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Priestley Duality for Bilattices

In memoriam Leo Esakia

Abstract. We develop a Priestley-style duality theory for different classes of algebras having a bilattice reduct. A similar investigation has already been realized by B. Mobasher, D. Pigozzi, G. Slutzki and G. Voutsadakis, but only from an abstract category-theoretic point of view. In the present work we are instead interested in a concrete study of the topological spaces that correspond to bilattices and some related algebras that are obtained through expansions of the algebraic language.

Keywords: bilattices, Priestley duality theory, bilattices with conflation, bilattices with implication, Brouwerian bilattices.

Introduction

Bilattices are algebraic structures introduced in 1988 by Matthew Ginsberg [14] as a uniform framework for inference in Artificial Intelligence. Since then they have found a variety of applications, sometimes in quite different areas from the original one. The interest in bilattices has thus different sources: among others, computer science and A.I. (see especially the works of Ginsberg, Arieli and Avron), logic programming (Fitting), lattice theory and algebra [16] and, more recently, algebraic logic [4, 5, 24]. An up-to-date review of the applications of this formalism and also of the motivation behind its study can be found in the dissertation [24].

In the present work we develop a Priestley-style duality theory for bilattices and some related algebras that are obtained by adding new operations to the basic algebraic language of bilattices. The main idea guiding our work is that it is possible to view bounded bilattices and related algebras as bounded lattices having two extra constants that satisfy certain properties. This approach will enable us to apply known results on duality theory for different classes of lattices to the study of bilattices and other algebras having a bilattice reduct.

A duality theory for bilattices has already been introduced by Mobasher et al. in [16]. However, while the point of view of [16] is abstract and

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category-theoretic, in the present paper instead we are interested in a concrete study of the topological spaces that correspond to bilattices and related algebras. We will briefly review the results of [16] and discuss the differences between their approach and ours in Sections 1.2 and 1.6.

The rest of the paper is organized as follows. In Section 1 we introduce some definitions and algebraic results on bilattices and language expansions thereof (bilattices with a dual negation operation, bilattices with implication) that will be needed to develop our duality theory. There are no new results here but it is included for ease of reference. Section 1.6 is crucial: it is here where we introduce the fundamental point of view with which we approach the topic. It is concluded with an overview of the various algebras that are considered in this paper.

In Section 2 we recall some known results on duality theories for De Morgan algebras and N4-lattices, on which we will base our treatment of various classes of bilattices in Section 3. We start in Section 3.1 with a duality theory for bilattices without negation (that we call "pre-bilattices") and in Section 3.2 for bilattices with negation. We extend this theory to bilattices with an additional negation-like unary operator ("conflation") in Section 3.3. Finally, in Section 3.4 we develop a duality theory for bilattices with an additional implication operation.

1. Algebraic preliminaries

In this section we fix the algebraic terminology adopted in this paper, introduce the main definitions and recall some known algebraic results that we will use to develop our duality theory for bilattices.

1.1. (Pre-)bilattices

The terminology concerning bilattices is not uniform. Following [2], we reserve the name "bilattice" for algebras that carry a "negation" operation, which are sometimes called "bilattices with negation" in the literature. Consequently, when there is no negation we use the term "pre-bilattice".

Definition 1.1. A *pre-bilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$ such that $\langle B, \wedge, \vee \rangle$ and $\langle B, \otimes, \oplus \rangle$ are both lattices.

The lattice $\langle B, \wedge, \vee \rangle$ is called the *truth lattice*, *t-lattice*, or *logical lattice*; its order is denoted by $\leq_t$ and is called the *truth*, *t*-, or *logical* order. The lattice $\langle B, \otimes, \oplus \rangle$ is called the *knowledge lattice*, *k-lattice*, or *information lattice* and its order $\leq_k$ the *knowledge*, *k*-, or *information* order.

If they exist, the bounds of the logical lattice are denoted by f and t. Similarly, $\bot$ and $\top$ refer to the minimum and maximum of the information lattice. When we speak of a *bounded* pre-bilattice we mean not only that the four bounds exist but also that they have become part of the algebraic signature.

One way of establishing a connection between the two orders of a pre-bilattice is to impose certain monotonicity properties on the lattice connectives, as in the following definition, due to Fitting [12].

A pre-bilattice $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus \rangle$ is *interlaced* whenever each one of the four operations $\{\wedge, \vee, \otimes, \oplus\}$ is monotonic with respect to both orders $\leq_t$ and $\leq_k$.

Pre-bilattices obviously form a variety, axiomatized by the lattice identities for the two lattices. In [2] it is proved that the class of interlaced pre-bilattices is also a variety, axiomatized by the identities for pre-bilattices plus the following ones:

$$(x \wedge y) \otimes z \leq_t y \otimes z \qquad (x \wedge y) \oplus z \leq_t y \oplus z$$
$$(x \otimes y) \wedge z \leq_k y \wedge z \qquad (x \otimes y) \vee z \leq_k y \vee z.$$

From an algebraic point of view, interlaced pre-bilattices form perhaps the most interesting subclass of pre-bilattices. Its interest comes mainly from the fact that any interlaced pre-bilattice can be represented as a special product of two lattices. This result is well-known for bounded pre-bilattices, and it has been more recently generalized to the unbounded case [17, 5].

The interlacing conditions may be strengthened through the following definition, due to Ginsberg [14]. A pre-bilattice is *distributive* when all possible distributive laws concerning the four lattice operations, i.e., all identities of the following form, hold: $x \circ (y \bullet z) = (x \circ y) \bullet (x \circ z)$ for every $\circ, \bullet \in \{\wedge, \vee, \otimes, \oplus\}$. The class of distributive pre-bilattices is a proper subvariety of interlaced pre-bilattices.

A second way of relating the two lattice orders of a pre-bilattice is by expanding the algebraic language with a unary operator. This is the method originally used by Ginsberg to introduce bilattices.

Definition 1.2. A *bilattice* is an algebra $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ such that the reduct $\langle B, \wedge, \vee, \otimes, \oplus \rangle$ is a pre-bilattice and the *negation* $\neg$ is a unary operation satisfying that for every $a, b \in B$,

(neg 1) if $a \leq_t b$, then $\neg b \leq_t \neg a$

(neg 2) if $a \leq_k b$, then $\neg a \leq_k \neg b$

(neg 3) $a = \neg\neg a$.

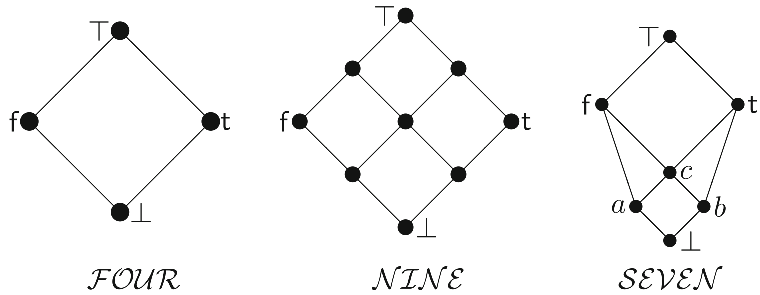

Figure 1. Some examples of (pre-)bilattices

The interlacing and distributivity properties extend to bilattices in the obvious way. We say that a bilattice is interlaced (respectively, distributive) when its pre-bilattice reduct is interlaced (distributive).

The following equations (De Morgan laws and "dual De Morgan" laws) hold in any bilattice:

$$\neg(x \wedge y) = \neg x \vee \neg y \qquad \neg(x \vee y) = \neg x \wedge \neg y$$
$$\neg(x \otimes y) = \neg x \otimes \neg y \qquad \neg(x \oplus y) = \neg x \oplus \neg y.$$

Moreover, if the bilattice is bounded, then $\neg\top = \top$, $\neg\bot = \bot$, $\neg\mathsf{t} = \mathsf{f}$ and $\neg\mathsf{f} = \mathsf{t}$. So, if a bilattice $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg\rangle$ is distributive, or at least the reduct $\langle B, \wedge, \vee\rangle$ is distributive, then $\langle B, \wedge, \vee, \neg\rangle$ is a structure known as a *De Morgan lattice*, i.e., a distributive lattice equipped with a unary order-reversing involution. It is also easy to check that the four De Morgan laws imply that the negation operator satisfies **(neg 1)** and **(neg 2)**. It follows that the class of bilattices is a variety. As in the case of pre-bilattices, distributive bilattices are a subvariety of interlaced bilattices, and this inclusion is proper.

Figure 1 shows the double Hasse diagrams of some of the best-known (pre-)bilattices. They should be read as follows: $a \leq_t b$ if there is a path from a to b which goes uniformly from left to right, while $a \leq_k b$ if there is a path from a to b which goes uniformly from the bottom to the top. The four lattice operations are thus uniquely determined by the diagram, while negation, if there is one, corresponds to reflection along the vertical axis connecting $\bot$ and $\top$. It is then clear that all the pre-bilattices shown in Figure 1 can be endowed with a negation in a unique way and turned in this way into bilattices. When no confusion is likely to arise, we use the same name to denote a concrete pre-bilattice and its associated bilattice; the names used in the diagrams are by now more or less standard in the literature. Let us note that $\mathcal{FOUR}$ and $\mathcal{NINE}$ are distributive, while $\mathcal{SEVEN}$ is not even interlaced.

1.2. Product (pre-)bilattices

A useful way of relating (and, to some extent, reducing) bilattice theory to lattice theory is provided by the following construction, due to Fitting [12].

Let $\mathbf{L_1} = \langle L_1, \sqcap_1, \sqcup_1 \rangle$ and $\mathbf{L_2} = \langle L_2, \sqcap_2, \sqcup_2 \rangle$ be lattices with associated orders $\leq_1$ and $\leq_2$. The *product pre-bilattice* $\mathbf{L_1} \odot \mathbf{L_2} = \langle L_1 \times L_2, \wedge, \vee, \otimes, \oplus \rangle$ is defined as follows. For all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in L_1 \times L_2$,

$$
\begin{aligned}
\langle a_1, a_2 \rangle \wedge \langle b_1, b_2 \rangle &:= \langle a_1 \sqcap_1 b_1,\ a_2 \sqcup_2 b_2 \rangle \\
\langle a_1, a_2 \rangle \vee \langle b_1, b_2 \rangle &:= \langle a_1 \sqcup_1 b_1,\ a_2 \sqcap_2 b_2 \rangle \\
\langle a_1, a_2 \rangle \otimes \langle b_1, b_2 \rangle &:= \langle a_1 \sqcap_1 b_1,\ a_2 \sqcap_2 b_2 \rangle \\
\langle a_1, a_2 \rangle \oplus \langle b_1, b_2 \rangle &:= \langle a_1 \sqcup_1 b_1,\ a_2 \sqcup_2 b_2 \rangle .
\end{aligned}
$$

The algebra $\mathbf{L_1} \odot \mathbf{L_2}$ is always an interlaced pre-bilattice, and it is distributive if and only if both $\mathbf{L_1}$ and $\mathbf{L_2}$ are distributive lattices. From the definition it follows immediately that

$$
\begin{array}{lllll}
\langle a_1, a_2 \rangle \leq_k \langle b_1, b_1 \rangle & \text{iff} & a_1 \leq_1 b_1 & \text{and} & a_2 \leq_2 b_2 \\
\langle a_1, a_2 \rangle \leq_t \langle b_1, b_1 \rangle & \text{iff} & a_1 \leq_1 b_1 & \text{and} & a_2 \geq_2 b_2 .
\end{array}
$$

Notice also that the bounds of the two lattice orders, if they exist, are: $\mathsf{f} = \langle 0_1, 1_2 \rangle$, $\mathsf{t} = \langle 1_1, 0_2 \rangle$, $\perp = \langle 0_1, 0_2 \rangle$, $\top = \langle 1_1, 1_2 \rangle$.

If $\mathbf{L_1}$ and $\mathbf{L_2}$ are isomorphic, then it is possible to define a negation in $\mathbf{L_1} \odot \mathbf{L_2}$, and we speak of the *product bilattice* instead of the product pre-bilattice. If $h : \mathbf{L_1} \cong \mathbf{L_2}$ is an isomorphism, then a negation is defined as

$$\neg \langle a_1, a_2 \rangle := \langle h^{-1}(a_2), h(a_1) \rangle.$$

In particular, if $\mathbf{L_1} = \mathbf{L_2}$, the definition applied to the identity isomorphism gives $\neg \langle a_1, a_2 \rangle := \langle a_2, a_1 \rangle$. Note that this negation is entirely independent of any negation that may or may not exist on the factor lattices $\mathbf{L_1}$ and $\mathbf{L_2}$.

The following results were proved by Avron [2] for bounded (pre-)bilattices, then generalized in [17, 5] to the unbounded case:

THEOREM 1.3 (Representation of pre-bilattices). *A (bounded) pre-bilattice* $\mathbf{B}$ *is interlaced if and only if there exist two (bounded) lattices* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *such that* $\mathbf{B} \cong \mathbf{L_1} \odot \mathbf{L_2}$. *Moreover,* $\mathbf{B}$ *is distributive if and only if both* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *are distributive lattices.*

As a special case one obtains the representation theorem for bilattices:

THEOREM 1.4 (Representation of bilattices). *A (bounded) bilattice* $\mathbf{B}$ *is interlaced if and only if there is a (bounded) lattice* $\mathbf{L}$ *such that* $\mathbf{B}$ *is isomorphic to* $\mathbf{L} \odot \mathbf{L}$*. Moreover,* $\mathbf{B}$ *is distributive if and only if* $\mathbf{L}$ *is a distributive lattice.*

As observed in [16, Theorems 10 and 13, Corollaries 11 and 14], these representation results easily extend to categorical equivalences between the following categories:

1. bounded distributive pre-bilattices and the product category of bounded distributive lattices with itself,
2. bounded distributive bilattices and bounded distributive lattices,

where the morphisms in each case preserve all algebraic structure.

Since the category of bounded distributive lattices is dually equivalent to the category of Priestley spaces, it is immediate to conclude that there are dual equivalences between the following categories:

1. bounded distributive pre-bilattices and the product category of Priestley spaces with itself;
2. bounded distributive bilattices and Priestley spaces.

This was proved in [16] and it is easy to see that, using the equivalences established in [4], analogous results can be obtained for other classes of algebras defined through expansions of the algebraic language of bilattices. However, as mentioned in the Introduction, we are interested in a more concrete description of the kind of spaces that correspond to (pre-)bilattices and related algebras. We are therefore going to follow a different strategy: instead of focusing on the lattice factors of a (pre-)bilattice (as given by the above representation theorems), we will focus on one of its lattice reducts. Although we will still make use of the (pre-)bilattice product representation, the key tool in our approach is the so-called *90-degree lemma*[1], well known from lattice theory:

LEMMA 1.5 (90-degree lemma). *Let* $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t} \rangle$ *be a bounded lattice with associated order* $\leq_t$ *and let* $\bot, \top \in B$ *be elements satisfying the following conditions:*

(b 1) $\top \vee \bot = \mathsf{t}$

[1] Adopting the terminology of [15].

(b 2) $\top \wedge \bot = \mathsf{f}$

(b 3) *for all* $a, b, c \in B$*, if* $\bot \in \{a, b, c\}$ *or* $\top \in \{a, b, c\}$*, then* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ *and* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$*.*

Then, defining

$$a \otimes b := (a \wedge \bot) \vee (b \wedge \bot) \vee (a \wedge b)$$
$$a \oplus b := (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b)$$

one has that $\langle B, \otimes, \oplus, \bot, \top \rangle$ *is a bounded lattice and* $\langle B, \wedge, \vee, \otimes, \oplus, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$ *is a bounded interlaced pre-bilattice. In case* $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t} \rangle$ *is a distributive lattice, then* $\langle B, \otimes, \oplus, \top, \bot \rangle$ *is distributive as well.*

Conversely, in any bounded interlaced pre-bilattice it holds that

$$a \otimes b = (a \wedge \bot) \vee (b \wedge \bot) \vee (a \wedge b)$$
$$a \oplus b = (a \wedge \top) \vee (b \wedge \top) \vee (a \wedge b).$$

As noted in [2, Theorem 5.3], the lemma can be formulated as follows:

Theorem 1.6. *The variety of bounded interlaced pre-bilattices and the variety of bounded lattices having two constants that satisfy the above conditions are termwise (or: definitionally) equivalent.*

1.3. Filters and ideals of interlaced (pre-)bilattices

As in lattice theory, the notions of filter and ideal are important for the study of (pre-)bilattices, especially in the context of topological duality. In this case one has to take both orders into account, which leads to the following definition, due to Arieli and Avron [1].

Definition 1.7. A *bifilter* of a (pre-)bilattice $\langle B, \wedge, \vee, \otimes, \oplus \rangle$ is a non-empty set $F \subseteq B$ such that F is a lattice filter of both orders $\leq_t$ and $\leq_k$. A bifilter F is *prime* if $F \neq B$ and $a \vee b \in F$ or $a \oplus b \in F$ implies that $a \in F$ or $b \in F$ for all $a, b \in B$.

There are of course other possibilities, as one could consider subsets that are simultaneously a t-filter and a k-ideal, or a t-ideal and a k-filter, or an ideal in both orders. Here we shall need just the first one, which allows us to use a somewhat condensed terminology:

Definition 1.8. A *filter-ideal* of a (pre-)bilattice $\langle B, \wedge, \vee, \otimes, \oplus \rangle$ is a non-empty set $G \subseteq B$ such that G is a lattice filter of $\leq_t$ and a lattice ideal of $\leq_k$. We say that G is *prime* if $G \neq B$ and $a \vee b \in G$ or $a \otimes b \in G$ implies that $a \in F$ or $b \in F$ for all $a, b \in B$.

All these notions have been studied in [5], to which we refer for proofs and further details on the results stated in this section.

For bounded (pre-)bilattices Definition 1.7 implies that $\mathtt{t}, \top \in F$ for any bifilter F. Analogously, $\mathtt{t}$ and $\bot$ are contained in any filter-ideal.

Notice also that, in an interlaced bilattice, a bifilter F cannot be prime w.r.t. to one order only. This is so because the interlacing conditions imply that $a \vee b \leq_k a \oplus b$ and $a \oplus b \leq_t a \vee b$, therefore we have that $a \vee b \in F$ if and only if $a \oplus b \in F$. A similar argument shows that the same holds for filter-ideals.

We know from Theorem 1.3 that interlaced pre-bilattices have the form $\mathbf{L_1} \odot \mathbf{L_2}$ where $\mathbf{L_1}$ and $\mathbf{L_2}$ are lattices. The following result [5, Proposition 3.18] will have special interest for our approach (cf. [20, Corollary 2.1]):

Proposition 1.9. *Let $\mathbf{L_1} \odot \mathbf{L_2}$ be an interlaced pre-bilattice and F a non-empty subset of $L_1 \times L_2$. Then:*

(i) *F is a (prime) bifilter of $\mathbf{L_1} \odot \mathbf{L_2}$ iff $F = \nabla \times L_2$ for some (prime) lattice filter ∇ of $\mathbf{L_1}$,*

(ii) *F is a (prime) filter-ideal of $\mathbf{L_1} \odot \mathbf{L_2}$ iff $F = L_1 \times \Delta$ for some (prime) lattice ideal Δ of $\mathbf{L_2}$.*

Using the previous proposition it is easy to prove the following:

Theorem 1.10. *In any interlaced bilattice $\mathbf{L_1} \odot \mathbf{L_2}$ the following structures are isomorphic: (i) the poset of (prime) bifilters of $\mathbf{L_1} \odot \mathbf{L_2}$ and the poset of (prime) filters of $\mathbf{L_1}$, (ii) the poset of (prime) filter-ideals of $\mathbf{L_1} \odot \mathbf{L_2}$ and the poset of (prime) ideals of $\mathbf{L_2}$.*

As mentioned above, it is our intention to view bounded (pre-)bilattices as bounded lattices with extra constants. To this end the following characterization will be useful:

Lemma 1.11. *Let $\mathbf{B}$ be a bounded interlaced pre-bilattice and $F \subseteq B$. Then:*

(i) *F is a (prime) bifilter iff F is a (prime) t-filter and $\top \in F$,*

(ii) *F is a (prime) filter-ideal iff F is a (prime) t-filter and $\bot \in F$.*

Proof. (i). Suppose F is a t-filter such that $\top \in F$ and let $a, b \in F$. By the interlacing conditions, we have that $a \wedge b \leq_t a \otimes b$ for all $a, b \in B$. So $a \otimes b \in F$ because F is an up-set w.r.t. $\leq_t$. Suppose there is $c \in B$ such that $a \leq_k c$. Then $c = a \oplus c$ and by Lemma 1.5 we know that $a \oplus c = (a \wedge \top) \vee (c \wedge \top) \vee (a \wedge c)$. So the result readily follows because $a \wedge \top \in F$. So F is a bifilter, and since

we are in an interlaced pre-bilattice, being prime w.r.t. one of the two lattice orders is equivalent to being prime w.r.t. both. The converse implication follows simply from the definition of bifilter.
(ii). Similar to the proof of the previous item, using the equality $a \otimes b = (a \wedge \bot) \vee (b \wedge \bot) \vee (a \wedge b)$ of Lemma 1.5. ■

Notice that in a bounded interlaced pre-bilattice any prime t-filter must contain either $\bot$ or $\top$, because $\bot \vee \top = \mathsf{t} \in F$ for any t-filter F, but it cannot contain both elements, because then $\bot \wedge \top = \mathsf{f} \in F$, so the F would not be proper. This immediately yields the following:

THEOREM 1.12. *In a bounded interlaced pre-bilattice every prime t-filter is either a bifilter or a filter-ideal.*

1.4. Bilattices with conflation

The algebraic signature considered above has been expanded in various ways and for different purposes in the literature on bilattices. For some of the algebras thus obtained representation theorems analogous to the one described in the previous section can be proved.

The first expansion we shall consider, due to Fitting [13], consists in adding an operator that behaves as a dual of the bilattice negation, called *conflation*.

DEFINITION 1.13. An algebra $\mathbf{B} = \langle B, \wedge, \vee, \otimes, \oplus, \neg, - \rangle$ is called a *bilattice with conflation* if the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is a bilattice and the *conflation* $- \colon B \to B$ is an operation satisfying, for all $a, b \in B$,

(conf 1) if $a \leq_k b$, then $-b \leq_k -a$

(conf 2) if $a \leq_t b$, then $-a \leq_t -b$

(conf 3) $a = --a$.

We say that $\mathbf{B}$ is *commutative* if it also satisfies the equation: $\neg - x = - \neg x$.

In any bounded bilattice with conflation $\mathbf{B}$ it holds that $-\top = \bot$, $-\bot = \top$, $-\mathsf{t} = \mathsf{t}$ and $-\mathsf{f} = \mathsf{f}$. If $\mathbf{B}$ is distributive, then both reducts $\langle B, \wedge, \vee, \neg \rangle$ and $\langle B, \otimes, \oplus, - \rangle$ are De Morgan lattices. If, in addition, $\mathbf{B}$ is commutative, then the composed operation $\neg \cdot -$ is involutive, so $\langle B, \wedge, \vee, \neg \cdot - \rangle$ and $\langle B, \otimes, \oplus, \neg \cdot - \rangle$ are De Morgan lattices as well.

The class of bilattices with conflation is a variety, axiomatized by the equations defining bilattices together with **(conf 3)** and the following ones:

$$-(x \otimes y) = -x \oplus -y \qquad -(x \oplus y) = -x \otimes -y$$
$$-(x \wedge y) = -x \wedge -y \qquad -(x \vee y) = -x \vee -y.$$

Adding the appropriate equations to a presentation of this class, we may define the varieties of interlaced (distributive) bilattices with conflation and commutative (interlaced, distributive) bilattices with conflation.

As shown in [24, 4], the variety of commutative distributive bilattices with conflation has exactly two proper subvarieties. The first is axiomatized by either of the following equations:

$$(x \wedge \neg - x) \wedge (y \vee \neg - y) = (x \wedge \neg - x)$$
$$(x \otimes \neg - x) \otimes (y \oplus \neg - y) = (x \otimes \neg - x).$$

We call its objects *Kleene bilattices* as they are closely related to a subclass of De Morgan lattices known as Kleene lattices (more on this later).

The second, contained in Kleene bilattices, is given by any of the following four the equations

$$x \wedge (y \vee \neg - y) = x \qquad x \otimes (y \oplus \neg - y) = x$$
$$x \vee (y \wedge \neg - y) = x \qquad x \oplus (y \otimes \neg - y) = x.$$

Following [1], we call its objects *classical bilattices*, highlighting their relationship to classical logic and Boolean algebras.

A representation theorem for bilattices with conflation can be obtained through the following construction, also due to Fitting. Let $\mathbf{L} = \langle L, \sqcap, \sqcup, ' \rangle$ be an *involutive lattice*, i.e., an algebra such that the reduct $\langle L, \sqcap, \sqcup \rangle$ is a lattice and the operation $'\colon A \to A$ satisfies, for all $a, b \in A$,

(inv 1) if $a \leq b$, then $b' \leq a'$

(inv 2) $a = a''$.

Notice that De Morgan lattices coincide with distributive involutive lattices. Kleene lattices, then, are De Morgan lattices satisfying the additional equation: $x \sqcap x' \leq y \sqcup y'$. Boolean algebras are also a subvariety of De Morgan lattices (the minimal one, in fact). They can be defined as De Morgan lattices satisfying the equation: $x \sqcap x' \leq y$.

Given an involutive lattice $\mathbf{L} = \langle L, \sqcap, \sqcup, ' \rangle$, we perform the product construction described in Section 1.2 and equip $\mathbf{L} \odot \mathbf{L}$ with the conflation $-\langle a, b \rangle = \langle b', a' \rangle$. It can be easily checked that $\mathbf{L} \odot \mathbf{L}$ is always a commutative interlaced bilattice with conflation.

The following result was proved by Fitting [13] for the case of bounded distributive bilattices, then generalized in [24, 4] to unbounded interlaced bilattices.

Theorem 1.14. *An algebra* $\mathbf{B}$ *is a (bounded) commutative interlaced bilattice with conflation if and only if there is a (bounded) involutive lattice* $\mathbf{L}$ *such that* $\mathbf{B}$ *is isomorphic to* $\mathbf{L} \odot \mathbf{L}$*.*

Furthermore, we have that (i) $\mathbf{B}$ is distributive if and only if $\mathbf{L}$ is a De Morgan lattice, (ii) $\mathbf{B}$ is a Kleene bilattice if and only if $\mathbf{L}$ is a Kleene lattice and (iii) $\mathbf{B}$ is a classical bilattice if and only if $\mathbf{L}$ is a Boolean algebra.

1.5. Brouwerian bilattices

The second way of expanding the bilattice language that we consider consists in adding a binary connective that plays (on a logical level) the role of an implication. These enriched algebras arose from the study developed in [24], then generalized in [4], of the algebraic models of the "logic of logical bilattices" introduced by Arieli and Avron [1].

Definition 1.15. A *Brouwerian bilattice* is an algebra $\langle B, \wedge, \vee, \otimes, \oplus, \supset, \neg \rangle$ such that the reduct $\langle B, \wedge, \vee, \otimes, \oplus, \neg \rangle$ is a bilattice and the following equations are satisfied:

(B1) $(x \supset x) \supset y = y$

(B2) $x \supset (y \supset z) = (x \wedge y) \supset z = (x \otimes y) \supset z$

(B3) $(x \vee y) \supset z = (x \supset z) \wedge (y \supset z) = (x \oplus y) \supset z$

(B4) $x \wedge ((x \supset y) \supset (x \otimes y)) = x$

(B5) $\neg(x \supset y) \supset z = (x \wedge \neg y) \supset z.$

Brouwerian bilattices obviously form a variety. An interesting subvariety, the minimal one, in fact, is the class of *classical implicative bilattices*, defined by the following equation: $((x \supset y) \supset x) \supset x = x \supset x$. This class was introduced and studied under the name of "implicative bilattices" in [24], where it is proved that they are the equivalent algebraic semantics (in the sense of [3]) of Arieli and Avron's logic of logical bilattices with implication. Brouwerian bilattices can be considered a natural generalization of the implicative ones, and the relation between the two classes is analogous, as we will see below, to the relation between (generalized) Heyting algebras and (generalized) Boolean algebras.

Any Brouwerian bilattice has a top element $\top$ w.r.t. the k-order, defined by the expression: $\top = (a \supset a) \oplus \neg(a \supset a)$. Moreover, the pre-bilattice reduct of a Brouwerian bilattice is distributive. One can then hope to represent Brouwerian bilattices as products of some kind. This can be done through the following construction.

Let $\mathbf{L} = \langle L, \sqcap, \sqcup, \to, 1\rangle$ be a Brouwerian lattice[2], i.e., an algebra such that $\langle L, \sqcap, \sqcup, 1\rangle$ is a lattice with maximum element 1, satisfying the following residuation condition: for all $a, b, c \in L$,

$$a \sqcap b \leq c \quad \text{if and only if} \quad b \leq a \to c.$$

Given a Brouwerian lattice $\mathbf{L} = \langle L, \sqcap, \sqcup, \to, 1\rangle$, we perform the product construction and equip $\mathbf{L} \odot \mathbf{L}$ with the operation $\supset$, defined by

$$\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle = \langle a_1 \to b_1, a_1 \sqcap b_2\rangle.$$

As shown in [4, Proposition 4.11], we get that $\mathbf{L}\odot\mathbf{L}$ is a Brouwerian bilattice. Conversely, any Brouwerian bilattice can be represented as a product of this kind:

THEOREM 1.16. *An algebra* $\mathbf{B}$ *is a Brouwerian bilattice if and only if there is a Brouwerian lattice* $\mathbf{L}$ *such that* $\mathbf{B}$ *is isomorphic to* $\mathbf{L} \odot \mathbf{L}$*.*

Furthermore, (i) $\mathbf{B}$ is a bounded Brouwerian bilattice if and only if $\mathbf{L}$ is a Heyting algebra, (ii) $\mathbf{B}$ is a classical implicative bilattice if and only if $\mathbf{L}$ is a classical implicative lattice (i.e., the 0-free subreduct of a Boolean algebra) and (iii) $\mathbf{B}$ is a bounded classical implicative bilattice if and only if $\mathbf{L}$ is a Boolean algebra.

1.6. Bounded interlaced bilattices

In order to develop our duality theory for bilattices, we will view a bounded interlaced pre-bilattice $\langle B, \wedge, \vee, \otimes, \oplus, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ as a bounded lattice $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ where $\bot$ and $\top$ are constants that satisfy the properties of the 90-degree Lemma 1.5. Thus, we will apply known results from Priestley duality to the logical lattice $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}\rangle$ and study the effect that the various additional structure has on the basic picture.

We focus on the t-lattice rather than on the k-lattice because our language extensions concern *logical* operations. For example, the $\{\wedge, \vee, \neg\}$-reduct of a distributive bilattice is a De Morgan lattice, while the $\{\otimes, \oplus, \neg\}$-reduct is not.

The purpose of this section, then, is to express the various algebra definitions introduced so far entirely with the aid of the two constants $\bot$ and $\top$. These will always be assumed to satisfy the conditions **(b 1)**, **(b 2)**,

[2]These algebras are also called *generalized Heyting algebras*, *Brouwerian algebras* [9], implicative lattices [18] or *relatively pseudo-complemented lattices* [23]. Note also that some authors call "Brouwerian lattices" structures that are dual to those defined above.

and **(b 3)** listed in Lemma 1.5 and we will express this fact simply by calling them a *complemented pair*.

Beginning with negation, we have to reformulate the conditions of Definition 1.2 as follows:

(**neg 1**) if $a \leq_t b$, then $\neg b \leq_t \neg a$

(**neg 2'**) $\neg\bot = \bot$ and $\neg\top = \top$

(**neg 3**) $a = \neg\neg a$.

Note that **(neg 1)** and **(neg 3)** guarantee that the t-lattice is *involutive*. It is now easy to check that we obtain a termwise equivalence between the class of bounded interlaced bilattices and the class of bounded involutive lattices with a complemented pair of constants.

Similarly, the condition **(conf 1)** on a conflation from Definition 1.13 can be recast as

(**conf 1'**) $-\bot = \top$

The case of Brouwerian bilattices is more interesting and we allow ourselves to present in some detail the genesis of the (itself rather simple looking) characterization presented in Theorem 1.19 below. As observed in [4, p. 17], the class of $\{\wedge, \vee, \supset, \neg\}$-subreducts of Brouwerian bilattices is a variety of algebras known as *N4-lattices* [18, 19].

Definition 1.17. An *N4-lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \supset, \neg\rangle$ such that:

(i) The reduct $\langle A, \wedge, \vee, \neg\rangle$ is a De Morgan lattice.

(ii) The relation $\preceq$, defined as $a \preceq b$ iff $a \supset b = (a \supset b) \supset (a \supset b)$, is a pre-ordering on $\mathbf{A}$.

(iii) The relation $\sim$, defined as $a \sim b$ iff $a \preceq b$ and $b \preceq a$, is a congruence w.r.t. $\wedge, \vee, \supset$ and the quotient algebra $\langle A, \wedge, \vee, \supset\rangle/\!\sim$ is a Brouwerian lattice.

(iv) $\neg(a \supset b) \sim a \wedge \neg b$ for all $a, b \in A$.

(v) $a \leq b$ iff $a \preceq b$ and $\neg b \preceq \neg a$ for all $a, b \in A$, where $\leq$ is the lattice order of $\mathbf{A}$.

Here we are interested in bounded N4-lattices, which means that the quotient algebra $\langle A, \wedge, \vee, \supset\rangle/\!\sim$ is in fact a Heyting algebra. In [18] it is shown that any N4-lattice is isomorphic to a subalgebra of a *twist-structure* $\langle L \times L, \wedge, \vee, \supset, \neg\rangle$ obtained from a Heyting algebra $\langle L, \sqcap, \sqcup, \rightarrow, 0, 1\rangle$ as follows: for all $a_1, a_2, b_1, b_2 \in L$,

(i) $\langle a_1, a_2\rangle \wedge \langle b_1, b_2\rangle = \langle a_1 \sqcap b_1, a_2 \sqcup b_2\rangle$

(ii) $\langle a_1, a_2\rangle \vee \langle b_1, b_2\rangle = \langle a_1 \sqcup b_1, a_2 \sqcap b_2\rangle$

(iii) $\langle a_1, a_2\rangle \supset \langle b_1, b_2\rangle = \langle a_1 \to b_1, a_1 \sqcap b_2\rangle$

(iv) $\neg\langle a_1, a_2\rangle = \langle a_2, a_1\rangle$.

It is obvious that the twist-structure construction can be seen as a special case of the Brouwerian bilattice product introduced above. The logical constants t and f of a twist-structure are, respectively, $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$. Using this fact, it is easy to check that if a pair of elements $a, b \in L \times L$ satisfies **(b 1)**, **(b 2)** and **(neg 2')**, then $a = \langle 0, 0\rangle$ and $b = \langle 1, 1\rangle$. If such elements exist in a twist-structure, then we can use the 90-degree lemma to obtain a distributive bilattice. Moreover, by Theorem 1.16, we know that the structure thus obtained is in fact a bounded Brouwerian bilattice.

Notice, however, that in a Brouwerian bilattice $\bot$ and $\top$ differ in their behaviour with respect to the $\supset$ operation. In other words, while in a general bilattice the k-order can be reversed to obtain again a bilattice, this is not the case for Brouwerian bilattices. So, if we want to extend the 90-degree lemma in order to obtain a Brouwerian bilattice starting from a bounded N4-lattice with a complemented pair, we need to be specific about which constant will play the role of $\top$ and which one will be $\bot$. This can be formalized through the following equations:

(imp $\top$) $\quad \top \supset \mathsf{f} = \mathsf{f}$, $\qquad$ **(imp $\bot$)** $\quad \bot \supset \bot = \mathsf{t}$.

Thus, we have the following result.

Lemma 1.18. *Let* $\mathbf{B}$ *be a bounded N4-lattice that is a subalgebra of the twist-structure* $\langle L \times L, \wedge, \vee, \supset, \neg\rangle$, *where* $\langle L, \sqcap, \sqcup, \to, 0, 1\rangle$ *is the underlying Heyting algebra. Let* $a \in B$ *be such that* $a = \neg a$. *Then,*

(i) *if* a *satisfies* **(imp $\top$)**, *then* $a = \langle 1, 1\rangle$,

(ii) *if* a *satisfies* **(imp $\bot$)**, *then* $a = \langle 0, 0\rangle$.

Proof. (i). If an element $\langle a, b\rangle \in B$ satisfies $\langle a, b\rangle = \neg\langle a, b\rangle = \langle b, a\rangle$, then $a = b$. Now if $\langle a, a\rangle$ satisfies **(imp $\top$)**, then $\langle a, a\rangle \supset \langle 0, 1\rangle = \langle a \to 0, a \sqcap 1\rangle = \langle 0, 1\rangle$. Thus $a = 1$, i.e., $\langle a, b\rangle = \langle 1, 1\rangle$.
(ii). If an element $\langle a, a\rangle$ satisfies **(imp $\bot$)**, then $\langle a, a\rangle \supset \langle a, a\rangle = \langle 1, a\rangle = \langle 1, 0\rangle$ therefore, $\langle a, a\rangle = \langle 0, 0\rangle$. ■

Thus we arrive at our reformulation of Brouwerian bilattices:

THEOREM 1.19. *The variety of bounded Brouwerian bilattices is termwise equivalent to the variety of bounded N4-lattices having two extra constants* $\bot, \top$ *such that*

(i) $\bot$ *and* $\top$ *satisfy* **(neg 2')**, *i.e.,* $\neg\bot = \bot$ *and* $\neg\top = \top$

(ii) $\top$ *satisfies* **(imp** $\top$**)**, *i.e.,* $\top \supset \mathsf{f} = \mathsf{f}$

(iii) $\bot$ *satisfies* **(imp** $\bot$**)**, *i.e.,* $\bot \supset \bot = \mathsf{t}$.

As in the previous cases, this result allows us to base our duality for bounded Brouwerian bilattice on the duality theory for N4-lattices that was developed in [20].

Notice that Lemma 1.18 also implies that a function $h\colon \mathbf{B} \to \mathbf{B}'$ between two bounded Brouwerian bilattices $\mathbf{B}, \mathbf{B}'$ is a bounded Brouwerian bilattice homomorphism if and only if h is a bounded N4-lattice homomorphism between the N4-lattice reducts of $\mathbf{B}$ and $\mathbf{B}'$.

(pre-)bilattice	(additional) operations	t-lattice reduct	(additional) axioms
bounded distributive pre-bilattice	$\wedge$, $\vee$, $\otimes$, $\oplus$, f, t, $\bot$, $\top$	bounded distributive lattice with complemented pair	$\top \vee \bot = \mathsf{t}$ $\top \wedge \bot = \mathsf{f}$
bounded distributive bilattice	$\neg$	bounded De Morgan lattice with complemented pair	$\neg\bot = \bot$ $\neg\top = \top$
bounded distributive bilattice with conflation	$\neg$, $-$	bounded De Morgan lattice with complemented pair and monotone involution	$-\bot = \top$
bounded Brouwerian bilattice	$\neg$, $\supset$	bounded N4-lattice with complemented pair	$\top \supset \mathsf{f} = \mathsf{f}$ $\bot \supset \bot = \mathsf{t}$

Table 1. Bilattices and their t-lattice reducts.

In Table 1 we give an overview of the various structures we have introduced, highlighting the relationship between bilattices and their t-lattice reducts. Since this will be required for Priestley duality in any case, we have restricted ourselves to bounded distributive bilattices.

2. Duality for De Morgan and N4-lattices

In this section, we recall the essential elements of Priestley duality for bounded distributive lattices, De Morgan lattices, and N4-lattices, which we will later use to develop a duality theory for corresponding classes of bilattices.

2.1. Priestley duality

Although we assume that the reader is familiar with basic definitions and results on Priestley duality for bounded distributive lattices [10, 21, 22], we give a quick overview, mainly in order to establish the notation for the rest of the paper.

Priestley duality concerns the category **DLat** of bounded distributive lattices and bounded lattice homomorphisms. To every bounded distributive lattice $\mathbf{A}$, one associates the set $X(\mathbf{A})$ of prime filters. On $X(\mathbf{A})$ one has the *Priestley topology* τ, generated by the sets $\varphi(a) := \{P \in X(\mathbf{A}) : a \in P\}$ and $\varphi'(a) := \{P \in X(\mathbf{A}) : a \notin P\}$, and the inclusion relation between prime filters as an order. The resulting ordered topological spaces are called *Priestley spaces*[3]. A homomorphism h between bounded distributive lattices $\mathbf{A}$ and $\mathbf{A}'$ gives rise to a function $X(h) : X(\mathbf{A}') \to X(\mathbf{A})$, defined by $X(h)(P') = h^{-1}[P']$, that is continuous and order preserving. Taking the functions with these properties, called *Priestley functions*, as morphisms between Priestley spaces one obtains the category **PrieSp**, and X is now readily recognized as a contravariant functor from **DLat** to **PrieSp**.

For a functor in the opposite direction, one associates to every Priestley space $\mathcal{X} = \langle X, \tau, \leq \rangle$ the set $L(\mathcal{X})$ of clopen up-sets. This is a bounded distributive lattice with respect to the set-theoretic operations $\cap, \cup, \emptyset$, and X. To a Priestley map $f : \mathcal{X} \to \mathcal{X}'$ one associates the function $L(f)$, given by $L(f)(U') = f^{-1}[U']$, which is easily seen to be a bounded lattice homomorphism from $L(\mathcal{X}')$ to $L(\mathcal{X})$. Together, then, L constitutes a contravariant functor from **PrieSp** to **DLat**.

The two functors are adjoint to each other with the units given by

$$\Phi_{\mathbf{A}} \colon \mathbf{A} \to L(X(\mathbf{A})) \qquad \Phi_{\mathbf{A}}(a) = \{P \in X(\mathbf{A}) : a \in P\}$$
$$\Psi_{\mathcal{X}} \colon \mathcal{X} \to X(L(\mathcal{X})) \qquad \Psi_{\mathcal{X}}(x) = \{U \in L(\mathcal{X}) : x \in U\}$$

[3] A *Priestley space* is defined as a compact ordered topological space $\langle X, \tau, \leq \rangle$ such that, for all $x, y \in X$, if $x \not\leq y$, then there is a clopen up-set $U \subseteq X$ with $x \in U$ and $y \notin U$. It follows that $\langle X, \tau \rangle$ is a Stone space.

One shows that these are the components of a natural transformation from the identity functor on **DLat** to $L \cdot X$, and from the identity functor on **PrieSp** to $X \cdot L$, respectively, satisfying the required diagrams for an adjunction. In particular, they are morphisms in their respective categories. Furthermore, they are *isomorphisms* and thus the central result of Priestley duality is obtained: *The categories* **DLat** *and* **PrieSp** *are dually equivalent.*

All dualities in the rest of this paper concern bounded distributive lattices with additional structure. In each case, the functors X and L are defined as above, and likewise for the units Φ and Ψ.

2.2. De Morgan and Kleene algebras

The duality theory for bounded De Morgan lattices[4] was developed in [7, 8], to which we refer for all proofs and further details.

Let $\mathbf{A} = \langle A, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}\rangle$ be a bounded De Morgan lattice and let $X(\mathbf{A})$ be the set of prime filters of $\mathbf{A}$. For any $P \in X(\mathbf{A})$, define

$$\neg P := \{a \in A : \neg a \in P\}.$$

For any $P \in X(\mathbf{A})$, we have that $\neg P$ is a prime ideal. So, defining

$$g(P) := A \backslash \neg P$$

we have that $g(P)$ is a prime filter. It is then easy to check that the map $g \colon X(\mathbf{A}) \to X(\mathbf{A})$ is an order-reversing involution on the poset $X(\mathbf{A})$, i.e., that $g^2 = id_{X(\mathbf{A})}$ and, for all $P, Q \in X(\mathbf{A})$,

$$P \subseteq Q \quad \text{iff} \quad g(Q) \subseteq g(P).$$

If we endow $X(\mathbf{A})$ with the Priestley topology, we have that the structure $\langle X(\mathbf{A}), \tau, \subseteq, g\rangle$ is a *De Morgan space*, defined as follows:

Definition 2.1. A *De Morgan space* is a structure $\mathcal{X} = \langle X, \tau, \leq, g\rangle$ where $\langle X, \tau, \leq\rangle$ is a Priestley space and $g \colon \mathcal{X} \to \mathcal{X}$ is an order-reversing homeomorphism such that $g^2 = id_{\mathcal{X}}$.

Conversely, given a De Morgan space $\mathcal{X} = \langle X, \tau, \leq, g\rangle$, one defines on the Priestley dual $\langle L(\mathcal{X}), \cap, \cup, \emptyset, X\rangle$ an operation $\neg$ as follows. For any $U \in L(\mathcal{X})$,

$$\neg U := X \setminus g[U].$$

---

[4] We use "bounded De Morgan lattice" and "De Morgan algebra" interchangeably.

One obtains that $\langle L(\mathcal{X}), \cap, \cup, \neg, \emptyset, X\rangle$ is a De Morgan algebra. One also shows that the unit maps $\Phi_{\mathbf{A}}$ preserve negation and thus that they are De Morgan algebra isomorphisms.

On the side of the spaces, one defines a *De Morgan function* $f\colon \mathcal{X} \to \mathcal{X}'$ to be a Priestley function for which $f \cdot g = g' \cdot f$. One then shows that the unit maps $\Psi_{\mathcal{X}}$ are in fact De Morgan functions and hence De Morgan isomorphisms (since the extra structure g can be viewed as a unary algebraic operation).

Thus one arrives at the result that *the category of De Morgan algebras and homomorphisms is dually equivalent to the category of De Morgan spaces and De Morgan functions*.

This duality specializes to one between the full subcategories of Kleene algebras (i.e., De Morgan lattices satisfying $x \wedge \neg x \leq y \vee \neg y$), and Kleene spaces, defined as follows. Given a De Morgan space $\langle X, \tau, \leq, g\rangle$, consider the sets

$$X^+ := \{x \in X : x \leq g(x)\}, \qquad X^- := \{x \in X : g(x) \leq x\}.$$

A *Kleene space* is then defined as a De Morgan space $\langle X, \tau, \leq, g\rangle$ such that $X = X^+ \cup X^-$.

Specialising Kleene algebras further to Boolean algebras one obtains the classical Stone duality by insisting that g be the identity.

2.3. N4-lattices

By definition, any bounded N4-lattice has a De Morgan algebra reduct. The duality theory for N4-lattices can thus be developed building on the theory for De Morgan algebras, as Odintsov does in [20], to which we refer for all proofs and further details.

Let $\langle A, \wedge, \vee, \supset, \neg\rangle$ be an N4-lattice. Using the pre-order relation introduced in Definition 1.17 (ii), we can identify special lattice filters that play an important role in the duality theory for these algebras. A subset $F \subseteq A$ is called a *special filter of the first kind (sffk)* if, for all $a, b \in A$,

1. $a, b \in F$ imply $a \wedge b \in F$
2. $a \in F$ and $a \preceq b$ imply $b \in F$.

$F \subseteq A$ is a *special filter of the second kind (sfsk)* if, for all $a, b \in A$,

1. $a, b \in F$ imply $a \wedge b \in F$
2. $a \in F$ and $\neg b \preceq \neg a$ imply $b \in F$.

Any special filter (of either kind) is a lattice filter, but not every lattice filter is a special filter.

Given the above-mentioned relationship between N4-lattices and Brouwerian bilattices, it may be worth noticing that special filters of the first kind correspond to bifilters of Brouwerian bilattices and special filters of the second kind correspond to filter-ideals of Brouwerian bilattices. In fact, if an N4-lattice is the reduct of a Brouwerian bilattice, then sffk coincide with bifilters and sfsk coincide with filter-ideals (this can be easily proved using [20, Corollary 2.1] together with our Proposition 1.9).

Any *prime* lattice filter is either a *sffk* or a *sfsk*. Thus, for any N4-lattice $\mathbf{A}$, we have that

$$X(\mathbf{A}) = X^1(\mathbf{A}) \cup X^2(\mathbf{A})$$

where $X^1(\mathbf{A})$ denotes the set of prime filters of the first kind and $X^2(\mathbf{A})$ the set of prime filters of the second kind. As we have seen with Kleene algebras, we can define

$$X^+(\mathbf{A}) := \{P \in X(\mathbf{A}) : P \subseteq g(P)\}$$
$$X^-(\mathbf{A}) := \{P \in X(\mathbf{A}) : g(P) \subseteq P\}.$$

In an N4-lattice it holds that:

1. $\langle X(\mathbf{A}), \tau, \subseteq, g\rangle$ is a De Morgan space
2. $g[X^1(\mathbf{A})] = X^2(\mathbf{A})$
3. $X(\mathbf{A}) = X^1(\mathbf{A}) \cup X^2(\mathbf{A})$ and $X^1(\mathbf{A}) \cap X^2(\mathbf{A}) = X^+(\mathbf{A}) \cap X^-(\mathbf{A})$
4. $X^1(\mathbf{A})$ is closed in τ and $X^1(\mathbf{A})$ with the induced topology is an Esakia space
5. for any $P \in X^1(\mathbf{A})$ and $Q \in X^2(\mathbf{A})$, if $P \subseteq Q$, then $P \in X^+(\mathbf{A})$, $Q \in X^-(\mathbf{A})$ and there exists $R \in X(\mathbf{A})$ such that $P, g[Q] \subseteq R \subseteq g[P], Q$
6. for any $P \in X^2(\mathbf{A})$ and $Q \in X^1(\mathbf{A})$, if $P \subseteq Q$, then $P \in X^+(\mathbf{A})$, $Q \in X^-(\mathbf{A})$ and $P \subseteq g[Q]$.

Recall that an *Esakia space* (also known as *Heyting space*) is a Priestley space such that, for any open set O, the downset $O{\downarrow}$ is also open [11, 22].

In [20], Odintsov defines an *N4-space* to be a tuple $\mathcal{X} = \langle X, X^1, \tau, \leq, g\rangle$ such that properties (1) to (6) are satisfied.

Given an N4-space $\langle X, X^1, \tau, \leq, g\rangle$, the structure $\langle L(\mathcal{X}), \cap, \cup, \neg, \emptyset, X\rangle$ defined as before is a De Morgan algebra. One defines an implication operation $\supset : L(\mathcal{X}) \times L(\mathcal{X}) \to L(\mathcal{X})$ as follows: for any $U, V \in L(\mathcal{X})$,

$$U \supset V \ := \ \big(X^1 \setminus ((U \setminus V) \cap X^1){\downarrow} \,\big) \cup \big(X^2 \setminus (g[U] \setminus V) \,\big).$$

One then shows that $\langle L(\mathcal{X}), \cap, \cup, \supset, \neg, \emptyset, X\rangle$ is a bounded N4-lattice.

To complete the picture, one defines an *N4-function* to be a function f between N4-spaces $\langle X, X^1, \tau, \leq, g\rangle$ and $\langle Y, Y^1, \tau', \leq', g'\rangle$ which satisfies:

1. f is a De Morgan function from $\langle X, \tau, \leq, g\rangle$ to $\langle Y, \tau', \leq', g'\rangle$
2. $f[X^1] \subseteq Y^1$
3. $f\colon X^1 \to Y$ is an Esakia function, i.e., for any open $O \in \tau'$,

$$f^{-1}[(O \cap Y^1){\downarrow}] \cap X^1 = (f^{-1}[O \cap Y^1]){\downarrow} \cap X^1.$$

With these definitions, Odintsov [20, Theorem 5.4] obtains that *the category of bounded N4-lattices with homomorphisms is dually equivalent to the category of N4-spaces with N4-functions.*

3. Bilattice dualities

3.1. Duality for pre-bilattices

In this section we refine Priestley duality to obtain a duality for bounded distributive pre-bilattices. Recall that — in the spirit of this paper — we represent bounded distributive pre-bilattices by their t-lattice reduct, augmented with a complemented pair. As stated in Theorem 1.6, the two concepts are algebraically equivalent, and we denote the resulting category by **DPreBiLat**.

Let $\mathbf{B} = \langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ be a bounded distributive lattice with complemented pair. As before, we denote by $X(\mathbf{B})$ the set of prime filters of the t-lattice $\langle B, \wedge, \vee, \mathsf{f}, \mathsf{t}\rangle$. By Theorem 1.12, a prime t-filter $P \in X(\mathbf{B})$ is either a prime k-filter (if $\top \in P$) or a prime k-ideal (if $\bot \in P$), that is, either a prime bifilter or a prime filter-ideal. Then, consistently with the notation used in the previous section for N4-lattices, we set

$$X^1(\mathbf{B}) := \{P \in X(\mathbf{B}) : P \text{ is a k-filter}\} = \{P \in X(\mathbf{B}) : \top \in P\}$$
$$X^2(\mathbf{B}) := \{P \in X(\mathbf{B}) : P \text{ is a k-ideal}\} = \{P \in X(\mathbf{B}) : \bot \in P\}.$$

We have then that

$$X^1(\mathbf{B}) \cap X^2(\mathbf{B}) = \emptyset \quad \text{and} \quad X^1(\mathbf{B}) \cup X^2(\mathbf{B}) = X(\mathbf{B}).$$

This tells us how the usual definitions of Priestley duality should be extended:

Definition 3.1. A *Priestley bispace* is a tuple $\mathcal{X} = \langle X, X^1, X^2, \tau, \leq\rangle$ such that:

(i) $\langle X, \tau, \leq\rangle$ is a Priestley space

(ii) $X^1, X^2 \subseteq X$ are clopen up-sets

(iii) $X^1 \cap X^2 = \emptyset$ and $X^1 \cup X^2 = X$.

A map f between Priestley bispaces $\langle X, X^1, X^2, \tau, \leq\rangle$ and $\langle Y, Y^1, Y^2, \tau', \leq'\rangle$ is called a *Priestley bifunction* if:

(i) f is continuous and order-preserving

(ii) $f[X^1] \subseteq Y^1$ and $f[X^2] \subseteq Y^2$.

We denote the category of Priestley bispaces and Priestley bifunctions by **PrieBiSp**.

We are ready to adjust the Priestley functor X to one from **DPreBiLat** to **PrieBiSp**. On objects, we assign to a bounded distributive lattice with complemented pair $\mathbf{B}$ the structure $\langle X(\mathbf{B}), X^1(\mathbf{B}), X^2(\mathbf{B}), \tau, \leq\rangle$, which we have seen to be a Priestley bispace. We must check that for h a homomorphism of bounded distributive lattices with complemented pair, $X(h)$ is indeed a Priestley bifunction: we know by Priestley duality that it is continuous and order-preserving; if P' is a bifilter in B', then by Lemma 1.11 it contains $\top'$, so $h^{-1}(P')$ contains $\top$ and therefore is a bifilter of B. The same simple argument shows that $X(h)$ maps filter-ideals to filter-ideals.

For a functor L in the other direction, let $\mathcal{X}$ be a Priestley bispace. Then let $L(\mathcal{X})$ be the bounded distributive lattice of clopen up-sets augmented with X^1 and X^2 for the complemented pair. Given a Priestley bifunction $f\colon \mathcal{X} \to \mathcal{Y}$, it is clear that $L(f)$ preserves the complemented pair and everything else works as in the classical case.

To complete the picture we check that the units are morphisms in their respective categories:

Proposition 3.2. *For any Priestley bispace $\mathcal{X} = \langle X, X^1, X^2, \tau, \leq\rangle$, the map $\Psi_{\mathcal{X}}\colon \mathcal{X} \to X(L(\mathcal{X}))$, $\Psi_{\mathcal{X}}(x) := \{U \in L(\mathcal{X}) : x \in U\}$, is a* **PrieBiSp** *isomorphism between $\mathcal{X}$ and $X(L(\mathcal{X}))$.*

Proof. We know from Priestley duality theory that $\Psi_{\mathcal{X}}$ is a homeomorphism between the underlying Priestley spaces of $\mathcal{X}$ and $X(L(\mathcal{X}))$, and an order isomorphism between $\langle X, \leq\rangle$ and $\langle X(L(\mathcal{X})), \subseteq\rangle$. Thus, we only need to show that $\Psi_{\mathcal{X}}[X^1] \subseteq X^1(L(\mathcal{X}))$ and $\Psi_{\mathcal{X}}[X^2] \subseteq X^2(L(\mathcal{X}))$. The analogous conditions for $\Psi_{\mathcal{X}}^{-1}$ follow automatically.

To prove that $\Psi_{\mathcal{X}}[X^1] \subseteq X^1(L(\mathcal{X}))$ assume $x \in X^1$. By Priestley duality we know that $\Psi_{\mathcal{X}}(x)$ is a prime t-filter of $L(\mathcal{X})$. Since X^1 is a clopen up-set, we have that $X^1 \in \Psi_{\mathcal{X}}(x)$ and, by Lemma 1.11 (i) we can conclude that $\Psi_{\mathcal{X}}(x)$ is a prime bifilter. The same reasoning shows that if $x \in X^2$ then $\Psi_{\mathcal{X}}(x)$ is a prime filter-ideal, that is, $\Psi_{\mathcal{X}}(x) \in X^2(L(\mathcal{X}))$. ■

On the algebraic side, too, we define the unit maps $\Phi_{\mathbf{B}} \colon \mathbf{B} \to L(X(\mathbf{B}))$ as usual: $\Phi_{\mathbf{B}}(a) := \{P \in X(\mathbf{B}) : a \in P\}$.

It is immediate from the definitions and Lemma 1.11 that $\Phi_{\mathbf{B}}(\top) = X^1(\mathbf{B})$ and $\Phi_{\mathbf{B}}(\bot) = X^2(\mathbf{B})$, and therefore that $\Phi_{\mathbf{B}}$ preserves the complemented pair. Since this is an algebraic condition, its inverse (which exists by Priestley duality) is a **DPreBiLat** morphism as well. Hence we have:

PROPOSITION 3.3. *For any bounded distributive lattice with complemented pair* $\mathbf{B}$ *we have that, in* **DPreBiLat**,

$$\Phi_{\mathbf{B}} \colon \mathbf{B} \cong \langle L(X(\mathbf{B})), \cap, \cup, \emptyset, X(\mathbf{B}), X^1(\mathbf{B}), X^2(\mathbf{B})\rangle.$$

Reverting back to the algebraically equivalent language of pre-bilattices, we conclude and summarize:

THEOREM 3.4. *The category of bounded distributive pre-bilattices* **DPreBiLat** *is dually equivalent to the category of Priestley bispaces* **PrieBiSp** *via the functors* X *and* L.

3.2. Duality for bilattices

We are now going to amalgamate the results obtained so far with those for De Morgan algebras in Section 2.2 to obtain a duality result for bounded distributive bilattices, exploiting that these are represented equivalently as De Morgan algebras with complemented pair $\langle B, \wedge, \vee, \neg, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$. We denote the corresponding category by **DBiLat**.

For a duality, we employ the construction introduced for De Morgan algebras, defining, for any prime t-filter $P \in X(\mathbf{B})$,

$$\neg P := \{\neg a : a \in P\} \quad \text{and} \quad g(P) := B \setminus \neg P.$$

We know from the theory of De Morgan algebras that $\langle X(\mathbf{B}), \tau, \subseteq, g\rangle$ is a De Morgan space, which implies that $g \colon X(\mathbf{B}) \to X(\mathbf{B})$ is an order-reversing bijection on the poset $X(\mathbf{B})$. Using Lemma 1.11, it is also easy to check that $P \in X^1(\mathbf{B})$ if and only if $g(P) \in X^2(\mathbf{B})$ and vice-versa. For instance, if $\top \in P$, then $\neg\top = \top \in \neg P$. Thus, $\top \notin g(P)$, which implies, by primeness of $g(P)$, that $\perp \in g(P)$. Therefore, we have that $g\colon X^1(\mathbf{B}) \to X^2(\mathbf{B})$ is an order-reversing isomorphism between the two posets.

The above considerations also imply that $\langle X(\mathbf{B}), \tau, \subseteq, g\rangle$ cannot be a Kleene space. In fact, this would imply that either $P \subseteq g(P)$ or $g(P) \subseteq P$ for any $P \in X(\mathbf{B})$. But this is absurd, because a prime bifilter cannot be contained in a prime k-ideal (otherwise it would not be proper) and vice versa.

We have therefore that, for any bounded distributive bilattice $\mathbf{B}$,

(i) $\langle X(\mathbf{B}), \tau, \subseteq, g\rangle$ is a De Morgan space

(ii) $\langle X(\mathbf{B}), X^1(\mathbf{B}), X^2(\mathbf{B}), \tau, \subseteq\rangle$ is a Priestley bispace

(iii) $g[X^1(\mathbf{B})] = X^2(\mathbf{B})$.

We take these three properties as our definition of the spaces dual to bounded distributive bilattices:

Definition 3.5. A *De Morgan bispace* is a tuple $\mathcal{X} = \langle X, X^1, X^2, \tau, \leq, g\rangle$ such that:

(i) $\langle X, \tau, \leq, g\rangle$ is a De Morgan space

(ii) $\langle X, X^1, X^2, \tau, \leq\rangle$ is a Priestley bispace

(iii) $g[X^1] = X^2$.

For a morphism f we require that it is a Priestley bifunction and a De Morgan function. We denote the resulting category by **DMBiSp**.

Notice that, for any De Morgan bispace $\mathcal{X}$, it holds that $X^+ = X^- = \emptyset$. To see this, assume, for instance, $x \in X^+$ for some $x \in X^1$. Then $x \leq g(x)$ and, since X^1 is an up-set with respect to the Priestley order, $g(x) \in X^1$. But by definition we have that $g[X^1] = X^2$, so $g(x) \in X^2$ and this contradicts the condition $X^1 \cap X^2 = \emptyset$ of Priestley bispaces.

From the definition it follows that, for any De Morgan bispace $\mathcal{X}$, the algebra $\langle L(\mathcal{X}), \cap, \cup, \emptyset, X, X^1, X^2\rangle$ is a bounded distributive pre-bilattice and $\langle L(\mathcal{X}), \cap, \cup, \neg, \emptyset, X\rangle$ is a De Morgan algebra where the negation is defined as in Section 2.2. Thus, we immediately have the following result.

Proposition 3.6. *The algebra $\langle L(\mathcal{X}), \cap, \cup, \neg, \emptyset, X, X^1, X^2\rangle$ is a bounded distributive bilattice for any De Morgan bispace $\mathcal{X}$.*

We also know that for a De Morgan bifunction h, $L(h)$ is a morphism of De Morgan algebras preserving the complemented pair, in other words, it is a bounded bilattice homomorphism. Thus we have that L is a contravariant functor from **DMBiSp** to **DBiLat**.

Combining in a similar way the results for pre-bilattices above and for De Morgan algebras in Section 2.2, we obtain that X, restricted to **DBiLat**, is a contravariant functor into **DMBiSp**.

The proofs that the units are homomorphisms in their respective categories are now based on the duality of De Morgan algebras rather than Priestley duality, but are otherwise the same as those for pre-bilattices, propositions 3.2 and 3.3.

Thus we may conclude:

THEOREM 3.7. *The category of bounded distributive bilattices* **DBiLat** *is dually equivalent to the category of De Morgan bispaces* **DMBiSp** *via the functors X and L.*

3.3. Duality for bilattices with conflation

Recall from Section 1.6 that the algebraic structure of bilattices with conflation can equivalently be captured by De Morgan algebras augmented with a complemented pair and a monotone involution "$-$" satisfying $-\bot = \top$. We denote the corresponding category by **DBiLatCon** and seek a duality that extends that of bounded distributed bilattices exhibited in the previous section. The task, obviously, is to capture the involution, so we begin by analysing its interaction with prime filters.

Let $\mathbf{B} = \langle B, \wedge, \vee, \neg, -, \mathsf{f}, \mathsf{t}, \bot, \top\rangle$ be a bounded distributive bilattice with conflation and let P be a prime filter of $\mathbf{B}$. Define

$$k(P) := \{a \in B : -a \in P\}.$$

It is easy to check that $k(P)$ is again a prime filter. Therefore, it is clear that the map $k\colon X(\mathbf{B}) \to X(\mathbf{B})$ is an order-preserving involution on the poset $X(\mathbf{B})$. Note also that it is a homeomorphism with respect to the usual Priestley topology on $X(\mathbf{B})$.

Since $\top \in P$ if and only if $-\top = \bot \in -P$, we have that $P \in X^1(\mathbf{B})$ if and only if $k(P) \in X^2(\mathbf{B})$. Thus, $k\colon X^1(\mathbf{B}) \to X^2(\mathbf{B})$ is an order-preserving isomorphism between the two posets.

The following definition is meant to capture these properties.

DEFINITION 3.8. A tuple $\mathcal{X} = \langle X, X^1, X^2, \tau, \leq, g, k\rangle$ is called a *conflation bispace* if:

(i) $\langle X, X^1, X^2, \tau, \leq, g\rangle$ is a De Morgan bispace

(ii) $k\colon X \to X$ is a continuous and order-preserving endomap satisfying $k^2 = \mathrm{id}_{\mathcal{X}}$ and $k(X^1) = X^2$.

A map between two such structures is called a *conflation bifunction* if it is a De Morgan bifunction commuting with the endomaps, that is, $f \cdot k = k' \cdot f$. The resulting category is denoted by **ConBiSp**.

Leading up to this definition we have presented everything necessary to show that X maps the objects of **DBiLatCon** to **ConBiSp**, and it remains to check the morphisms.

PROPOSITION 3.9. *Let $h\colon B \to B'$ be a homomorphism between bounded distributive bilattices with conflation $\mathbf{B}$ and $\mathbf{B}'$. Then $X(h)\colon X(\mathbf{B}') \to X(\mathbf{B})$ is a conflation bifunction.*

PROOF. Only the interaction with the endomaps needs to be checked, for which we compute: $X(h) \cdot k'(P) = h^{-1}[k'(P)] = h^{-1}[\{a \in B' : -'a \in P\}] = \{b \in B : -'h(b) \in P\} = \{b \in B : h(-b) \in P\} = \{-b \in B : h(b) \in P\} = k[\{b \in B : h(b) \in P\}] = k \cdot h^{-1}[P] = k \cdot X(h)(P)$. ∎

Let us now check whether L is a functor in the other direction. Definition 3.8 (i), together with the results of the previous section, implies that, for any conflation bispace $\mathcal{X}$, the algebra $\langle L(\mathcal{X}), \cap, \cup, \neg, \emptyset, X, X^1, X^2\rangle$ is a bounded distributive bilattice. For any $U \in L(\mathcal{X})$, we define

$$-U := \{k(x) : x \in U\}$$

which is guaranteed to yield another element of $L(\mathcal{X})$. It is also easy to check that the map $-\colon L(\mathcal{X}) \to L(\mathcal{X})$ satisfies conditions **(conf 1')** to **(conf 3)** introduced in Definition 1.13 and Section 1.6. Therefore we have the following for the object part of L:

PROPOSITION 3.10. *For any conflation bispace $\mathcal{X}$, we have that he algebra $\langle L(\mathcal{X}), \cap, \cup, \neg, -, \emptyset, X, X^1, X^2\rangle$ is a bounded distributive bilattice with conflation (more precisely, but equivalently, a De Morgan algebra with conflation).*

For the morphisms we show:

PROPOSITION 3.11. *Let $f\colon X \to Y$ be a conflation bifunction between two conflation bispaces $\mathcal{X}$ and $\mathcal{Y}$. Then $L(f)\colon L(\mathcal{Y}) \to L(\mathcal{X})$ is a homomorphism of bounded bilattices with conflation.*

PROOF. By the results of the previous section we already know that $L(f)$ is a bounded bilattice homomorphism. It remains to show that, for all

$U \in L(\mathcal{Y})$: $L(f)(-' U) = -\, L(f)(U)$, where $-$ and $-'$ denote the conflation operation in $L(\mathcal{X})$ and $L(\mathcal{Y})$, respectively. Applying the definitions, we have that $x \in L(f)(-' U) = f^{-1}[-' U]$ iff $f(x) \in -' U$ iff $k' \cdot f(x) \in U$. By assumption $f \cdot k = k' \cdot f$, so we have that $k' \cdot f(x) \in U$ iff $f \cdot k(x) \in U$ iff $k(x) \in f^{-1}[U] = L(f)(U)$ iff $x \in -\, L(f)(U)$. We have thus proved that $L(f)$ preserves the conflation operation. ■

Next we check that the units $\Phi_{\mathbf{B}}$ and $\Psi_{\mathcal{X}}$, defined as before, preserve the extra structure of their respective categories. For $\Phi_{\mathbf{B}}$ this is straightforward; one shows that, for any $a \in B$: $\Phi_{\mathbf{B}}(-\, a) = -\, \Phi_{\mathbf{B}}(a)$.

For $\Psi_{\mathcal{X}}$ we show:

PROPOSITION 3.12. *For a conflation bispace $\mathcal{X}$, the map $\Psi_{\mathcal{X}} \colon \mathcal{X} \to X(L(\mathcal{X}))$ is a conflation bifunction.*

PROOF. Let $\mathcal{X} = \langle X, X^1, X^2, \tau, \leq, g, k\rangle$. We know from the previous section that $\Psi_{\mathcal{X}}$ is a morphism between the underlying De Morgan bispaces $\langle X, X^1, X^2, \tau, \leq, g\rangle$ and $\langle X(L(\mathcal{X})), X^1(L(\mathcal{X})), X^2(L(\mathcal{X})), \tau', \subseteq, g'\rangle$. So, we only need to show that $\Psi_{\mathcal{X}} \cdot k = k' \cdot \Psi_{\mathcal{X}}$. Let then $x \in X$ and notice that, since k is involutive, we have that $x \in -\, U$ if and only if $k(x) \in U$ for all $U \in L(\mathcal{X})$. Then,

$$\begin{array}{rcl} \Psi_{\mathcal{X}} \cdot k(x) & = & \{U \in L(\mathcal{X}) : k(x) \in U\} \\ & = & \{U \in L(\mathcal{X}) : x \in -\, U\}] \\ & = & k'[\{U \in L(\mathcal{X}) : x \in U\}] \\ & = & k' \cdot \Psi_{\mathcal{X}}(x). \end{array}$$

■

Thus, we have:

THEOREM 3.13. *The category of bounded distributive bilattices with conflation* **DBiLatCon** *is dually equivalent to the category of conflation bispaces* **ConBiSp** *via the functors X and L.*

We end this section with some considerations on how the above results specialize to the case of bounded commutative distributive bilattices with conflation. As we have seen in Theorem 1.14, any algebra in this variety can be represented as a product $\mathbf{L} \odot \mathbf{L}$, where $\mathbf{L}$ is a De Morgan algebra. Then, we know by Theorem 1.10 that the subspaces $X^1(\mathbf{B})$ and $X^2(\mathbf{B})$ are homeomorphic (as Priestley spaces) to $X(\mathbf{L})$. Also, as noted in Section 1.4, the algebra $\langle B, \wedge, \vee, \neg \cdot -, \mathsf{f}, \mathsf{t}\rangle$ is itself a De Morgan algebra in which the involution operation is given by the composition of negation and conflation.

In this case, the function $g \cdot k \colon X(\mathbf{B}) \to X(\mathbf{B})$ is also an order-reversing involutive homeomorphism and we have that $\langle X(\mathbf{B}), \tau, \subseteq, g \cdot k \rangle$ is a De Morgan space. The subspaces $\langle X^1(\mathbf{B}), \tau, \subseteq, g \cdot k \rangle$ and $\langle X^2(\mathbf{B}), \tau, \subseteq, g \cdot k \rangle$ are also De Morgan spaces homeomorphic to the De Morgan space associated with $\mathbf{L}$.

If $\mathbf{B} = \mathbf{L} \odot \mathbf{L}$ is a Kleene bilattice, then $\langle X(\mathbf{B}), \tau, \subseteq, g \cdot k \rangle$ is a Kleene space and, as in the previous case, $X^1(\mathbf{B})$, $X^2(\mathbf{B})$ and $X(\mathbf{L})$ are homeomorphic Kleene spaces. Similarly, if $\mathbf{B}$ is a classical bilattice, then $X^1(\mathbf{B})$, $X^2(\mathbf{B})$ and $X(\mathbf{L})$ are isomorphic Stone spaces and $\langle X(\mathbf{B}), \tau, \subseteq, g \cdot k \rangle$ is a Stone space as well.

It is not difficult to see that the categorical duality results proved in this section specialize to the full subcategories of commutative bilattices with conflation, Kleene and classical bilattices.

3.4. Duality for Brouwerian bilattices

In this section we present a duality for the category **BrBiLat** of bounded Brouwerian bilattices based on Odintsov's duality for N4-lattices, recalled in Section 2.3. As mentioned at the end of Section 1.6, we view bounded Brouwerian bilattices as bounded N4-lattices having a complemented pair $\top, \bot$ satisfying conditions **(neg 2')**, **(imp** $\top$**)** and **(imp** $\bot$**)**. As noted there, a function $h \colon \mathbf{B} \to \mathbf{B}'$ between two bounded Brouwerian bilattices $\mathbf{B}, \mathbf{B}'$ is a homomorphism if and only if h is a bounded N4-lattice homomorphism between the N4-lattice reducts of $\mathbf{B}$ and $\mathbf{B}'$. The upshot of this is that whereas in the previous cases we *extended* the classical dualities, we are here *specializing* to an equationally defined *full* subcategory, and as long as the dualities work for the objects, the morphisms will take care of themselves.

Given a bounded Brouwerian bilattice $\mathbf{B} = \langle B, \wedge, \vee, \supset, \neg, \mathsf{f}, \mathsf{t}, \bot, \top \rangle$, we know that $\langle X(\mathbf{B}), X^1(\mathbf{B}), \tau, \subseteq, g \rangle$ is an N4-space and, moreover, that $\langle X(\mathbf{B}), X^1(\mathbf{B}), X^2(\mathbf{B}), \tau, \subseteq, g \rangle$ is a De Morgan bispace. We could take these two conditions as our definition of Brouwerian bispace, but the following simpler definition will be enough.

Definition 3.14. A tuple $\mathcal{X} = \langle X, X^1, X^2, \tau, \leq, g \rangle$ is a *Brouwerian bispace* if $\langle X, X^1, X^2, \tau, \leq, g \rangle$ is a De Morgan bispace and X^1 with the induced topology is an Esakia space.
A map between two such structures is called a *Brouwerian bifunction* if it is an N4-function between the corresponding N4-spaces $\langle X, X^1, \tau, \leq, g \rangle$ and $\langle Y, Y^1, \tau', \leq', g' \rangle$. The resulting category is denoted by **BrBiSp**.

Some comments on this definition are in order. Given a Brouwerian bispace $\mathcal{X} = \langle X, X^1, X^2, \tau, \leq, g \rangle$, it is easy to check that $\langle X, X^1, \tau, \leq, g \rangle$

is an N4-space. Let us take a look at properties (1) to (6) that define N4-spaces (Section 2.3). We have that (1), (2) and (4) are satisfied by definition. As to (3), a stronger property holds for De Morgan bispaces, namely that $X^1 \cap X^2 = X^+ = X^- = \emptyset$. Finally, conditions (5) and (6) are trivially satisfied, since for any $x \in X^1$ and $y \in X^2$ it holds that $x \not\leq y$ and $y \not\leq x$.

Regarding the maps it may be surprising that the component X^2 is ignored. Note, however, that $f[X^1] \subseteq Y^1$ is part of the definition of an N4-function. Furthermore, f is required to respect the De Morgan structure $\langle X, \tau, \leq, g\rangle$ and since $g[X^1] = X^2$ in a De Morgan bispace, we get $f[X^2] \subseteq Y^2$ for free. So analogously to the algebraic side, the spaces form a full subcategory of that of N4-spaces, characterized by equation-like conditions.

We know from [20] that the structure $\langle L(\mathcal{X}), \cap, \cup, \supset, \neg, \emptyset, X\rangle$, with the implication $\supset$ defined as in Section 2.3, is a bounded N4-lattice. Moreover, we have seen in Section 3.1 that $\langle L(\mathcal{X}), \cap, \cup, \neg, \emptyset, X, X^1, X^2\rangle$ is a bounded distributive bilattice. Hence, by Theorem 1.19, we only need to check that **(imp $\top$)** and **(imp $\bot$)** are satisfied in order to be able to conclude that it is, in fact, a Brouwerian bilattice.

Proposition 3.15. *For any Brouwerian bispace $\mathcal{X}$, we have that the algebra $\langle L(\mathcal{X}), \cap, \cup, \supset, \neg, -, \emptyset, X, X^1, X^2\rangle$ is a bounded Brouwerian bilattice.*

Proof. Applying the definition of $\supset$, for **(imp $\top$)** we need to check that $X^1 \supset \emptyset = \emptyset$. We have

$$\begin{aligned} X^1 \supset \emptyset &= \big(X^1 \setminus ((X^1 \backslash \emptyset) \cap X^1){\downarrow} \big) \cup \big(X^2 \setminus (g[X^1] \backslash \emptyset) \big) \\ &= \big(X^1 \setminus X^1{\downarrow} \big) \cup \big(X^2 \setminus X^2 \big) \\ &= \emptyset \cup \emptyset = \emptyset. \end{aligned}$$

As for **(imp $\bot$)**, we have $\emptyset \supset \emptyset = \big(X^1 \setminus ((\emptyset \backslash \emptyset) \cap \emptyset){\downarrow} \big) \cup \big(X^2 \setminus (g[\emptyset] \backslash \emptyset) \big) = X^1 \cup X^2 = X$. ∎

Having established that the functors X and L between the categories of N4-lattices and N4-spaces restrict to bounded Brouwerian bilattices and Brouwerian bispaces, we can rely on the duality properties for the former to establish:

Theorem 3.16. *The category of bounded Brouwerian bilattices* **BrBiLat** *is dually equivalent to the category of Brouwerian bispaces* **BrBiSp** *via functors X and L.*

By Theorem 1.16, any bounded Brouwerian bilattice $\mathbf{B}$ is isomorphic to a product $\mathbf{L} \odot \mathbf{L}$, where $\mathbf{L}$ is a Heyting algebra. By Theorem 1.10, it is then clear that $X^1(\mathbf{B})$ and $X^2(\mathbf{B})$ are Esakia spaces homeomorphic to the Esakia space associated with $\mathbf{L}$. As observed at the end of Section 1.5, if $\mathbf{B}$ is a bounded classical implicative bilattice, then $\mathbf{L}$ is a Boolean algebra. Then we know that $X^1(\mathbf{B})$, $X^2(\mathbf{B})$ and $X(\mathbf{L})$ are homeomorphic Stone spaces. If we specialize Definition 3.14 (ii) by requiring that X^1 with the induced topology be a Stone space, then we obtain a class of "Stone bispaces" that correspond to bounded classical implicative bilattices. Again, we can restrict the existing duality to obtain a categorical duality for this case without further difficulties.

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