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IDENTITY LOGICS

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In this paper* we prove the completeness of three logical systems **IL1**, **IL2** and **IL3**. **IL1** deals solely with identities ($a = b$), and its deductions are the direct deductions constructed with the three traditional rules: (T) from $a = b$ and $b = c$ infer $a = c$, (S) from $a = b$ infer $b = a$ and (A) infer $a = a$ (from anything). **IL2** deals solely with identities and inidentities ($a \neq b$) and its deductions include both the direct and the indirect deductions constructed with the three traditional rules. **IL3** is a hybrid of **IL1** and **IL2**: its deductions are all direct as in **IL1** but it deals with identities and inidentities as in **IL2**. **IL1** and **IL2** have a high degree of naturalness. Although the hybrid system **IL3** was constructed as an artifact useful in the mathematical study of **IL1** and **IL2**, it nevertheless has some intrinsically interesting aspects.

The main motivation for describing and studying such simple systems is pedagogical. In teaching beginning logic one would like to present a system of logic which has the following properties. First, it exemplifies the main ideas of logic: implication, deduction, non-implication, counter-argument (or countermodel), logical truth, self-contradiction, consistency, satisfiability, etc. Second, it exemplifies the usual general metaprinciples of logic: contraposition and transitivity of implication, cut laws, completeness, soundness, etc. Third, it is simple enough to be thoroughly grasped by beginners. Fourth, it is obvious enough so that its rules do not appear to be arbitrary or purely conventional. Fifth, it does not invite confusions which must be unlearned later. Sixth, it involves a minimum of presuppositions which are no longer accepted in mainstream contemporary logic. These are vague conditions which are satisfied to a greater or lesser extent by propositional logic, **PL**, and by Aristotelian logic (or syllogistic), **AL**, and the majority of contemporary beginning logic students are

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presented either with **PL** or **AL** as a first system. Of course, versions of Boolean logic are often presented as first systems but Boolean logic satisfies the above conditions to a much lesser extent than **PL** or **AL**.

In the opinion of the authors, **IL1** and **IL2** satisfy the above conditions better than either **PL** or **AL**. In regard to the first two conditions per se **IL1** and **IL2** are probably no better than **PL** or **AL**. In regard to the third condition, both are superior to **PL** and **AL** in that the identity logics involve fewer logical primitives, viz., only identity and non-identity, and the logical primitives which they involve are less apt to be confused with near-neighbors in ordinary English. There is no "material implication", no "inclusive or", etc., as in **PL** and no "quantification" as in **AL**. In regard to the fourth condition, "naturalness", it must be admitted that **PL** can be formulated in a way that makes it equally natural. But the natural formulations of **PL** involve many complexities (subproofs, conditionalization, etc.) which are not present in identity logic. In its original formulation, **AL** has a degree of naturalness comparable to identity logic. The fifth and sixth conditions, involvement of possible confusions and presuppositions, seem to be satisfied better by the identity logics than by **PL**. At least identity logics do not suggest that there are atomic propositions wholly devoid of structure, nor does it suggest that validity is merely a matter of truth-tables. Likewise, comparing identity logic now with **AL**, there is no problem of "existential import" nor is there any suggestion that validity is merely a matter of relations between classes (Venn diagrams).

We do not mean to suggest that **PL** or **AL** should not be taught in beginning logic. Our only point is that the general framework of metalogical concepts and principles is more clearly and more simply exemplified in a system of identity logic than in **PL** or **AL**. Moreover, in moving from an identity logic to **PL**, to **AL**, or to one of the other elementary systems (e.g., Boolean logic or monadic logic) there is nothing to be abandoned or unlearned. Besides, the sooner the logic student learns to use "identity" carefully and precisely the better.[1]

1 *Preliminaries* Let PN be a countably infinite set of *proper names* (or *individual constants*): $p_1, p_2, \ldots, p_n, \ldots$. A *sentence of* $L=$ is a string '$a = b$' or '$a \neq b$', with a, b in PN. An *interpretation* (or *model*) for $L=$ is a (denotation) function D defined on PN (and taking values in an arbitrary set). If Da and Db are identical then '$a = b$' is *true in* D and '$a \neq b$' is *false in* D, and if Da and Db are distinct then '$a = b$' is *false in* D and '$a \neq b$' is *true in* D.

Let P, Q and R be subsets of $L=$ and let p, q, r, and s be members of $L=$. P is *satisfied by* an interpretation D if every member of P is true in D. If every interpretation which satisfies P satisfies p then P *implies* p ($P \models p$). If P is satisfied by some interpretation then P is *satisfiable*, otherwise P is *unsatisfiable*.

$P + Q$ is the union of P and Q, and $P + p$ is the union of P and the unit set of p.

In a few places below we will have occasion to refer to *the contradictory* $\overline{p}$ of a sentence p. This is defined as follows: $\overline{a = b}$ is $a \neq b$, $\overline{a \neq b}$ is $a = b$.

2 *Semantic results*[2] The main purpose of this section is to prove some results about the semantics of identities and inidentities which will reveal one set of rules of inference sufficient for constructing a deduction for each implication $(P \vDash p)$. We begin by setting up some technical apparatus.

By an *identity Iab* we mean one of the two sentences $a = b$ or $b = a$. By an *inidentity Nab* we mean one of $a \neq b$ or $b \neq a$. If a and b are distinct then by an *identity chain* $Ia \ldots b$ we mean a set which contains for some distinct $c_1, c_2, \ldots, c_n$ at least one perhaps both of each of the following: an Iac_1, an Ic_1c_2, ... an $Ic_{n-1}c_n$, an Ic_nb (and nothing else). A single identity Iab is also an *identity chain* $Ia \ldots b$. The null set is an identity chain $Ia \ldots a$. Thus every identity chain $Ia \ldots b$ is an identity chain $Ib \ldots a$, and when a and c are distinct the union of an $Ia \ldots b$ and an $Ib \ldots c$ contains an $Ia \ldots c$. Thus, if P is an arbitrary set of sentences the relation on PN of being "linked" by some identity chain contained in P is an equivalence relation.

Using this equivalence relation we define a unique function fP from PN to PN, for each P: fPa is the first constant b in PN with an identity chain $Ia \ldots b$ in P.

Sublemma *$fPa = fPb$ if and only if some $Ia \ldots b$ is in P.*

Proof: If $fPa = fPb$ then there is a unique first constant c, viz. fPa, such that an $Ia \ldots c$ and an $Ic \ldots b$ are both in P. The union is then in P. But the union contains an $Ia \ldots b$. Conversely, if some $Ia \ldots b$ is in P, then every c with an $Ia \ldots c$ in P also has an $Ic \ldots b$ in P and vice versa. Thus the first constant c with an $Ia \ldots c$ in P *is* the first constant c in PN with an $Ib \ldots c$ in P. QED

The following lemma and its corollaries concern the satisfiability and the consequences of an arbitrary set P of sentences of $L=$.

Lemma 1 *If for no a, b is there both an Nab and an $Ia \ldots b$ in P, then fP satisfies P.*

Proof: Suppose the hypothesis. If $a \neq b$ is in P, then by hypothesis no $Ia \ldots b$ is in P. Thus by the sublemma $fPa \neq fPb$ and so $a \neq b$ is true in fP. If $a = b$ is in P, then trivially an $Ia \ldots b$ is in P and by the sublemma $fPa = fPb$. Thus $a = b$ is true in fP. QED

Corollary XS *P is unsatisfiable if and only if for some a, b both an Nab and an $Ia \ldots b$ are in P.*

Corollary IS *If $P \vDash a = b$ then either an $Ia \ldots b$ is in P or P is unsatisfiable.*

Proof: Suppose $P \vDash a = b$. If P is unsatisfiable the conclusion follows. Assume that P is satisfiable. Then by Lemma 1 fP satisfies P and by the sublemma $fPa = fPb$ if and only if some $Ia \ldots b$ is in P. Since $P \vDash a = b$ and fP satisfies P, $fPa = fPb$ and so there is an $Ia \ldots b$ in P. QED

Corollary NS *If* $P \models a \neq b$ *then either for some* c, d *an* $Ic \ldots a$, *an* $Id \ldots b$ *and an* Ncd *are all in* P *or* P *is not satisfiable.*

Proof: Suppose $P \models a \neq b$. If P is unsatisfiable the conclusion follows. Assume that P is satisfiable. Then no identity chain $Ia \ldots b$ is in P, because otherwise $a = b$ and $a \neq b$ both follow from P. But $P + a = b$ is unsatisfiable. Thus for some c, d, an Ncd and an $Ic \ldots d$ are both in $P + a = b$. The $Ic \ldots d$ must contain $a = b$. (Otherwise the Ncd and the $Ic \ldots d$ are both in P, but P is satisfiable.) This means that $Ic \ldots d$ is $a = b$ plus a union of an $Ic \ldots a$ and an $Ib \ldots d$. Since the Ncd must be in P, the conclusion follows. QED

3 *Deductions in* **IL3** For pedagogical reasons and in order to reflect deductive practice it is useful to mark the assumptions and the goal of a deduction. Accordingly all deductions begin with a finite list of sentences of $L=$ prefixed with A (for assumption) followed by a sentence prefixed with ? (the goal). Thereafter each line is added according to one of the following six rules (until the goal is reached[3]).

Rules of Group 3:

T	Affirmative Transitivity:	$a = b,\ b = c/a = c$
NT	Negative Transitivity:	$a = b,\ b \neq c/a \neq c$
S	Affirmative Symmetry:	$a = b/b = a$
NS	Negative Symmetry:	$a \neq b/b \neq a$
X	Contradiction:	$a \neq a/p$
A	Axiom:	$/a = a$

The last two rules, of course, countenance (1) the addition of an arbitrary line once a self-contradiction has been inferred and (2) the addition of any logical identity $a = a$ at any point.

Any list constructed according to the above rules is called *a deduction in* **IL3**. If there is a deduction in **IL3** whose assumptions are all in P and whose goal is p, we write $P \vdash_3 p$ and where it is clear from the context that **IL3** is under discussion we write simply $P \vdash p$.

Corollary ID *If an* $Ia \ldots b$ *is in* P *then* $P \vdash_3 a = b$.

Proof: List the members of $Ia \ldots b$ prefixed with A and followed with $?a = b$. If a and b are identical the deduction is completed using the axiom. Assume that a and b are not identical. Using symmetry one can add the following intermediate conclusions: $a = c_1$, $c_1 = c_2, \ldots c_{n-1} = c_n$, $c_n = b$, where of course $c_1, \ldots c_n$ are the other constants in $Ia \ldots b$. Then using transitivity repeatedly one obtains: $a = c_2, a = c_3, \ldots, a = c_n, c_n = b, a = b$. The deduction is thus constructed. QED

Corollary ND *If* P *contains, for some* c, d, *an* $Ic \ldots a$, *an* $Id \ldots b$, *and an* Ncd *then* $P \vdash_3 a \neq b$.

Proof: List all of the mentioned as assumptions and put $?a \neq b$ as the goal. By the reasoning of the previous corollary $a = c$ and $b = d$ can be added. By negative symmetry $d \neq c$ can be added (if it is not already present). By negative transitivity $b \neq c$ is added, then $c \neq b$ is added by *NS*. Using *NT* again, add $a \neq b$. QED

Corollary XD *If P contains both an Nab and an Ia . . . b, then for every s in L =,* $P \vdash_3 s$.

Proof: Set down the mentioned sentences as assumptions followed by $?s$. By the reasoning of corollary ID one can add intermediate conclusions ending with $a = b$. By *NS* one can add $b \neq a$ (if such is not already an assumption). From the last two sentences mentioned one can add $a \neq a$ using *NT*. And by Rule X, s can be added. QED

4 *Completeness and soundness of* **IL3** By combining the semantic results with the results concerning the deductions one easily obtains completeness and soundness of **IL3**.

Theorem C3 *If* $P \models s$ *then* $P \vdash_3 s$.

Proof: If P is unsatisfiable use corollaries XS and XD. Assume that P is satisfiable. Then s is either '$a = b$' or '$a \neq b$'. If s is the former use corollaries IS and ID. If the latter use *NS* and *ND*. QED

Theorem S3 *If* $P \vdash_3 s$ *then* $P \models s$.

Proof: Suppose $P \vdash_3 s$. Let π be an arbitrary deduction in **IL3** of s from P. All of the assumptions in π follow from P. After the goal, each line was added by one of the six rules, each of which only permits the addition of consequences of its operands. Since s is the last line added, $P \models s$. QED

5 *Indirect deductions in* **IL2** An indirect deduction begins with assumptions and a goal, $?s$, just like the direct deductions, but in an indirect deduction the line following the goal is the *reductio* assumption of $\overline{s}$, the contradictory of s, prefixed with R to signify its role. (Recall that $\overline{a = b}$ is $a \neq b$ and $\overline{a \neq b}$ is $a = b$.) The remaining lines of an indirect deduction in **IL2** are added by means of rules of Group 2 (below) until a pair of contradictories ($a = b$, $a \neq b$) have both been reached. A direct deduction in **IL2** is constructed as in **IL3** except that only rules in Group 2 may be used. We use $P \vdash_2 s$ to indicate deducibility in **IL2**.

Rules of Group 2:

T: $a = b, b = c / a = c$
S: $a = b / b = a$
A: $/ a = a$

Corollary ID2 *If an Ia . . . b is in P then* $P \vdash_2 a = b$.

Corollary ND2 *If for some c, d, an Ic . . . a, an Id . . . b and an Ncd are in P then* $P \vdash_2 a \neq b$.

Corollary XD2 *If for some a, b, an Ia . . . b and an Nab are in P then for all s in L =,* $P \vdash_2 s$.

Proofs: *ID2* is proved like *ID*. To see *ND2*, set up the assumptions and the goal $?a \neq b$ and then put $Ra = b$ as the reductio assumption. Note that $a = b$ plus $Ic \ldots a$ plus $Id \ldots b$ is an Icd. Thus using T and S one can construct a chain of identities ending with $c = d$. Either $c = d$ is the contradictory of the Ncd or such can be added by S. QED To see *XD2*, set up the premises and the goal $?s$. Put $R\overline{s}$ as the reductio assumption. Using S and T one can add a list of intermediate inferences ending with $a = b$. This or $b = a$ is the contradictory of the Nab. QED

Theorem C2 *If* $P \models s$ *then* $P \vdash_2 s$.

Proof: Similar to that of Theorem C3.

Theorem S2 *If* $P \vdash_2 s$ *then* $P \models s$.

Proof: Assume $P \vdash_2 s$. Let B be a deduction of s from P. If B is direct the conclusion follows by Theorem S3. If B is indirect the reasoning of Theorem S3 gives us that $P + \overline{s} \models q$ and $P + \overline{s} \models \overline{q}$ where $q, \overline{q}$ is the contradictory pair ending the deduction. Thus $P + \overline{s}$ is unsatisfiable. Thus $P \models s$. QED

It should be clear that **IL2** is in effect the "natural deduction" system one normally uses when dealing with identities or, more precisely, that **IL2** is a subsystem of the usual logic of mathematics. The particular completeness proof given here illustrates the way metamathematical considerations suggest replacing a natural system with a mathematically more convenient one. In particular, this example can be used to show the kind of motivation that exists for replacing a natural deduction system by a Hilbert-type linear/axiomatic system.

6 *The pure identity logic* **IL1** The language of this system is the set of identities. There are no inidentities. The deductions are the direct deductions using the rules of Group 2. The soundness and completeness of this system are immediate corollaries of results obtained above. However, the value of **IL1** as a first system is enhanced if completeness and soundness are proved for **IL1** itself without reference to a larger system. Soundness is too obvious to warrant discussion here.

Completeness is proved by means of two lemmas which illustrate interesting techniques in a context simple enough to be grasped by very inexperienced classes. The first lemma is essentially corollary ID (above), viz., if P contains an $Ia \ldots b$ then there is a deduction of $a = b$ from P in **IL1**. The second lemma is as follows: if P contains no identity chain $Ia \ldots b$ then there is an interpretation which satisfies P but in which $a = b$ is false. To see this let Q be the union of all of the identity chains in P which involve a. If $a = a$ is in P put it in Q also. Let R be the rest of the sentences in P. Q and R have no constants in common, a is not in R and b is not in Q (otherwise there is an identity chain $Ia \ldots b$ in P). Let $Dx = 1$ for $x = a$ and for all x in Q. Let $Dx = 2$ for the rest of the constants x in PN. D satisfies P and $a = b$ is false in D.

Completeness is proved by putting these two lemmas together as follows. Suppose P implies $a = b$. Then by the second lemma P contains an identity chain $Ia \dots b$. By the first lemma then there is a deduction from P to $a = b$.

7 *Identity logic without necessary truths and self-contradictions* Some teachers prefer to teach the ideas of logical consequence (implication), of establishing implications by deduction, and of establishing non-implications by counterarguments (or countermodels) before introducing the somewhat heady ideas of necessary truths and self-contradictory sentences. Moreover, the original Aristotelian system lacks both sorts of sentences.[4] The system **IL** presented here is essentially the result of deleting the sentences $a = a$ and $a \neq a$ (all a in *PN*) from the language of **IL2**, and leaving everything else the same.

In particular: the language of **IL** consists in all of the identities and inidentities between distinct members of *PN*, the deductions of **IL** and the direct and indirect deductions constructed using only transitivity and symmetry, and the interpretations of **IL** are the same as above.

By suitable modification of the lemmas and corollaries in the proofs of completeness and soundness for **IL2**, the completeness and soundness of **IL** can be proved. This highlights the otherwise plausible fact that the logical axiom $a = a$ plays no role in a **IL2** other than to ensure the provability of the logical truths $a = a$.

8 *Simple metalogical principles* One of the main purposes in presenting a simple logical system is to illustrate some of the more common metalogical principles. **IL1** and **IL2** can be used to illustrate a surprisingly large group of such principles. Following are a selection of principles for **IL1** and **IL2** and a selection of principles which apply only to logics which contain the equivalent of negation. We use $\Box p$ to mean that p is logically true. The proofs of these principles are, of course, very easy.

8.1 *Metalogical principles for* **IL1** *and* **IL2**

1. Premise Addition: If $P \vDash p$ then $P + Q \vDash p$
2. Theorem Deletion[5]: If $P + q \vDash r$ and $P \vDash q$ then $P \vDash r$
3. Logical Truth Deletion: If $P + q \vDash r$ and $\Box q$ then $P \vDash r$
4. Transitivity[6]: If $p \vDash q$ and $q \vDash r$ then $p \vDash r$
5. Cut[7]: If $P + q \vDash r$ and $Q \vDash q$ then $P + Q \vDash r$
6. Chaining: If $P \vDash p_1$, $P \vDash p_2$, $\dots$, $P \vDash p_n$ and $[p_1, \dots, p_n] \vDash q$; then $P \vDash q$
7. Logical truths: $\Box p$ if and only if for all P, $P \vDash p$.

8.2 *Metalogical principles for* **IL2**

1. Reductio[8]: If $P + \overline{q} \vDash r$ and $P + \overline{q} \vDash \overline{r}$ then $P \vDash q$
2. Dilemma[9]: If $P + q \vDash r$ and $P + \overline{q} \vDash r$ then $P \vDash r$

2a. Dilemma: If $q \vDash r$ and $\overline{q} \vDash r$ then $\Box r$

3. Contraposition[10]: If $p \vDash q$ then $\overline{q} \vDash \overline{p}$
4. General Contraposition[11]: If $P + r \vDash q$ then $P + \overline{q} \vDash \overline{r}$
5. Self contradiction: If $p \vDash \overline{p}$ then $\Box \overline{p}$
6. Self contradiction: If $\Box \overline{p}$ then, for all q, $p \vDash q$
7. Logical truths: if $\overline{p} \vDash p$ then $\Box p$.

9 *Glimpses beyond* There are several directions which may be pursued with a class which knows **IL1** and/or **IL2**. For example, one can put a propositional logic on top of **IL2** by adding the connectives, their truth-functions and their rules. There is some advantage in adding the connectives one by one. Another possibility is to add predicates and relations to show the interaction of identities, inidentities and atomic sentences.

Another possibility is to put an algebraic logic (complex terms and the universal quantifier) on top of **IL1**. This would be a good move for a teacher who wanted to present elementary algebraic theories (semigroups, groups, Boolean Algebra, etc.) early in the course.

In philosophy courses where Aristotelian logic is usually presented one could present Aristotelian logic as a separate system having the same general form as **IL** or one could put Aristotelian logic in with **IL**. The latter has the disadvantage of compounding the problems of separating particulars from classes in the minds of beginning students but it has the advantage of showing that both can be discussed using the same formal language.

In courses where foundational questions are to be discussed or in courses where some metamathematics is to be treated, one can supply interesting but trivial proof procedures and decision procedures for these systems.

NOTES

1. A secondary motivation of the paper is to give yet another example of a logic which does not contain propositional logic as a part. It has been widely asserted that propositional logic is "prior to" all other logics and that propositional logic is "more fundamental than" other logics. The presentation of identity logics should invite those who hold the above views to revise them—at least to the extent of defining what they mean by "prior to" and "more fundamental than".
2. The methods and results of sections **2-4** are due to Ziewacz.
3. For pedagogical purposes one might want to require that "QED" or some other "deduction completed" sign be added after the goal has been reached.
4. See for example, Corcoran, J., *The Journal of Symbolic Logic*, vol. 37 (1972), pp. 696-702.
5. This was known by Stoic logicians.
6. This was known by Aristotle.
7. This is a Stoic principle.
8. This was known by Aristotle.
9. These two principles have been attributed to the Stoics by ancient writers.
10. This was known by Aristotle. In the *Prior Analytics* he used this in connection with transitivity to show that a proposition implied by both of a pair of contradictories is implied by its own contradictory.
11. This is in Galen's *Institutio Logica.*

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