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An investigation into modal subintuitionistic logic

Abstract. Subintuitionistic logics are the weakenings of intuitionistic propositional calculus. While intuitionistic logic receives a lot of attention, the logics that are obtained from structural weakening, dropping the intuitionistic restrictions, which are persistence, reflexivity, and transitivity on the Kripke frames, are not widely studied. For the first time, we aim to enrich the language of subintuitionistic logic with modal operators. We take an approach to model-theoretic semantics by equipping the models with two relations $<*$ and R, where $<*$ acts like the accessibility relations in intuitionistic logics, and substructural logics, and R acts like the accessibility relations in modal logic. This way we can strengthen and weaken the logic both at the intuitionistic level, and the modal level by changing the conditions on the accessibility relation. We give a natural deduction system for these logics. Soundness and Completeness results are proven for the logics that are generated, and we prove that the logics that are generated are distinct. We observe that there does not seem to be good introduction, and elimination rules for the intuitionistic if-then connective in our logic that corresponds to Kripke frames without reflexivity, transitivity, and persistence using standard techniques of formalizing natural deduction. We propose a different deductive calculus for this logic by marking our formulas with "step-markers", either 0 or 1, where 0 indicates the formula is actually true, and 1 indicates the formula is true in every accessible world.

1 Introduction

There has been little research into the field of subintuitionistic logics, which can be viewed as weakenings of intuitionistic propositional calculus. In this paper, we study modal subintuitionistic logics both model-theoretically, and by giving a natural deduction system. A modal subintuitionistic logic is an attempt to capture how subintuitionistic propositional logic interacts with normal modal logics of various strengths. In this case, we examine the logics on modal frames that correspond to the K,T,4 axioms. We take an approach to the model-theoretic semantics by equipping the models with two relations $< *$, and R, where $< *$ acts like the accessibility relations in intuitionistic logics, and substructural logics, and R acts like the accessibility relations in modal logic. This way we can strengthen and weaken the logics both at the intuitionistic level, and the modal level by changing the conditions on the accessibility relation. Soundness, and Completeness results are proven for the logics that are generated, and we prove that the logics generated are distinct. We have two connectives for if-then $\supset^*$, which acts classically, and $\supset$, which acts intuitionistically. This way we are allowed to capture the if-then in normal modal logics, which has a classical character, and if-then connective in intuitionistic logics. We make the observation that subintuitionist logics do not seem to have nice introduction, and elimination rules for the intuitionistic if-then connective taking an ordinary approach to formalizing deductions. We aim to ease this problem by proposing new deductive rules through a process called "looking under the hood" of the logic introduced by Hodes [1], [2]. In this process, we develop a new logic SIK+ that has nice introduction and elimination rules for the intuitionistic if-then for subintuitionistic logics. We then extend the base logic SIK+ adding rules that correspond to Reflexivity, Transitivity, and Persistence: the usual restrictions on intuitionistic Kripke frames.

1: Model-Theoretic Semantics

1.1. Preliminaries

We have our logical constants $\mathscr{LC}=\{\vee,\wedge,\supset,\supset^*,\bot,\Box\}$, and give us a countable set of propositional variables **S**, and freely generate the formula as usual, and let **Fml** be the formula set.

DEFINITION 1.1. A *subintuitionistic frame* will be $\mathscr{F}=<W,R,< *>$, where R is meant to play the standard accessibility relation in modal logics, and $< *$ is meant to play the role of the orderings in sub-structural logics, and intuitionistic logics. W is a non-empty set of worlds, and $R \subseteq WxW$, and $< * \subseteq WxW$. We do not place any further restrictions on $< *$, and R at first, but we will strengthen our frames to obtain stronger logics later in the paper. It is worth noting that we need both $< *$, and R because we may want to strengthen our subintuitionistic base logic, and not the modal logic that connects to it, and vice versa.

DEFINITION 1.2. A model $\mathscr{M}$ is a quadruple $<\mathscr{F},V>$, where V is a function $V{:}S \rightarrow P(W)$ We define $\models$ inductively as usual.
$\mathscr{M}$,w $\models$ P iff w$\in$V(P) (where P is atomic)
$\mathscr{M}$,w $\models \theta \wedge \phi$ iff $\mathscr{M}$,w $\models \theta$ and $\mathscr{M}$,w $\models \phi$
$\mathscr{M}$,w $\models \theta \vee \phi$ iff $\mathscr{M}$,w $\models \theta$ or $\mathscr{M}$,w $\models \phi$
$\mathscr{M}$,w $\models \theta \supset \phi$ iff for every w', where w$< *$w', we have that $\mathscr{M}$,w $\models \theta$ implies $\mathscr{M}$,w* $\models \phi$
$\mathscr{M}$,w $\models \theta \supset^* \phi$ iff $\mathscr{M}$,w $\models \theta$ implies that $\mathscr{M}$,w $\models \phi$
$\mathscr{M}$,w $\models \Box\theta$ iff for every w', where wRw', we have that $\mathscr{M}$,w' $\models \theta$
$\mathscr{M}$,w $\not\models \bot$

The valuation funtion V can be extended in the natural way.
$V(\bot) = \emptyset$
$V(\phi\wedge\psi)=V(\phi)\cap V(\psi)$
$V(\phi\vee\psi)=V(\phi)\cup V(\psi)$
$V(\phi\supset\psi)=\{x\in W{:}\forall y(x< *y \wedge y\in V(\phi) \text{ implies } y\in V(\psi))\}$
$V(\phi\supset^*\psi)=\{x\in W{:}\ x \in V(\phi) \text{ implies } x \in V(\psi)\}$

V(□ϕ)={x∈W:∀y(xRy implies y∈ V(ϕ)}

DEFINITION 1.3. $\mathscr{M}$ is a *subintuitionsitic model* iff it is a model, and $\mathscr{F}$ is a subintuitionistic frame.

REMARK 1.1 We should note that we have two connectives for if-then: ⊃ which acts intuitionistically, and ⊃* which acts classically. For these semantics, we are trying to study enhancing subintuitionistic logic with modal logics. For the normal modal logics K, T, S4, etc., the if-then connective acts classically. A lot would be lost in the system if we did not have an idea of the character of classical if-then connective.

2: A Natural Deduction System

This approach to deductions can be found [1]. The following definitions are very similar to the ones found there.

DEFINITION 2.1. A string is a function in ω. Let $\frown$ be the concatenation of strings. For any string s_1, s_0 is an initial segment of s_1 iff for some s, s_1=$s_0$$\frown$s denote this s_0<′s_1. $\mathscr{T}$ is a naked tree iff $\mathscr{T}$ is a non-empty set of strings of natural numbers (1) closed under taking initial segments, or equivalently for any s∈$\mathscr{T}$ then if t<′s then t∈$\mathscr{T}$, and (2) for any m,n,s if s$\frown$[m]∈$\mathscr{T}$ then and n<m, we must have if s$\frown$[n]∈$\mathscr{T}$. We say that s is the leaf of a tree iff s∈$\mathscr{T}$ and for every s'∈$\mathscr{T}$ and s<′s' then s=s'. For a $\mathscr{T}$ that meets those conditions, we define $\mathscr{T}_s$={t|s$\frown$t∈T}.

DEFINITION 2.2. A labeled tree is a function whose domain is a naked tree. For every labeled tree $\mathscr{T}$, let $\mathscr{T}_s$ be the labeled tree with dom($\mathscr{T}$)$_s$ such that $\mathscr{T}_s$(t)= $\mathscr{T}$(s$\frown$t) for each of the t∈ dom($\mathscr{T}$)$_s$

DEFINITION 2.3. A tagged formula, which is a primitive type assignment, is a symbol of the form v:θ for θ∈Fml;v is the tag.

DEFINITION 2.4. C is a context iff C is a single-valued set of tagged formulas, which means that for any v,θ, and θ* if v:θ,v:θ*∈ C then θ is θ*. The dom(C) = the set of the variables occurring on the left-side of the members of C, and the ran(C)= the set of formulas occurring on the right-side of the members of C. We say a set of contexts is coherent iff the union of the contexts is coherent.

Approach to Deductions:

We follow Hodes in how he defined deductions [1]. We formally define our logic SIK inductively through a type assignment $\Longrightarrow_{SIK}$. But, the idea is that "C $\Longrightarrow_{SIK}$$\mathscr{D}$:$\theta$" means that relative to the context C, $\mathscr{D}$ is an SIK-deduction with the conclusion being θ. A deduction will be a labeled tree. The leaves of the dom($\mathscr{D}$) will be labeled by tagged formulas, and the non-leaves in dom($\mathscr{D}$)

will be labeled by a formula, or an ordered pair of a formula followed by a tagged formula, or an ordered triple of a formula followed by 2 tagged formulas. In the last 2 cases mentioned, the left-most is the formula label while the other components are supposed to indicate the discharging of tagged formulas at that string. We will let the metalanguage variables v_0, and v_1 to represent distinct variables. We will use "C,v:$\theta \implies_{SIK}\mathscr{D}$:$\theta$" abbreviate that v is not in dom(C) and either C∪{v:θ} $\implies_{SIK}\mathscr{D}$:$\theta$, and C $\implies_{SIK}\mathscr{D}$:$\theta$, and extend this to C,$v_1$,..,$v_n \implies_{SIK}\mathscr{D}$:$\theta$. We will represent the deductions pictorially, where when we discharge the tagged formula, we will represent that by putting square brackets around the label, and we will superscript the formula being discharged. Finally, we will define a dependency sets dpd($\mathscr{D}$) for $\mathscr{D}$, which is meant to have the informal meaning that the set of leaves of dom($\mathscr{D}$) on which $\mathscr{D}$ will depend along with $\implies_{SIK}$. Finally, we will abbreviate $\implies_{SIK}$ with $\implies$

Expanding previous systems:

Along with using Hodes approach to deductions, we note that our rules for subintuitionistic logic which will expand with rules for a previously proposed logic. Dösen has characterized a logic which takes as it set of theorems the formulas valid in every Kripke frame with the intuitionist meaning of the if-then connective. Although Dösen's logic has as its set of theorems the formulas valid in every Kripke frame with intuitionistic semantics, its entailment relation does not line up with the local consequence relation defined by the class of all Kripke frames. Furthermore, Celani weakens this logic [3] to obtain weak Dösen logic which does axiomitize the local consequence relationship of all Kripke frames. We give a natural deduction system rather than using the Hilbert system, and axioms, proposed by Celani. This system will act as our base subintuitionistic logic, in which we add minimal modal rules corresponding to the modal logic K, and also rules guiding the classical $\supset^*$. From there on we expand it by adding more axioms that correspond to strengthenings on the $< *$, and R relations in our Kripke frames.

Inductive definition of deduction:

Base case: For v∈Var, and $\theta\in$ **Fml**, v:$\theta \implies \mathscr{D}$:$\theta$ for $\mathscr{D}$={<[ ],v:θ>}, i.e, $\mathscr{D}$ is [ ] labeled by v:θ, and dpd($\mathscr{D}$)=dom($\mathscr{D}$)=[ ].

Inductive clauses:

∨**E)** If $C_2 \implies \mathscr{D}_2$:($\theta_0\vee\theta_1$), and for i∈{0,1}, C_i,v_i:$\theta_i \implies \mathscr{D}_i$;$\sigma$, where v_i∈Var, and we have {C_0,C_1,C_2} is coherent then the $C_0\cup C_2\cup C_3 \implies \mathscr{D}$:$\sigma$. We picture.

$$\dfrac{\begin{matrix} & v_0{:}\theta_0 & v_1{:}\theta_1 \\ \mathscr{D}_2 & \mathscr{D}_0 & \mathscr{D}_1 \\ (\theta_0\vee\theta_1) & \sigma & \sigma \end{matrix}}{\sigma^{v_0,v_1}}$$

The dpd($\mathscr{D}$)=$\bigcup_{i\in\{1,2\}}$ {[i+1] $\frown$ s | s $\in$ dpd($\mathscr{D}_i$), s is not v_i:θ_i}$\cup${[0]$\frown$ s| s$\in$dpd($\mathscr{D}$) }

$\vee$**I)** If i $\in$ {0,1}, C $\Longrightarrow$ $\mathscr{D}_0$:θ_i with $\theta_i$$\in$ Fml. Picture the deduction

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \theta_i\end{array}}{(\theta_1 \vee \theta_2)}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}

$\bot$**E)** If C $\Longrightarrow$ $\mathscr{D}_0$:$\bot$ then C $\Longrightarrow$ $\mathscr{D}$:θ. Then, we picture $\mathscr{D}$ as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \bot\end{array}}{\theta}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}

$\supset$**I)** If {},v:σ $\Longrightarrow$ $\mathscr{D}_0$:θ then {} $\Longrightarrow$ $\mathscr{D}$:(σ $\supset$$\theta$).
Note: We do not allow vacuous discharging. For example v:θ $\Longrightarrow$ $\tau \supset \theta$ is not valid with the tag q:τ
Then, we picture $\mathscr{D}$ as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \text{v: } \sigma\\ \theta\end{array}}{(\sigma\supset\theta)^v}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$) is not v:σ}

W) If C_0 $\Longrightarrow$ $\mathscr{D}_0$:$\theta\supset\phi$, and C_1 $\Longrightarrow$ $\mathscr{D}_1$:$\theta\supset\sigma$. Then, if we have that {C_0,C_1} is coherent then we have that $C_0$$\cup$$C_1$ $\Longrightarrow$ $\mathscr{D}$:$\theta\supset(\phi\wedge\sigma)$. We can picture it as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \theta\supset\phi\end{array} \qquad \begin{array}{c}\mathscr{D}_1\\ \theta\supset\sigma\end{array}}{\theta\supset(\phi \wedge \sigma)}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}$\cup$[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

$\supset$**-)** If C_0 $\Longrightarrow$ $\mathscr{D}_0$:$\theta\supset\phi$, and C_1 $\Longrightarrow$ $\mathscr{D}_1$:$\sigma\supset\phi$. Then, if we have that {C_0,C_1} is coherent then we have that $C_0$$\cup$$C_1$ $\Longrightarrow$ $\mathscr{D}$:$(\theta\vee\sigma)\supset\phi$. We can picture it as follows.

$$\begin{array}{c}\mathscr{D}_0\\ \theta\supset\phi\end{array} \qquad \begin{array}{c}\mathscr{D}_1\\ \sigma\supset\phi\end{array}$$

$$\frac{}{(\theta\vee\sigma)\supset\phi}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}$\cup$[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

$\supset$+) If $C_0 \Longrightarrow \mathscr{D}_0$:$\theta\supset\phi$, and $C_1 \Longrightarrow \mathscr{D}_1$:$\phi\supset\sigma$. Then, if we have that {C_0,C_1} is coherent then we have that $C_0\cup C_1 \Longrightarrow \mathscr{D}$:$\theta \supset\sigma$. We can picture it as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \theta\supset\phi\end{array} \qquad \begin{array}{c}\mathscr{D}_1\\ \phi\supset\sigma\end{array}}{(\theta \supset\sigma)}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}$\cup$[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

$\wedge$E) Let i$\in$ {1,2}If C $\Longrightarrow \mathscr{D}_0$:$\theta_0\wedge\theta_1$ then C $\Longrightarrow \mathscr{D}$:$\theta_i$. Then, we picture $\mathscr{D}$ as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ (\theta_0 \wedge \theta_1)\end{array}}{\theta_i}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}

$\wedge$I) If $C_0 \Longrightarrow \mathscr{D}_0$:$\theta$, and $C_1 \Longrightarrow \mathscr{D}_1$:$\phi$. Then, if we have that {C_0,C_1} is coherent then we have that $C_0\cup C_1 \Longrightarrow \mathscr{D}$:$\theta \wedge \phi$. We can picture it as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \theta\end{array} \qquad \begin{array}{c}\mathscr{D}_1\\ \phi\end{array}}{(\theta \wedge \phi)}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}$\cup$[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

MP) If $C_0 \Longrightarrow \mathscr{D}_0$:$\theta\supset^*\phi$, and $C_1 \Longrightarrow \mathscr{D}_1$:$\theta$. Then, if we have that {C_0,C_1} is coherent then we have that $C_0\cup C_1 \Longrightarrow \mathscr{D}$:$\phi$. We can picture it as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \theta\supset^*\phi\end{array} \qquad \begin{array}{c}\mathscr{D}_1\\ \theta\end{array}}{\phi}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}$\cup$[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

$\supset$I*) If C,v:$\sigma \Longrightarrow \mathscr{D}_0$:$\theta$ then C $\Longrightarrow \mathscr{D}$:($\sigma \supset^*\theta$). **Note:** here we allow vacuous discharging.
Then, we picture $\mathscr{D}$ as follows.

v:σ

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \theta\end{array}}{(\sigma\supset^*\theta)^v}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$) is not v:σ}

Modal rules:

Nec) If { } $\Longrightarrow \mathscr{D}_0$:$\theta$ then we have that { } $\Longrightarrow \mathscr{D}_0$: $\Box\theta$
Then, we picture $\mathscr{D}$ as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \theta\end{array}}{\Box\theta}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$)}

K)

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \Box(\theta\supset^*\phi)\end{array}\qquad\begin{array}{c}\mathscr{D}_1\\ \Box\theta\end{array}}{\Box\phi}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$)}∪[1] ⌢ s | s ∈ dpd($\mathscr{D}_1$)}

K*) If C $\Longrightarrow \mathscr{D}_0$:($\Box(\theta_1)\wedge$, . . . , $\wedge\Box(\theta_n)$) with $\theta_i\in$ Fml. then we have that C $\Longrightarrow \mathscr{D}$:$\Box(\theta_1\wedge$, . . , $\wedge\theta_n)$. Picture the deduction.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \Box(\theta_1)\wedge\ ,\ .\ .\ .\ ,\ \wedge\Box(\theta_n)\end{array}}{\Box(\theta_1\wedge\ ,\ .\ .\ ,\ \wedge\theta_n)}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$)}

And nothing else is a deduction. Note we may use $\vdash\theta$ as shorthand for {} $\Longrightarrow$ $\mathscr{D}$:θ, which stands for relative to the empty context $\mathscr{D}$ is a deduction of type θ. This can natural extend out to what it means for $\Gamma\vdash\theta$

OBSEVRATION 2.0. K*, the inference often used for showing completeness in K, will be an admissible as one would expect. We just state it as a rule so we can invoke it in our Completeness theorem. ■

3: Soundness for SIK

DEFINITION 3.1. We say that θ is a local consequence from Γ relative to frame $\mathscr{F}$, in other words $\Gamma \models_{LF} \theta$, iff for every model $\mathscr{M}$ based on $\mathscr{F}$, V(Γ) $\subseteq$ V(θ). We say that θ is a global consequence from Γ relative to frame $\mathscr{F}$, in other words $\Gamma \models_{GF} \theta$, iff for every model $\mathscr{M}$ based on $\mathscr{F}$, V(Γ)=W implies V(θ)=W.

DEFINITION 3.2. Let $<\Gamma,\theta>$ be an inference, where Γ is a subset of the formula set, and θ is a formula. We say this inference SIK-valid, if θ is a local consequence of Γ.

THEOREM 1. *Soundness: for every inference $<\Gamma,\theta>$, $<\Gamma,\theta>$ is SIK-valid*

PROOF. We prove that for every inference $<\Gamma,\theta>$ that $<\Gamma,\theta>$ is SIK-valid. We consider an SIK model $\mathscr{M}$, with an SIK-frame $\mathscr{F}$ for any C, $\mathscr{D}$, and θ, if C $\Longrightarrow \mathscr{D}:\theta$ then $<$A($\mathscr{D}$),$\theta>$ is SIK-valid. Here A($\mathscr{D}$) is the range(C). The proof will be done on the induction of the height of the derivation.
Base Case: We have an inference of the form v:$\theta \implies \theta$. Obviously, if $\mathscr{M}$,w$\models$ θ then $\mathscr{M}$,w$\models$ θ.

Assume the obvious inductive hypothesis (i.e) if we have are $\leq$ n-th stage of $\Longrightarrow$ then we for a context C, and $\mathscr{D}$, and θ that if C $\Longrightarrow \mathscr{D}:\theta$ then $<$A(D),$\theta>$ is SIK-valid. Assume [ ] was entered in by a use of $\wedge$I. By, the (IH), we have that (A($\mathscr{D}_0$),θ) is SIK-valid, and (A($\mathscr{D}_1$),ϕ) is SIK-valid because A($\mathscr{D}_i$)$\subseteq$ A($\mathscr{D}$) where i=0,1. Then, if we have a $\mathscr{M}$,w $\models$A($\mathscr{D}$) then $\mathscr{M}$,w$\models$ A($\mathscr{D}_0$), and A($\mathscr{D}_1$), so $\mathscr{M}$,w $\models\phi$ and $\mathscr{M}$,w$\models\theta$, and $\mathscr{M}$,w$\models(\theta\wedge\phi)$. Most of the cases are trivial like this so I will do 3 more of the harder ones, and leave the rest as an exercise to the reader.

Suppose that [ ] was entered by $\supset$-. Then by the (IH) we have that (A($\mathscr{D}_0$),$\theta\supset\phi$) is SIK-valid, and (A($\mathscr{D}_1$),$\sigma\supset\phi$), assume that $\mathscr{M}$,w$\models$ A($\mathscr{D}$) then $\mathscr{M}$,w$\models$ A($\mathscr{D}_0$), and $\mathscr{M}$,w $\models$ A($\mathscr{D}_1$), so by the (IH) we get $\mathscr{M}$,w $\models$ $\theta\supset\phi$ and $\mathscr{M}$,w$\models$ $\sigma\supset\phi$. Then we know for all w$<*$v we have $\mathscr{M}$,v$\models$ θ implies $\mathscr{M}$,v$\models$ ϕ, and $\mathscr{M}$,v$\models$ σ implies $\mathscr{M}$,v$\models$ ϕ. Then, if $\mathscr{M}$,v$\models$ $\theta \vee \sigma$ then M,v $\models \phi$ because either θ or σ is satisfied, and will imply it so $\mathscr{M}$,w$\models$ $(\theta \vee \sigma) \supset \phi$, so we are done with this case. Suppose that [ ] was entered by $\supset$I. Then, we have by (IH) that (v:σ,θ) is SIK-valid, then if $\mathscr{M}$,v$\models$ σ then $\mathscr{M}$,v$\models$ θ. So, if we fix a model $\mathscr{M}$, and a world w. $\mathscr{M}$,w$\models(\sigma\supset\theta)$ because if we have w$<*$v if v$\models\sigma$ implies v$\models\theta$. The rest of the cases are left as an exercise to the reader. ■

4: Completeness for SIK

We use the proof ideas that can be found in Celani, Dosen, and Ahmee [3],[4],[5] respectively. We must first introduce some notions, which can be found in Dösen .

DEFINITION 4.1. Let a set of formulas Γ be called a theory of SIK if it is closed under the relation $\vdash_{SIK}$. A theory will be called consistent if there is some θ such that $\theta \notin \Gamma$, and it is prime if it is consistent, and for all formulas θ and ϕ, if $(\theta \wedge \phi) \in \Gamma$ then either θ or ϕ is in Γ.

Definition 4.2. For Γ, and $\triangle \subseteq \Sigma$ where Σ is a prime theory, we say (Γ,$\triangle$) is a consistent pair iff for all $\{\theta_i\}_i^n \subseteq \Gamma$ and $\{\phi_i\}_i^m \subseteq \triangle$ then $\nvdash \wedge_i^n \theta_i \supset^* \vee_i^m \phi_i$

Proposition 4.1. *If (Γ,$\triangle$) is a consistent pair then there is a prime theory Γ' such that $\Gamma \subseteq \Gamma$', and Γ' $\cap \triangle = \emptyset$.*

Proof. Let S denote the subsets containing Γ and are disjoint from $\triangle$, and let it be partially ordered by inclusion. If $\{\Gamma\}_i$ is a chain let the union be the upper bound. The union must be consistent, or else one of the $\{\Gamma\}_i$ would be inconsistent, which is impossible. The rest of the of the proof is essentially Lindebuam's Lemma. [5] ■

1. Let W_c={Γ: Γ is a prime theory of SIK}, and define the binary relations W_c, and R_c as follows:

2. $\Gamma < *_c \Sigma$ iff $\forall\theta$, $\phi(\theta \supset \phi \in \Gamma$ and $\theta \in \Sigma$ implies that $\phi \in \Sigma)$

3. ΓR_c Σ iff $\Box\phi \in \Gamma$ implies that $\phi \in \Sigma$.

Let V_C(p)={Γ:p$\in$ Γ}. We call the frame $\mathscr{F}$=<W_c,R_c,$< *_c$> the canonical frame, and the model $\mathscr{M}$=<W_c,R_c,$< *_c$,V_c> the canonical model.

Proposition 4.2. For every prime theory Γ, and any formula ϕ , we have that $\Gamma \in$ V(ϕ) iff ϕ is in Γ.

Proof. We do this by induction on the complexity of the formula. We only do the harder cases, which are $\phi \supset \theta$, and $\Box\theta$.

If $\phi \supset \theta \in \Gamma$, and $\Gamma < *\Sigma$ where $\phi\in\Sigma$ we must have $\theta\in\Sigma$ by definition of $< *$ thus $\Gamma \in$ V($\phi \supset \theta$).

Next, assume that we have $\phi \supset \theta \notin \Gamma$, we show that there is a $\Sigma\in W_c$ such that $\Gamma < *\Sigma$, and $\phi \in \Sigma$ while $\theta \notin \Sigma$. Consider the following set:

S={Σ:Σ is a theory, and $\phi \in \Sigma$, and $\theta\notin \Sigma$ and $\forall$A,B(A$\supset$B$\in\Gamma\wedge$ A$\in\Sigma$ implies B$\in\Sigma$).

We observe that **S** is non-empty, as we can consider the theory Γ*={B: $\phi \supset$ B $\in \Gamma$}. First, we check whether we have closure under $\Longrightarrow_{SIK}$. Suppose we have $\Gamma^* \Longrightarrow_{SIK} \mu$. Then, we want to show that $\mu \in \Gamma^*$. For, S_i in Γ^*, we know that $\phi \supset S_i \in \Gamma$. Since Γ is closed under derivation, we have that $\Gamma \Longrightarrow_{SIK} \phi \supset S_i$. Then it is easy to show $\Gamma \Longrightarrow_{SIK} (\phi \supset S_1) \wedge , ... , \wedge (\phi \supset S_n)$ by $\wedge$I n-times. Then, one can show by rule applications of rule W we have $\Gamma \Longrightarrow_{SIK} \phi \supset (S_1 \wedge , ... , \wedge S_n)$. Then, We have that $(S_1 \wedge , . . . , \wedge S_n) \Longrightarrow_{SIK} \mu$, so by $\supset$I, we have that $\Longrightarrow_{SIK} (S_1 \wedge , ... , \wedge S_n) \supset \mu$ then by $\supset$+, we observe that $\Gamma \Longrightarrow_{SIK} \phi \supset \mu$, so $\mu \in \Gamma^*$. It is obvious $\phi \in \Gamma$

because $\Gamma \Longrightarrow_{SIK} \phi \supset \phi$.

We then apply Zorn's lemma to find a maximum element of **S**. Call this element $\triangle$. We then prove that $\triangle$ is prime. Assume that we have some A v B $\in \triangle$, and A $\notin \triangle$, and B $\notin \triangle$. Consider the following set:

$\mathbf{F}(\triangle,\tau)=\{\epsilon$: $\exists\beta\in\triangle$, $(\beta \wedge \tau) \supset \epsilon \in \Gamma\}$

is a theory and will make true (+) $\forall$A,B((A$\supset$B$\in\Gamma\wedge$ A$\in\mathbf{F}(\triangle,\tau)$) $\supset$ B$\in\mathbf{F}(\triangle,\tau)$). In addition, it includes $\triangle\cup\{\tau\}$. To see this, consider that Q $\in \mathbf{F}(\triangle,\tau)$ then we want to show that $\mathbf{F}(\triangle,\tau) \Longrightarrow_{SIK}$ Q then Q $\in \mathbf{F}(\triangle,\tau)$. Assume that if-clause then if we have B_i involved in the deduction set of $\mathbf{F}(\triangle,\tau)$, we must $(a_i \wedge \tau)$ $\supset B_i \in \Gamma$, so by closure under deduction $\Gamma \Longrightarrow_{SIK} (a_i \wedge \tau) \supset B_i$, where $a_i \in \triangle$. We note that $((a_1 \wedge , . . . , \wedge a_n) \wedge \tau) \supset (a_i \wedge \tau))$, where 1<=i<=n by $\supset$I. and we know that $\Gamma \Longrightarrow_{SIK} (a_i \wedge \tau) \supset B_i$, so we have that $\supset +$ rule we have $((a_1 \wedge , . . . , \wedge a_n) \wedge \tau) \supset B_i$, and then it is clear $((a_1 \wedge , . . . , \wedge a_n) \wedge \tau)$ $\supset (B_1 \wedge , ... , \wedge B_n)$ by rule W. Then since we have $(B_1 \wedge , ... , \wedge B_n)$ $\Longrightarrow_{SIK}$ Q, we have $\Longrightarrow_{SIK} (B_1 \wedge , ... , \wedge B_n) \supset$ Q. Then by rule $\supset$+, we get that $((a_1 \wedge , ... , \wedge a_n) \wedge \tau) \supset$ Q, and since $(a_1 \wedge , . . . , \wedge a_n) \in \triangle$ by closure under deduction, we have that Q$\in\mathbf{F}(\triangle,\tau)$. It's not hard to observe that (+) is a true condition. Assume that A$\supset$B$\in\Gamma$, and also assume that A$\in$F($\triangle,\tau$)). Then we have that for some $\beta \in \triangle$, $(\beta \wedge \tau) \supset$ A$\in \Gamma$. Then by rule $\supset$+ we can observe that $(\beta \wedge \tau) \supset$ B$\in \Gamma$, so B $\in$ F($\triangle,\tau$)). It is obvious that $\triangle\cup$ τ is in F($\triangle,\tau$)).

Also consider:

$\mathbf{F}(\triangle,\alpha)=\{\epsilon$: $\exists\beta\in \triangle$ $(\beta \wedge \alpha) \supset \epsilon \in \Gamma\}$.

this too will be a theory, which can be checked like in the previous case, and it will include $\triangle\cup\{\alpha\}$, and satisfies $\forall$A,B(A$\supset$B$\in\Gamma\wedge$ A$\in\mathbf{F}(\triangle,\alpha)$ implies B$\in\mathbf{F}$($\triangle,\alpha$)).

We now can consider: $\sigma_1\in \triangle$, and $\sigma_2 \in \triangle$ such that we have (1) $(\sigma_1 \wedge \tau)$ $\supset \theta \in \Gamma$, and (2) $(\sigma_2 \wedge \alpha) \supset \theta \in \Gamma$, and thus we have $((\sigma_1 \wedge \sigma_2 \wedge \tau) \vee ((\sigma_1$ $\wedge\ \sigma_2 \wedge \alpha) \supset \in \Gamma$, which of course entails $((\sigma_1 \wedge \sigma_2) \wedge (\tau \vee \alpha)) \supset \theta \in \Gamma$ by closure under deductions, which would imply $(\sigma_1 \wedge \sigma_2)$,$(\tau \vee \alpha) \in \triangle$, and so θ $\in \triangle$, which is impossible.

For the $\Box\theta$ case consider if $\Box\theta\in \Gamma$ then we have that if $\Gamma R_c \Sigma$ $\theta \in \Sigma$, so $\Sigma\in$V$_c(\theta)$, thus we have $\Gamma \in$V$_c(\Box\theta)$. For the reverse direction, consider that we have a $\Box\gamma\notin\Gamma$. We want to find a Σ such that ΓRΣ, $\gamma\notin \Sigma$. Consider the set $\triangle=\{\theta$:$\Box\theta \in \Gamma\}$. We construct Σ. We note that $(\triangle,\gamma)$ is consistent because otherwise we would have $\Box\gamma\in \Gamma$. We need the following:
(1*) $\Box$ (A $\supset^*$ B) $\Longrightarrow_{SIK}$ ($\Box$ A $\supset^*$ $\Box$ B).

Start by recognizing $\Box$ A, $\Box$ (A $\supset^*$ B) $\Longrightarrow_{SIK}$ $\Box$ B by rule K. Then, observe rule $\supset^*$ I will get us where we want.

Assuming inconsistency we have $\Longrightarrow_{SIK} (\theta_1 \wedge , ... , \wedge \theta_n) \supset^* \gamma$. By nec$\Box$ $\Longrightarrow_{SIK} \Box((\theta_1 \wedge ...\wedge \theta_n) \supset^* \gamma)$. Then, by (1*) we have $\Longrightarrow_{SIK} \Box(\theta_1$ $\wedge , ... , \wedge \theta_n) \supset^* \Box(\gamma)$. We have $\Box\ \theta_i \in \Gamma$, so $\Gamma \Longrightarrow_{SIK} (\Box\theta_1 \wedge , ... , \wedge \Box$

θ_n), which by Rule (K*) means $\Gamma \implies_{SIK} \Box(\theta_1 \wedge , . . . , \wedge \theta_n)$ then we must have $\Gamma \implies_{SIK} \Box\, \gamma$ then by closure under deduction we have $\Box\, \gamma \in \Gamma$, which is not possible. Applying Proposition 4.1 we can take Σ to be the prime theory that extends Γ. such that $\gamma \notin \Sigma$. And, by definition of R, we have that $\Gamma R \Sigma$, which is what we set out to prove. The other cases are left as an exercise, and aren't hard. ■

PROPOSITION 4.3. *$\triangle \implies_{SIK} \theta$ iff for every prime theory Γ such that $\triangle \subseteq \Gamma$ then we have that $\theta \in \Gamma$.*

PROOF. Suppose that $\triangle \nvdash \theta$ then I claim that θ does not belong to the theory of $\triangle$. We argue by using Zorn's lemma on the following ordering F=$\{\Gamma$: Γ is a theory such that $\triangle \subseteq \Gamma$ and $\theta \notin \Gamma\}$. By Zorn's lemma there is a maximal element Γ^*. We show that Γ^* is a prime theory. Assume A $\vee$ B $\in \Gamma^*$, and A,B $\notin \Gamma^*$. Consider the following theories, $\triangle_1$ be the theory generated by $\Gamma^* \cup \{A\}$, and $\triangle_2$ be the theory generated by $\Gamma^* \cup \{B\}$. We show $\Gamma^* \cup \{A\}$ is consistent. If it were not then $\Gamma^* \cup \{A\} \implies_{SIK} B$, and thus $\Gamma^* \cup \{A \vee B\} \implies_{SIK} B$ which is an easy deduction, and thus $\Gamma^* \implies_{SIK} B$, which is impossible so it is consistent. The same method works for showing $\Gamma^* \cup \{B\}$ is consistent. Applying maximality, we have that Γ^*, $\theta \in \triangle_1$, and $\theta \in \triangle_2$. Then, we have that $\Gamma^* \cup \{B\} \implies \theta$, and $\Gamma^* \cup \{A\} \implies \theta$, so its easy to show $\Gamma^* \cup \{A \vee B\} \implies \theta$, but that means $\Gamma^* \implies \theta$, which is absurd. The other direction is trivial. ■

THEOREM 2. *if <A($\mathscr{D}$)),θ> is SIK-valid then A($\mathscr{D}$) $\implies_{SIK} \theta$*

PROOF. is a consequence of previous Proposition. ■

OBSERVATION 4.1. $\{P, P \supset Q\} \implies_{SIK} Q$ is not valid. We utilize Completeness. Consider the w1$< *$w2, and R empty, and let V(P)=$\{$w1,w2$\}$, and let V(Q)=$\{$w2$\}$. Then, P holds in w1, and so does P$\supset$Q because Q is true in w2, but Q does not hold in w1. ■

5: Strengthenings of SIK

5.1: Motivation: We have set up our model theory with two relations $< *$, which corresponds to the relations in intuitionistic logics, and substructural logics, and R which corresponds to the relations in modal logics. The way we obtain stronger logics is by strengthening the conditions on R, and $< *$. Let N denote a normal modal logic. It is well known that modal system N+T corresponds to reflexive Kripke frames, and N+4 corresponds to transitive Kripke frames. In addition, in standard Intuitionistic semantics, we have that the Kripke frames, are reflexive, and transitive, and the valuation is persistence. To obtain the logics of different strengths on the "modal level" we vary the conditions on R. Say, we add reflexivity, we add transitivity, or both. To obtain the logics of different strengths on the "intuitionistic level" we vary the conditions on $< *$, adding the conditions that are on usual intuitionistic Kripke frames. We will

define the logics this way.
Let M$\in$ {T,4}, and K $\in$ {R (Reflexive), T* (Transitive), P (Persistent)}
SIK frame + an M frame + K frame will generate the conditions for our new logic. In other words our logics will be combinations of {T,4}, and combinations possible in {R,T*,P}. We will show that many pairs of these logics will be distinct later in the paper.

5.2: Soundness, Completeness, and Distinctness of extensions of SIK

LEMMA 5.2.1 T Soundness: $\{\} \Longrightarrow_{SIK} \Box P \supset^* P$ is valid on SIK+T frames. ∎

LEMMA 5.2.2 T Completeness: if $\{\} \Longrightarrow_{SIK} \Box P \supset^* P$ then Reflexivity holds on the Canonical model. ∎

LEMMA 5.2.3 T* Soundness: $\{\} \Longrightarrow_{SIK} \Box P \supset^* \Box \Box P$ is valid on SIK+4 frames. ∎

LEMMA 5.2.4 T* Completeness: If $\{\} \Longrightarrow_{SIK} \Box P \supset^* \Box \Box P$ then transitivity holds on the Canonical model. ∎

PROOF. Lemma 5.2.1-5.2.4 follow from well-know frame correspondence results in modal logic, and they are also easy to check. ∎

We have axioms, or one-line-inferences corresponding to the modal frame conditions. We can add to strengthening and on $<*$, which are supposed to correspond to an intuitionistic accessibility relation. Greg Restall has proposed the following characteristic formulas that correspond to reflexivity, transitivity, and persistence on frames [6], but unfortunately with our semantics we can not apply his formulas directly. Greg Restall had brought in the idea of "forcing at a base world". Meaning the true formulas are the formulas the base world makes true in every model, where the base world can access every other world. Many others have used this strategy as well [7],[8]. However, we take a standard approach to Kripke semantics, which seems less common in the literature on subintutionistic logics, so we have to be a little clever with our characteristic formulas. The trick is to use the fact we have classical $\supset^*$ in our language.

Consider the formulas Greg Restall gives:

1) $((P \wedge (P \supset Q)) \supset Q)$ corresponds to reflexivity
2) $((A \supset B) \supset ((B \supset C) \supset (A \supset C)))$ corresponds to transitivity
3) $(A \supset (B \supset A))$ corresponds to persistence.

Consider the modified formulas:

1*) $((P \wedge (P \supset Q)) \supset^* Q)$ corresponds to reflexivity

2*) ((A ⊃ B)⊃*((B⊃C) ⊃ (A⊃C))) corresponds to transitivity
3*) (A ⊃* (B ⊃ A)) corresponds to persistence.

Observation 5.0. If we had allowed for vacuous discharging on ⊃I we could derive 3). Because v:A $\Longrightarrow_{SIK}$ B ⊃ A discharging q:B, and then $\Longrightarrow_{SIK}$ A ⊃ (B ⊃ A) discharging v:A works. We could also derive 3*) by using ⊃*I on the last step. ■

Observation 5.1. Let R be any accessibility relation, and V be any valuation function, and $< *$ be the empty set. Then consider all Kripke models $\mathscr{M}$= <W,V,R,$< *$>. We see that in any model 1) hold because no world can access any other world. ■

The previous observations highlights a strength of the characteristic formulas we are using. 1*) is not trivially true if the worlds can not access anywhere. This sort of work does not seem to be possible to do with purely intuitionist if-then. We also will come to observe that we get Completeness results without vastly changing the structure of the semantics we are using like many writers on subintuitionistic logics. The point being this. Subintutionistic logics are weak, with the intuitionistic if-then failing to validate modus ponens on non-reflexive Kripke frames, and also failing to validate certain prefixing formulas [6]. However, since our language includes classical implication as well, we can see the important completeness results without changing the structure of the logic.

Lemma 5.2.5 *Soundness for 1*)*:

Proof. Suppose that $< *$ is reflexive. Fix a model $\mathscr{M}$, and a world w $\mathscr{M}$,w⊨ ((P ∧ (P⊃ Q)) then $\mathscr{M}$,w⊨ P, and $\mathscr{M}$,w⊨ P⊃ Q. If wRw, we must have $\mathscr{M}$,w⊨ Q. ■

Lemma 5.2.6 *Completeness for 1*)*:

Proof. Assume that we have $\Longrightarrow_{SIK}$ ((P ∧ (P⊃ Q)) ⊃* Q). Let P∈ Σ, and (P⊃ Q)∈ Σ, and then we have (P∧ (P⊃ Q))∈Σ, and since ((P ∧ (P⊃ Q)) ⊃* Q)∈ Σ, by MP we must have that Q∈Σ. Then by definition of $< *_c$ on the canonical model we have, Σ$< *$Σ. ■

Lemma 5.2.7 *Soundness for 2*)*:

Proof. Suppose that we have $< *$ is transitive. Then, if we have $\mathscr{M}$, and a world w, and $\mathscr{M}$,w⊨(A ⊃ B), and $\mathscr{M}$,w⊨(B ⊃ C), and w$< *$v$< *$r by transitivity we would have w$< *$r, so $\mathscr{M}$,r⊨A implies $\mathscr{M}$,r⊨B and $\mathscr{M}$,r⊨B implies $\mathscr{M}$,r⊨C. So, it is easy to see that and $\mathscr{M}$,v⊨(A ⊃ C). ■

Lemma 5.2.7 *Completeness for 2*)*:

PROOF. Assume that $\Longrightarrow_{SIK}$ ((A ⊃ B)⊃*((B⊃C) ⊃ (A⊃C))). Let Σ< ∗△< ∗Γ (A ⊃ B)∈Σ. We know (B⊃B)∈△. And ((A ⊃ B)⊃*((B⊃B) ⊃ (A⊃B))∈Σ. Then by MP we obtain ((B⊃B) ⊃ (A⊃B))∈Σ. So we have that since Σ< ∗△ (B⊃B) ∈ △ implies (A⊃B)∈ △ , so (A⊃B)∈ △ then we have that A ∈ Γ implies B ∈ Γ by definition of $< *_c$ we have that Σ< ∗Γ. ■

PERSISTENCE LEMMA. *If we have a persistent relation, i.e for every propositional variable we have that if w ∈ V(p), and wRv then v ∈ V(p) then for any formula θ if* $\mathscr{M}$*,w ⊨ θ implies* $\mathscr{M}$*,v ⊨ θ provided that < ∗, and R are both transitive.*

PROOF. The base cases, and the case of ∧,∨ are trivial, so we only do the cases for ⊃, and □. Assume that $\mathscr{M}$,w⊨ (A⊃B) if we have w< ∗v and v< ∗r by transitivity we obtain w< ∗r, so if r⊨A then we must have r⊨B then it is clear $\mathscr{M}$,v⊨ A ⊃B. For the □A case assume that $\mathscr{M}$,w⊨□A. Then if wRv, and vRu then by transitivity we have wRu, and so u⊨A, and then we have that v⊨ □A. ■

DEFINITION 5.1. We will refer to a relation as persistent, or hereditary from now on if we have transitivity on R, and < ∗, and the above condition that if w ∈ V(p), and wRv then v∈ V(p).

LEMMA 5.2.8 *Soundness for 3*)*:

PROOF. Of course here was assume R, and < ∗ are transitive. Fix a $\mathscr{M}$, and a world w such that $\mathscr{M}$,w ⊨ A, and if w< ∗u then we have if $\mathscr{M}$,u ⊨ B we have that $\mathscr{M}$,u ⊨ A by a use of the Persistence Lemma. ■

LEMMA 5.2.9 *Completeness for 3*)*:

PROOF. Assume that we have Σ< ∗Γ (A ⊃* (B ⊃ A))∈Σ. And, A∈Σ. And since we have at least some theorem in Γ say B, we can do the following. By MP we get that (B ⊃ A)∈Σ, since (A ⊃* (B ⊃ A))∈Σ. So, B∈Γ implies A∈Γ, and by definition of $< *_c$ this completes the proof. ■

THEOREM 3. *All the logics in the SIK frame + an M frame + K frame where M∈ {T,4}, and K ∈ {R (Reflexive), T* (Transitive), P (Persistent)} are Sound and Complete. One, caveat, if we have P, we must also have transitivity on R,< ∗, so we must have 4, and T*.*

PROOF. Use Soundness and Completeness for SIK, and the previous 9 Lemmas. ■

We now prove that the logics are distinct. The idea is to split all the logics into groups. Group 1 will lack reflexivity on < ∗. Group 2 will lack transitivity on < ∗. Group 3 will lack heredity on < ∗. Group 4 will lack reflexivity on R. Group 5 will lack Transitvity on R. The proof comes down to showing that all the logics in the groups we are referring to fail to validate the characteristic formula on the frame condition they lack.

PROPOSITION 5.2.10. *Distinctness for R*

PROOF. If any SIK extension frame is not reflexive on $<*$ $(P \wedge (P \supset Q)) \supset^{*} Q$ is not valid. Consider the countermodel $\mathscr{M}=<\{w,u\},R,<*,V>$, and let $w<*u$. Let us also Let R be universal. Consider $V(P)=\{w,u\}$, and $V(Q)=\{u\}$ Then, we have Persistence. Then, $\mathscr{M},w \models P \wedge (P \supset Q)$ because P is true at v, and $u \models P$ implies that $u \models Q$, but Q is not true at w. ■

PROPOSITION 5.2.11. *Distinctness for T**

PROOF. If any SIK extension is not transitive on $<*$ then we will not have 2) $((A \supset B) \supset^{*} ((B \supset C) \supset (A \supset C)))$. Note, we have said persistence requires transitivity on $<*$ so we do not require the countermodel be persistent. Consider the countermodel $\mathscr{M}=<\{w,u,v\},R,<*,V>$. As usual assume that R is universal, and consider a chain such that $w<*u$, and $u<*v$, and $u<*v$ is not true. Ensure that $<*$ is reflexive. Let $V(A)=\{v\}$, and $V(B)=\emptyset$, and $V(C)=\emptyset$. It is also obvious $w \models (A \supset B)$, and $w \models (B \supset C)$. But then, we should have $v \models A$ implies $v \models C$. We do not. ■

PROPOSITION 5.2.12. *Distinctness for P*

PROOF. If any SIK extension does not have the a persistent relationship then $(A \supset^{*} (B \supset A))$ is not valid. Consider the countermodel $\mathscr{M}=<\{w,u\},R,<*,V>$, where R is universal, ensure we have reflexivity on $<*$, and let $v<*u$ be a witness for the lack of a hereditary. Also, note that it is obviously transitive. Let $V(A)=\{v\}$, and let $V(B)=\{u\}$. Then, we have that $v \models A$, and $v<*u$, and $u \models B$, but $u \not\models A$, so $v \not\models (A \supset^{*} (B \supset A))$. ■

PROPOSITION 5.2.13. *Distinctness for 4*

PROOF. If any SIK extension does not have the transitivity on R then $\Longrightarrow$ $\Box A \supset^{*} \Box\Box A$ is not valid. Since we defined persistence in a way that requires we have transitivity on R, and $<*$, we do not require this countermodel to be persistent. Consider the countermodel $\mathscr{M}=<\{w,u,v\},R,<*,V>$. Let $<*$ be universal. And let wRu, and uRv, and let R be reflexive. We let $V(A)=\{w,u\}$.Then it is easy to see $w \models \Box A$, but $w \not\models \Box\Box A$ since wRu and uRv, but $v \not\models A$ ■

PROPOSITION 5.2.14. *Distinctness for T*

PROOF. If any SIK extension is not reflexive on R then $\models \Box P \supset^{*} P$ is not valid. This is easy. Let Let $<*$ be universal. Consider the countermodel $\mathscr{M}=<\{w\},R,<*,V>$ and consider a we have, and $R=\emptyset$, and the $V(P)=\emptyset$, so is persistent. $u \models \Box P$ because u does not access any worlds with R, and P is not true at u. ■

Theorem 4. *The following logics are distinct up to reordering, where the logics are defined by the frame conditions they correspond to. SIK+ M $\in$ {T,4}, + K $\in$ {R (Reflexive), T* (Transitive), P (Persistent)}, where you can choose SIK plus any combination, with the restrictions that we only have P when we have 4, and T*.*

Proof. Appeal to propositions 5.2.10-5.2.14. ■

Proposition 5.2.15. *It is easy to see the number of logics that generated is 13.*

Proof. By the above condition {T,4} gives us 3 free choices, and {R (Reflexive), T* (Transitive), P (Persistent)}, and if we have Persistence we must have T*, and 4, which means there are 4 possibilities with Persistence, the other 3 are for adding T, and R, and both. Then, {R,T*} together give us another 3 free choices, with the free choices from {T,4}, we obtain 9 more possible logics. That leaves us with 13 logics. ■

5.3: Extending our natural deduction system SIK

From a proof-theoretic perspective, we want better deduction rules that correspond to reflexivity, transitivity, and persistence than the one-line theorems that were given. We propose the follow rules will suffice:

Reflexivity (MP) If $C_0 \implies \mathscr{D}_0$:$\theta\supset^*\phi$, and $C_1 \implies \mathscr{D}_1$:$\theta$. Then, if we have that $\{C_0,C_1\}$ is coherent then we have that $C_0\cup C_1 \implies \mathscr{D}$:$\phi$. We can picture it as follows.

$$\dfrac{\begin{matrix}\mathscr{D}_0 \\ \theta\supset\phi\end{matrix} \qquad\qquad \begin{matrix}\mathscr{D}_1 \\ \theta\end{matrix}}{\phi}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}$\cup$[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

Persistence) If C $\implies \mathscr{D}_0$:θ then C $\implies \mathscr{D}$:($\phi\supset\theta$). Then, we picture $\mathscr{D}$ as follows.

$$\dfrac{\begin{matrix}\mathscr{D}_0 \\ \theta\end{matrix}}{(\phi\supset\theta)}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}

Transitivity) If C $\implies \mathscr{D}_0$:($\theta\supset\phi$) then C $\implies \mathscr{D}$:($\phi\supset\tau$)$\supset$($\theta\supset\tau$). Then, we picture $\mathscr{D}$ as follows.

$$\begin{matrix}\mathscr{D}_0 \\ (\phi\supset\theta)\end{matrix}$$

$$\frac{}{(\phi\supset\tau)\supset(\theta\supset\tau)}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}

Proposition 5.3.1. *The above natural deduction calculus is equivalent to the one-line inference scheme given for these logics modulo SIK*

Proof. For reflexivity, assume that ((P $\wedge$ (P$\supset$ Q)) $\supset^*$ Q) is a theorem. Then, if we have a deduction of (P$\supset$ Q) and a deduction of P. We have a deduction of (P $\wedge$ (P$\supset$ Q)). Then by the axiom we can deduce Q with modus ponens for $\supset^*$. Assume that Reflexivity (MP) is valid. Then if we have (P$\supset$ Q), and P. We must have Q. So it is clear that $\supset^*$I we get ((P $\wedge$ (P$\supset$ Q)) $\supset^*$ Q) is a theorem. For Persistence, assume A$\supset^*$(B$\supset$A) is a theorem. Then if we have a deduction of A. We must have (B$\supset$A) by MP. Conversely, suppose Persistence) is valid. Then if we have a deduction of A, we must have a deduction of (B$\supset$A) by $\supset^*$I, we get what we want. For Transitivity, the proof will be similar to the other two cases. ■

Proposition 5.3.2. *Completeness, and Soundness for the natural deduction.*

Proof. Follows from Proposition 5.3.1, along with the fact SIK plus its extensions are Sound and Complete. ■

6: Looking "Under the Hood" of the $\supset$ rules of SIK

In intuitionistic logic $\supset$ has nice elimination, and introduction rules. They can be pictured as follows.

Int$\supset$I If C,v:$\sigma \Longrightarrow \mathscr{D}_0$:$\theta$ then C $\Longrightarrow \mathscr{D}$:($\sigma \supset\theta$). **Note:** here we allow vacuous discharging.

Then, we picture $\mathscr{D}$ as follows.

$$\frac{\begin{array}{c}\text{v}:\sigma\\ \mathscr{D}_0\\ \theta\end{array}}{(\sigma\supset\theta)^v}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$) is not v:σ}

MP) If $C_0 \Longrightarrow \mathscr{D}_0$:$\theta\supset\phi$, and $C_1 \Longrightarrow \mathscr{D}_1$:$\theta$. Then, if we have that {C_0,C_1} is coherent then we have that $C_0 \cup C_1 \Longrightarrow \mathscr{D}$:$\phi$. We can picture it as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \theta\supset\phi\end{array} \qquad \begin{array}{c}\mathscr{D}_1\\ \theta\end{array}}{\phi}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}$\cup$[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$))}

For SIK we have a rule that somewhat resembles Int⊃I, which we called ⊃I, but we do not have any rule that resembles modus ponens. For, Int⊃I there is two things to note. We have vacuous discharging which would not fly for SIK. Remember that if we allowed for it, we could derive $\implies$ (A $\supset^*$ (B $\supset$ A)), which is only valid on the hereditary Kripke models. Secondly, in SIK we talk about the context v:σ only for ⊃I meaning we cannot use this rule for deductions with a larger context which we can do for intuitionistic logic.

Observation 6.0. If we have C,v:$\theta \implies_{SIK} \sigma$ then we have C $\implies_{SIK} \theta \supset \sigma$ is not valid in SIK. We use Soundness. v_0:P,v_1:P$\supset^*$Q $\implies_{SIK}$Q by MP, and then by the rule above we get v_0:(P$\supset^*$Q) $\implies_{SIK}$ P $\supset$ Q. By the Soundness, theorem we must have that (P⊃Q) is a local consequence of (P$\supset^*$Q). But, consider the model with w1, and w2, and w1$<*$w2, R is universal, and V(P)={w2}, and V(Q)=$\emptyset$. Then w1 $\models$ (P$\supset^*$Q) because P does not hold. But w1Rw2, and w2 $\models$ P, but w2 $\not\models$ Q, so w1 $\not\models$ (P⊃Q), which is a contradiction. ■

The above observation shows that ⊃I is not a satisfactory introduction rule. It would be much more satisfying to have a rule that allows us to talk about introducing ⊃ with more than a one-element context.

More problematic is that there is no rule that resembles modus ponens at all, without reflexivity on the Kripke frame. Under the perspective that the meaning of the logical connectives proof-theoretically is guided by our elimination, and introduction rules, we do not seem to have an elimination rule for ⊃. The problem seems to be that ⊃ "looks in the future" and we can not really say much about what is happening at a particular world at least without some conditions like heredity, reflexivity, and transitivity. And, this is extremely problematic because we are trying to describe the local consequence relation. The same issue appears to arise when giving introduction, and elimination rules for modal operators. Hodes observes that for modal logics rule K, and Nec, which have already been laid out in Section 2 do not function like introduction of elimination rules. For example, necessitation is not a rule of inference, it is a rule of proof, because the deduction must have no assumptions. And K does not look like an elimination rule for □. Hodes has suggested that our ordinary systems fail to capture some important work happening "under the hood" of these modal logics [1],[2].

Hodes attempt to solve this issue by considering marked formulas $\mathbf{m}\theta$, where **m** is either **0**, or **1**, and indicates the "mode of acceptance". **0** marks the formulas which have the mode of acceptance of being actually true, and **1** marks the mode of acceptance of being true in a possible world that is accessible. This analysis may really help us to further understand subintuitionistic logics because our problem was not knowing enough about the worlds that are accessible. In what comes to follow, I will attempt to look "under the hood" of some of the rules for subintuitionistic logic like Hodes does for modal systems. Except, **1** marks the mode of acceptance of being true in every possible world that is accessible, not just some world that is accessible. For, what follows, we will drop the modal character of our language, and just analyze the positive implication fragment of SIK. Our new logical connectives $\mathscr{LC}=\{\vee,\wedge,\supset,\top\}$.

DEFINITION 6.1. Let a frame $\mathscr{F}=<W,R>$, where W is a non-empty set of worlds, and R is an accessibility relationship such that $R\subseteq WxW$. Give us a countable set of propositional variables, and freely generate the formulas with the $\mathscr{LC}$, and call this set **Fml**. A model $\mathscr{M}$ is a triple $<\mathscr{F},V>$, where V is a function $V:S \to P(W)$. Let a marked formula be of the form $\mathbf{m}\theta$, where m=0,1 for $\theta\in$ **Fml** We define $\models$ as follows.

$\mathscr{M}$,w $\models$ P iff w$\in$V(P) (where P is atomic)
$\mathscr{M}$,w $\models \top$
$\mathscr{M}$,w $\models \theta \wedge \phi$ iff $\mathscr{M}$,w $\models \theta$ and $\mathscr{M}$,w $\models \phi$
$\mathscr{M}$,w $\models \theta \vee \phi$ iff $\mathscr{M}$,w $\models \theta$ or $\mathscr{M}$,w $\models \phi$
$\mathscr{M}$,w $\models \theta \supset \phi$ iff for all wRv $\mathscr{M}$,w $\models \theta$ implies $\mathscr{M}$,w $\models \phi$
$\mathscr{M}$,w $\models \theta \supset^* \phi$ iff $\mathscr{M}$,w $\models \theta$ implies $\mathscr{M}$,w $\models \phi$
$\mathscr{M}$,w $\models \mathbf{m}\phi$ iff $\mathscr{M}$,w $\models \phi$, and $\mathbf{m}$=0
$\mathscr{M}$,w,u $\models \mathbf{m}\phi$ iff (1) $\mathbf{m}$=1 and for all wRu we have that $\mathscr{M}$,u $\models\mathbf{0}\phi$, or (2) we have $\mathbf{m}$=0, and $\mathscr{M}$,w $\models \phi$
$\mathscr{M}$,w $\models \Gamma$ iff for all $\beta \in \Gamma$, we have $\mathscr{M}$,w $\models \beta$
$\mathscr{M}$,w,u $\models \Gamma$ iff for all $\beta \in \Gamma$, we have that for all u such that wRu, $\mathscr{M}$,w,u$\models\beta$

We use **n** as a marker throughout a proof it can be replaced by 0 or 1. In other words, we have 2 rules in one.

Consider the new rules:

Base case: For v$\in$Var, and $\theta\in$Fml, v:$\theta \implies \mathscr{D}$:$\theta$ for $\mathscr{D}=\{<[\,],v:\theta>\}$, i.e, $\mathscr{D}$ is [ ] labeled by v:θ, and dpd($\mathscr{D}$)=dom($\mathscr{D}$)=[ ].
Inductive cases:
$\supset$I) If C,v:$\mathbf{1}\sigma \implies \mathscr{D}_0$:$\mathbf{1}\theta$ then C $\implies \mathscr{D}$:$\mathbf{0}(\sigma \supset\theta)$. **Note:** here we do not allow vacuous discharging.
Then, we picture $\mathscr{D}$ as follows.

$$\dfrac{\begin{array}{c} v:\mathbf{1}\sigma \\ \mathscr{D}_0 \\ \mathbf{1}\theta \end{array}}{\mathbf{0}(\sigma\supset\theta)^v}$$

The dpd($\mathscr{D}$)=$\{[0] \frown s \mid s \in dpd(\mathscr{D}_0)$ is not v:$\sigma\}$

Rule T) If v:$\mathbf{0}\sigma \implies \mathscr{D}_0$:$\mathbf{0}\theta$ then {} $\implies \mathscr{D}$:$\mathbf{0}(\sigma \supset\theta)$. **Note:** here we do not allow vacuous discharging.
Then, we picture $\mathscr{D}$ as follows.

v:$\mathbf{0}\sigma$

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \mathbf{0}\theta\end{array}}{\mathbf{0}(\sigma\supset\theta)^v}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$) is not v:σ}

⊃E) If $C_0 \Longrightarrow \mathscr{D}_0$:$\mathbf{0}(\theta\supset\phi)$, and $C_1 \Longrightarrow \mathscr{D}_1$:$\mathbf{1}\theta$. Then, if we have that $\{C_0,C_1\}$ is coherent then we have that $C_0 \cup C_1 \Longrightarrow \mathscr{D}$:$\mathbf{1}\phi$. We can picture it as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \mathbf{0}(\theta\supset\phi)\end{array} \qquad \begin{array}{c}\mathscr{D}_1\\ \mathbf{1}\theta\end{array}}{\mathbf{1}\phi}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$)}∪[1] ⌢ s | s ∈ dpd($\mathscr{D}_1$)}

Swap1) If $C_0 \Longrightarrow \mathscr{D}_0$:$\mathbf{0}(\theta\supset\phi)$, and if we have that $\{C_0,C_1\}$ is coherent then we have that $C_0 \cup C_1 \Longrightarrow \mathscr{D}$:$\mathbf{1}(\theta\supset^*\phi)$. We can picture it as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \mathbf{0}(\theta\supset\phi)\end{array}}{\mathbf{1}(\theta\supset^*\phi)}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$)}

Swap2) If $C_0 \Longrightarrow \mathscr{D}_0$:$\mathbf{1}(\theta\supset^*\phi)$, and if we have that $\{C_0,C_1\}$ is coherent then we have that $C_0 \cup C_1 \Longrightarrow \mathscr{D}$:$\mathbf{0}(\theta\supset\phi)$. We can picture it as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \mathbf{1}(\theta\supset^*\phi)\end{array}}{\mathbf{0}(\theta\supset\phi)}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$)}

⊃*I If v:$\mathbf{n}\sigma \Longrightarrow \mathscr{D}_0$:$\mathbf{n}\theta$ then $\{\} \Longrightarrow {}^*\mathscr{D}$:$\mathbf{n}(\sigma \supset\theta)$. **Note:** here we do allow vacuous discharging.
Then, we picture $\mathscr{D}$ as follows.

$$\frac{\begin{array}{c}\text{v:}\mathbf{n}\sigma\\ \mathscr{D}_0\\ \mathbf{n}\theta\end{array}}{\mathbf{n}(\sigma\supset^*\theta)^v}$$

The dpd($\mathscr{D}$)={[0] ⌢ s | s ∈ dpd($\mathscr{D}_0$) s is not v:nσ}

⊃*E) If $C_0 \Longrightarrow \mathscr{D}_0$: $\mathbf{n}(\theta\supset\phi)$, and $C_1 \Longrightarrow \mathscr{D}_1$:$\mathbf{n}\theta$. Then, if we have that $\{C_0,C_1\}$ is coherent then we have that $C_0 \cup C_1 \Longrightarrow \mathscr{D}$:$\mathbf{n}\phi$. We can picture it as follows.

$\mathscr{D}_0$ $\qquad\qquad$ $\mathscr{D}_1$

$$\frac{\mathbf{n}(\theta \supset^* \phi) \qquad \mathbf{n}\theta}{\mathbf{n}\phi}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}$\cup$[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

The rules for conjunction, disjunction will be roughly the same as before. No need to reinvent the wheel. However, looking at $\supset^*$E a similar thing will happen. We prefix all the formulas in the deduction by an **n**. Denote this logic SIK+.

PROPOSITION 6.0. *All the following $\supset$ rules corresponding to weak Dösen logic [3], or in other words SIK without modal rules, are derivable in this system.*
1)v_0:$\mathbf{0}$(P$\supset$Q),v_1:$\mathbf{0}$(Q$\supset$R) $\Longrightarrow$ $\mathbf{0}$(P$\supset$R)
2)v_0:$\mathbf{0}$(P$\supset$Q),v_1:$\mathbf{0}$(P$\supset$R) $\Longrightarrow$ $\mathbf{0}$(P$\supset$(Q$\wedge$R))
3)v_0:$\mathbf{0}$(P$\supset$R),v_1:$\mathbf{0}$(Q$\supset$R) $\Longrightarrow$ $\mathbf{0}$((P$\vee$Q)$\supset$R)
4) if $v_0\mathbf{0}$P $\Longrightarrow$ $\mathbf{0}$Q then $\Longrightarrow$ $\mathbf{0}$(P $\supset$ Q).

PROOF.
For 1)

$$\dfrac{\dfrac{\dfrac{\dfrac{v_0{:}\mathbf{0}(P\supset Q)}{\mathbf{1}(P\supset^* Q)}\text{Swap1} \quad t{:}\mathbf{1}P}{\mathbf{1}Q}\supset^*E \qquad \dfrac{v_1{:}\mathbf{0}(Q\supset R)}{\mathbf{1}(Q\supset^* R)}\text{Swap1}}{\mathbf{1}R}\supset^*E}{\mathbf{0}(P\supset R)^t}\supset I$$

For 2)

$$\dfrac{\dfrac{\dfrac{\dfrac{v_0{:}\mathbf{0}(P\supset Q)}{\mathbf{1}(P\supset^* Q)}\text{Swap1} \quad t{:}\mathbf{1}P}{\mathbf{1}Q}\supset^*E \qquad \dfrac{\dfrac{v_1{:}\mathbf{0}(P\supset R)}{\mathbf{1}(P\supset^* R)}\text{Swap1} \quad t{:}\mathbf{1}P}{\mathbf{1}R}\supset^*E}{\mathbf{1}(Q\wedge R)}\wedge 1I}{\mathbf{0}(P \supset (Q\wedge R))^t}\supset I$$

For 3)
We leave this as an exercise for the reader to practice. The proof idea is similar to the previous two cases.
For 4) This is just **Rule T**. ■

Exercise. Prove that **Swap2**, and $\supset$E are admissible rules. ■

Definition 6.2. Let $<\Gamma,\theta>$ be an inference. Given any $\mathscr{M}$ and w$\in$W, $<\Gamma,\theta>$ is valid at w iff (1) w is a dead-end, can not access any worlds, and if $\mathscr{M}$,w$\models\Gamma$ then $\mathscr{M}$,w$\models\theta$ (2) for every v, $\mathscr{M}$,w,v$\models\Gamma$ implies $\mathscr{M}$,w,v$\models\theta$. $<\Gamma,\theta>$ is $\mathscr{M}$-valid iff $<\Gamma,\theta>$ is valid at every world. $<\Gamma,\theta>$ is SIK+-valid iff for every SIK model $\mathscr{M}$ $<\Gamma,\theta>$ is valid.

Definition 6.3. Let Γ be a subset of marked formulas, $\mathbf{m}^{-1}\Gamma=\{\theta{:}\mathbf{m}\theta\in\Gamma\}$,

Theorem 5. *When $A(\mathscr{D}) \Longrightarrow_{SIK+}\theta$ then $<A(\mathscr{D}),\theta>$ is SIK+-valid.*

Proof. The base case is trivial, and we will just do the hard cases. And then the rest of the cases will be left as an exercise. Assume the obvious IH, and suppose that [ ] was entered in by $\supset$**I**. Then by the IH we know $<$A($\mathscr{D}_0\cup\{\mathbf{1}\sigma\}$, $\mathbf{1}\theta>$ is SIK+-valid. Fix, a $\mathscr{M}$, and a world w. Suppose that w is a dead end. $\mathscr{M}$,w$\models$ $\mathbf{0}(\sigma\supset\theta)$ is trivial. Suppose w is not a dead end then $\mathscr{M}$,w,u$\models$ A($\mathscr{D}$)$_0\cup\{\mathbf{1}\sigma\}$ implies that $\mathscr{M}$,w,u$\models$ $\mathbf{1}\theta$. If wRu, we must have u$\models\mathbf{1}^{-1}$A($\mathscr{D}_0$) implies u$\models\mathbf{1}\theta$ given $\mathscr{M}$,w,u$\models$ $\mathbf{0}^{-1}$A($\mathscr{D}_0$). Then, if we have $\mathscr{M}$,w,u$\models$ A($\mathscr{D}$), we have $\mathscr{M}$,w,u$\models\mathbf{0}^{-1}$ A($\mathscr{D}_0$) then we have - $\mathscr{M}$,w$\models$ $\mathbf{0}(\sigma\supset\theta)$ because if wRu $\mathscr{M}$,u$\models$ $\mathbf{0}\sigma$ then we have $\mathscr{M}$,u$\models\mathbf{1}^{-1}$ A($\mathscr{D}_0$) which implies that $\mathscr{M}$,u$\models\mathbf{0}\theta$. Completing the case.

Suppose that [ ] was entered by $\supset$E. Then by the (IH) we have that $<$A($\mathscr{D}_0$), $\mathbf{0}(\theta\supset\phi)>$ is a SIK-valid inference and $<$A($\mathscr{D}_1$), $\mathbf{1}\theta>$ is also one. Then since A($\mathscr{D}_i$)$\subseteq$A($\mathscr{D}$). So, assume that we have a model $\mathscr{M}$, and a world u. And, u is a dead end. Then $\mathbf{1}\theta$ is trivially satisfied. Suppose u is not a dead end. Then, assume we have $\mathscr{M}$,w,u$\models$ A($\mathscr{D}$). Then we have that $\mathscr{M}$,w,u$\models$ A($\mathscr{D}_0$), and $\mathscr{M}$,w,u$\models$ A($\mathscr{D}_1$), so we have $\mathscr{M}$,w,u$\models\mathbf{0}(\theta\supset\phi)$, and $\mathscr{M}$,w,u$\models\mathbf{1}\theta$. Then we have that $\mathscr{M}$,u$\models\mathbf{0}\theta$. And we have if wRu, u$\models\mathbf{0}\theta$ implies that u$\models\mathbf{0}\phi$, so $\mathscr{M}$,u$\models\mathbf{0}\phi$, and so $\mathscr{M}$,w,u$\models\mathbf{1}\phi$.

Suppose that [ ] was entered in by Swap2). Fix a model $\mathscr{M}$, and a world w. Suppose that w is a dead end, again the conclusion will be trivial. Suppose that w is not a dead end $\mathscr{M}$,w,u$\models$A($\mathscr{D}_0$) so $\mathscr{M}$,w,u$\models$A($\mathscr{D}$). And that will imply $\mathscr{M}$,w,u$\models\mathbf{1}(\theta\supset^*\phi)$. Then for all wRu, we have u$\models(\theta\supset^*\phi)$, meaning u$\models\theta$ implies u$\models\phi$, so $\mathscr{M}$,w,u$\models\mathbf{0}(\theta\supset\phi)$.

Suppose that [ ] was entered by $\supset^*$E. Fix a model $\mathscr{M}$, and a world w. If w is a dead end, and n=1 the conclusion is trivial. If n=0, by IH, $\mathscr{M}$,w$\models$ A($\mathscr{D}_0$), which is the same as $\mathscr{M}$,w$\models\mathbf{0}^{-1}$ A($\mathscr{D}_0$), implies that $\mathscr{M}$,w$\models\mathbf{0}(\theta\supset^*\phi)$, and $\mathscr{M}$,w$\models\mathbf{0}^{-1}$ A($\mathscr{D}_1$) implies $\mathscr{M}$,w$\models\mathbf{0}\theta$, since $\mathbf{0}^{-1}$A($\mathscr{D}_i$)$\subseteq\mathbf{0}^{-1}$A($\mathscr{D}$), we see if $\mathscr{M}$,w$\models$ A($\mathscr{D}$) then $\mathscr{M}$,w$\models\mathbf{0}^{-1}$A($\mathscr{D}$), which will of course imply we have $\mathscr{M}$,w$\models\mathbf{0}\phi$. If w is not a dead end. Then assume we have $\mathscr{M}$,w,u$\models$A($\mathscr{D}$). Then again A($\mathscr{D}_i$)$\subseteq$A($\mathscr{D}$). So we have for n=1 case $\mathscr{M}$,w,u$\models\mathbf{1}(\theta\supset^*\phi)$, and $\mathscr{M}$,w,u$\models\mathbf{1}\theta$. It is easy to see $\mathscr{M}$,w,u$\models\mathbf{1}\phi$. Similarly, for the n=0 case.
The other cases aren't much harder. This is left as an exercise to the reader. ■

Proposition 6.1.*It is not true Γ,v:$\phi \Longrightarrow_{SIK}\theta$ iff $\Gamma \Longrightarrow_{SIK}(\phi\supset\theta)$*

Proof. Appeal to Observation 6.0. ■

Proposition 6.2. *the Semi-Strong Deduction theorem* Γ,*v:*$\mathbf{1}\phi \Longrightarrow \mathbf{1}\theta$ *iff* $\Gamma \Longrightarrow \mathbf{0}(\phi\supset\theta)$.

Proof. Assume that we have Γ,v:$\mathbf{1}\phi \Longrightarrow \mathbf{1}\theta$. Then, if we have Γ, and v:$\mathbf{1}\phi$ then we can conclude $\mathbf{1}\theta$, and by $\supset$*I, we get $\Gamma \Longrightarrow \mathbf{1}(\phi\supset^*\theta)$. and by Swap2, we get that $\Gamma \Longrightarrow$:$\mathbf{0}(\phi\supset\theta)$. Conversely assume that we have $\Gamma \Longrightarrow \mathbf{0}(\phi\supset\theta)$, and if we have v:$\mathbf{1}\phi$, by a use of $\supset$E we get $\mathbf{1}\theta$. ■

Proposition 6.3. *We know that 1)* $\Longrightarrow_{SIK}((A \wedge (A\supset B)) \supset^* B)$, *2)* $\Longrightarrow_{SIK}((A \supset B)\supset^*((B\supset C) \supset (A\supset C)))$,*and 3)* $\Longrightarrow_{SIK} (A \supset^* (B \supset A))$ *are not derivable in SIK without transitivity, reflexivity, or heredity added to* $<*$, *but the following are derivable in SIK+.We have that 1*)* $\Longrightarrow_{SIK+}\mathbf{0}$ $(A \wedge(A\supset^*B))\supset B)$. *We have that 2*)* $\Longrightarrow_{SIK+}\mathbf{0}((A\supset B)\supset((B\supset^*C)\supset(A\supset^*C)))$ *We have that 3*)* $\Longrightarrow_{SIK+}\mathbf{0}(A \supset (B \supset^* A))$

Proof. For 1*) assume v:$\mathbf{1}$(A $\wedge$ (A $\supset^*$ B)), it is easy to see we can derive $\mathbf{1}$A, and $\mathbf{1}$ (A $\supset^*$ B). Then simply use $\supset$I to get $\Longrightarrow_{SIK+}\mathbf{0}$(A $\wedge$(A$\supset^*$B))$\supset$B) discharging v. For 2*) assume that we have v:$\mathbf{0}$(A$\supset$B), and we also assume t:$\mathbf{1}$(B$\supset$C), by **Swap 1** applied to v, we get v:$\mathbf{1}$(A$\supset^*$B), now assume that we have s:$\mathbf{1}$A. Then by a of $\supset$*E then a use of $\supset$E, first applied to v: then to t:, it is easy to get $\mathbf{1}$C, and then using $\supset$*I we obtain $\mathbf{1}$(A$\supset^*$C) discharging s. Then, by a use of $\supset$I, we obtain $\mathbf{0}$((B$\supset^*$C) $\supset$ (A$\supset^*$C)) discharging t, and finally by a use of **Rule T**, it is easy to see $\Longrightarrow_{SIK+}\mathbf{0}$((A$\supset$B)$\supset$((B$\supset^*$C)$\supset$(A$\supset^*$C))) For 3*) assume v:$\mathbf{0}$A. By a use of $\supset$*I, we get that v:$\mathbf{0}$A $\Longrightarrow_{SIK+}$ $\mathbf{0}$(B$\supset^*$A), and here we vacuously discharge q: $\mathbf{0}$B, and by a use of **Rule T**, we obtain $\Longrightarrow_{SIK+}$ $\mathbf{0}$(A $\supset$ (B $\supset^*$ A)). ■

The result of "lifting the hood" was profound. On the ordinary approach, it was hard to see how we could have rules that acted like modus ponens, or conditional introduction for the subintuitionist $\supset$. But, we discovered that $\supset$I in SIK+ works like the typical if-then introduction rule, and $\supset$E acts exactly like a rule for modus ponens. Also, we should note, before "lifting the hood" the rule of inference that looked most similar to the standard intuitionist $\supset$I required that we only work with a context of v:σ. In the new system, the equivalent rule would be **Rule T**. But, this hardly captures the introduction rule, as you cannot make inferences with larger contexts. The real introduction rule is $\supset$**I**, which required that we have marked formulas that package information on future states of formulas. Conveniently, we were also able to write intuitionistic implication in terms of classical implication, with rules **Swap1**, and **Swap2**. And, it is no surprise $\supset$E is an admissible rule because we have rules that guide $\supset^*$ which $\supset$ can be written in terms of. Although, we do not have a strong deduction theorem, we obtain something like it with marked formulas. In addition, we see similar formulas that characterize the conditions on the intuitionist frames are valid in the subintuitionist logic with no conditions on the Kripke frame. The trick to see this result was to think of the formulas in terms of both intuitionist implication, and classical implication; this perspective is often overlooked likely because the subintuitionist typically do not have a symbol for classical implication. In the end, these rules give us a nice perspective of what nice introduction and elimination rules would look like for subintutionist logic. Since we can describe all the important rules that guide the if-then connective in weak Dösen logic, our system is strong enough to say what we want to say. In the future it would be nice to have a Complete system using the aforementioned inference rules. It would also be nice to lift the hood of the modal rules in our system, as Hodes did for classical and intuitionist modal logics [2], as they do not have nice introudction, and elimination properties either.

7: Strengthenings of SIK+

In this section, I aim to show that we have nice rules that correspond to the reflexivity, transitivity, and persistence on Kripke frames using marked formulas. Perhaps, even more natural then the rules we gave for the extensions of SIK in section 5. We will take the rules of SIK+, and expand them, with the resulting systems being SIKR+, SIKT+, SIKP+ for reflexivity, transitivity, and persistence respectively.

The rule for reflexivity is:

Reflexivity) If $C_0 \Longrightarrow \mathscr{D}_0{:}\mathbf{0}(\theta\supset\phi)$, and $C_1 \Longrightarrow \mathscr{D}_1{:}\mathbf{1}\theta$. Then, if we have that $\{C_0,C_1\}$ is coherent then we have that $C_0 \cup C_1 \Longrightarrow \mathscr{D}{:}\mathbf{0}\phi$. We can picture it as follows.

$$\dfrac{\begin{matrix}\mathscr{D}_0 \\ \mathbf{0}(\theta\supset\phi)\end{matrix} \qquad\qquad \begin{matrix}\mathscr{D}_1 \\ \mathbf{1}\theta\end{matrix}}{\mathbf{0}\phi}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}∪[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

The rule for transitivity is:

Transitvity) If $C_0 \Longrightarrow \mathscr{D}_0{:}\mathbf{0}(\theta\supset\phi)$, and $C_1 \Longrightarrow \mathscr{D}_1{:}\mathbf{1}(\phi\supset\beta)$. Then, if we have that $\{C_0,C_1\}$ is coherent then we have that $C_0 \cup C_1 \Longrightarrow \mathscr{D}{:}\mathbf{1}(\theta\supset\beta)$. We can picture it as follows.

$$\dfrac{\begin{matrix}\mathscr{D}_0 \\ \mathbf{0}(\theta\supset\phi)\end{matrix} \qquad\qquad \begin{matrix}\mathscr{D}_1 \\ \mathbf{1}(\phi\supset\beta)\end{matrix}}{\mathbf{1}(\theta\supset\beta)}$$

The dpd($\mathscr{D}$)={[0] $\frown$ s | s $\in$ dpd($\mathscr{D}_0$)}∪[1] $\frown$ s | s $\in$ dpd($\mathscr{D}_1$)}

The rule for persistence is:

Persistence)
If $\mathscr{D}_0$:θ then $\mathscr{D}$:$\mathbf{1}\theta$. Then, we picture $\mathscr{D}$ as follows.

$$\frac{\begin{array}{c}\mathscr{D}_0\\ \mathbf{0}\theta\end{array}}{\mathbf{1}\theta}$$

The dpd($\mathscr{D}$)=$\{[0] \frown s \mid s \in \text{dpd}(\mathscr{D}_0)\}$

Proposition 7.1 *Soundness for Reflexivity*

Proof. Suppose [ ] was in by the Reflexivity rule. We know w is not a dead end as there are none in reflexive models. So, if we have $\mathscr{M}$,w,u$\models$ A($\mathscr{D}$) since A($\mathscr{D}_i$)$\subseteq$A($\mathscr{D}$) for i=0,1. We must have that $\mathscr{M}$,w,u$\models$$\mathbf{0}(\theta\supset\phi)$, and $\mathscr{M}$,w,u$\models$$\mathbf{1}\theta$, since wRw, we have $\mathscr{M}$,w,w$\models$$\mathbf{1}\theta$, so $\mathscr{M}$,w$\models$$\mathbf{0}\theta$ then it follows $\mathscr{M}$,w$\models$$\mathbf{0}\phi$ ■

Proposition 7.2 *Soundness for Transitivity*

Proof. Suppose [ ] was in by the Transitivity rule. Then, of course fix a $\mathscr{M}$, and w. If w is a dead end, it is trivial, so suppose w is not. Then, as usual we have A($\mathscr{D}_i$)$\subseteq$A($\mathscr{D}$) for i=0,1. Then suppose $\mathscr{M}$,w,u $\models$A($\mathscr{D}$) It is easy to see by the (IH) we must have $\mathscr{M}$,w,u $\models$ $\mathbf{0}(\theta\supset\phi)$, and $\mathscr{M}$,w,u $\models$ $\mathbf{1}(\phi\supset\beta)$. Then, if uRv, we have wRv by transitivity, and we know v$\models$$\phi$ implies v$\models$$\beta$. And we know v$\models$ θ implies v$\models$ ϕ then it is easy to see v$\models$$\theta$ implies v$\models$ β, so $\mathscr{M}$,w,u $\models$ $\mathbf{1}(\theta\supset\beta)$. ■

Proposition 7.3 *Soundness for Persistence*

Proof. easy. ■

8: Further Research

It would also be very nice to prove Completeness theorems for SIK+, and to also offer modal introduction and elimination rules, which hopefully can be done with marked formulas as with Hodes' semantics. We have, however, laid the groundwork for what the inference rules guiding the intuitionistic if-then connective in the subintuitionistic logics, and the rules that relate to the added conditions on the Kripke frames. Hopefully, this provides a useful starting point for further research in the field.

As it stands a lot of the work done with subintuitionistic logic works by having a world that can access every world, and defining the consequences of the logic as those forced by the base world in every model. [6], [7], [8]. Celani departs from this approach, and instead uses standard notion of what it means to be a semantic consequence in a Kripke frame. In particular, we look at the formulas which are local consequences in every Kripke model. I would argue this approach is philosophically more meaningful as it is how semantic consequences are generally looked at in intuitionistic logic. So, preserving this property of the logic is a virtue, which are logics have. We have noticed that the characteristic formulas, and the inference rules that were used to prove Completeness were well-behaved because we had a symbol for the classical if-then connective. Greg Restall had been particularly concerned with the fact that if we did not have a omniscient base world which could see every other world in our model, it would be rather difficult to prove completeness as we do not have certain prefixing formulas, or modus ponens. However, if we enrich our language with classical if-then there are ways to obtain similar principles that allow us to prove completeness, as used in this paper. I encourage further researchers to enrich their subintutionist lanaguage with a connective for classical if-then to get the attractive features of the logics we developed. By attractive features, I am referring to the Completeness results, and the system SIK+ where we are able to obtain theorems with intuitionistic $\supset$ that are close to ones which were invalid in a standard subintuitionistic logic, and we were able to write intutionistic if-then in terms of classical if-then, which leaves us with a more famailar proof theory.

9: Conclusion

In this paper, we analyze how subintuitionistic logics interact with modal logics through a natural deduction system, and model-theoretic semantics, which we prove Soundness and Completeness theorems for logics of different strengths both on the intuitionistic level, and the modal level. We characterize the local consequence relation of all Kripke frames for the subintuitionistic logic enriched with a language strong enough to talk about classical modal logics as well. We notice the intuitionistic $\supset$ does not seem to be nicely guided by introduction, and elimination rules in SIK, which is a phenomenon that seems common with modal logic. We locate the problem being that there is not enough information in the formulas to conclude what will happen in future states. We follow Hodes' approach of using marked formulas to provide nice introduction and elimination rules for the intuitonistic $\supset$ in our subintuitionist logics. We go further with this approach and talk about rules for that characterize the reflexivity, transitivity, and persistence on the Kripke frames with marked formulas. We hope in the future completeness theorems can be proved with these rules, as well as with marked formulas for the modal operators in our language. We also hope to see more research experiment with a connective for classical if-then in their subintuitionistic logic.

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