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EXTENDED GERGONNE SYLLOGISMS

ABSTRACT. Syllogisms with or without negative terms are studied by using Gergonne's ideas. Soundness, completeness, and decidability results are given.

1. Background and motivation

Gergonne [2] relates the familiar A, E, I, and O sentences without negative terms to five basic sentences that express the "Gergonne relations." These relations are: exclusion, identity, overlap, proper containment, and proper inclusion. What makes these relations especially interesting is that for any pair of non-empty class terms exactly one of them holds.

Faris [1] develops a formal system that takes the Gergonne relations as basic. His system takes advantage of Łukasiewicz's [4], which attempts to formalize the Aristotelian syllogistic. The following paper results from two ideas: 1) If Gergonne had been interested in studying A, E, I, and O sentences *with* negative terms, the count of Gergonne relations would be seven rather than five; and 2) The most Aristotelian way to develop a syllogistic system based on the these seven relations is by following Smiley's [5] rather than Łukasiewicz's [4].

After developing the Aristotelian "full syllogistic" based on seven relations, we will discuss a subsystem that is adequate for representing AEIO-syllogisms with or without negative terms.

2. The system

Sentences are defined by referring to:

terms: $\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots$

simple quantifiers: $=, =^{-}, \subset^{++}, \subset^{+-}, \subset^{-+}, \subset^{--}, \mathrm{Z}$

comma: ,

$\mathrm{Q}_1, \ldots, \mathrm{Q}_n$ is a *quantifier* provided i) each Q_i $(1 \leq i \leq n)$ is a simple quantifier, ii) Q_i precedes Q_j if $i < j$, where precedence among simple quantifiers is indicated by the above ordering of simple quantifiers, and iii) at least one quantifier is not a Q_i. No expressions are *quantifiers* other than those generated by the above three conditions. So, for example, $=, \subset^{++}$ is a quantifier but $\subset^{++}, =$ is not. Qab is a *sentence* iff Q is a quantifier and a and b are distinct terms. So, for example, $=, \subset^{++}$AB and $=^{-}, \subset^{--}$, ZAB are sentences, but $=, \subset^{++}$AA is not. Qab is a *simple sentence* iff Qab is a sentence and Q is a simple quantifier. Read simple sentences as follows: $=ab$ as “The a are the b,” $=^{-}ab$ as “The a are the non-b,” $\subset^{++}ab$ as “The a are properly included in the b,” $\subset^{+-}ab$ as “The a are properly included in the non-b,” $\subset^{-+}ab$ as “The non-a are properly included in the b,” $\subset^{--}ab$ as “The non-a are properly included in the non-b,” and Zab as “Some a are b, some a are non-b, some non-a are b, and some non-a are non-b.” Read $Q_1, \ldots, Q_n ab$ by putting “or” between sentences that correspond to $Q_i ab$. So, read $=, \subset^{++}ab$ as “The a are the b, or the a are properly included in the b” (or “All a are b.”) $=^{-}, \subset^{+-}, \subset^{-+}, \subset^{--}, \mathrm{Z}ab$ may be read as “Some a are not b.”

The deducibility relation $(\vdash)$, relating sets of sentences to sentences, is defined recursively. Read “X $\vdash y$” as “y is deducible from X.” Set brackets are omitted in the statement of the following definition. “X, y” is short for “X $\cup \{y\}$” and “x, y” is short for “$\{x\} \cup \{y\}$.” “a”, “b”,... range over terms; and “p”, “q”,... range over “+”, and “−”. p^* is “+” iff p is “−”. $cd(\mathrm{P}ab) = \mathrm{Q}ab$ iff every quantifier that does not occur in P occurs in Q. Read “*cd*” as “the contradictory of.”

(B1) $=ab \vdash =ba$

(B2) $=^{-}ab \vdash =^{-}ba$

(B3) $\subset^{pq}ab \vdash \subset^{q^*p^*}ba$

(B4) $\mathrm{Z}ab \vdash \mathrm{Z}ba$

(B5) $=ab, \mathrm{Q}bc \vdash \mathrm{Q}ac$, where Q is $=, =^{-}$, or $\subset^{pq}$

(B6) $=^{-}ab, =^{-}bc \vdash =ac$

(B7) $=^{-}ab, \subset^{pq}bc \vdash \subset^{p^*q}ac$

(B8) $\subset^{pq}ab, \subset^{qr}bc \vdash \subset^{pr}ac$

(R1) If X $\vdash y$ and $y, z \vdash w$ then X, $z \vdash w$

(R2) If X, $y \vdash \mathrm{P}ab$ then X, $\mathrm{Q}ab \vdash cd(y)$ if no quantifier in P is a quantifier in Q

(R3) If X, $\mathrm{P}ab \vdash y$ and X, $\mathrm{Q}ab \vdash y$ then X, $\mathrm{R}ab \vdash y$ if each quantifier in R is in P or Q

(L1) $\mathrm{X} \vdash y$ iff $\mathrm{X} \vdash y$ in virtue of B1–R3.

So, for example, $=^{-}\mathrm{AB}$, $\subset^{++}\mathrm{BC} \vdash \subset^{-+}\mathrm{AC}$ (by B7) and $\subset^{-+}\mathrm{AC}$, $\subset^{+-}\mathrm{CD} \vdash \subset^{--}\mathrm{AD}$ (by B8). So $=^{-}\mathrm{AB}$, $\subset^{++}\mathrm{BC}$, $\subset^{+-}\mathrm{CD} \vdash \subset^{--}\mathrm{AD}$ (by R1). So $=^{-}\mathrm{AB}$, $\subset^{++}\mathrm{BC}$, $\subset^{+-}, \mathrm{ZAD} \vdash =, =^{-}, \subset^{++}, \subset^{-+}, \subset^{--}$, ZCD (by R2).

THEOREM 1. (D1) *If* $\mathrm{X}, y \vdash \mathrm{P}ab$ *then* $\mathrm{X}, y \vdash cd(\mathrm{Q}ab)$ *if no simple quantifier occurs in both* P *and* Q. (D2) *If* $\mathrm{X}, y \vdash cd(\mathrm{P}ab)$ *and* $\mathrm{X}, y \vdash cd(\mathrm{Q}ab)$ *then* $\mathrm{X}, y \vdash cd(\mathrm{R}ab)$ *if each quantifier in* R *is in* P *or* Q. (D3) *If* $\mathrm{X}, y \vdash z$ *and* $v, w \vdash y$ *then* $\mathrm{X}, v, w \vdash z$.

Proof. Begin each proof by assuming the antecedent. (D1) Then $\mathrm{X}, \mathrm{Q}ab \vdash cd(y)$ (by R2). Then $\mathrm{X}, y \vdash cd(\mathrm{Q}ab)$ (by R2). (D2) Then $\mathrm{X}, \mathrm{P}ab \vdash cd(y)$ and $\mathrm{X}, \mathrm{Q}ab \vdash cd(y)$ (by R2). Then $\mathrm{X}, \mathrm{R}ab \vdash cd(y)$ (by R3). Then $\mathrm{X}, y \vdash cd(\mathrm{R}ab)$ (by R2). (D3) Then $\mathrm{X}, cd(z) \vdash cd(y)$ and $v, cd(y) \vdash cd(w)$ (by R2). Then $\mathrm{X}, v, cd(z) \vdash cd(w)$ (by R1). Then $\mathrm{X}, v, w \vdash z$ (by R2). □

A *model* is a quadruple $\langle \mathrm{W}, \nu_{+}, \nu_{-}, \nu \rangle$, where i) W is a non-empty set, ii) ν_{+} and ν_{-} are functions that assign non-empty subsets of W to terms such that $\nu_{+}(a) \cup \nu_{-}(a) = \mathrm{W}$ and $\nu_{+}(a) \cap \nu_{-}(a) = \varnothing$, and iii) ν is a function that assigns t or f to sentences such that the following conditions are met:

(i) $\nu(=ab) = t$ iff $\nu_{+}(a) = \nu_{+}(b)$

(ii) $\nu(=^{-}ab) = t$ iff $\nu_{+}(a) = \nu_{-}(b)$

(iii) $\nu(\subset^{pq}ab) = t$ iff $\nu_{p}(a) \subset \nu_{q}(b)$

(iv) $\nu(\mathrm{Z}ab) = t$ iff $\nu_{p}(a) \cap \nu_{q}(b) \neq \varnothing$ for each p and q

(v) $\nu(\mathrm{Q}_1, \ldots, \mathrm{Q}_n ab) = t$ iff for some i $(1 \leq i \leq n)$ $\nu(\mathrm{Q}_i ab) = t$

y is a semantic consequence of X ($\mathrm{X} \vDash y$) iff there is no model $\langle \mathrm{W}, \ldots, \nu \rangle$ such that ν assigns t to every member of X and ν assigns f to y. X is *consistent* iff there is a model $\langle \mathrm{W}, \ldots, \nu \rangle$ such that ν assigns t to every member of X. X is *inconsistent* iff X is not consistent.

THEOREM 2 (Soundness). *If* $\mathrm{X} \vdash y$ *then* $\mathrm{X} \vDash y$.

Proof. Straightforward. (For B1, note that for any model $\langle \mathrm{W}, \ldots, \nu \rangle$, if $\nu_+(a) = \nu_+(b)$ then $\nu_+(b) = \nu_+(a)$. For R2, suppose no quantifier in P is a quantifier in Q, and suppose that X, Q$ab \nvDash cd(y)$. Then there is a model $\langle \mathrm{W}, \ldots, \nu \rangle$ in which ν assigns t to every member of X, $\nu(\mathrm{Q}ab) = t$, and $\nu(cd(y)) = f$. Note that $\nu(cd(y)) = f$ iff $\nu(y) = t$. And note that since no quantifier in P is a quantifier in Q, $\nu(\mathrm{P}ab) = f$. So X, $y \nvDash \mathrm{P}ab$.) □

A *chain* is a set of sentences whose members can be arranged as a sequence $\langle \mathrm{Q}_1[a_1a_2], \mathrm{Q}_2[a_2a_3], \ldots, \mathrm{Q}_n[a_na_1] \rangle$, where $\mathrm{Q}_i[a_ia_j]$ is either $\mathrm{Q}_ia_ia_j$ or $\mathrm{Q}_ia_ja_i$ and where $a_i \neq a_j$ if $i \neq j$. So, for example, $\{=\mathrm{AB}, =^-, \subset^{++}\mathrm{CB}, \mathrm{ZCA}\}$ is a chain. A pair $\langle \mathrm{X}, y \rangle$ is a *syllogism* iff $\mathrm{X} \cup \{y\}$ is a chain. So $\langle \{=\mathrm{AB}, =^-, \subset^{++}\mathrm{CB}\}, \mathrm{ZCA} \rangle$ is a syllogism.

A *normal chain* is a set of sentences whose members can be arranged as a sequence $\langle \mathrm{Q}_1a_1a_2, \mathrm{Q}_2a_2a_3, \ldots, \mathrm{Q}_na_na_1 \rangle$, where $a_i \neq a_j$ if $i \neq j$. A *simple normal chain* is a normal chain in which each quantifier is simple. So, for example, $\{=, =^-\mathrm{AB}, =\mathrm{BA}\}$ is a normal chain. And $\{=\mathrm{AB}, =\mathrm{BA}\}$ is a simple normal chain.

By definition, $e(=ab)$ is $=ba$, $e(=^-ab)$ is $=^-ba$, $e(\subset^{pq}ab)$ is $\subset^{q^*p^*}ba$, and $e(\mathrm{Z}ab)$ is $\mathrm{Z}ba$.

$\{\mathrm{Q}_1ab, \mathrm{Q}_2bc\}$ *a-reduces to* Q_3ac iff the triple $\langle \mathrm{Q}_1ab, \mathrm{Q}_2bc, \mathrm{Q}_3ac \rangle$ is recorded on the following Table of Reductions:

Q_2bc

Q_1ab	$=$	$=^-$	$\subset^{qr}$
$=$	$=$	$=^-$	$\subset^{qr}$
$=^-$	$=^-$	$=$	$\subset^{q^*r}$
$\subset^{pq}$	$\subset^{pq}$	$\subset^{pq^*}$	$\subset^{pr}$

Q_3ac

So, for example, $\{=\mathrm{AB}, =\mathrm{BC}\}$ a-reduces to $=\mathrm{AC}$, and $\{\subset^{++}\mathrm{AB}, \subset^{+-}\mathrm{BC}\}$ a-reduces to $\subset^{+-}\mathrm{AC}$.

If X_1 is a simple chain then a sequence of chains $\mathrm{X}_1, \ldots, \mathrm{X}_m$ $(=\mathrm{Y}_1)$, $\ldots, \mathrm{Y}_n$ *is a full reduction of* X_1 *to* Y_n iff: i) X_m is a normal chain and if $m > 1$ then, for $1 \leq i < m$, if X_i has form $\{\mathrm{Q}ab\} \cup \mathrm{Z}$ then X_{i+1} has form $\{e(\mathrm{Q}ab)\} \cup \mathrm{Z}$, and ii) there is no pair in Y_n that a-reduces to a sentence and if $n > 1$ then, for $1 \leq i < n$, if Y_i has form $\{\mathrm{Q}_1ab, \mathrm{Q}_2bc\} \cup \mathrm{Z}$ then Y_{i+1} has form $\{\mathrm{Q}_3ac\} \cup \mathrm{Z}$. X *fully reduces to* Y iff there is a full reduction of X to Y.

THEOREM 3. *Every simple chain fully reduces to a simple normal chain.*

Proof. Assume X_1 is a simple chain. We construct a sequence of chains that is a full reduction of X_1 to Y_n. Step 1: If X_1 is a simple normal chain let $X_1 = Y_1$ and go to Step 2. If X_1 is not a simple normal chain find the alphabetically first pair of sentences in X_1 of form $\langle Qab, Qcb\rangle$ and replace Qcb with $e(Qcb)$, forming X_2. Repeat Step 1 (with "X_j" in place of "X_1"). Step 2: If no pair of sentences in Y_1 a-reduces to a sentence, then X_1 fully reduces to Y_1. If a pair of sentences in Y_1 a-reduces to a sentence x find the alphabetically first pair that a-reduces to x and form Y_2 by replacing this pair with x. Repeat Step 2 (with "Y_j" in place of "Y_1"). □

So, for example, given the sequence $\langle\{=AB\}, \{=AB, =BA\}\rangle$, $\{=AB\}$ fully reduces to $\{=AB, =BA\}$. And, given the sequence $\langle\{\subset^{++}AB, \subset^{--}CB, \subset^{++}CA\}, \{\subset^{++}AB, \subset^{++}BC, \subset^{++}CA\}, \{\subset^{++}AC, \subset^{++}CA\}\rangle$, $\{\subset^{++}AB, \subset^{--}CB, \subset^{++}CA\}$ fully reduces to $\{\subset^{++}AC, \subset^{++}CA\}$. Some chains fully reduce to themselves. $\{\subset^{++}AB, \subset^{--}BC, ZCA\}$ is an example.

$\{P_1[a_1a_2], \ldots, P_n[a_na_1]\}$ *is a strand of* $\{Q_1[a_1a_2], \ldots, Q_n[a_na_1]\}$ iff each P_i is a simple quantifier in Q_i and a_i is the first term in $P_i[a_ia_{i+1}]$ iff a_i is the first term in $Q_i[a_ia_{i+1}]$, where $P[ab]$ is Pab or Pba. So, for example, $\{=AB, =^{-}AB\}$ is a strand of $\{=, \subset^{++}AB, =^{-}, \subset^{++}AB\}$.

A simple normal chain is a *cd-pair* iff it has one of the following forms:

$$\{=ab, =^{-}ba \text{ (or } \subset^{pq}ba \text{ or } Zba)\}, \{=^{-}ab, \subset^{pq}ba \text{ (or } Zba)\},$$
$$\text{or } \{\subset^{pq}ab, \subset^{qr}ba \text{ (or } Zba)\}.$$

THEOREM 4 (Syntactic decision procedure). *If* $\langle X, y\rangle$ *is a syllogism then* $X \vDash y$ *iff every strand of* $X \cup \{cd(y)\}$ *fully reduces to a cd-pair.*

Proof. Assume $\langle X, y\rangle$ is a syllogism. We use Lemmas 1–3, below. (If) Suppose every strand of X, $cd(y)$ fully reduces to a cd-pair. Then by Lemmas 1 and 2, X, $cd(y)$ is inconsistent. Then $X \vDash y$. (Only if) Suppose some strand of X, $cd(y)$ does not fully reduce to a cd-pair. Then, by Theorem 3, some strand of X, $cd(y)$ fully reduces to a simple normal chain that is not a cd-pair. Then, by Lemmas 1 and 3, X, $cd(y)$ is consistent. Then $X \nvDash y$. □

LEMMA 1. *A chain is inconsistent iff each of its strands is inconsistent.*

Proof. Note that a model satisfies $\{Q_1ab\} \cup X$ and $\{Q_2ab\} \cup X$ iff it satisfies $\{Q_3ab\} \cup X$, where the quantifiers in Q_3 are the quantifiers in Q_1 and Q_2. □

LEMMA 2. *If a simple chain* X *fully reduces to a cd-pair, then* X *is inconsistent*.

Proof. Use the following three lemmas, whose proofs will be omitted since they are easily given. □

LEMMA 2.1. *Each cd-pair is inconsistent.*

LEMMA 2.2. *If a simple normal chain* $\{\mathrm{Q}_3ac\} \cup \mathrm{X}$ *is inconsistent and* $\{\mathrm{Q}_1ab, \mathrm{Q}_2bc\}$ *a-reduces to* Q_3ac*, then* $\{\mathrm{Q}_1ab, \mathrm{Q}_2bc\} \cup \mathrm{X}$ *is inconsistent.*

LEMMA 2.3. *If a simple chain* $\{\mathrm{Q}ab\} \cup \mathrm{X}$ *is inconsistent, then* $\{e(\mathrm{Q}ab)\} \cup \mathrm{X}$ *is inconsistent.*

LEMMA 3. *If a simple chain* X *fully reduces to a simple normal chain that is not a cd-pair, then* X *is satisfied in an m-model, where* $m \leq n+2$ *and* n *is the number of terms in* X.

Proof. Use the following three lemmas. □

LEMMA 3.1. *If a simple chain fully reduces to a simple normal chain* X *that is not a cd-pair, then* X *is satisfied in an m-model, where* $m \leq n+2$ *and* n *is the number of terms in* X.

Proof. Assume the antecedent. We consider three cases determined by the number of occurrences of "Z" in X.

Case 1: "Z" does not occur in X. If either "=" or "$=^-$" occurs in X then X has form $\{=ab, =ba\}$ or $\{=^-ab, =^-ba\}$. Use $\langle\{1,2\}, \ldots, \nu\rangle$, where, for each term $x, \nu_+(x) = \{1\}$. If neither "=" or "$=^-$" occurs in X then X has form $\{\subset^{p_1p_2}a_1a_2, \ldots, \subset^{p_{2i-1}p_{2i}}a_ia_{i+1}, \ldots, \subset^{p_{2n-1}p_{2n}}a_na_1\}$, where $p_{2i} = p^*_{2i+1}$, for $1 \leq i < n$, and $p_{2n} = p^*_1$. We use induction on the number n of terms in X to show that X is satisfied in a 3-model. Basis step: $n = 2$. X has form $\{\subset^{p_1p_2}a_1a_2, \subset^{p^*_2p^*_1}a_2a_1\}$. Use $\langle\{1,2,3\}, \ldots, \nu\rangle$, where $\nu_{p_1}(a) = \{1\}$, and, for terms x other than a, $\nu_q(x) = \{1,2\}$. Induction step: $n > 2$. By the induction hypothesis $\{\subset^{p_1p_4}a_1a_3, \ldots, \subset^{p_{2i-1}p_{2i}}a_ia_{i+1}, \ldots, \subset^{p_{2n-1}p_{2n}}a_na_1\}$ is satisfied in a 3-model $\langle W, \ldots, \nu\rangle$, where $p_{2i} = p^*_{2i+1}$, for $2 \leq i < n$, and $p_{2n} = p^*_1$. Construct a model $\langle W, \ldots, \nu'\rangle$, $\nu'_{p_2}(a_2) = \nu_{p_1}(a_1) \cup \nu_{p^*_4}(a_3)$, and, for other terms $x, \nu'_+(x) = \nu_+(x)$. Then $\nu'(\subset^{p_1p_2}a_1a_2) = t$. $\nu'_{p^*_2}(a_2) = \nu_{p_4}(a_3) - \nu_{p_1}(a_1)$ and $p^*_2 = p_3$. So $\nu'(\subset^{p_3p_4}a_2a_3) = t$.

Case 2: "Z" occurs exactly once in X. Then X has at least three members and has form $\{\mathrm{Z}ab\} \cup \{\subset^{pq}bc, \ldots, \subset^{rs}da\}$. We use induction on the number of terms in X to show that X is satisfied in a 4-model. Basis step: $n = 3$. X has form $\{\mathrm{Z}ab\} \cup \{\subset^{pq}bc, \subset^{q^*r}ca\}$. Construct a model $\langle\{1,2,3,4\}, \ldots, \nu\rangle$, where $\nu_r(a) = \{1,2\}$, $\nu_p(b) = \{1,3\}$, and, for other terms $x, \nu_q(x) = \{1,3,4\}$. Induction step: $n > 3$. Follow the model construction in the induction step in Case 1.

Case 3: "Z" occurs at least twice in X. Then X has form $\{Zab, \dots, Zcd, \dots\}$. We use induction on the number n of terms in X. Basis step: $n = 2$. X has form $\{Zab, Zba\}$. Use $\langle\{1,2,3,4\}, \dots, \nu\rangle$, where $\nu_+(a) = \{1,2\}$ and, for other terms $x, \nu_+(x) = \{1,3\}$. Induction step: $n > 2$. X has form $\{Zab, Qbc, \dots, Zde, \dots\}$. By the induction hypothesis, $\{Zac, \dots, Zde\}$ is satisfied in an m-model, where $m \leq n+2$ and n is the number of terms in X. Suppose Q is "=". Construct model $\langle W, \dots, \nu'\rangle$, where $\nu'_+(b) = \nu_+(c)$ and, for terms x other than b, $\nu'_+(x) = \nu_+(x)$. Suppose Q is "$=^-$". Construct model $\langle W, \dots, \nu'\rangle$, where $\nu'_+(b) = \nu_-(c)$ and, for terms x other than b, $\nu'_+(x) = \nu_+(x)$. Suppose Q is "Z". Construct a model $\langle W, \dots, \nu'\rangle$, where $\nu'_+(b) = (\nu_+(a) \cap \nu_+(c)) \cup (\nu_-(a) \cap \nu_-(c))$, and, for other terms x, $\nu'_+(x) = \nu_+(x)$. Finally, suppose that Q is "$\subset^{pq}$". The strategy is to construct a model $\langle W', \dots, \nu'\rangle$ such that X is satisfied in it, where $W' = W \cup \{M\}$, and $\nu'_+(a) \cap \nu'_q(c)$ has at least two members, including M. Then we construct a second model $\langle W', \dots, \nu''\rangle$, such that X is satisfied in it by letting $\nu''_p(b) = \nu'_q(c) - \{M\}$, and, for terms x other than b, $\nu''_+(x) = \nu'_+(x)$. Then $\nu''(Zab) = t$ and $\nu''(\subset^{pq}bc) = t$.

We construct $\langle W', \dots, \nu'\rangle$. If a and c are the only terms in X, let $\alpha = \nu_+(a) \cap \nu_q(c)$ (and, thus, α has at least one member). If terms $d_1, \dots, d_n$ occur in X, where these terms are other than "a" or "c", pick $p_1 - p_n$ such that α has at least one member, where $\alpha = \nu_+(a) \cap \nu_q(c) \cap \nu_{p_1}(d_1) \cap \dots \cap \nu_{p_n}(d_n)$. Let $W' = W \cup \{M\}$, where $M \notin W$. Let $\nu'_+(x) = \nu_+(x) \cup \{M\}$ if $\alpha \subseteq \nu_+(x)$; otherwise, let $\nu'_+(x) = \nu_+(x)$. Then $\nu'_-(x) = \nu_-(x) \cup \{M\}$ if $\alpha \subseteq \nu_-(x)$; otherwise, $\nu'_-(x) = \nu_-(x)$. Note that $\nu'_+(a) \cap \nu'_q(c)$ has at least two members and $M \in \nu'_+(a) \cap \nu'_q(c)$. We show that X is satisfied in $\langle W', \dots, \nu'\rangle$. Suppose $\nu(Qde) = t$. Suppose Q is "=". Then $\nu'_+(d) = \nu_+(d) \cup \{M\}$ and $\nu'_+(e) = \nu_+(e) \cup \{M\}$ or $\nu'_+(d) = \nu_+(d)$ and $\nu'_+(e) = \nu_+(e)$. Then $\nu'(=de) = t$. Suppose Q is "$=^-$". Then $\nu'_+(d) = \nu_+(d) \cup \{M\}$ and $\nu'_-(e) = \nu_-(e)$ or $\nu'_+(d) = \nu_+(d)$ and $\nu'_-(e) = \nu_-(e) \cup \{M\}$. Then $\nu'(=^-de) = t$. Suppose Q is "$\subset^{pq}$". If $\alpha \subseteq \nu_p(d)$ then $\nu'_p(d) = \nu_p(d) \cup \{M\}$ and $\nu'_q(e) = \nu_q(e) \cup \{M\}$. If $\alpha \nsubseteq \nu_p(d)$ then $\nu'_p(d) = \nu_p(d)$ and either $\nu'_q(e) = \nu_q(e)$ or $\nu'_q(e) = \nu_q(e) \cup \{M\}$. Then $\nu'(\subset^{pq}de) = t$. Finally, suppose Q is "Z". Then, for any p and q, $\nu_p(d) \cap \nu_q(e) \subseteq \nu'_p(d) \cap \nu'_q(e)$. Then $\nu'(Zde) = t$. □

LEMMA 3.2. *If a simple chain* $\{Q_3ac\} \cup X$ *is satisfied in an n-model* $\langle W, \dots, \nu\rangle$, *where* n *is the number of terms in* $\{Q_3ac\} \cup X$, *and if* $\{Q_1ab, Q_2bc\}$ *a-reduces to* Q_3ac, *then* $\{Q_1ab, Q_2bc\} \cup X$ *is satisfied in an m-model, where* $m \leq n$ *and* n *is the number of terms in* $\{Q_1ab, Q_2bc\} \cup X$.

Proof. Assume the antecedent. Suppose Q_1 is "=". Construct $\langle W, \dots, \nu'\rangle$, where $\nu'_+(b) = \nu_+(a)$, and, for terms x other than b, $\nu'_+(x) = \nu_+(x)$. Suppose Q_1 is "$=^-$". Construct $\langle W, \dots, \nu'\rangle$, where $\nu'_+(b) = v_-(a)$, and, for terms x other than b, $\nu'_+(x) = \nu_+(x)$. Use similar constructions if Q_2 is "=" or "$=^-$". So, the only a-reduction left is this: $\{\subset^{pq}ab, \subset^{qr}bc\}$ a-reduces to $\subset^{pr}ac$. Construct a model $\langle \mathrm{W}', \ldots, \nu' \rangle$ such that $\mathrm{W}' = \mathrm{W} \cup \{\mathrm{M}\}$, $\mathrm{M} \notin \mathrm{W}$, and $\nu'_{p^*}(a) \cap \nu'_r(c)$ has at least two members, including M. To do this follow the procedure in Case 3 of Lemma 3.1. Then construct a model $\langle \mathrm{W}', \ldots, \nu'' \rangle$ such that $\nu''(b) = \nu'_p(a) \cup \{M\}$ and, for other terms x, $\nu''_+(x) = \nu'_+(x)$. □

LEMMA 3.3. *If a simple chain* $\{\mathrm{Q}ab\} \cup \mathrm{X}$ *is satisfied in an n-model, where* n *is the number of terms in* $\{\mathrm{Q}ab\} \cup \mathrm{X}$*, then* $\{e(\mathrm{Q}ab)\} \cup \mathrm{X}$ *is satisfied in an n-model, where* n *is the number of terms in* $\{e(\mathrm{Q}ab)\} \cup \mathrm{X}$.

Proof. Straightforward. □

THEOREM 5 (Semantic decision procedure). *If* $\langle \mathrm{X}, y \rangle$ *is a syllogism then* $\mathrm{X} \vDash y$ *iff* X, $cd(y)$ *is not satisfied in an m-model, where* $m \leq n+2$ *and* n *is the number of terms in* X.

Proof. Assume $\langle \mathrm{X}, y \rangle$ is a syllogism. (Only if) Immediate. (If) Assume X, $cd(y)$ is not satisfied in an m-model, where $m \leq n+2$ and n is the number of terms in X. Then every strand of X, $cd(y)$ is not satisfied in an m-model where $m \leq n+2$ and n is the number of terms in X, $cd(y)$. Then every strand of X, $cd(y)$ fully reduces to a cd-pair (by Theorem 3 and Lemma 3 of Theorem 4). Then $\mathrm{X} \vDash y$ (by Theorem 4).

Given Theorem 5, it is natural to ask whether, for any n, there is an n-termed syllogism that requires an $n+2$ model to show that it is invalid. The answer is Yes. If $n = 2$, use $\langle \{\mathrm{Z}a_1a_2\}, cd(\mathrm{Z}a_2a_1) \rangle$. If $n > 2$, use $\langle \{\mathrm{Z}a_1a_2, \subset^{++}a_2a_3, \ldots, \subset^{++}a_{n-1}a_n\}, cd(\mathrm{Z}a_na_1) \rangle$. Consider a model $\langle \mathrm{W}, \ldots, \nu \rangle$ in which $\{\mathrm{Z}a_1a_2, \subset^{++}a_2a_3, \ldots, \subset^{++}a_{n-1}a_n, \mathrm{Z}a_na_1\}$ is satisfied. Note that $\nu_+(a_1)$ has at least two members, since $\nu(\mathrm{Z}a_1a_2) = t$. So $\nu_+(a_n)$ has at least n members. $\nu_-(a_n)$ has at least two members since $v(\mathrm{Z}a_na_1) = t$. □

THEOREM 6 (Completeness). *If* $\langle \mathrm{X}, y \rangle$ *is a syllogism and* $\mathrm{X} \vDash y$ *then* $\mathrm{X} \vdash y$.

Proof. Assume the antecedent. Then, by Theorem 4, every strand of X, $cd(y)$ fully reduces to a cd-pair. So, by Lemmas 1–4, below, $\mathrm{X} \vdash cd(cd(y))$. That is $\mathrm{X} \vdash y$. □

LEMMA 1. *If* $\{x, y\}$ *is a cd-pair, then* $x \vdash cd(y)$.

Proof. 1) $=ab \vdash =ba$ (by B1). So $=ab \vdash cd(=^-ba)$ (and $cd(\subset^{pq}ba)$ and $cd(\mathrm{Z}ba)$) (by D1). 2) $=^-ba \vdash =^-ab$ (by B2). So $=^-ba \vdash cd(=ab)$ (by D1). And $=^-ab \vdash =^-ba$ (by B2). So $=^-ab \vdash cd(\subset^{pq}ba)$ (and

$cd(\mathrm{Z}ba)$) (by D1). 3) $\subset^{pq}ba \vdash \subset^{q^*p^*}ab$ (by B3). So $\subset^{pq}ba \vdash cd(=ab)$ (and $cd(=^-ab)$) (by D1). $\subset^{pq}ab \vdash \subset^{q^*p^*}ba$ (by B3). So $\subset^{pq}ab \vdash cd(\subset^{qr}ba)$ (and $cd(\mathrm{Z}ba)$) (by D1). $\subset^{qr}ba \vdash \subset^{r^*q^*}ab$ (by B3). So $\subset^{qr}ba \vdash cd(\subset^{pq}ab)$ (by D1). 4) $\mathrm{Z}ba \vdash \mathrm{Z}ab$ (by B4). So $\mathrm{Z}ba \vdash cd(=ab)$ (and $cd(=^-ab)$ and $cd(\subset^{pq}ab)$) (by D1). □

LEMMA 2. *If* $\mathrm{X} = \{\mathrm{Q}_3ac\} \cup \mathrm{Z}$, $\mathrm{Y} = \{\mathrm{Q}_1ab, \mathrm{Q}_2bc\} \cup \mathrm{Z}$, $\{\mathrm{Q}_1ab, \mathrm{Q}_2bc\}$ *a-reduces to* Q_3ac, *and* $\mathrm{X} - \{x\} \vdash cd(x)$, *for every* x *such that* $x \in \mathrm{X}$, *then* $\mathrm{Y} - \{y\} \vdash cd(y)$, *for every* y *such that* $y \in \mathrm{Y}$.

Proof. Assume the antecedent. Case 1: $y \in \mathrm{Z}$. $\{\mathrm{Q}_3ac\} \cup \mathrm{Z} - \{y\} \vdash cd(y)$. We use

LEMMA 2.1. *If* $\{\mathrm{Q}_1ab, \mathrm{Q}_2bc\}$ *a-reduces to* Q_3ac *then* Q_1ab, $\mathrm{Q}_2bc \vdash \mathrm{Q}_3ac$.

Proof. Given B5–B8, we only need to show that: i) $=^-ab$, $=bc \vdash =^-ac$; ii) $\subset^{pq}ab$, $=bc \vdash \subset^{pq}ac$; and iii) $\subset^{pq}ab$, $=^-bc \vdash \subset^{pq^*}ac$. For i), $=bc \vdash =cb$ (by B1) and $=^-ab \vdash =^-ba$ (by B2). $=cb$, $=^-ba \vdash =^-ca$ (by B5). So $=^-ab$, $=bc \vdash =^-ca$ (by D3). $=^-ca \vdash =^-ac$ (by B2). So $=^-ab$, $=bc \vdash =^-ac$ (by R1). Use similar reasoning for ii) and iii).

So $\mathrm{Q}_1ab, \mathrm{Q}_2bc \vdash \mathrm{Q}_3ac$ (by Lemma 2.1). So $\{\mathrm{Q}_1ab, \mathrm{Q}_2bc\} \cup \mathrm{Z} - \{y\} \vdash cd(y)$ (by D3).

Case 2: $y = \mathrm{Q}_1ab$. $\mathrm{Z} \vdash cd(\mathrm{Q}_3ac)$. $\mathrm{Q}_2bc, cd(\mathrm{Q}_3ac) \vdash cd(\mathrm{Q}_1ab)$ (by Lemma 2.1 and R2). So Z, $\mathrm{Q}_2bc \vdash cd(\mathrm{Q}_1ab)$ (by R1).

Case 3: $y = \mathrm{Q}_2bc$. Use reasoning similar to that for Case 2. □

LEMMA 3. *If* $\mathrm{X} = \{\mathrm{Q}ab\} \cup \mathrm{Z}$, $\mathrm{Y} = \{e(\mathrm{Q}ab)\} \cup \mathrm{Z}$, *and* $\mathrm{X} - \{x\} \vdash cd(c)$, *for every* x *such that* $x \in \mathrm{X}$, *then* $\mathrm{Y} - \{y\} \vdash cd(y)$, *for every* y *such that* $y \in \mathrm{Y}$.

Proof. Assume the antecedent. Case 1: $y \in \mathrm{Z}$. $\{\mathrm{Q}ab\} \cup \mathrm{Z} - \{y\} \vdash cd(y)$. $e(\mathrm{Q}ab) \vdash \mathrm{Q}ab$ (by B1–B4). So $\{e(\mathrm{Q}ab)\} \cup \mathrm{Z} - \{y\} \vdash cd(y)$ (by D3). Case 2: $y = e(\mathrm{Q}ab)$. $\mathrm{Z} \vdash cd(\mathrm{Q}ab)$. $cd(\mathrm{Q}ab) \vdash cd(e(\mathrm{Q}ab))$ (by B1–B4 and R2). So $\mathrm{Z} \vdash cd(e(\mathrm{Q}ab))$ (by R1). □

LEMMA 4. *If each strand* $\mathrm{Y} \cup \{z\}$ *of* $\mathrm{X} \cup \{y\}$ *is such that* $\mathrm{Y} \vdash cd(z)$, *then* $\mathrm{X} \vdash cd(y)$.

Proof. Use D2 and R3. (The proof is illustrated below.) □

The proof of the above theorem provides a mechanical procedure for showing that $\mathrm{X} \vdash y$ given that $\mathrm{X} \models y$. We illustrate by showing that $=, \subset^{++}\mathrm{AB}, =\mathrm{BC} \vdash cd(=^-, \subset^{+-}\mathrm{AC})$. First, fully reduce the following strands as indicated: i) $\{=\mathrm{AB}, =\mathrm{BC}, =^-\mathrm{AC}\}$ to $\{=\mathrm{AB}, =\mathrm{BC}, =^-\mathrm{CA}\}$ to $\{=\mathrm{AC}, =^-\mathrm{CA}\}$; ii) $\{=\mathrm{AB}, =\mathrm{BC}, \subset^{+-}\mathrm{AC}\}$ to $\{=\mathrm{AB}, =\mathrm{BC}, \subset^{+-}\mathrm{CA}\}$

to $\{=\mathrm{AC}, \subset^{+-}\mathrm{CA}\}$; iii) $\{\subset^{++}\mathrm{AB}, =\mathrm{BC}, =^{-}\mathrm{AC}\}$ to $\{\subset^{++}\mathrm{AB}, =\mathrm{BC}, =^{-}\mathrm{CA}\}$ to $\{\subset^{++}\mathrm{AC}, =^{-}\mathrm{CA}\}$; and iv) $\{\subset^{++}\mathrm{AB}, =\mathrm{BC}, \subset^{+-}\mathrm{AC}\}$ to $\{\subset^{++}\mathrm{AB}, =\mathrm{BC}, \subset^{+-}\mathrm{CA}\}$ to $\{\subset^{++}\mathrm{AC}, \subset^{+-}\mathrm{CA}\}$. By the proof of Lemma 1: $=\mathrm{AC} \vdash cd(=^{-}\mathrm{CA})$; $=\mathrm{AC} \vdash cd(\subset^{+-}\mathrm{CA})$; $\subset^{++}\mathrm{AC} \vdash cd(=^{-}\mathrm{CA})$; and $\subset^{++}\mathrm{AC} \vdash cd(\subset^{+-}\mathrm{CA})$. By the proof of Lemma 2: $=\mathrm{AB}, =\mathrm{AC} \vdash cd(=^{-}\mathrm{CA})$; $=\mathrm{AB}, =\mathrm{AC} \vdash cd(\subset^{+-}\mathrm{CA})$; $\subset^{++}\mathrm{AB}, =\mathrm{BC} \vdash cd(=^{-}\mathrm{CA})$; and $\subset^{++}\mathrm{AB}, =\mathrm{BC} \vdash cd(\subset^{+-}\mathrm{CA})$. By the proof of Lemma 3: $=\mathrm{AB}, =\mathrm{AC} \vdash cd(=^{-}\mathrm{AC})$; $=\mathrm{AB}, =\mathrm{AC} \vdash cd(\subset^{+-}\mathrm{AC})$; $\subset^{++}\mathrm{AB}, =\mathrm{BC} \vdash cd(=^{-}\mathrm{AC})$; and $\subset^{++}\mathrm{AB}, =\mathrm{BC} \vdash cd(\subset^{+-}\mathrm{AC})$. By D2, $=\mathrm{AB}, =\mathrm{AC} \vdash cd(=^{-}, \subset^{+-}\mathrm{AC})$ and $\subset^{++}\mathrm{AB}, =\mathrm{BC} \vdash cd(=^{-}, \subset^{+-}\mathrm{AC})$. By R3, $=, \subset^{++}\mathrm{AB}, =\mathrm{AC} \vdash cd(=^{-}, \subset^{+-}\mathrm{AC})$.

3. GERGONNE SYLLOGISMS

Faris [1] is motivated by an interest in providing a decision procedure for Gergonne syllogisms. Faris construes syllogisms as sentences, following Łukasiewicz's [4], rather than as inferences, as in Smiley's [5]. For us, a *Gergonne syllogism* is a syllogism consisting of Gergonne sentences, which are defined as follows, using Gergonne's symbols in [2]. The *Gergonne-quantifiers* are: $\mathbf{H} =_{\mathrm{df}} =^{-}, \subset^{+-}$; $\mathbf{X} =_{\mathrm{df}} \subset^{-+}, \mathrm{Z}$; $| =_{\mathrm{df}} =$; $\subset =_{\mathrm{df}} \subset^{++}$, and $\supset =_{\mathrm{df}} \subset^{--}$. A *Gergonne-sentence* is any sentence of form $\mathrm{Q}_1, \ldots, \mathrm{Q}_m ab$, where Q_i is a Gergonne-quantifier. So Theorem 4 above gives an alternative solution to the problem that motivated Faris' [1], since every Gergonne syllogism may be expressed in our system. Note, for example, that "$\mathbf{H}, \mathbf{X}\mathrm{AB}$" is expressed as "$=^{-}, \subset^{+-}, \subset^{-+}, \mathrm{ZAB}$".

4. SYSTEM B

In this section we develop a subsystem B which expresses no sentences other than those that may be expressed by using sentences of form "All... are − − −", "No... are − − −", "Some... are − − −", or "Some... are not − − −", where the blanks are filled by expressions of form x or non-x (the "A, E, I, and O sentences, respectively, with or without negative terms.")

The *B-quantifiers* ("B" for "basic") are: $=, \subset^{++}(\mathrm{A}^{++})$; $=^{-}, \subset^{+-}(\mathrm{A}^{+-})$; $=^{-}, \subset^{-+}(\mathrm{A}^{-+})$; $=, \subset^{--}(\mathrm{A}^{--})$; $=^{-}, \subset^{+-}, \subset^{-+}, \subset^{--}, \mathrm{Z}(\mathrm{O}^{++})$; $=, \subset^{++}, \subset^{-+}, \subset^{--}, \mathrm{Z}(\mathrm{O}^{+-})$; $=, \subset^{++}, \subset^{+-}, \subset^{--}, \mathrm{Z}(\mathrm{O}^{-+})$; and $=^{-}, \subset^{++}, \subset^{+-}, \subset^{-+}, \mathrm{Z}(\mathrm{O}^{--})$. $\mathrm{Q}ab$ is a *B-sentence* iff $\mathrm{Q}ab$ is a sentence and Q is a B-quantifier. So, for example, $\mathrm{A}^{++}\mathrm{AB}$ is a B-sentence. And a *B-syllogism* is a syllogism composed of B-sentences.

We define y is *B-deducible from* X ($\mathrm{X} \vdash_{\mathrm{B}} y$), where X, y is a set of B-sentences, and where $ct(\mathrm{A}^{pq}ab) = \mathrm{A}^{pq^*}ab$, $cd(\mathrm{A}^{pq}ab) = \mathrm{O}^{pq}ab$, and $cd(\mathrm{O}^{pq}ab) = \mathrm{A}^{pq}ab$:

(B_1) $\mathrm{A}^{pq}ab \vdash_{\mathrm{B}} \mathrm{A}^{q^*p^*}ba$

(B_2) $\mathrm{A}^{pq}ab, \mathrm{A}^{qr}bc \vdash_{\mathrm{B}} \mathrm{A}^{pr}ac$

(R_1) If $\mathrm{X} \vdash_{\mathrm{B}} y$ and $y, z \vdash_{\mathrm{B}} w$ then $\mathrm{X}, z \vdash_{\mathrm{B}} w$

(R_2) If $\mathrm{X}, y \vdash_{\mathrm{B}} ct(z)$ or $cd(z)$ then $\mathrm{X}, z \vdash_{\mathrm{B}} cd(y)$

(L_1) If $\mathrm{X} \vdash y$, then $\mathrm{X} \vdash y$ in virtue of B_1–R_2.

THEOREM 7. (D_1) *If* $\mathrm{X}, y \vdash_{\mathrm{B}} z$ *and* $u, v \vdash_{\mathrm{B}} y$ *then* $\mathrm{X}, u, v \vdash_{\mathrm{B}} z$.
Proof. Use the reasoning for the proof of Theorem 1. □

THEOREM 8 (Soundness). *If* $\mathrm{X} \vdash_{\mathrm{B}} y$ *then* $\mathrm{X} \vDash y$.
Proof. Straightforward. □

By definition, $e(\mathrm{A}^{pq}ab)$ is $\mathrm{A}^{q^*p^*}ba$ and $e(\mathrm{O}^{pq}ab)$ is $\mathrm{O}^{q^*p^*}ba$. And, by definition, a set X of sentences *b-reduces to* a sentence y iff $\langle \mathrm{X}, y\rangle$ has form $\langle\{\mathrm{A}^{pq}ab, \mathrm{A}^{qr}bc\}, \mathrm{A}^{pr}ac\rangle$.

If X_1 is a chain of B-sentences then a sequence of chains $\mathrm{X}_1, \ldots, \mathrm{X}_m$ $(=\mathrm{Y}_1), \ldots, \mathrm{Y}_n$ *is a full B-reduction of* X_1 *to* Y_n iff: i) X_m is a normal chain and if $m > 1$, then, for $1 \leq i < m$, if X_i has form $\{\mathrm{Q}ab\} \cup \mathrm{Y}$, then X_{i+1} has form $\{e(\mathrm{Q}ab)\} \cup \mathrm{Y}$; and ii) there is no pair of sentences in Y_n that b-reduces to a sentence and if $n > 1$ then, for $1 \leq i < n$, Y_i has form $\{\mathrm{A}^{pq}ab, \mathrm{A}^{qr}bc\} \cup \mathrm{X}$ and Y_{i+1} has form $\{\mathrm{A}^{pr}ac\} \cup \mathrm{X}$. X *fully* B*-reduces to* Y iff there is a full B-reduction of X to Y.

THEOREM 9. *Every chain of* B*-sentences fully* B*-reduces to a normal chain of* B*-sentences*.
Proof. Imitate the proof of Theorem 3. □

A normal chain of B-sentences is a cd-B-pair iff it has one of the following forms: $\{\mathrm{A}^{pq}ab, \mathrm{A}^{qp^*}ba\}$ or $\{\mathrm{A}^{pq}ab, \mathrm{O}^{q^*p^*}ba\}$.

THEOREM 10 (Syntactic decision procedure). *If* $\langle \mathrm{X}, y\rangle$ *is a* B*-syllogism then* $\mathrm{X} \vDash y$ *iff* $\mathrm{X}, cd(y)$ *fully* B*-reduces to a cd-*B*-pair.*
Proof. Assume $\langle \mathrm{X}, y\rangle$ is a B-syllogism. We use Lemmas 1 and 2, below. (If) Suppose $\mathrm{X}, cd(y)$ fully B-reduces to a *cd*-B-pair. Then, by Lemma 1, $\mathrm{X}, cd(y)$ is consistent. Then $\mathrm{X} \nvDash y$. (Only if) Suppose $\mathrm{X} \vDash y$. Then $\mathrm{X}, cd(y)$ is inconsistent. Then $\mathrm{X}, cd(y)$ fully B-reduces to a *cd*-B-pair (by Lemma 2 and Theorem 9). □

LEMMA 1. *If a chain* X *of* B*-sentences fully* B*-reduces to a cd*-B*-pair then* X *is inconsistent*.

Proof. Imitate the proof of Lemma 2 of Theorem 4. □

LEMMA 2. *If a chain* X *of* B*-sentences fully* B*-reduces to a normal chain of* B*-sentences that is not a cd*-B*-pair, then* X *is satisfied in a* 3*-model*.

LEMMA 2.1. *If a chain of* B*-sentences fully* B*-reduces to a normal chain of* B*-sentences* X *that is not a cd*-B*-pair, then* X *is satisfied in a* 3*-model*.

Proof. Assume the antecedent. We consider three cases determined by the number of occurrences of "O" in X.

Case 1: "O" does not occur in X. We use induction on the number n of terms in X. Basis step: $n = 2$. Then X has form $\{\mathrm{A}^{pq}ab, \mathrm{A}^{qp}ba\}$ or form $\{\mathrm{A}^{pq}ab, \mathrm{A}^{q^*p^*}ba\}$. If $p = q$, use $\langle\{1,2,3\},\ldots,\nu\rangle$, where $\nu_+(x) = \{1\}$. If $p \neq q$, use $\langle\{1,2,3\},\ldots,\nu\rangle$, where $\nu_+(a) = \{1\}$, and, for terms x other than a, $\nu_+(x) = \{2,3\}$. Induction step: $n > 2$. Then X has form $\{\mathrm{A}^{p_1p_2}a_1a_2,\ldots,\mathrm{A}^{p_{2i-1}p_{2i}}a_ia_{i+1},\ldots,\mathrm{A}^{p_{2n-1}p_{2n}}a_na_1\}$, where $p_{2i} = p^*_{2i+1}$. By Case 1 of Lemma 3.1 $\{\subset^{p_1p_2}a_1a_2,\ldots,\subset^{p_{2i-1}p_{2i}}a_ia_{i+1},\ldots,\subset^{p_{2n-1}p_{2n}}a_na_1\}$, where $p_{2i} = p^*_{2i+1}$, for $1 \leq i < n$, and $p_{2n} = p^*_1$, is satisfied in a 3-model. So X is satisfied in a 3-model.

Case 2: "O" occurs exactly once in X. Suppose there are exactly two terms in X. Then X has form $\mathrm{A}^{pq}ab$, $\mathrm{O}^{q^*p}ba$ (or $\mathrm{O}^{qp}ab$ or $\mathrm{O}^{qp^*}ba$). 3-models are easily constructed to show that X is consistent. Suppose there are more than two terms in X. We use induction on the number n of terms in X to show that X is satisfied in a 3-model. Basis step: $n = 3$. Then X has form $\{\mathrm{O}^{pq}ab, \mathrm{A}^{rs}bc, \mathrm{A}^{s^*u}ca\}$. So there is a strand of X with one of the following forms: $\{\subset^{pq^*}ab, \subset^{rs}bc, \subset^{s^*u}ca\}$, $\{\subset^{p^*q}ab, \subset^{rs}bc, \subset^{s^*u}ca\}$, and $\{\subset^{p^*q^*}ab, \subset^{rs}bc, \subset^{s^*u}ca\}$. So, by Case 1 of Lemma 3.1 of Theorem 4, X is consistent if $p = u$ or $q = r$. Suppose $p \neq u$ and $q \neq r$. Then X has form $\{\mathrm{O}^{pq}ab, \mathrm{A}^{q^*s}bc, \mathrm{A}^{s^*p^*}ca\}$. If $p = q$, there is a strand of X with form $\{=^-ab, =bc, =^-ca\}$ or form $\{=^-ab, =^-bc, =ca\}$. If $p \neq q$, there is a strand of X with form $\{=ab, =bc, =ca\}$ or form $\{=ab, =^-bc, =^-ca\}$. Each of these four chains can easily be shown to be satisfied in a 3-model. Induction step: $n > 3$. X has form $\mathrm{O}^{pq}ab, \mathrm{A}^{rs}bc, \mathrm{A}^{s^*u}cd, \ldots$. By the induction hypothesis, $\mathrm{O}^{pq}ab, \mathrm{A}^{r^*u}bd, \ldots$ is satisfied in a 3-model $\langle \mathrm{W},\ldots,\nu\rangle$. Construct $\langle \mathrm{W},\ldots,\nu'\rangle$, where $\nu'_s(c) = \nu_r(b)$, and, for terms x other than c, $\nu'_+(x) = \nu_+(x)$. Note that $\nu'(\mathrm{A}^{rs}bc) = t$, since $\nu'_r(b) = \nu'_s(c)$, and $\nu'(\mathrm{A}^{s^*u}cd) = t$, since $\nu'_{s^*}(c) = \nu'_{r^*}(b)$.

Case 3: "O" occurs at least twice in X. We use induction on the number of terms n in X. Basis step: $n = 2$. X has form $\{\mathrm{O}^{pq}ab, \mathrm{O}^{rs}ba\}$. It is easy to show that X is satisfied in a 3-model. Induction step: $n > 2$. X has form $\{\mathrm{O}^{pq}ab, \mathrm{Q}^{rs}bc, \ldots, \mathrm{O}^{uv}de, \ldots\}$. Suppose Q is "A" and $r = s$ or Q is "O" and $r \neq s$. By the induction hypothesis, $\{\mathrm{O}^{pq}ac, \ldots, \mathrm{O}^{uv}de, \ldots\}$ is satisfied in a 3-model $\langle \mathrm{W}, \ldots, \nu\rangle$. Construct 3-model $\langle \mathrm{W}, \ldots, \nu'\rangle$, where $\nu'_q(b) = \nu_q(c)$, and, for terms x other than c, $\nu'_+(x) = \nu_+(x)$. Suppose Q is "A" and $r \neq s$ or Q is "O" and $r = s$. By the induction hypothesis, $\{\mathrm{O}^{pq^*}ac, \ldots, \mathrm{O}^{uv}de, \ldots\}$ is satisfied in a 3-model $\langle \mathrm{W}, \ldots, \nu\rangle$. Construct 3-model $\langle \mathrm{W}, \ldots, \nu'\rangle$, where $\nu'_q(b) = \nu_{q^*}(c)$, and, for terms x other than c, $\nu'_+(x) = \nu_+(x)$. □

LEMMA 2.2. *If* $\{\mathrm{A}^{pr}ac\} \cup \mathrm{Y}$ *is satisfied in a 3-model and if term* b *does not occur in a member of* Y, *then* $\{\mathrm{A}^{pq}ab, \mathrm{A}^{qr}bc\} \cup \mathrm{Y}$ *is satisfied in a 3-model.*

Proof. Assume that $\{\mathrm{A}^{pr}ac\} \cup \mathrm{Y}$ is satisfied in a 3-model $\langle \mathrm{W}, \ldots, \nu\rangle$. Construct $\langle \mathrm{W}, \ldots, \nu'\rangle$, where $\nu'_p(b) = \nu_q(a)$, and, for terms x other than b, $\nu'_+(x) = \nu_+(x)$. □

LEMMA 2.3. *If* $\{\mathrm{Q}ab\} \cup \mathrm{Y}$ *is satisfied in a 3-model, then* $\{e(\mathrm{Q}ab)\} \cup \mathrm{Y}$ *is satisfied in a 3-model.*

Proof. Straightforward. □

THEOREM 11 (Semantic decision procedure). *If* $\langle \mathrm{X}, y\rangle$ *is a* B-*syllogism then* $\mathrm{X} \vDash y$ *iff* $\mathrm{X}, cd(y)$ *is not satisfied in a 3-model.*

Proof. Assume $\langle \mathrm{X}, y\rangle$ is a B-syllogism. (Only if) Immediate. (If) Suppose $\mathrm{X}, cd(y)$ is not satisfied in a 3-model. Then, by Theorem 9 and Lemma 2 of Theorem 10, $\mathrm{X}, cd(y)$ fully B-reduces to a *cd*-B-pair. So, by Theorem 10, $\mathrm{X} \vDash y$. □

Theorem 11 extends the result in Johnson's [3]. There it is shown, in effect, that any invalid syllogism constructed by using B-sentences other than those of form $\mathrm{A}^{-+}ab$ or $\mathrm{O}^{-+}ab$ is satisfied in a 3-model. There are invalid B-syllogisms that require a domain with at least three members to show their invalidity. This is an example: $\langle\{\mathrm{A}^{+-}\mathrm{AB}, \mathrm{A}^{+-}\mathrm{BC}\}, \mathrm{O}^{+-}\mathrm{AC}\rangle$.

THEOREM 12 (Completeness). *If* $\langle \mathrm{X}, y\rangle$ *is a* B-*syllogism and* $\mathrm{X} \vDash y$ *then* $\mathrm{X} \vdash_{\mathrm{B}} y$.

Proof. Assume the antecedent. Then, by Theorem 10, $\mathrm{X} \cup \{cd(y)\}$ fully B-reduces to a *cd*-B-pair. Use the following three lemmas. □

LEMMA 1. *If* $\{x, y\}$ *is a cd*-B-*pair, then* $x \vdash_{\mathrm{B}} cd(y)$ *and* $y \vdash_{\mathrm{B}} cd(x)$.

Proof. (1) $\mathrm{A}^{qp^*}ba \vdash_{\mathrm{B}} \mathrm{A}^{pq^*}ab$, that is, $ct(\mathrm{A}^{pq}ab)$ (by B_1). So $\mathrm{A}^{pq}ab \vdash_{\mathrm{B}} cd(\mathrm{A}^{qp^*}ba)$ (by R_2). So $\mathrm{A}^{qp^*}ba \vdash_{\mathrm{B}} cd(\mathrm{A}^{pq}ab)$ (by R_2). (2) $\mathrm{A}^{pq}ab \vdash_{\mathrm{B}}$

$A^{q^*p^*}ba$, that is, $cd(O^{q^*p^*}ba)$ (by B_1). So $O^{q^*p^*}ba \vdash_B cd(A^{pq}ab)$ (by R_2). □

LEMMA 2. *If* $X = \{A^{pr}ac\} \cup Z$, $Y = \{A^{pq}ab, A^{qr}bc\} \cup Z$, *and* $X - \{x\} \vdash_B cd(x)$, *for each sentence* x *in* X, *then* $Y - \{y\} \vdash_B cd(y)$, *for each sentence* y *in* Y.

Proof. Imitate the proof of Lemma 2 of Theorem 6. □

LEMMA 3. *If* $X = \{Qab\} \cup Z$, $Y = \{e(Qab)\} \cup Z$, *and* $X - \{x\} \vdash_B cd(x)$, *for each sentence* x *in* X, *then* $Y - \{y\} \vdash_B cd(y)$, *for each sentence* y *in* Y.

Proof. Imitate the proof of Lemma 3 of Theorem 6. □

5. Conclusion

Our interest has been in extending the Aristotelian syllogistic. But, in conclusion, we mention Smiley's classic result in [5] about the Aristotelian syllogistic, which follows from the results obtained above. First, delete sentences of form $A^{-+}ab$ and $O^{-+}ab$ from system B. Let $Aa - b = \varnothing$ if $a = b$; otherwise, let $Aa - b$ be a set of sentences that can be arranged as follows: $\langle A^{++}a_1a_2$ (or $A^{--}a_2a_1), \ldots, A^{++}a_na_{n+1}$ (or $A^{--}a_{n+1}a_n\rangle$, where $a_1 = a$ and $a_{n+1} = b$. Then, by Theorem 10, a chain of sentences in this subsystem is inconsistent iff it has one of the following forms: i) $Aa - b$, $O^{++}ab$ ($O^{--}ab$); ii) $Aa - b$, $A^{+-}bc$, $Ac - a$; or iii) $Aa - b$, $A^{+-}bc$, $Ad - c$, $O^{+-}da$ (or $O^{+-}ad$). Next, delete sentences of form $A^{--}ab$ and $O^{--}ab$ from this system. The resulting system can express all of the Aristotelian syllogisms. So, as Smiley [5] says, an Aristotelian syllogism $\langle X, y\rangle$ is valid iff $X, cd(y)$ has one of the following forms: i′) $Aa - b$, $O^{++}ab$, ii), or iii). (Smiley uses A, E, I, and O instead of our $A^{++}, A^{+-}, O^{+-}, O^{++}$, respectively.) So, for example, "A^{++}BC, A^{++}BA; so O^{+-}AC" (Darapti) is valid since "A^{++}BC, A^{++}BA, A^{+-}AC" has form ii).

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