reasoning, beginning with Tolman's controversial proposal that a "principle of similitude"—a principle asserting that global scale transformations of length quantities are dynamical and empirical symmetries—ought to be the foundational principle of dimensional analysis. Bridgman, inspired to clear up the mass of confusion he saw in the ensuing debate, wrote the first book in English on the topic: *Dimensional Analysis* (originally published 1922, with a revised edition published in 1931), which coined what is now the standard name for the method.[3] This correspondence has yet to have been published or referred to in the literature on dimensional analysis and its history. With its publication I include this editorial introduction and some exegetical and contextualizing notes. This correspondence is not only significant because it clarifies some of the—largely metaphysical—issues left unsettled by the original debate between Tolman, Bridgman, and others, but it also highlights the *practical* significance of these issues for physicists in the early 20th century who were working to standardize the unit system used in the teaching and practices of physics and engineering—especially with respect to electromagnetic units.[4] A

[1]Richard Chace Tolman (1881-1948) was Professor of Physical Chemistry and Mathematical Physics at the California Institute of Technology. Besides being one of the central figures in debates regarding the foundations of dimensional analysis, he was one of the first disseminators of relativity theory in the United States and served as a scientific advisor for the Manhattan Project. Tolman in fact first suggested the implosion method that used in the "Fat Man" bomb on Nagasaki (Monk 2014, 364).

[2]Percy Williams Bridgman (1882-1961) was Higgins University Professor at Harvard. He is well known for his high pressure experiments, for which he won the 1946 Nobel Prize in Physics. He is also well known as the father of the philosophy of science known as operationalism (*locus classicus*: Bridgman 1927).

[3]See Tolman (1914) for the original paper which initiated the debate. See Skow (2017) for a contemporary discussion of the metaphysical issues that arise for Tolman. See Jalloh (Forthcoming) for an account of the debate prior to Bridgman's book, and see Walter (1990) for a survey of the debate(s) both before and after Bridgman's book.

[4]This fits into a larger pattern of standardization and professionalization in (and beyond) American physics in the first half of the twentieth century, see Kevles (1995). It is also remarkable that the initial systematization

direct connection is thereby drawn from philosophical controversies regarding metaphysics and convention to scientific practicalities.

The Tolman-Bridgman correspondence was initiated by a request from Fred W. Warburton,[5] on behalf of the American Association of Physics Teachers Committee on Units, for Tolman's advice regarding the number of fundamental units needed for electromagnetism. Tolman's opinion as of 1917 was taken to be that three units were needed for mechanics (one each for mass, length, and time) and an additional unit each for electromagnetism and thermodynamics. Warburton cites then recent work in support of the contrary view, shared by Bridgman, that the number of fundamental units is purely conventional, depending only on the number of "coordination ideas", conventional rules of coordination[6] that define derived units in terms of basic units sans proportionality constants (e.g. $F = ma$ defines the dyne). Tolman's original position is based on a metaphysical realism regarding substantival natural kinds of quantities. Each fundamental unit is needed to measure each fundamental kind of "stuff" that our physics describes.

Though Tolman believed a "conceptual map" of physics ought not reduce the number of *quantity* dimensions, he agreed with the conventionalists that the number of basic *units* (and so *unit* dimensions) could be reduced by the adoption of unit defining equations. This method of unit reduction (e.g. the elimination of time units by adopting $ct = x$ as a definition of a length unit for time) is most exhaustively put forward by Raymond T. Birge (1935a, 1935b). In the AAPT committee's interim report (in which both Tolman and Birge, among others, are acknowledged), they adopt a conventionalist attitude:

> The subject of the dimensions of physical quantities is little understood by the

of the dimensional calculus by Maxwell and others was also driven by concerns with unit systems that integrated electrical and magnetic quantities, see Mitchell (2017).

[5] Fred William Warburton (1898-1969) was Assistant Professor of Physics at University of Kentucky from 1931 to 1935 and an Associate Professor from 1935 to 1946. From 1946 to 1963 he was Professor of Physics at the University of Redlands. He continued to do research at the U.S. Naval Ordnance Laboratory in Silver Spring, MD after retiring from academia. He received his A.B. (1922) and his PhD (1927) from Cornell.

[6] This is closer to the original terminology of Abraham (1933) that Warburton references.

> average physicist; or it may be better to say, what one person means by dimensions may be quite different from what another means. The subject is controversial and philosophical. The number and choice of the primary quantities is quite arbitrary. (Michener 1935, 90)

However, the committee elides the distinction between dimensions of quantities and dimensions of units favored by Tolman and Birge.[7] Setting that issue aside for a moment, it is important to note that this conventionalism accommodates the many unit systems for electromagnetism then in use. Particularly, there is a divide between the "absolute" unit systems, e.g. Gaussian units, in which electromagnetic quantities are *reduced* to mechanical dimensions and extended systems which introduce either a *basic* electrical or a *basic* magnetic dimension (or both). In the interim report, the committee expresses a desire for a unified electromagnetic dimensional system:

> But in electricity usage varies greatly. Since the dimensions of a quantity depend upon the primary quantities chosen and the series of definitions or defining equations leading up to the quantity considered, it is advisable to have generally accepted conventions covering the whole field of physics, and departure from these conventions by any writer should be made clear. (Michener 1935, 91)

In a later report (Little et al. 1938), the committee recommends the meter-kilogram-second-ampere (m.k.s.a.) system and the Gaussian (or Heavyside-Lorentz) system, the former for engineers and beginning students and (either of) the latter for more advanced theoretical work. Allowing for the convenience of having multiple systems, the committee retained hope that their endorsement would lead to the desuetude of many of the other systems in use. The majority of the 1938 report concerns the m.k.s.a. system, which was then recently adopted by the International Electrotechnical Commission.

[7] As far as I know, this distinction is first made in Tolman (1917), but Abraham (1933) was also influential in making this distinction. Birge thinks *only* units can properly be said to have dimension, as dimensions are only devices for unit conversion, compare Bridgman (1916, 1931) and see Mitchell (2017).

The final section of the committee report addresses the vexed matter of dimensions. With the exception of the eliding of distinction between quantity dimensions and unit dimensions, the committee largely follows Birge (1935a, 1935b). Objections to the metaphysically substantive notion of dimensions—according to which they describe the "nature of physical quantities"—are rehearsed (see Jalloh Forthcoming):

(1) Dimension underdetermines the nature of a quantity. Distinct quantities like energy, work, and moment of force have the same dimension.

(2) Dimension makes spurious distinctions among quantities. Density and specific gravity both correspond to the same concept, but have different dimension owing to the use of a reference standard in the case of the latter (making specific gravity dimensionless). The dimension of a quantity also varies across different unit systems.

The consequence of these arguments is a conventionalism regarding the dimensions of quantities:

> This illustration serves to show that the dimensions of a quantity depend upon the primary quantities chosen in terms of which the dimensions are to be expressed, the defining equations used, and also certain arbitrary assumptions regarding factors of proportionality. (Little et al. 1938, 150)

The increased degrees of freedom in establishing the conventional dimensions of electromagnetic units (or quantities) was blamed for the confusion that had occurred in their establishment theretofore. In the m.k.s.a. system endorsed by the committee, the introduction of a fourth fundamental unit (the ampere) has utility in distinguishing a greater number of quantities by dimension. On the other hand the Gaussian (or Heavyside-Lorentz) system has the advantage of simplification through the replacement of dimensional constants ϵ_0 and μ_0 with the single dimensional constant c.

Before leaving the historical context for these letters behind, let me attempt a clarification of the issues regarding whether dimensions are attributable to quantities or units and what sort of distinction there is between the two understandings of "dimension". The ambiguity that we have between what I will call ontic dimensions and formal dimensions was a source of much confusion and debate in the early half of the twentieth century. This ambiguity is even invited by Fourier's original statement of the principle of dimensional homogeneity:

> It must now be remarked that every undetermined magnitude or constant has one *dimension* proper to itself, and that the terms of one and the same equation could not be compared, if they had not the same *exponent of dimension*. [...] it is derived from primary notions on quantities; for which reason, in geometry and mechanics, it is the equivalent of the fundamental lemmas which the Greeks have left us without proof. (Fourier 1878, sec. 160)

Here it is easy to read into Fourier's statement an ontic conception of dimension: each quantity has some intrinsic character that is its dimension. From the broader context it is clear that whether or not Fourier meant for dimension to be a metaphysical notion it is certainly a formal one. Specifications of the formal notion of dimension can be found in Ehrenfest-Afanassjewa (1916) and Bridgman (1931), among other places, but let me borrow from the analysis of Abraham (1933) who pinpoints the source of ambiguity in what is often taken to be an innocent shift in formalism.

Let $A, B, C \dots$ be the set of basic quantities and let $a, b, c \dots$ be their corresponding units. In a dimensionally coherent system of units, the units of any derived quantity U will be defined in terms of the units of the basic quantities. If we wish to change to another unit system in the dimensional system,[8] the following equation holds:

$$\frac{u'}{u} = \left(\frac{a'}{a}\right)^{\alpha} \cdot \left(\frac{b'}{b}\right)^{\beta} \cdot \left(\frac{c'}{c}\right)^{\gamma} \cdot \ldots$$

$\alpha, \beta, \gamma, \ldots$ are U's exponents of dimension. Formal dimensions, identified with these exponents, are merely devices to guide units transformations between purely conventional unit systems. Given a different kind of unit system (e.g. one with a different number of basic units) the dimensions of any derived quantity may differ. Therefore, this formal notion of dimension cannot be identified with any deep metaphysical character of some quantity (which ought to be invariant under changes of our choice of convention).

The slide to metaphysics comes from a natural reading of an equivalent equation:

$$[U] = [A]^{\alpha} \cdot [B]^{\beta} \cdot [C]^{\gamma} \cdot \ldots$$

$[U]$ is usually read as "the dimension of U". Here dimensions seem to be properties of quantities—a quantity and its dimension stand in a functional relation. Abraham (1933) argues that, since this equation is nothing more than shorthand for the one before it, it cannot have any metaphysical implications which outstrip it.

I do not propose to settle the issue in this introduction. It seems clear to me that there are these two notions of dimension and that, while the formal notion is well understood, the ontic notion was—and remains—vague. Whether there is any reason to believe in an ontic counterpart to formal dimensions remains to be conclusively established (however see, Jalloh Forthcoming). It is my intention that by bringing the following correspondence into public view I may stimulate some interest in these philosophical issues arising from the very foundations of scientific practice.

[8] For example, the c.g.s. and m.k.s. systems belong to the same dimensional system as they are related by scalar transformations on the basic length and mass units. Ellis (1964) calls units related by such scalar transformations "similar scales". Non-coherent, but still similar, unit systems are still captured by such dimensional formulae but involve additional dimensionless scale factors.

2 Notes on Preparation of the Text

Some notes on style: I have ignored tabs or indents in the text and have adopted the usual current style for such indents, emphasis has been rendered by *italics*, and the subsections of Tolman's letter of August 22nd have been rendered in **bold**. All footnotes are mine, numbered by Arabic numerals, except for three in Tolman's letter of August 22nd, which are given distinct markings: asterisks and daggers. All mistakes that are not merely grammatical, corrected by hand, or typographical are left unchanged—certainly including those which I did not notice. Blanks that were presumably filled in with pen, missing from the available copy of the Warburton letter, are filled with best guesses, as explained in a footnote. In this case no original was available to crosscheck. Generally, insertions are marked by square brackets, with the exception of these mentioned variables, as that would conflict with the use of square brackets to mark the dimensions of a quantity.

These letters are available at the Percy Bridgman papers in the Harvard Archives, HUG4234.10, Box 5. I thank the staff at the Harvard Archives for permission to publish this material.

3 Tolman to Bridgman, August 24th, 1934

California Institute of Technology

Pasadena

Gates Chemical Laboratory

Professor Percy W. Bridgman,

Physics Laboratory,

Harvard College,

Cambridge, Mass.

Dear Bridgman,

I am enclosing a copy of a letter from Professor Warburton and my reply. It all has to do with the dimensions of quantities and I thought you would be interested. I should enjoy very much knowing to what extent you agree and disagree with what I have said. It would be fine if we could talk together sometime, long enough to be using the same language, or at least to think that we were doing so. I have always thought that I was understanding your language better than you were mine, which can only mean that your language is the better.

With best wishes,

Sincerely,

Richard C. Tolman

4 Warburton to Tolman, June 14th, 1934

University of Kentucky,

Lexington, Ky.

Professor Richard C. Tolman

California Institute of Technology

Pasadena, California.

My dear Professor Tolman:

In an article published in the Physical Review, Vol. IX, 1917,[9] you apparently accepted the view held by a considerable number of physicists that four fundamental units are necessary in electricity and magnetism – including temperature five basic units in all – and you quote Sir A. W. Rucker[10] as showing this necessity. On the other hand Professor Henri Abraham,

[9]Tolman (1917).

[10]Rücker (1888).

Nat. Res. Bull., No. 93, p. 17 1933,[11] and others have shown that the number of fundamental units is quite arbitrary, being the difference between the number of quantities defined and the number of rules of coordination ideas. Your opinion carried weight with members of the American Physics Teachers' Committee on Units. Those who fear that a vague but vital something is lost by reducing the number of fundamental units in electricity and magnetism from four to three quote your 1917 paper.

We would appreciate a statement from you giving your views as of 1934, first, as to whether the Heavyside-Lorentz and Gaussian systems are in error or suffer because they use but three fundamental units, making ϵ and μ dimensionless and $v = c/(\epsilon\mu)^{1/2}$ in preference to $v = 1/(\epsilon\mu)^{1/2}$ with ϵ and μ carrying unknown but not independent dimensions as in systems assuming four fundamental units; and second, concerning the advantages and disadvantages of reducing from four to three basic units.[12]

I shall be glad to transmit to the members of this committee whatever statement you care to make. Thanking you in advance, I am

Respectfully yours,

(Signed) F. W. Warburton, Member of the A. A. P. T. Committee on Units

5 Tolman to Warburton, August 22, 1934

Professor F. W. Warburton,

Department of Physics,

[11] Abraham (1933).

[12] In the copy of the letter available to me only the left-hand side of these equations ("v") and the numerators of the right-hand side ("1/" and "c/") are typed, with the denominators and the variables so involved ("ϵ" and "μ") presumably being written in pen on the originals (as occurs in other letters). The involvement of ϵ and μ is confirmed by the later letters. These equations define the speed of propagation of an electromagnetic wave—or equivalently, light—in a medium. ϵ is the permittivity of the medium and μ is the permeability of the medium. In a vacuum $\epsilon = \epsilon_0$, $\mu = \mu_0$, and $v = c$—three fundamental constants of electromagnetism. See discussion in Birge (1934).

University of Kentucky,

Lexington, Ky.

My dear Professor Warburton,

Your friendly letter of June 14th has been on my desk for a long time but I have had great inertia in answering it. This is perhaps partly because I have not thought about the dimensions of physical quantities for a long time, and perhaps partly because I no longer have the same youthful assurance that my own way of looking at things is the only right one. In addition, discussions of dimensions often seem to arouse violent differences of opinion, which are I suspect due to unrecognized differences in the underlying philosophy tacitly assumed by the different contenders. I will try, however, to answer your questions as well as I can, although I am afraid my answer will seem a long and elaborate essay. Here goes.

1. Meaning of the Term "Dimensions". It seems to me that the main source of disagreement in dimensional considerations lies in the fact that different physicists – or for that matter individual physicists at different times – do not always use the term "the dimensions of a physical quantity" in the same sense. For example, in Bridgman's beautiful book[*] on dimensional analysis, the dimensional formula of a quantity is used in the sense of a short-hand restatement of the rules of operation by which the number to be assigned to a so-called *secondary* quantity can be obtained from numbers provided by the measurement of certain *primary* quantities associated with the quantity in question. On the other hand, in the article[†] of mine which you mention, the dimensional formula of a quantity is used in the sense of a short-handed restatement of the method of definition of a *defined* kind of quantity in terms of the kinds of quantity which have been chosen as *fundamental*.[‡]

It will be profitable to discuss these two possible uses of the term "dimensions" in some

*[*Dimensional Analysis*], New Haven, 1922.
†Phys. Rev., 9, 237 (1917).
‡Note that throughout the discussion, the consistent use of the terms *primary* and *secondary* quantity in connection with the first sense, and *fundamental* and *defined* quantity in connection with the second sense.

detail, since they are sufficiently alike as to be easily confused, and yet sufficiently different so that disagreement will arise unless it is known which sense is intended.

2. Dimensional Formulae as a Short-hand for Certain Operations. The use of dimensional formulae as a short-hand for certain operations on numbers is inspired by a desire to keep as close as possible to a description of the actual behaviour of physicists when they make measurements in their laboratories and computations in their studies. These measurements and computations, carried out in accordance with certain rules of operation, are found to lead to numbers which are assigned as the values of the various quantities considered. The later attempt can of course then be made to find mathematical relations connecting the values obtained for the different quantities involved when a given kind of phenomenon is investigated under varying conditions.

On examination it is found that the quantities involved in any given experiment fall with reasonable lack of ambiguity into two classes, primary quantities to which the numbers are assigned as a direct result of the rules of measurement, and secondary quantities to which numbers are assigned by measuring certain primary quantities – connected therewith – and then combining the numbers so obtained in accordance with certain rules to obtain the number for the secondary quantity in question. Furthermore, it is found that the rules actually employed for making such combinations require that the numbers corresponding to the primary quantities are to be raised to certain powers, as given by a specified set of exponents, and then multiplied together.

In accordance with this line of thought the dimensional formula of a secondary quantity is then regarded as a short-hand statement of the rule by which the number to be assigned to the quantity in question is computed. For example, consider an experiment in which the velocity of a particle is taken as a secondary quantity to be obtained from the measurement of two primary quantities, a certain length – that traveled by the particle – and a certain time – that needed for the journey. The dimensional formula for velocity

$$[v] = [lt^{-1}] \tag{1}$$

would then indicate that the number assigned to the velocity will involve a multiplication of the number assigned by direct measurement to the length by the reciprocal of that assigned by direct measurement to the time.

Such dimensional formulae are useful in two ways.

In the first place, since they indicate how the numbers belonging to different primary quantities are combined together to give the number assigned to the secondary quantity, they tell us what change in the number assigned to a secondary quantity will result from a change in the units used in measuring the primary quantities. For instance, in the above example, if we change to a new unit of time one-half the original one, it is evident that the number assigned to the measured time will be twice as great, and hence since the formula shows us that this number will occur in the final result to the minus first power, it is evident that the number finally assigned to the velocity will be one half as great as before.

In the second place, such formulae are useful in the dimensional analysis of physical problems. In making such an analysis we desire to get some information as to the solution of a problem without actually carrying out the complete process of writing down and solving the fundamental equations which are involved. To carry out such an analysis we assume that we are familiar enough with the field in question to know what primary and secondary quantities and dimensional constants would occur in the fundamental equations and in their solution. We also assume that the fundamental equations are written – as is usual and possible – in a so-called "complete" form so that they, and hence also the desired solution, will hold independent of the units used in the measurements of the primary quantities. We then know that the final solution can in any case be written down in the form

$$F(\pi_1, \pi_2, \dots) = 0 \tag{2}$$

where F is some function of the independent dimensionless products $\pi_1, \pi_2, \ldots$ which can be built from the primary quantities, secondary quantities and dimensional constants under consideration, in accordance with the dimensional formulae given to them.

It should be specially noted that the foregoing procedure, for obtaining and making use of dimensional formulae, may permit considerable flexibility in the number and kinds of quantity which are chosen as primary when any given physical situation is to be investigated – either actually or conceptually.

For example, in considering a given problem involving the measurement of a volume, it might turn out to be possible to regard volume as a secondary quantity to be computed from the measurements of certain lengths in accordance with the dimensional formula $[l^3]$, or to regard volume as a primary quantity with dimensions of its own $[v]$ to be directly measured by comparison with some standard volume. In Chapter [VI] of Bridgman's book an actual problem in dimensional analysis is treated in which volume is first regarded as a secondary and then as a primary quantity.[13] The results obtained by the two treatments are of course compatible, although that obtained when volume is taken in the usual way, as a secondary quantity with the dimensions $[l^3]$, is the *less* specific in form.[14]

To take a more extreme illustration of the flexibility which is compatible with the foregoing treatment of dimensional formulae, let us consider a set of conceptual experiments in which length is treated as a secondary quantity and time as a primary quantity, with the rule of operation that the number to be assigned to any given length is to be taken as the measured time necessary for light to travel in vacuo from one end of the length to the other. In such a treatment length will have the dimensions of time

$$[l] = [t] \tag{3}$$

[13]Tolman here mistakenly refers to the chapter VII. The double treatment of this problem begins on page 59 of both the 1922 and the 1931 editions.

[14]This is a form of the Rayleigh-Riabouchinsky paradox, see Rayleigh (1915), Riabouchinsky (1915), Palacios (1964), and Gibbings (2011).

since the number assigned to any length will vary with the unit chosen for the measurement of the primary quantity time in the way indicated by this formula. Such a treatment is, I think, entirely logical and illustrates somewhat forcibly that the use of dimensional formulae – in the way described above – as a short-hand for certain rules by which we decide to assign numbers to secondary quantities from the measurement of related primary quantities does not in the least imply that two quantities which in a given treatment are taken as having the same dimensions are also to be regarded as necessarily being the same kind of quantity or as having the same physical character.

3. Dimensional Formulae as a Short-hand for Certain Definitions. Turning now to the use of dimensional formulae as a short-hand for certain definitions, we find that this alternative procedure is inspired by a desire to build up a conceptual structure – a universe of discourse – which physicists can use as a kind of map, to help in finding their way, when making measurements in their laboratories and computations in their studies. The attempt to construct such a universe of discourse gives explicit recognition to the fact that physics is a highly abstract science, concerned with very limited aspects of reality, and treating a subject matter which has in any case come from the raw material of experience only after a deep-seated process of selection, interpretation and conceptualization.

In constructing such a universe of discourse we start with a set of indefinables, definitions and postulates. The indefinables and definitions provide the subject matter in the universe of discourse, and the postulates, together with the theorems derived from them with the help of logic or other disciplines more fundamental than that of the field of interest, provide the significant assertions which can be made concerning the subject matter and assertions in our universe of discourse and the elements and regularities of behaviour observed in the world of phenomena.

Proceeding along such lines in the case of physics, we shall desire to include quantities among the elements of subject matter in out universe of discourse. We shall not wish, however, to take all the different kinds of quantity as indefinables but to select some kinds as fundamental from which the rest can be obtained with the help of certain mathematical operations which will also be included among the elements of our construct. In order to carry out this program in a manner to agree with the actual conceptual background tacitly employed in classical physics, it proves desirable to pick out *five* kinds of quantity as fundamental, corresponding to the concepts – space, time, matter, electricity and statistical probability of configuration – which are present in the five component branches – geometry, kinematics, mechanics, electrodynamics and thermodynamics – which may be regarded as the field of classical physics. As a satisfactory choice of the five fundamental kinds of quantity we may take quantities of length, time, mass, charge, and entropy.

Having chosen the five fundamental kinds of quantity we may then define further kinds of quantity with the help of certain operations such as ordinary multiplication, the two kinds of vector multiplication, differentiation, etc. which will also be included among the elements of our construct. Thus a quantity of area may be defined as the outer product of two quantities of length

$$\bar{A} = \bar{l}_1 \times \bar{l}_2 \tag{4}$$

where the outer product is understood in the sense of Grassmann as being the actual parallelogram determined by the two vectors $\bar{l}_1$ and $\bar{l}_2$.[15] Similarly a quantity of velocity may be defined by the equation

$$\bar{v} = \frac{d\bar{l}}{dt} \tag{5}$$

where the operation of differentiation is here understood as giving the actual vector quantity of interest. In this way some success at least can be achieved in defining all the quantities of interest.

It should be specially emphasized that such equations of definition are intended in the present treatment as a symbolism which indicates not only the mere rules by which the number

[15]Tolman is here making reference to Hermann Grassmann's *Ausdehnungslehre* which presents an early and very general form of vector analysis. See Grassmann (1995) for an English translation and Crowe (1994) for an account of Grassmann's work in historical context.

assigned to the derived quantity is obtained from the numbers assigned to the fundamental quantities, but the full physical relation by which the quantities on the right-hand side determine those on the left. Thus equation (4) is regarded as telling us not only that the magnitudes of areas may be calculated from the magnitudes of the lengths involved by the purely numerical equation

$$|A| = |l_1||l_2| \sin\theta \tag{6}$$

where θ is the included angle, but as also telling us that a unique area with a specific spatial orientation and location is as a matter of fact actually determined by two lengths diverging from a common point. Similarly equation (5) tells us not only that the number assigned to a velocity can be calculated from the numbers assigned to certain changes in length and time by a limiting process expressed by

$$|v| = \frac{d|l|}{dt} = \lim_{\Delta t \to 0} \frac{\Delta|l|}{\Delta t} \tag{7}$$

but also tells us that a unique velocity actually is completely determined by specifying a given $d\bar{l}$ and dt.

In the treatment that we are considering, dimensional formulae may now be introduced as a short-hand to remind us of the above equations of definition by which we can obtain the derived quantities from the fundamental ones. These formulae will be similar in form to the ones used in the previous treatment but unfortunately will now have a somewhat different significance. In the first place, the dimensional formulae now used will differ from the previous ones in being based on a definite selected set of five fundamental kinds of quantity, instead of depending on a variable choice – to be specified at the beginning of each consideration – as to the number and kinds of quantity taken as primary. In the second place the formulae will now be used to remind us not only of the way the numbers assigned to a derived quantity are dependent on the numbers assigned to the corresponding fundamental quantities, but also to remind us of the more complete relation by which the fundamental kinds of quantity do determine the defined quantity. It is this latter aspect which we have in mind when we say that the dimensions of a quantity are a partial expression of its physical nature.

In concluding the description of this mode of procedures, is should be noted – since the dimensional formulae do continue to express as part of their content a possible way of choosing the relations between the numbers to be assigned to various quantities – that they can still be used for the purposes of determining the dependence of assigned numbers on size of units and of carrying out dimensional analyses.

4. Remarks on the Two Procedures. What can we now say about the advantages and disadvantages of the two procedures?

It would be very convenient if we could say that one of these procedures was right and the other wrong. It does not seem to me, however, that such can be the case. The first procedure endeavours to describe what physicists actually do with the numbers that they obtain in the laboratory, and the second procedure endeavours to construct a conceptual map which physicists will find useful in the laboratory.

We might object to the first procedure as follows. Physicists do not treat numbers in this or that respect the way you say. All right, I will modify my description at those points. Physicists do not treat their numbers the way they ought to; watch out and you will find the next generation doing something else. All right, I will modify my description when I find them doing something else. Physicists make use of an enormous conceptual background, what Bridgman (p. 51) calls "the experience of all the ages", and your first procedure gives no adequate account of this. All right, I never said it did, I have purposely drawn a line between the definite operations which physicists do carry out with numbers and which I can describe with clarity and assurance, and those treacherous fields bordering on metaphysics where I might search for an account and evaluation of the conceptual structure which has arisen from "the experience of all the ages."

On the other hand we might object to the second procedure as follows. Your universe of discourse is faulty at this or that point, it doesn't serve as a reliable map. All right, I will modify my it at those points. Your universe of discourse is not at all what it ought to be; watch out, the next generation will have a very different set of concepts. All right, I know even better than you that my universe of discourse is even now a partial and faulty structure, and I am modifying and improving it and shall continue to do so in the future. Your whole idea of building a universe of discourse is ridiculous, the resulting map is no help at all either in the laboratory or in the study or anywhere else. All right, I am sorry you don't like the idea, but the procedure seems to give explicit expression to a part of what physicists actually are doing anyway, and there are a great many who do find the universe of discourse helpful.

It would be equally convenient if we could get physicists to agree to abandon one or the other of the two senses in which dimensional formulae have been used. This, however also seems improbable, although they are not always a very good short-hand when used in the second sense described above.

The first use of dimensional formulae reminds us merely of the way that the number assigned to a secondary quantity depends with a particular method of procedure on the numbers obtained in the measurement of associated primary quantities. This use is too convenient to be abandoned, and flexibility which it permits in the choice as to number and kinds of primary quantities corresponds to an actual and entirely logical flexibility in the behaviour of physicists.

The second use of dimensional formulae reminds us not only of numerical relations but also of physical relations between quantities. The latter use is also too convenient to be abandoned in the case of simple enough quantities so that it does provide a good short-hand to give symbolic expression to a portion of the conceptual background of physics, which the first treatment leaves unanalyzed. Indeed it would be very difficult to force a use of dimensional formulae solely in the sense of the first treatment. Among the possible sets of dimensional formulae permitted by the first treatment will be one in which the primary and secondary quantities are chosen so that they agree with a natural choice which we might make for fundamental and defined quantities. It will then be very difficult to prevent all use of the resulting dimensional formulae in the sense of the second treatment. For example, the dimensional formulae for area, velocity, momentum and current $[l^2]$, $[lt^{-1}]$, $[mlt^{-1}]$ and $[qt^{-1}]$ cannot fail to remind us that the conceptual background of physics is such that two lengths do give us a unique area, a change of length with time does give use a velocity, a mass moving with a velocity does five us a momentum, and a passage of charge with time does give us a current. On the other hand, in the case of defined quantities which depend on the fundamental kinds of quantity in a complicated way, dimensional formulae provide a very inadequate short-hand for reminding us of the full nature of the definitions. Thus the electric inductivity ε of a medium is a complicated statistical result of the behaviour of many individual electrons and the conceptual nature of this quantity is very poorly symbolized by the dimensional formula

$$[\varepsilon] = [q^2 m^{-1} l^{-3} t^2]. \tag{8}$$

A few further critical remarks concerning the two procedures will be helpful.

The first procedure has the advantages of allowing any choice as to the number and kind of quantities chosen as primary and secondary that physicists do employ, and of sticking so close to a description of the actual behaviour of physicists that there will be common agreement as to the success with which the program is carried out. The procedure has the disadvantages that the choice made for primary and secondary quantities must be specifically restated each time, and that the justification that the choice is a possible one must be found in an uncodified "background of a great deal of physical experience", "experience of all the ages", "instinct whether a problem is suitable for mechanical treatment", etc.

The secondary procedure has the advantage of trying to start in the direction of constructing a conceptual map which will provide some codification of the background of physical experience. It has, however, some very serious disadvantages. The indefinables, definitions and postulates for a universe of discourse can be chosen in a variety of ways even when the resulting conceptual constructs are equivalent. Hence there will be lack of agreement as to the best attack. The suggested programme has been carried out only very imperfectly and incompletely and needs continual tinkering as physics progresses.

5. Answers to Specific Questions. I apologize for the length of the foregoing. In the light of the discussion, however, I think that we can find answers to the specific questions in your letter. I will make my answers correspond to the successive sentences in the first two paragraphs of your letter.

1st sentence. It is my view that it is desirable to take five fundamental kinds of *quantities* (not units) in order to build a conceptual map of classical physics including geometry, kinematics, mechanics, electrodynamics and thermodynamics.

2nd sentence. I agree with Professor Abraham that the number of fundamental *units* may be chosen arbitrarily provided we give a sufficient number of rules of coordination by which we define units for all the quantities of interest from those chosen as fundamental. For example we may choose the unit of time as fundamental and define the unit of length as equal to the distance traveled by sound in air under standard conditions in unit time.

3rd and 4th sentences. Something "vital" and I hope no longer "vague" is lost, not by reducing the number of fundamental *units* from four to three in electricity and magnetism, but in thinking that this means that three instead of four kinds of *quantity* could be taken as fundamental for purposes of defining the contents of a universe of discourse in the sense of the second procedure described above. When we add the concept of electricity to our previous concepts of space, time and matter we shall wish to add a new fundamental kind of quantity for the purpose of defining that content. The *units* for measuring this quantity, however, can be defined if desired in terms of a conceptual process in which *direct* measurements are made only on the *three* quantities of length, time and mass.

5th sentence. I do not think that the Heavyside-Lorentz and Gaussian systems are in error because they take three instead of four kinds of *units* as primary. This is not equivalent to taking three kinds of quantity as *fundamental* in the sense of the second treatment described above, but only equivalent to taking three kinds of quantity as *primary* in the sense of the first treatment described above.

The electrostatic dimensional formula for charge based on three *primary* quantities

$$[q] = [m^{1/2} l^{3/2} t^{-1}] \tag{9}$$

is an entirely correct one when used in the sense of the first procedure described above, provided we regard it as a short-hand to remind us of the fact that we are taking mass, length and time as primary quantities, have already agreed to assign numbers to the secondary quantity force in the way usual in mechanics, and have then agreed to measure a charge by assigning a number which is proportional to the square root of the product of the measured force which it exerts in vacuo on an equal charge multiplied by the square of the distance between them.

Similarly the electromagnetic dimensional formula for charge

$$[q] = [m^{1/2} l^{1/2}] \tag{10}$$

is correct if we agree to the same choice of primary quantities but agree to assign numbers to charges by a procedure which involves the magnetic force that they exert when in motion in vacuo.

On the other hand, treatments in which inductivity $[\varepsilon]$ and permeability $[\mu]$ are introduced as additional primary quantities are also logical. They lead to the dimensional formulae

$$[q] = [\varepsilon^{1/2} m^{1/2} l^{3/2} t^{-2}] \tag{11}$$

and

$$[q] = [\mu^{-1/2} m^{1/2} l^{1/2}] \tag{12}$$

and correspond to the previous rules of assigning numbers to charge with a dropping of the requirement that the measurements are to be carried out in vacuo. It should be noted, however, that inductivity and permeability are not a very pleasing choice as primary quantities, when we look at dimensional formulae from the point of view of the rules for assigning numbers to secondary quantities, since the rules for the direct measurement of inductivity and permeability are not transparent.

Some persons might prefer (11) and (12) to (9) and (10) as a starting point for the definition of electrical units, being influenced by a desire to give recognition not only to the use of dimensional formulae as a short-hand for the relation between the numbers assigned to primary and secondary quantities but also to their use as a short-hand for the relation between fundamental and defined quantities. Nevertheless, it is only the first use that is important when we are defining *units*. Furthermore, although (11) and (12) each contain an additional kind of quantity as would be desirable for the second use of dimensional formulae, neither inductivity nor permeability would be a good choice as a fundamental kind of quantity owing to their complicated conceptual nature. Indeed, as already remarked in connection with equation (8), quantities such as these are so complicated that dimensional formulae containing them may no longer serve as a convenient short-hand for remembering the whole nature of the process of definition.

If you have read this far I thank you for your patience and sign myself

Cordially yours,

Richard C. Tolman

Prof. Physical Chemistry and Mathematical Physics

6 Bridgman to Tolman, September 9, 1934

Randolph, N. H. Sept. 9, 1934

Dear Tolman;

It was good to get your friendly letter. There seems to be a good deal of flutter recently on the subject of dimensions, and various echos of it had reached me before your letter came. I had not been stimulated to doing any fresh thinking on the subject, however, so that I am afraid that I am in much the same position that you were and my reactions to your letter can signify only the present state of the precipitate left in my mind from a previous condition of activity.

In general, it seems to me that we are closer together than we have been before, and that I understand much better what your position is. In fact, I think that I would admit that you have a *right* to do nearly everything that you want to do, and that our differences have reduced largely to matters of taste. At the same time there do still seem to be real differences of taste, and I cannot for the life of me understand why you want to do some of the things that you apparently do.

I do no believe that there is as clean cut a separation between the two ways of looking at the subject as you would suggest. I suppose that you intend your description of the first way of looking at dimensional analysis, that is, the attitude which regards a dimensional formula as a compact way of summarizing certain operations which have been performed on numbers, as essentially a statement of my own attitude. But my attitude in the face of a dimensional formula contains much more than a consciousness of the way in which certain numbers were obtained- it contains also as a vital part of the background a consciousness of the physical operations involved in the measurements and also a realization, obtained from “the experience of all the ages”, that it is useful to describe our experience in terms of these operations. In fact, it seems to me that in my background is embedded all that is to be found in you second way of looking at the matter, and more too. I will admit that if one wants as *comprehensive* a view as possible of all physics one may perhaps do well to adopt the classical division into geometry, kinetics, mechanics, electrodynamics, and thermodynamics, and therefore use five fundamental kinds of quantity in his definition. But for the purposes for which dimensional analysis is used one does not usually want to be reminded of the most comprehensive possible point of view, but instead one wants the most special one possible. For example, in treating a problem in ordinary elasticity, like the bending of a beam, one is saying more about nature and embodying a more extensive experience in defining force as a fourth kind of unit of its own kind, instead of defining it as mass times acceleration. It is saying more about nature, and it is more pertinent to say, that the problems with which we are confronted can be split up into subgroups, each capable of its own method of treatment, than it is to say that all phenomena can be treated under five grand subdivisions. And when you get right down to it, the five is not very significant, because we cannot yet describe biological and mental phenomena in terms of these five. In other words, in my experience the occasions are so rare on which I have found it profitable to be reminded that that selected subgroup of experience which we have chosen to call physics can be treated with complete generality under five main subdivisions, that I do not care to clutter my tool box by attaching special importance to this scheme of description, and I don't quite see why you do either. I think you will find this to be a fair description if you think over the uses made of dimensional analysis. I should class these as: changing units, checking equations, finding necessary relations in complicated experimental situations so as to reduce the number of necessary observations, as in model experiments, and perhaps in certain theoretical arguments as in the last chapter of my book. I think that all these purposes are served with a flexible system of definition as well if not better than with a system of five, and certainly the use in model experiments demands the maximum possible flexibility.

I think that I would be inclined to disagree with your position that your two ways of looking at dimensional analysis describes the attitude of actual physicists. It seems to me that there is still an awful lot of messy thinking about the whole thing, as in the classical statement of Eddington that a length of 10^{-41} cm must be the key to some essential physical structure.

I was glad to have you say what you did about the unattractiveness of ϵ and μ as fundamental kinds of quantity.[16] Leigh Page, who spends the summer next to me, has been writing an article with Adams, which I hope will be published in the *American Physics Teacher*,[17] in which he points out some disconcerting relations in this connection which are not ordinarily realized.[18]

I too wish that we might have chances to talk together oftener. You will probably want to talk more about this and I know that there are oodles of things about relativity theory that I am itching to get straight from you. Haven't you any inclination to spend another summer in the East? Randolph, only ten miles from the top of Mt. Washington, is very attractive in the summer, there are lots of nice cottages which you might rent, and we would be tickled to pieces to add you to our community.

Most sincerely,

[Percy Bridgman]

[16]These variables were missing from the copy available to me.
[17]The name of *The American Physics Teacher* was changed to *American Journal of Physics* in 1940.
[18]This presumably came to be Page and Adams (1935).

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