# Existential Import : an Extensional Approach

# Existential Import: an Extensional Approach

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## Introduction

*Existential import* is a notion researchers often mention, not explaining it in depth[1], but interests us. We approach it by beginning with the traditional diagrams, namely Euler diagrams and Venn diagrams (ch.1). As to $\exists x(Fx \wedge Gx)$, the Venn diagram is adopted (i.e. (1)). As to $\forall x(Fx \rightarrow Gx)$, the Euler diagram is adopted (i.e. (3)). By contrasting the two, we find the inference from the universal sentence to the particular sentence impossible (§2). Exactly here, existential import steps into the picture.

## Chapter 1. Diagrammatic Approaches[2]

First of all, we see $\{\forall x(Fx \rightarrow Gx)\} \nvdash \exists x(Fx \wedge Gx)$.[3] While $\forall x(Fx \rightarrow Gx)$ can be true vacuously, $\exists x(Fx \wedge Gx)$ cannot. It follows from this that the former cannot imply the latter. We see it by contrasting two diagrams, the Venn diagram and the Euler diagram.

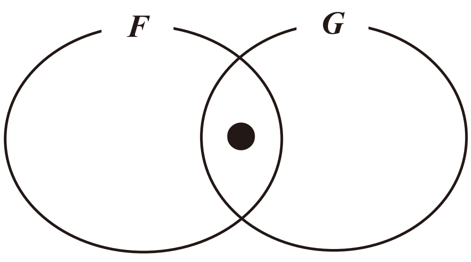
### §1. Venn diagrams and Euler Diagrams[4]

The explanations using diagrams help beginners so much. But something inadequate remains. We watch this, tracing a normal course of illustration.

#### 1.1. The Venn diagram

*Venn diagrams* work well for beginners of logic. It enables beginners to grasp the ideas so easily. To take an example:

(1)　A Venn diagram[5]　$\exists x(Fx \wedge Gx)$



This is a Venn diagram for *the particular sentence*[6]. Beginners may wonder why "Some *F* is a *G*."[7] is formulated as $\exists x(Fx \wedge Gx)$. Then, this diagram will help.

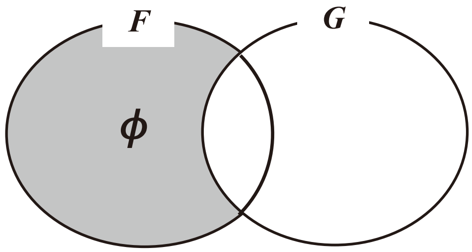
The intersection ensures the conjunction in $\exists x(Fx \wedge Gx)$. The point ● ensures *the existential character* of the particular sentence[8].

**1.2. The Euler diagram**

We make sure Venn is John Venn (1834-1923), a notable English mathematician.

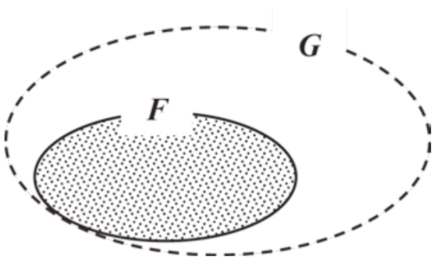
In contrast with the particular sentence, however, the Venn diagram begins malfunctioning in illustrating the universal sentence:

(2) A Venn diagamm[9] $\forall x(Fx \rightarrow Gx)$




The part colored gray means $\phi$, an empty set. This blockage (or emptiness) appears surplus[10], because we only need such an indication of inclusion $F \subset G$ as the following:

(3) An Euler diagram[11] $\forall x(Fx \rightarrow Gx)$




This is much better than (2). Note that the part colored gray does not mean $\phi$ this time, but emphasizes the realm of *F*.

As is well known, diagram (3) is called *an Euler diagram*. We make sure Euler is Leonhard Euler (1707-1783), a Swiss mathematician.

### 1.3. A Detour through Set Theory

The broken line in diagram (3) will not be discussed. It expresses the Aristotelian notion of *being undistributed* (cf. Kondo et al. 1979, pp.23-25). *Distribution*, a typically old-fashioned idea, is ascribed to Euler (cf. Sudo 1947, pp.51-58), who had no exact notion of sets (or *classes*). Set theory was initiated long after his death[12].

Venn actually abolished the notion of distribution. We follow it not to touch on distribution any further[13].

## §2. A Syntactical Approach

We saw Venn diagrams and Euler diagrams. By contrasting the two, we find $\{\forall x(Fx \to Gx)\} \nvdash \exists x(Fx \land Gx)$, at which our discussion centers.

### 2.1. A Collapse of Inference

In Aristotelian logic, the universal sentence supersedes the particular sentence. The former naturally implies the latter. So it seems. But that is not the case, actually.

See the preceding two diagrams, (3) and (1). The crossover of the Venn diagram and the Euler diagram is unproblematic, since neither Venn nor Euler knew set theory back then (§1.3).

The problem is $\{\forall x(Fx \to Gx)\} \nvdash \exists x(Fx \land Gx)$. $\exists x(Fx \land Gx)$ has ● in (3) while in (1), $\forall x(Fx \to Gx)$ does not. This slight difference makes the inference from $\forall x(Fx \to Gx)$ to $\exists x(Fx \land Gx)$ collapse.

### 2.2. A Vacuous Truth

Let us see the collapse of inference in more detail. Let $F$ be something empty, such as *a round triangle*. $F = \phi$, then. It makes $F \cap G = \phi$, so (1) at least never holds.

By contrast, in the same setting as $F = \phi$, (3) holds, since $F \subset G$ even if $F = \phi$. This is simply because $\phi \subset G$ for any $G$.

The so-called vacuous truth has a hand in this logic[14]. Define $\phi \subset G$ as $\forall x(x \in \phi \to x \in G)$[15]. And we can say for any $x$, $x \notin \phi$.[16] Based on these, $\forall x(x \in \phi \to x \in G)$ gets true vacuously.

### 2.3. A Carnapean-Kantian Approach

The preceding argument can be put in syntax. We may call it *a Carnapean-Kantian approach*, because it reminds us of the two philosophers.

Let $F$ be "a round triangle" as in the previous section (§2.2). And let $G$ be "square." Then, $\forall x(Fx \to Gx)$ is read, as follows:

(4) $\forall x$(($x$ is a round triangle) → ($x$ is square)) [vacuously true]

This says, “Every round triangle is square.” It is vacuously true, because its antecedent “($x$ is a round triangle)” is contradictory.

This contradiction, to be recognized, requires something extralogical, which Rudolf Carnap (1891-1970) would specify under the name of a *meaning postulate*:[17]

(5) $\forall x((x \text{ is round}) \rightarrow \neg(x \text{ is a triangle}))$ [meaning postulate]

By this postulate, the antecedent “($x$ is a round triangle)” in (4) is calculated[18] to be found contradictory[19].

## §3. Existential Import

$\forall x(Fx \rightarrow Gx)$ can be true vacuously, which collapses the inference $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$. At the end of this chapter, we see how logicians overcame this problem.

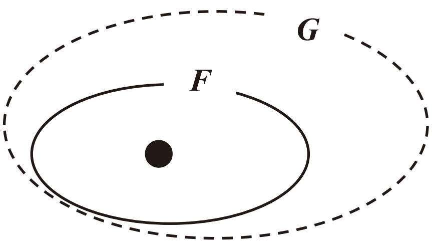
### 3.1. Existential Import

“Every round triangle is square.” It is vacuously true. But there is not a round triangle which is square. So $\{\forall x(Fx \rightarrow Gx)\} \nvdash \exists x(Fx \wedge Gx)$.

This appears odd to Aristotelian logicians thinking the universal sentence supersedes the particular sentence. Some refused, therefore, the existential character of the particular sentence. That is, “Some round triangle is square.” need not be “There is a round triangle which is square.” Among them were Euler[20] and Gottfried Leibnitz (1646-1716)[21].

Others thought the opposite. According to them, something extralogical is needed to ensure the inference from the universal sentence to the particular sentence; that is ●, which was lacking in (3), the diagram of $\forall x(Fx \rightarrow Gx)$.

(6) Existential import for $F$ in $\forall x(Fx \rightarrow Gx)$




This is how ● was brought in (3) as well, which logicians called *existential import*. Note that the broken line, which expresses being undistributed, is no longer discussed (§1.3).

### 3.2. $F^e$

Finally, we have reached existential import, the initial problem of this article. This idea is often expressed syntactically:

(7)　$\forall x(F^e x \to Gx)$　[existential import]

The superscript "*e*" indicates existential import[22].

$F^e$ makes sure of the existence of an element in *F*. In other words, the antecedent *F* must be something meaningful, not such a contradictory predicate as "(*x* is a round triangle)."

$F^e$ prevents the universal sentence from being vacuously true.

$\{\forall x(F^e x \to Gx)\} \vdash \exists x(Fx \land Gx)$ is graphically recognized. Compare (6) with (1). Clearly (6) implies (1). But not vice versa; $\exists x(Fx \land Gx)$ is compatible with $\exists x(Fx \land \neg Gx)$, which, i.e. $\exists x(Fx \land Gx) \land \exists x(Fx \land \neg Gx)$, contradicts $\forall x(Fx \to Gx)$.

## Chapter 2. Semantics

This is how logicians made up for the collapse of the inference, $\{\forall x(Fx \to Gx)\} \nvdash \exists x(Fx \land Gx)$, with the help of existential import. It secretly referred to semantics, however. We look into this aspect of the problem in succession.

### §4. $F^e$ vs. *F*

Existential import was indicated with $F^e$. But this notation is suspicious. We investigate it in the sequel.

### 4.1. The Problem of $F^e$

Syntax has no room to allow $F^e$, to tell the truth. For all syntax can do is prove. To prove (a theorem or an inference), all that we can use is axioms (or proof figures)[23]. Thus, $F^e$ cannot but be lain outside syntax[24].

One who sees $\forall x(F^e x \to Gx)$ images something like (6). But diagram (6) lies outside syntax, because in syntax, all we can do is prove; all we can use is axioms (or proof figures). Then, what is $F^e$, a sign accompanied by a diagrammatical image?

### 4.2. Amalgamated Expressions

$F^e$ is ungrammatical, lying outside syntax. This pushes us back to look over $\forall x(Fx \to Gx)$ again. We have checked how it gets true vacuously (§2.2). For that, we used such *an amalgamated expression*[25] as "*x* is a round triangle." Can we use such a expression in the first place? This query about amalgams turns our eyes to the actualities of logic.

### 4.3. Schematic Letters

In logic, we use *schematic letters*, such as *p, q*, $c_1$, $c_2$, *F, G,* etc. Amalgams, such as "*x* is a round triangle," are transformations of predicates among them, so to speak. On the other hand, we must think of schematic letters as constants, as are individual constants, such as $c_1$, $c_2$, etc. Predicates like *F, G,* etc. should be *constants*, The same is true of *p, q,* etc.[26]

As constants, schematic letters constitute a language, into which ordinary expressions, such as "a round triangle," are translated.

This is the actualities of logic. In the structure, i.e. symbolic (artificial) languages of logic vs. ordinary (natural) languages, we have no room to allow amalgamated expressions.

## §5. Immature Semantics

We have reflected on the actualities of logic. From the angle, we look over the previous argument again.

### 5.1. Interpretations

It is in the Carnapean-Kantian approach (§2.3) that we used amalgams, such as "*x* is a round triangle." But we should deny it, reflecting on the actualities of logic (§4.3).

What did we do back then? The answer will be that we made *interpretations*.

Amalgamated expressions are products of immature interpretations or, as it were, *immature semantics*. In terms of *mature semantics*, namely model theoretic semantics[27], the preceding arguments are corrected in the following way:

(8) The origin of an amalgam

(i) $\forall x(Fx \rightarrow Gx)$ [a formal expression in logic]

(ii) $I_1(F) = \{X \mid X \text{ is a round triangle}\}$[28], $I_1(G) = \{X \mid X \text{ is square}\}$. [*an interpretation*]

(iii) $\forall x((x \text{ is a round triangle}) \rightarrow (x \text{ is square}))$ [amalgam (4)]

This is how amalgam (4) originated from $\forall x(Fx \rightarrow Gx)$. Although the process is complicated, we merely reflected model theoretic interpretations of *F* and *G* into amalgam (4), or (8-iii) above[29].

The kernel is an interpretation, which is model theoretic mappings of predicates, *F* and $G$, onto[30] sets, or more precisely, subsets of a universe of discourse $D_1$ in model $M_1 = \langle D_1, I_1 \rangle$[31], as indicated in (8-ii)[32].

### 5.2. A Translation

An interpretation, reversed, becomes *a translation*, which is also found in the Carnapean-Kantian approach (§2.3). Let us examine it, too.

We decomposed (4), namely (8-iii) above, into $\forall x((x \text{ is round}) \wedge (x \text{ is a triangle}) \rightarrow (x \text{ is square}))$[33]. Behind it, we did a translation; that is, we translated an ordinary expression “a round triangle” into two schematic letters $F_1$ and $F_2$.

(9) The decomposition of an amalgam

(i) $\forall x((x \text{ is a round triangle}) \rightarrow (x \text{ is square}))$ [(8-iii), (4)]

(ii) $\forall x( F_1x \wedge F_2x \rightarrow Gx )$ [*a translation*]

(iii) $I_2(F_1) = \{X \mid X \text{ is round}\}$, $I_2(F_2) = \{X \mid X \text{ is a triangle}\}$. [an interpretation]

(iv) $\forall x((x \text{ is round}) \wedge (x \text{ is a triangle}) \rightarrow (x \text{ is square}))$ [amalgam]

$F_1$ and $F_2$ are, again, interpreted as “round” and “a triangle,” for which we made an interpretation.

This procedure of (9) was unnecessary, however. We can deduce (9-iv) from meaning postulate (5) directly, since $\{\forall x(F_1x \rightarrow \neg F_2x)\} \vdash \forall x(F_1x \wedge F_2x \rightarrow Gx)$, which is a mere logical truth[34]. We do not deal with this aspect of the argument any further.

## §6. Mature Semantics

In terms of model theoretic semantics, awkwardness or immaturity of the preceding discussions is to be revealed. We dig into this in succession.

### 6.1. A Reference Tree

The true nature of the amalgamated expressions lies in immature semantics (§5). Amalgamated expressions disappear if we apply correct semantics. As such, we take up a refutation tree, to begin with[35].

(10) A refutation tree[36]

1. $\forall x(Fx \rightarrow Gx)$ ✓
2. $\neg\exists x(Fx \wedge Gx)$ ✓
3. $Fa_1 \rightarrow Ga_1$ [1.R∀] ✓✓[37]
4. $\forall x \neg(Fx \wedge Gx)$ [2.R¬∃] ✓
5. $\neg(Fa_1 \wedge Ga_1)$ [4.R∀] ✓

6. $\neg Fa_1$ [5.R¬∧] | $\neg Ga_1$ [5.R¬∧]

7. $\neg Fa_1$ [3.R→] | $Ga_1$ [3.R→] | $\neg Fa_1$ [3.R→] | $Ga_1$ [3.R→]

8. | | | × [6,7]

This is a refutation tree of $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$. We can realize from this why we were driven to refer to such an amalgam as "*x* is a round triangle."

### 6.2. Counter Models

On the basis of (10), we can make two models to counter the inference, $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$.

First, see the left branch of (10). It tells us that $D_3$ having that object which is not *F* but *G* becomes a model to counter $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$. We name this model $M_3 = \langle D_3, I_3 \rangle$[38].

Second, see the right branch of (10). It tells us that $D_4$ having that object which is not *G* nor *F* becomes a model to counter $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$. We name this model $M_4 = \langle D_4, I_4 \rangle$.

The point common to $M_3$ and $M_4$ is that the object designated by $a_1$ is $\neg F$, that is, not *F*. This, i.e. $\neg Fa_1$, is a clue to realize "*x* is a round triangle." Jumping to the gist, we have to make *F* empty the way any object cannot satisfy it.

### 6.4. $M_3$

The essential point of counter models is to make *F* empty. We form $M_3$ as such.

(11) $M_3 = \langle D_3, I_3 \rangle$: $D_3 = \{a_1\}$, $I_3(F) = \phi$, $I_3(G) = \{a_1\}$, $\eta_1(a) = a_1$.

As to the free variable, we use "*a*" alone, which is enough since we deal with $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$ alone. Again, there is only one object, $a_1$, in $D_3$, so we have only to take up one evaluation, $\eta_1$.

Let us see $M_3$ countering $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$. We must begin with the form of the logical consequence $\{\forall x(Fx \rightarrow Gx)\} \vDash \exists x(Fx \wedge Gx)$.[39] Following the definition, it is said that for any model[40] $M_i$ and for any evaluation $\eta_j$, if $M_i \vDash_{\eta j} \forall x(Fx \rightarrow Gx)$, then $M_i \vDash_{\eta j} \exists x(Fx \wedge Gx)$.

In the present case, we begin with the middle part of this logic, having recourse to readers' knowledge:

(12) $M_3 \vDash \forall x(Fx \rightarrow Gx)$[41]
$\Longleftrightarrow$ for any evaluation $\eta_j$, $M_3 \vDash_{\eta j} Fa \rightarrow Ga$
$\Longleftrightarrow$ for any evaluation $\eta_j$, $M_3 \nvDash_{\eta j} Fa$ or $M_3 \vDash_{\eta j} Ga$
$\Longleftrightarrow$ for any evaluation $\eta_j$, $\eta_j(a) \notin I_3(F)$ or $\eta_j(a) \in I_3(G)$
$\Longleftrightarrow$ for any evaluation $\eta_j$, $\eta_j(a) \notin \phi$ or $\eta_j(a) \in \{\alpha_1\}$.

As $\eta_j$, we have $\eta_1$ alone in $M_3$. $\eta_1(a) = \alpha_1 \notin \phi$ as a matter of course[42], while $\eta_1(a) = \alpha_1 \in \{\alpha_1\}$. It follows from this that both disjuncts hold (although it is not necessary), so condition (12) is met; therefore, $M_3 \vDash \forall x(Fx \rightarrow Gx)$.

### 6.3. Satisfaction Conditions Continued

Next, we see the conclusion part of $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$.

(13) $M_3 \vDash \exists x(Fx \wedge Gx)$
$\Longleftrightarrow$ for some evaluation[43] $\eta_j$, $M_3 \vDash_{\eta j} Fa \wedge Ga$
$\Longleftrightarrow$ for some evaluation $\eta_j$, $M_3 \vDash_{\eta j} Fa$ and $M_3 \vDash_{\eta j} Ga$
$\Longleftrightarrow$ for any evaluation $\eta_j$, $\eta_j(a) \in I_3(F)$ and $\eta_j(\mathrm{a}) \in I_3(G)$
$\Longleftrightarrow$ for any evaluation $\eta_j$, $\eta_j(a) \in \phi$ and $\eta_j(a) \in \{\alpha_1\}$.

As $\eta_j$, we have $\eta_1$ alone in $M_3$. It is not the case that $\eta_1(a) = \alpha_1 \in \phi$ even though $\eta_1(a) = \alpha_1 \in \{\alpha_1\}$. So the conjunction, as a whole, does not hold; that is, $M_3 \nvDash \exists x(Fx \wedge Gx)$.

The premise holds, but the conclusion does not; therefore, $\{\forall x(Fx \rightarrow Gx)\} \nvDash \exists x(Fx \wedge Gx)$. $M_3$ counters $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$ for sure.

## §7. A Landing Point

Mature semantics, model theoretic semantics plus a refutation tree, gives us a clue to analyze $F^e$ and amalgams. We focus on it, summarizing the preceding discussions.

### 7.1. The Emptiness of *F*

Mature semantics tells us that $I_3(F) = \phi$, the emptiness of *F*, is a sufficient condition for $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$ collapsing. This point has already been grasped (§2.2), to tell the truth. Back then, however, we went in a wrong direction, taking the point as something verbal, namely such an amalgam as "*x* is a round triangle"[44]. It was quite unnecessary. We had only to make *F* empty; $I_3(F) = \phi$, which would do for everything in our course of discussions.

### 7.2. Extensionality

Below is the correct interpretation taking the place of (8) above.

(14) An extensional interpretation

(i) $\forall x(Fx \rightarrow Gx)$ [the universal sentence]

(ii) $I_3(F) = \phi$, $I_3(G) = \{a_1\}$. [*an extensional interpretation*]

This is all we had to know about the vacuous truth of $\forall x(Fx \rightarrow Gx)$ and the collapse of $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$. We can call this type of interpretation, which has no references to verbal contents of signs, *extensional*.

### 7.3. Existential Import Reformulated

$F^e$, the initial understanding of existential import, was originally directed to (14), $I_3(F) = \phi$ especially. $F^e$ was a measure to guard the universal sentence from the interpretation making $F$ empty. From this, in turn, we can have a better formulation of existential import than $F^e$.

(15) Existential import To secure $\{\forall x(Fx \rightarrow Gx)\} \vdash \exists x(Fx \wedge Gx)$, we keep $F$ non-empty; that is, $I_i(F) \neq \phi$ for any $I_i$.

This is the correct formulation of existential import. With this, we also realize the true intention of diagram (6).

### 7.4. A Concluding Remark

Beginning with diagrammatic approaches, we saw the problem of $\{\forall x(Fx \rightarrow Gx)\} \nvdash \exists x(Fx \wedge Gx)$, and found it solved on the premise of existential import (ch.1).

But the notion of existential import included something extralogical. To exclude that, we scrutinized amalgamated expressions to see that they are closely linked with immature semantics (§4-6).

In the end, we reached the extensional formulation of existential import with the help of model theoretic semantics (§6-7).

### Appendix 1. Intensionality

We have ended up with a model theoretic formulation of existential import. In the following appendices, we add a few points to the main discussions, which will lead to a further development.

## §8. Intension

A crucial point in this article is that we have done away with those amalgams which refer to verbal expressions. We dig into it a little further.

### 8.1. Past Logicians

Beginners tend to rewrite $\forall x(Fx \rightarrow Gx)$ into $\forall x((x \text{ is a round triangle}) \rightarrow (x \text{ is square}))$, to realize symbols. It is this approach that we have criticized in this article under the name of amalgams.

Amalgamated expressions, or approaches referring to them, had been inevitable, it seems, before model theoretic semantics was established. Even Venn (§1.1-1.2) and Euler (§1.2-1.3) had no clear image of mode theoretic semantics[45]. Thus, we had to say their approaches were immature. So was Leibniz's (§3.1)

### 8.2. Carnap

We nay count Carnap (§2.3) among them, namely such immature semanticists.

Carnap wrote three books on semantics under the common title of *Studies in Semantics* (Carnap 1942, p.ix, p.255; Carnap 1947, p.iii, p.251): *Introduction to Semantics* (published in 1942), *Formalization of Logic* (published in 1943), and *Meaning and Necessity* (published in 1947).

None of these books, however, became a correct showcase of model theoretic semantics, it seems[46].

### 8.3. Expressions of Meanings

Carnap's endeavor is said to have opened up a new path to logic or semantics, on the other hand. Today, it is called *intensional logic*[47].

(16) (i) The extension of a predicator is the corresponding class.
(ii) The intension of a predicator is the corresponding property.

This is Carnap's provisions of logic (1942, p.19). However, we cannot accept these, because we see a shadow of amalgams behind Carnap's locution of the property.

Verbal expressions of meanings, which we have seen on the letters of amalgams, are unnecessary, whether you may call them properties or intensions.

If a model, $M_i = \langle D_i, I_i \rangle$, has $n$ objects, i.e. $D_i = \{a_1, ..., a_n\}$, each of $F$, $G$, etc. has $2^n$ interpretations potentially[48]. All of these are fully expressed *extensionally*: $\phi$, $\{a_1\}$, $\{a_1, a_3\}$, etc. Intentional expressions like $\{X \mid ... X ...\}$ are not required.

**Appendix 2. The λ-calculus**

On the extended line of Carnap's logic, we find Church's λ-calculus as well. Let us touch on it.

## §9. An Intensional Descendant

The origin of intensional logic can be traced back to the logical school. We want to clarify it in addition to the preceding discussion.

### 9.1. Church

In 1941, Alonzo Church (1903-1995) published the book titled *The Calculi of Lambda-Conversion*. This book is the first to have developed what we call the *λ-calculus* today.

The λ-calculus is a computer-friendly system[49]. In logic as we have dealt with it so far, *F* is distinguished from its formula[50], such as *Fa*, to be thought of as an independent expression[51].

However[52], in distinction from that, which we may call *formalism*, the school propounded by David Hilbert (1862-1943)[53], there was another school, *the logical school* began by Gottlob Frege (1848-1925)[54] and by Bertrand Russel (1872-1970)[55] separately. According to it, *F* is regarded as a *propositional function*. Church's λ-calculus was descendent from this school.

### 9.2. The Function

The heart of the λ-calculus is to reserve the intensional aspect of the function. In set theory, we usually learn a functions as a set of ordered pairs:

(17) $I_4(S) = \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, \cdots\}$

This is an interpretation of the successor function *Sx*.[56] Church's λ-calculus is an antithesis to this extensional interpretation[57]. He persisted in interpreting the function as it is, in other words, as *a rule of correspondence* (Church 1941, p.1)[58].

(18)[59] $(\lambda x.\ Sx)$

This incarnates Church's idea in the most straightforward way. With it, he *abstracted*[60] function *S* from *Sx*, preserving its character as a role of correspondence.

### 9.3. The β-conversion

That characteristic as a rule of correspondence which Church thought the function has is realized with the following expression:

(19) $(\lambda x.\ Sx)0$

This means, “function $S$ is *applied*[61] to 0 as its argument.” The following follows this:

(20) $(\lambda x.\ Sx)0 \Rightarrow S0 \Rightarrow 1.$

This is the so-called *β-conversion*[62].

## §10. The Problem Left

The intensional descendant can be connected with the problem of existence unexpectedly. We confirm this point at the end of this article.

### 10.1. An Intensional Viewpoint

Let us make clearer the intensional aspect of Church’s λ-calculus behind it by contrasting it with model theoretic semantics.

When model theoretic semantics deals with $S0$, we read behind it $\langle 0, 1\rangle \in \{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, ...\}$, which stand for $S0 = 1$. That is, we eliminate the notation $S$, reducing it into the extensional notion, or set $\{\langle 0, 1\rangle, \langle 1, 2\rangle, \langle 2, 3\rangle, ...\}$, as we saw earlier (i.e. (17)).

In contrast, the λ-calculus preserves $S$, as we saw in (20). This is the difference and why the λ-calculus is thought intensional[63].

### 10.2. The Propositional Function

Church preserved the verbal, intensional aspect of the function (i.e. (19)) as opposed to our model theoretic semantics. As noted earlier, Church’s view is descendent from the logical school (§8.1).

Russel already invented the notation of the *circumflex* ^ to show the propositional function. His notation of $S\hat{x}$, for example, is the predecessor of $(\lambda x.\ Sx)$[64].

Our interest here is in Russel’s view on *the predicate*. As opposed to our distinction between predicate $F$ and formula $Fa$ (§8.1), Russel equated the two to adopt the notation $F\hat{x}$. This view is found in Church (1941, p.2) as well[65]. A predicate is a propositional function that receives an individual constant as its argument to return a truth value[66].

Our concern is the vision next to this. The existential quantifier is laid above this understanding, as it were; it is thought a second-order function.

### 10.3. The Problem of the Quantifier

According to Russel and Church, the amalgamated expression, such as “$x$ is a round triangle,” is kept in the form of ($\hat{x}$ is a round triangle) or ($\lambda x.\ x$ is a round triangle). This appears reasonable but raised another problem, which called in Frege's idea of the quantifier as a second-order function:

(21) [...] Existenz [ist] Eigenschaft des Begriffes[.] (Frege 1884, sec.53)

By saying this, Frege notoriously introduced the existential quantifier as *a second-order function*[67]. Church followed it:

(22) [T]he existential quantifier [...] is a function for which the range of arguments consists of propositional functions, and the range of values consists of truth values. (Church 1941, p.2)

The notation incarnating his idea will be as follows:

(23)[68] $(\lambda F.\ \exists x Fx)$

This corresponds to (18) above. From this perspective, the existential quantifier is considered a second-order function that receives $F$ as its argument to return a truth value.

It might be this idea that we must tackle as the genuine issue coming after our debates over existential import. Existence remains a problem even now, long after mathematicians established symbolic logic.

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1 Ishiguro 1984, pp.196f.; Nomoto 1990, p.31; Kondo et al. 1979, p.55.

2 This article consists of chapters, sections, subsections and one appendix. Chapters are divided into sections, and sections are divided into subsections. Chapters and sections have short summaries under their headings respectively. Sections and subsections are referred to, for example, by " § 1" and " § 1.1" respectively, to be distinguished from "sec." for References. Chapters are referred to by "ch."

3 Notations in logic are owed to Kaneko 2021.

4 Section numbers are counted continuously regardless of chapters changing.

5 Cf. Kaneko 2021, p.49 fig.76; Kaneko 2019, p.88 fig.(142); Nolt et al. 2011, p.116 fig.5-3; Kondo et al, 1979, p.56 fig.13.

6 We take it for granted that sentences in logic are divided into three groups: *singular sentences, particular sentences*, and *universal sentences*. See Kaneko 2019, sec.106; Kaneko 2021, sec.90, sec.95.

7 Most textbook writers say, "Some *F*('s) are *G*('s)," stating the particular sentence (Nolt et al. 2011, pp.111f.), but it is wrong. The origin of the particular sentence is eventually found in sentences having an indefinite pronoun "somebody" and "something" (Kaneko 2021, sec.102). In this sense, the particular sentence does not mean, "Several *F*'s are *G*'s." but "Some *F* is a *G*."

(cf. Badesa 2004, p.6). Here "some" expresses an unknown person or thing (cf. Konishi et al. 2006, p.1820; Swan 2005, p.547). From this perspective, "some" in the particular sentence is equated with the indefinite article "a (an)." "Some *F* is a *G*." is equated with "An *F* is a *G*." In terms of logic, "some" is equated with the so-called *witnessing constant* (cf. Kaneko 2021, sec.189).

8 The existential character of the particular sentence can be discussed from several angles (Kaneko 2021, sec.94, pt.VII ch.1; Kaneko 2019, sec.112-113).

9 Cf. Nolt et al. 2011, p.116 fig.5-2 ; Kondo et al. 1979, p.55 fig.10.

10 According to Aristotelian logic, *F* is *distributed* in the universal sentence "Every *F* is *G*," so the blockage by $\phi$ in diagram (2) is technically not blamable. See Kondo et al. 1979, pp.24-25.

11 Cf. Kaneko 2021, p.49 fig.75; Kaneko 2019, p.88 fig.(143); Sudo 1947, pp.53-54; Kondo et al. 1979, p.25.

12 By Georg Cantor (1845-1918) and his followers (cf. Kaneko 2021, pp.92f.).

13 The relationship between set theory and Venn diagrams (or Euler diagrams) is a touchy topic, which most mathematical books disregarded (Chart Inst. 2011, pp.268f.; Matsuzaka 1990a, pp.745f.; Matsuzaka 1990b, pp.1261f.). In addition, it seems that Venn and Euler discussed those pure *concepts* like "animal" and "human being" which were not confined to mathematical terms, affected by the Aristotelian tradition, so that the border line between set theory and philosophical debates (over concepts) was blurred.

14 This article presupposes readers' knowledge of the vacuous truth (Kaneko 2021, sec.56).

15 This is a normal course of discussion in set theory. See Kaneko 2021, p.94 n.76.

16 It follows from the axiom of the empty set that $\forall x(x \notin \phi)$. For further details, see Kaneko 2021, sec.191.

17 A meaning postulate was originally introduced by Carnap to analyze the Kantian notion of *the analytic truth* (Carnap 1956, p.222; Kant 1787, B11). Kant was wrong in ascribing analyticity to the concept that the subject of each sentence has (cf. Kaneko 2004, sec.2; Ishikawa 1995, p.102). Rather, the analytic truth is syntactic, calling for something extralogical that we may call *a proper axiom* (Kaneko 2019, sec.207; Kaneko 2021, p.202 n.22). That axiom is none other than the meaning postulate mentioned here (in the text).

18 Note that modern logic can often be mentioned as *calculation* to be distinguished from *psychological inferences* or the like (cf. Kaneko 2019, sec.27).

19 First, we decompose "($x$ is a round triangle)" into "($x$ is round) $\wedge$ ($x$ is a triangle)." Second, independently of that, we gain (*) below from (5) through the same logic as $\{p \rightarrow \neg q\} \vdash p \longleftrightarrow p \wedge \neg q$:

(*) $\forall x$(($x$ is round) $\longleftrightarrow$ ($x$ is round) $\wedge$ $\neg$($x$ is a triangle))

Third, we apply (*) to the conjunct "($x$ is round)" in "($x$ is round) $\wedge$ ($x$ is a triangle)." Then, "($x$ is round) $\wedge$ $\neg$($x$ is a triangle) $\wedge$ ($x$ is a triangle)" is gained, which is a contradiction. ∎

20 Cf. Kondo et al. 1979, p.25; Sudo 1947, p.55.

21 Cf. Ishiguro 1984, pp.196f.

22 Cf. Nomoto 1990, p.31.

23 For further details, see Kaneko 2021, pt.I.

24 Similarly, Iida (1987, p.34) adopted such a notation as $\forall_M x \exists_M y Rxy$ to express the domain "M" the bound variables, $x$ and $y$, range over. Yet, for the same reason as we refuse $F^e$, those nota-

tions, $\forall_M$ and $\exists_M$, are refused.

[25] More precisely, "an expression amalgamated with another expression belonging to a different language." Sometimes it is called an amalgam, too. Cf. Kaneko 2021, sec.15.

[26] The views in this section ( § 4.3) are owed to Kaneko 2021, sec.23.

[27] The distinction between immature semantics and mature semantics is peculiar to this article alone. As to model theoretic semantics, see Kaneko 2021, pt.VI.

[28] "$X$" is a bound variable used only in a meta-language. See Kaneko 2021, sec.137.

[29] As the abbreviation for (8) (iii), for example, we write (8-iii).

[30] "Mappings of $A$ onto $B$" means *surjections* (cf. Enderton 1977, p.43).

[31] Number 1 of $M_1$ is arbitrary.

[32] As to semantic notations, see Kaneko 2021, pt.VI.

[33] See also n.19 above.

[34] The same logic as $\{p\rightarrow\neg q\}\vdash(p\wedge q)\rightarrow r$.

[35] Strictly speaking, the refutation tree is a mixture of syntax and semantics (Kaneko 2021, sec.126).

[36] As to how we write a refutation tree, see Kaneko 2021, pt.V.

[37] Usually we think of the sentence (or formula) checked *once* as removed from the list (Kaneko 2021, p.66), but in the case where the tree has a branching part, such as 6 in (10), it is admitted to check the same sentence (or formula) *twice*.

[38] We used number 1 for (8), and 2 for (9), so the number here is 3.

[39] As to the notion of the logical consequence, see Kaneko 2021, sec.154.

[40] The locution around this, namely "for any model $M_i$," "for every model $M_i$," etc., is touchy in logic. The author made references to several writers, among whom Enderton (1972, p.71, p.83, p.89) is the best.

[41] As to *sentences*, not formulas, we can begin with the so-called *truth*, not specifying an evaluation. See Kaneko 2021, sec.145.

[42] See n.16 above.

[43] As to "some," see n.7 above. The locution around this, namely "for some evaluation $\eta_j$," "for an evaluation $\eta_j$," etc., is touchy in logic. The author owes the phrase in the text to Enderton (1972, p.71, p.84, p.85, p.87).

[44] Interestingly enough, Frege (1884, sec.53) took a similar approach when he talked about logic.

[45] Calixto Badesa, who wrote a book on the history of model theoretic semantics (2004), never mentioned Euler while he touched on Venn's work slightly (Badesa 2004, p.13 n.18). He said major predecessors of semantics are George Boole (1815-1864), Charles Peirce (1839-1914), and Ernst Schröder (1841-1902); he named their studies in semantics *the algebraic study of logic* (Badesa 2004, p.ix; Kaneko 2019, sec.9).

[46] For further details, see Kaneko 2022, glossary, sec.196, etc.

[47] Cf. Iida 1987, p.144 n.36.

[48] The number of subsets $\{\alpha_1, ..., \alpha_n\}$ has (Matsuzaka 1990a, p.704).

[49] RPI (2022, p.22) provides an actual *code*.

[50] Cf. Kaneko 2021, sec.92.

[51] Frege's notion of *unsaturatedness* (*Ungesättigkeit*) of the predicate clearly conflicts with this idea (cf. Frege 1906, S.192; Iida 1987, pp.62f.). So we classify him into the logical school. This

point is developed soon in the text.

52 The following classification is owed to Ramsey 1925.

53 See Kaneko 2021, sec.170.

54 Iida 1987, pp.48f.; Alama et al. 2018, sec.3.

55 Whitehead et al. 1910, p.15; Linsky 2016, sec.4.

56 If you do not know *the successor function*, see Kaneko 2021, sec.173. We remove parentheses from the function, so notation $Sx$, not $S(x)$,

57 See Kaneko 2019, pp.163-164; Enderton 1977, pp.42f.; Alama et al. 2018, intro., sec.1.2.

58 See also Alama et al. 2018, intro., sec,1.2.

59 Church (1941, p.1) kept the outermost parentheses, like "($S0$)," to denote the value of the function. With this idea, "($\lambda x.\ Sx$)" is considered a dummy, so Church named (18) *abstraction* in turn (1941, p.7). "($\lambda x.\ Sx$)" is also written as "($\lambda x[Sx]$)" or as "$\lambda x.\ Sx$". See Alama et al. 2018, sec.2.2.

60 See n.57 above.

61 As to the notion of application, see Hudak 2008, p.1; RPI 2022, p.20.

62 Cf. Hudak 2008, p.1; RPI 2022, p.21; see also Church 1941, p.12.

63 Carnap (1942, p.3) seems to have paid attention to this intensional aspect of the λ-calculus.

64 Whitehead et al. 1910, p.15 ; Linsky 2016, sec.2, sec.4.

65 That is, the procedure is a propositional function that receives an individual constant as its argument to return a truth value.

66 Noya (1994, pp.90-91) said a propositional function receives an individual constant and returns a proposition, but this explanation is not acceptable.

67 See also Klement 2022, sec.2; Iida 1987, pp.56-57; Kato 2021, p.111. Church endorsed it (1941, p.2).

68 Zalta (2018) used this notation.

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