LINGUISTIC SEMILINEAR ALGEBRAS AND LINGUISTIC SEMIVECTOR SPACES

W.B.VASANTHA KANDASAMY
K. ILANTHENRAL
FLORENTIN SMARANDACHE

This book introduces algebraic structures on linguistic sets associated with a linguistic variable. The linguistics with single closed binary operations are only semigroups and monoids. Authors feel it is not possible to define the notion of linguistic groups. We describe the new notion of linguistic semirings, linguistic semifields, linguistic semivector spaces and linguistic semilinear algebras defined over linguistic semifields. We also define algebraic structures on linguistic subsets of a linguistic set associated with a linguistic variable.

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US $40.00

ISBN 978-1-59973-752-2

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W. B. Vasantha Kandasamy
Ilanthenral K
Florentin Smarandache

Editorial Global Knowledge
2022

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ISBN-13: 978-1-59973-752-2
EAN: 9781599737522

Printed in the United States of America

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PREFACE

Algebraic structures on linguistic sets associated with a linguistic variable are introduced. The linguistics with single closed binary operations are only semigroups and monoids. Authors feel it is not possible to define the notion of linguistic groups.

We describe the new notion of linguistic semirings, linguistic semifields, linguistic semivector spaces and linguistic semilinear algebras defined over linguistic semifields. We also define algebraic structures on linguistic subsets of a linguistic set associated with a linguistic variable. We define the notion of linguistic subset semigroups, linguistic subset monoids and their respective substructures. We also define as in case of deals in classical semigroups, linguistic ideals in linguistic semigroups and linguistic monoids.

This concept of linguistic ideals is extended to the case of linguistic subset semigroups and linguistic subset monoids. In chapter two, we define and describe the notion of all linguistic structures with single binary operations. We also define linguistic substructures.

Chapter three of this book defines linguistic semilinear algebras over linguistic semifields. Clearly, linguistic semifields are linguistic algebraic structures with two binary operations.

We define linguistic semirings, linguistic semifields, linguistic subset semirings and linguistic subset semifields in this chapter.

In the case of linguistic semifields, we cannot define them. So the notion of the characteristic of the linguistic semifield.

We also define linguistic operators on these linguistic semivector spaces and linguistic semilinear algebras. We also define linguistic idempotent operators on these structures.

This book gives examples and problems for the reader to familiarise themselves with these concepts.

We acknowledge Dr K.Kandasamy with gratitude for his continuous support.

W.B.VASANTHA KANDASAMY
K. ILANTHENRAL
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Chapter One

BASIC CONCEPTS

In this chapter, we introduce some basic concepts of linguistic variables, the notion of matrix linear algebras, matrix semilinear algebras and their vector spaces and semivector spaces analogue. When we say L is a linguistic variable it can be represented as set of linguistic terms / words.

In this book we do not use the notion of associating with a linguistic variable linguistic sentences. We by default of notation use the term linguistic set or linguistic interval or linguistic continuum to represent a linguistic variable.

Suppose we have a linguistic variable 'age' of people.

The linguistic terms associated with the linguistic variable 'age' we will be denoted this by a linguistic set L_S or L or S.

L_S will denote a finite collection of linguistic terms qualifying the age in this case say;

L_S = {very old, old, oldest, young, very young, youngest, middle age, just middle age}.

However it is only a finite set of linguistic terms / words.

In this book we more often use the terminology linguistic terms.

L_S will be a finite set of linguistic terms (words) associated with the linguistic variable 'age'.

But if we want to realize or visualize the linguistic variable by a linguistic interval we denote it by I_L = [youngest, oldest]; I_L is the linguistic interval or linguistic continuum representing the linguistic variable age.

We can also show that the linguistic set L_S associated with the linguistic variable 'age' is always a totally ordered set.

For we see

youngest < very young < young < just middle age < middle age < old < very old < oldest.

This is a totally ordered set. Further we see the youngest has the least or the smallest value for its age where as the oldest will have the highest or the largest value of age.

So the ordering of these linguistic terms is in keeping with that of the numerical values.

Similarly the linguistic interval I_L = [youngest, oldest] has the numerical interval [0, 100] as the linguistic variable corresponds to age and age of a person in years is from 0 to 100.

As the numerical values is a totally ordered set so is I_L, the linguistic interval.

Thus in this book a linguistic set with linguistic variable or a linguistic interval variable or a linguistic interval associated with the linguistic variable we assume the set w(L) and I_L are totally ordered sets.

We can using this concept of total ordering on these linguistic sets define both min and max operations.

For in the next chapter we prove {w(L), min} and {(w(L), max} are linguistic semigroups of finite order where as {I_L, max} and {I_L, min} are linguistic semigroups of infinite order.

We will illustrate these situations by some examples.

Example 1.1. Let L be a linguistic variable associated with the "growth" of a plant. The linguistic words / terms associated with L denoted by w(L) = {very good, excellent, good, just good, normal, bad, very bad, worst}.

Now w(L) is a totally ordered set given by the following total order

excellent > very good > good > just good > normal > bad > very bad > worst.

Now min {good, good} = good and

min {very bad, normal} = very bad, so on and so forth.

Likewise we can define min on every pair of elements in w(L).

Now we define max on w(L) in the following;

max{good, good} = good,

max{very bad, normal} = normal and

max{very bad, worst} = very bad.

This is the way max operator is defined on w(L),

In fact for every pair of linguistic terms in w(L) we have max to be a well defined operator.

Further max {x, y} $\neq$ min {x, y} for x $\neq$ y; x, y $\in$ w(L).

Suppose L be the linguistic variable measuring the age of a person. The continuum defined for the linguistic variable age is I_L = [oldest, youngest]. Clearly I_L is a totally ordered set.

We see for any pair of elements in I_L, max and min operations are well defined.

For take just old and very old in I_L;

max {just old, very old} = very old as very old > just old and

min {just old, very old} = just old, thus

max {x, y} $\neq$ min {x, y} if x $\neq$ y and x, y $\in$ I_L.

Now we can easily prove {w(L), max}, {w(L), min}, {I_L, max} and {I_L, min} are monoids of finite and infinite order.

For more about studying in this direction please refer to [22-5]. Several interesting properties about these linguistic structures are carried out in [22-5].

Next, we briefly recall some of the properties about classical maps and linguistic maps about these linguistic sets with distinct domain and range space as well as same linguistic set. That is the range and the domain space are the same.

This study is very vital for we develop the notion of linguistic relational equations which is one of the important linguistic models that uses linguistic relational equations.

To this effect we briefly describe some special types of maps.

For more literature about these maps please refer [22-4].

The notions described in the following are very basic for us to introduce the concept of linguistic relational equations and applications to real world problems and in computer science like A.I, soft computing and so on.

Next we proceed onto describe and define all types of maps from linguistic set S to itself (as we are dealing with matrices we make an additional assumption that the linguistic set S is always a totally ordered set and the linguistic set is finite or finite. However most of the results are true even in case of infinite linguistic interval continuum).

What are the type of maps we can have

i) General maps which are not weighted or marked with any linguistic term.
ii) The domain and range linguistic sets may be same or different. There can be more than one map; that is these linguistic maps; that is these linguistic maps in general need not be only bipartite linguistic maps they can also be n-partite linguistic maps.

We will first describe this situation briefly for more literature can be found in [24].

First we provide an example of a bipartite linguistic graph got by mapping a linguistic set S to itself.

Example 1.2: Let S = {efficient, just efficient, not that efficient, not efficient, very efficient, least efficient} be the linguistic set measuring the efficiency of teachers in managing the students in the class rooms.

We order this set S;

very efficient > efficient > just efficient > not that efficient > least efficient > not efficient

We can have any number of usual maps from S to S some may be meaningful some very observed and one will be the identity map.

We provide a few of them.

i)

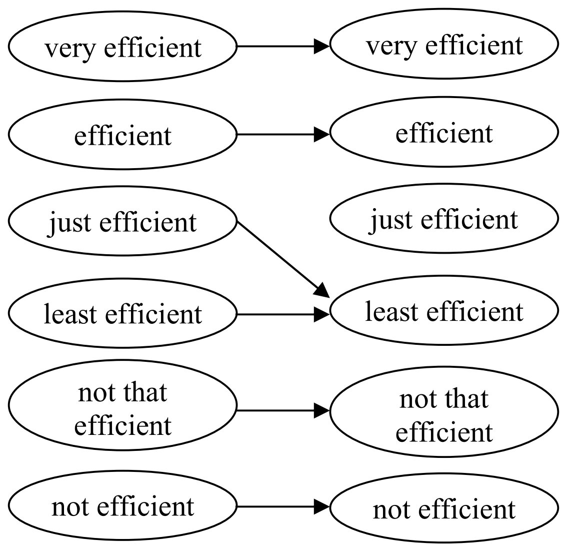

Figure 1.1

This map is given by an expert may not think or take a view to be an absurd one.

Some what reasonable or tolerable for he / she may not like to desect the efficiency of a teacher in that way.

ii) Now we give an absurd map from S to S.

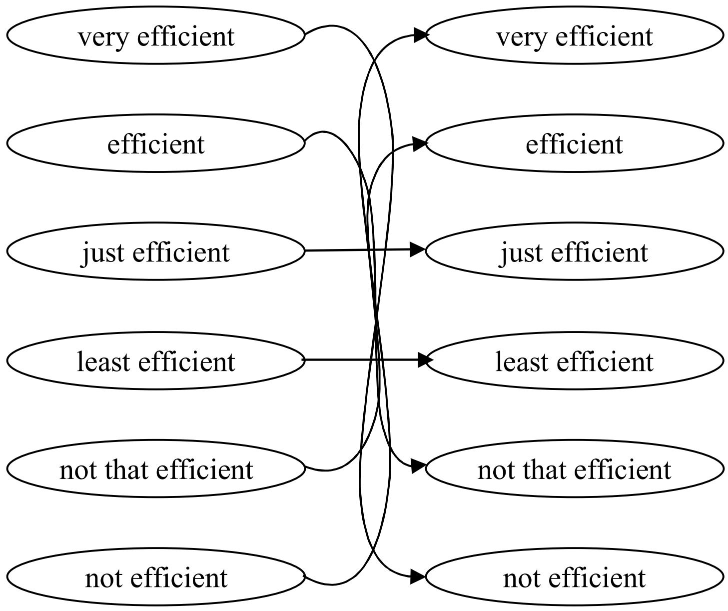

Figure 1.2

This map will be considered even by a lay man as an absurd map.

iii) Now we give an identical map

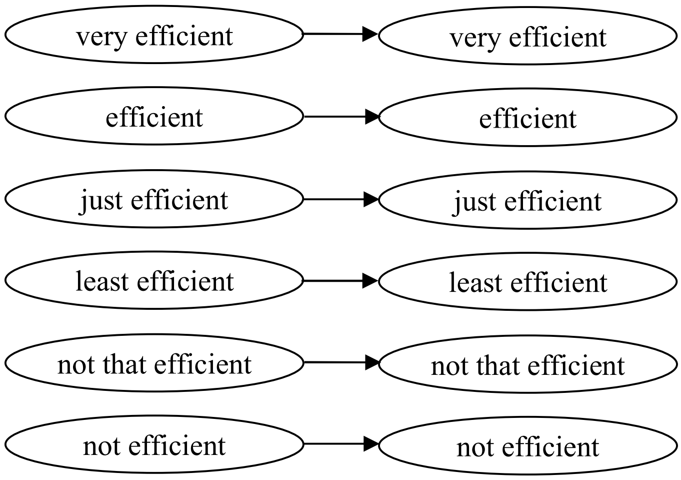

Figure 1.3

This map is defined as in case of classical maps / functions as the identity map which is unique by all means.

We see 2 of the maps from S to S are one to one whereas the first map is not one to one.

iv) We give a very special type of map from S to S which we roughly call as clustering or grouping.

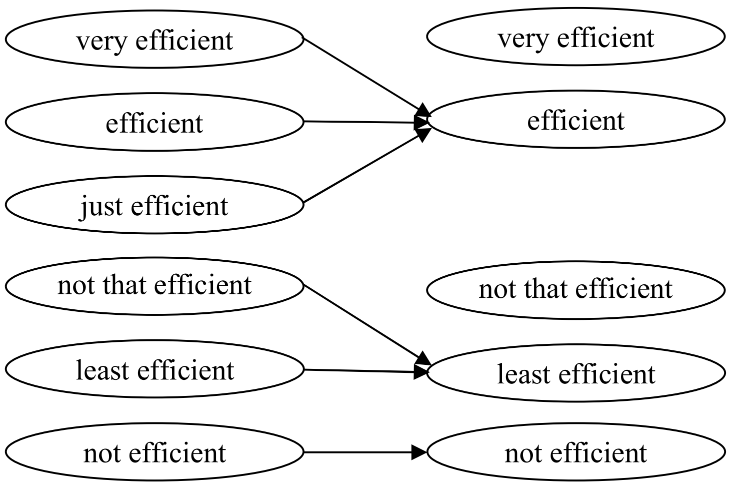

Figure 1.4

This sort of mapping is very important, as this clusters of these linguistic terms which are close to each or not very deviant from each other.

Next we discuss about classical maps of two linguistic sets S and R by an example.

Example 1.3. Let S and R be two linguistic sets where S = {good, very good, best, poor, very poor, fair, just fair} depicts, the performance of a student and

R = {very efficient, good, just good, efficient, in different, very devoted mediocre, very bad, bad} is associated with the teachers credentials.

Now the classical map between the linguistic from R to S (and S to R) are obtained in the following.

The classical map from R to S

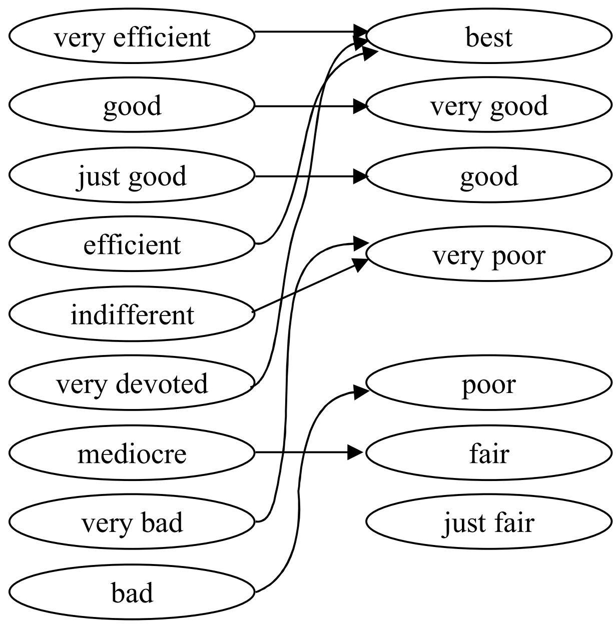

Figure 1.5

We see keeping in mind that student is one who has been passing in classes and happens to be one who can be made better or best by the proper teacher.

However this map does not consider the extreme case of students who were failures in every class and over aged one, in that class.

In the following we give the map (classical map) from the linguistic set S to R.

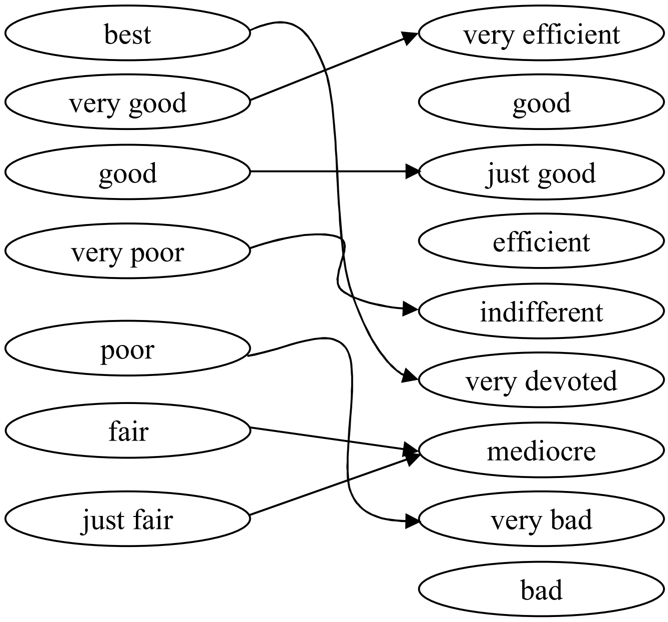

Figure 1.6

We do not call them as bipartite graphs. They are classical maps from two sets.

The only problem is bad that we have no option to map one linguistic term in the domain space to more than one linguistic term in the range space.

Next we proceed onto describe linguistic maps from a linguistic set S to itself and linguistic set S to another linguistic set R by the following examples.

Example 1.4. Let

S = {tallest, tall, very tall, medium height, short, very short, shortest} be the linguistic set associated with the linguistic variable height of plants in a farm.

Now we give the linguistic map from S to S. This map is not the classical map.

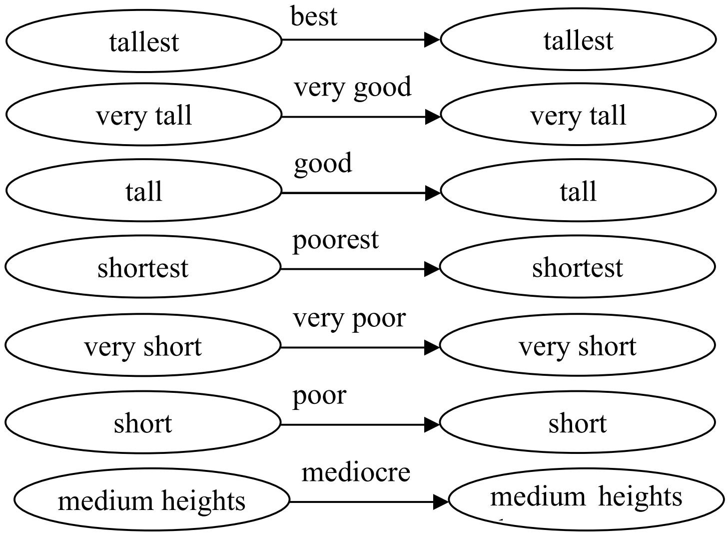

Figure 1.7

The map or the relational edges / lines are marked as {best growth, good growth, very good growth, mediocre growth, poor growth, very poor growth, poorest growth}.

Thus associated with the linguistic variable growth are rewritten as {poorest, very poor, poor, mediocre, best good and very good}.

Some other expert may give the linguistic map as follows.

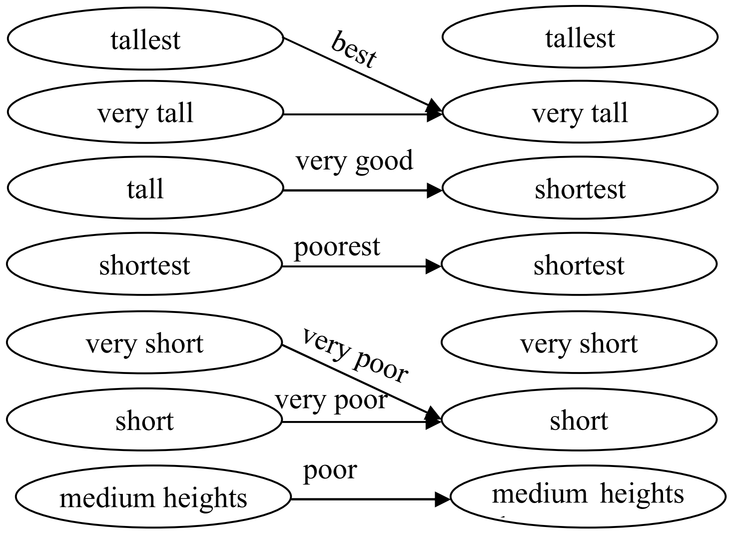

Figure 1.8

The linguistic map is at the hands of the expert.

Some other expert may use for the linguistic variable growth the following linguistic terms

{stunted, normal, very normal, super, very super}.

The following linguistic map is obtained using the linguistic terms of the linguistic variable growth.

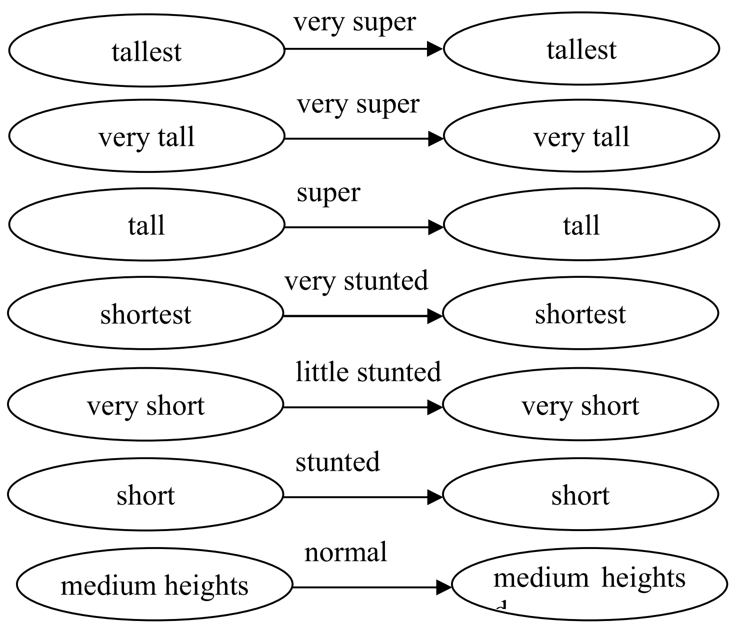

Figure 1.9

We have the flexibility to choose the linguistic terms associated with the linguistic variable, "growth of a plant".

However we wish to record at this juncture that the examples given here are just illustrations and the data do not pertain to any real world problem.

In fact these can be used in real, world problems.

When the domain linguistic set and the range linguistic set are distinct, we give an example how the linguistic map is defined in that case.

We give the linguistic binary relation (which are not in general functions).

We first give a classical sagittal diagram with linguistic set S where range and domain are the same.

Example 1.5. Let S = $\{x_1, \ldots, x_6\}$ be the linguistic terms, representing some linguistic variable L; that is S = w(L).

Consider the sagittal diagram of a binary relation.

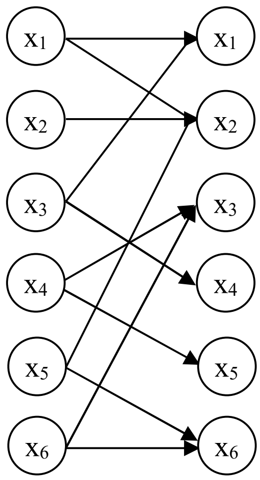

Figure 1.10

Now for the two different linguistic sets S = $\{x_1, \ldots, x_8\}$ and R = $\{y_1, y_2, \ldots, y_6\}$ we give the sagittal diagram where R and S are associated with two distinct linguistic variables in the following.

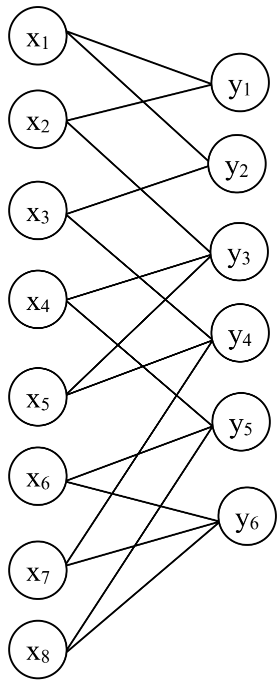

Figure 1.11

Finally we provide linguistic sagittal diagram for binary relation in the following. Here S is associated with the functioning of the teacher in teaching.

S = {good, bad, very good, indifferent, very bad, devoted, lazy}.

Let R denote the students knowledge.

R = {best, very bad, good, bad, worst, very good, average}.

Now the expert connects the linguistic sets S and R by the following relational linguistic maps with linguistic values / terms

{very hard working, hard working, just mediocre, not working, not hard working, evade work}.

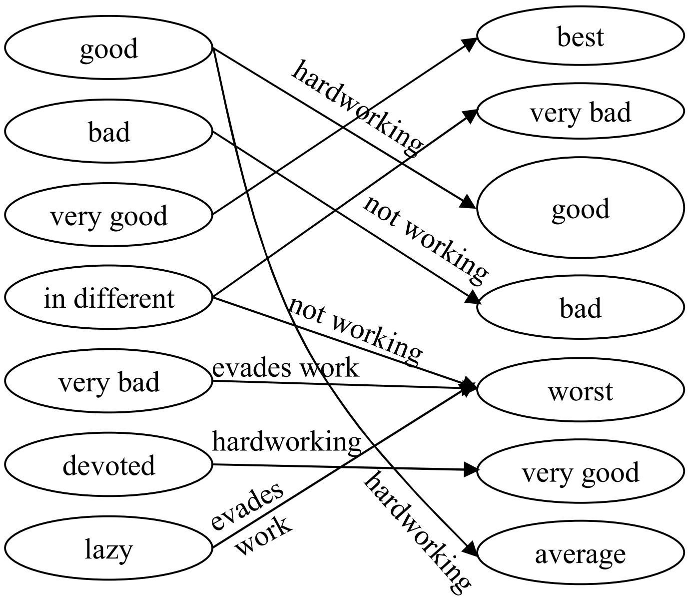

Figure 1.12

We can write the corresponding linguistic matrix associated with sagittal binary linguistic relation as follows

	best	very bad	good
good	ϕ	ϕ	hard working
bad	ϕ	ϕ	ϕ
very good	very hard working	ϕ	ϕ
in different	ϕ	not working	ϕ
very bad	ϕ	ϕ	ϕ
devoted	ϕ	ϕ	ϕ
lazy	ϕ	ϕ	ϕ

bad	worst	very good	average
ϕ	ϕ	ϕ	ϕ
not working	ϕ	ϕ	ϕ
ϕ	ϕ	ϕ	ϕ
ϕ	evades work	ϕ	ϕ
ϕ	not working	ϕ	ϕ
ϕ	ϕ	hard working	ϕ
ϕ	evades work	ϕ	ϕ

This is the way the linguistic matrix is formed.

Now we proceed onto recall the definition of semigroup, semiring, semivector spaces and semilinear algebras.

We also provide examples of them.

Further we assume the reader is familiar with the notion of classical matrices and their properties.

Definition 1.1. *Let (S, *) be a non empty set, with a binary operation * defined on it.*

*We say (S, *) is a semigroup, if the following conditions are true.*

*i) For every x, y $\in$ S we have x * y and y * x is in S (closure axions)* *ii) For every x, y, z $\in$ S we have x * (y * z) = (x * y) * z (associative axion)*

*If in addition for every x, y $\in$ S, if we have x * y = y * x we say (S, *) is a commutative semigroup.*

*Suppose S is a finite ordered set then we say (S, *) is a finite semigroup or a semigroup of finite order.*

*If S has infinite number of elements then we say (S, *) is a semigroup of infinite order.*

Suppose in S we have an element say e such that

*x * e = e * x = x for all x $\in$ S*

*we say (S, *) is semigroup with identity or is a monoid.*

We will first illustrate this situation by some examples.

Example 1.6: Let S = $Z^+ \cup \{0\}$ be the set of positive integers. (S, +) is monoid of infinite order. For one can easily verify all the conditions of a monoid are satisfied. 0 is the identity element as x + 0 = 0 + x = x for all x $\in Z^+ \cup \{0\}$ = S.

Now {S, $\times$} is a commutative monoid with 1 as its multiplicative identity. S is also of infinite order.

We have seen examples of monoids (commutative) semigroup of infinite order.

Now we give examples of non commutative monoids of infinite order.

Example 1.7: Let M = $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ / a, b, c, d $\in Z^+ \cup \{0\}\}$ be the collection of all 2×2 matrices with entries from $Z^+ \cup \{0\}$.

Clearly for every A = $\begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 1 \\ 0 & 6 \end{pmatrix} \in M$

we have $A \times B = \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 6 & 33 \end{pmatrix} \in M$ and

$$B \times A = \begin{pmatrix} 2 & 1 \\ 0 & 6 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 18 & 30 \end{pmatrix} \in M$$

Clearly both $A \times B$ and $B \times A \in M$ but

$A \times B \neq B \times A$ as $\begin{pmatrix} 2 & 1 \\ 6 & 33 \end{pmatrix} \neq \begin{pmatrix} 5 & 5 \\ 18 & 30 \end{pmatrix}$ and

$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M$ serves as the multiplicative identity for $A \times I_2 = I_2 \times A = A$.

That is $\begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} = A$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix} = A.$$

Thus $\{M, \times\}$ is a non commutative monoid of infinite order.

Suppose N = $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ / a, b, c, d $\in Z^+\}$ be the collection of all 2 × 2 matrices with entries from Z^+ then {N, +} is only a semigroup of infinite order. {N, +} has no additive identity viz.

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ = (0).

So, {N, +} is only a commutative semigroup of infinite order which is not a monoid.

Now we provide examples of finite order semigroups which are commutative.

Example 1.8: Let W = $\{Z_{12}, \times\}$ be the set of integers modulo 12. Z_{12} under × modulo 12 is a semigroup and 1 $\in Z_{12}$ acts as the multiplicative identity. $\{Z_{12}, \times\}$ is a commutative monoid of order 12.

If 3, 4 $\in Z_{12}$ 3 × 4 = 4 × 3 ≡ 12 = 0 (mod 12) $\in Z_{12}$.

1 × Z_{12} acts as the multiplicative identity.

Consider V = $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ / a, b, c, d $\in Z_{12}\}$ be the collection of all 2 × 2 matrices with entries from Z_{12}.

{V, ×} is a monoid which is non commutative and is of finite order.

If $x = \begin{pmatrix} 2 & 0 \\ 1 & 7 \end{pmatrix}$ and $y = \begin{pmatrix} 6 & 1 \\ 0 & 9 \end{pmatrix} \in V$; we see

$$x \times y = \begin{pmatrix} 2 & 0 \\ 1 & 7 \end{pmatrix} \times \begin{pmatrix} 6 & 1 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 6 & 4 \end{pmatrix} \text{ is in V.}$$

$$\text{Now } y \times x = \begin{pmatrix} 6 & 1 \\ 0 & 9 \end{pmatrix} \times \begin{pmatrix} 2 & 0 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ 9 & 3 \end{pmatrix} \text{ is in V.}$$

However $x \times y \neq y \times x$ so $\{V, \times\}$ is a non commutative monoid of finite order.

Now we proceed onto define first the notion of semirings.

Definition 1.2. *Let $\{S, +, \times\}$ be a non empty set S together with the two closed binary operations satisfying the following conditions is defined as a semiring.*

i) *$\{S, +\}$ is a commutative monoid.*

ii) *$\{S, \times\}$ is a semigroup.*

iii) *$a \times (b + c) = a \times b + a \times c$ and*
$(b + c) \times a = b \times a + c \times a$ for all $a, b, c \in S$

We call $\{S, +, \times\}$ to be a semiring of infinite order.

If $\{S, \times\}$ is a commutative monoid then we call $\{S, +, \times\}$ a commutative semiring. A commutative semiring with no zero divisors that is $x \times y \neq 0$ if $x \neq 0$ and $y \neq 0$ or equivalentity $x \times y = 0$ if and only if one of x or y is zero.

Then we define $\{S, +, \times\}$ to be a semifield.

We give examples of both semifields and semirings in the following.

Example 1.9. Let S = $\{Z^+ \cup \{0\}, +, \times\}$. S is a semifield of infinite order.

It is easily verified both $\{Z^+ \cup \{0\}, +\}$ and $\{Z^+ \cup \{0\}, \times\}$ are both commutative monoid and $\{Z^+ \cup \{0\}, \times\}$ has no zero divisors hence the claim.

Example 1.10. Let N = $\{(a_1, a_2, a_3) / a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 3\}$ be the collection of all 1 × 3 row matrices. {N, +, ×} is only a semiring of infinite order which is not a semifield.

For {N, ×} is a commutative monoid but has nontrivial zero divisors. Take x = (3, 0, 9) and y = (0, 9, 0) ∈ N;

x × y = (3, 0, 9) × (0, 9, 0) = (0, 0, 0). Hence the claim.

(0, 0, 0) is the additive identity in N.

N is only a semiring.

Having the definition and examples of semirings and semifields we now proceed onto defined semivector sapces.

Definition 1.3. *A semivector space V over the semifield S is the set of elements called vectors with the two laws of combination called vector addition and scalar multiplication satisfying the following conditions.*

1. *To every pair of vectors, α, β in V there is an associated a vector in V called then sum by $\alpha + \beta$.*

2. *Addition is associative that is* $(\alpha + \beta) + \gamma = \alpha(\beta + \gamma)$ *for all* $\alpha, \beta, \gamma \in V$.

3. *There exists* α *vector which we denote by zero such that* $\alpha + 0 = 0 + \alpha = \alpha$ *for all* $\alpha \in V$

4. *Addition is commutative for* $\alpha + \beta = \beta + \alpha$ *for all* $\alpha, \beta \in V$.

5. *If* $0 \in S$ *and* $\alpha \in V$ *we have* $0.\ \alpha = 0$.

6. *To every scalar* $s \in S$ *and every vector* $v \in V$ *there is a unique vector called the product s.v which is denoted by sv is in V.*

7. *For all* $a, b \in S$ *and for all* $\alpha \in V$ *we have* $(ab)\ \alpha = a(b\alpha)$.

8. *Scalar multiplication is distributive with respect to vector addition* $a(\alpha + \beta) = a\alpha + a\beta$ *for all* $a \in S$ *and* $\alpha, \beta \in V$.

9. *Scalar multiplication is distributive with respect to scalar addition* $(a + b)\ \alpha = a\alpha + b\alpha$ *for all* $a, b \in S$ *and* $\alpha \in V$.

10. $1.\ \alpha = \alpha$ *(where* $1 \in S$*) and* $\alpha \in V$.

We will illustrate this by examples. However for more about these concepts please refer [20, 32-8].

We give some examples of them.

Example 1.11. Let $\{(a_1, a_2, \ldots, a_9) / a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 9\}$ be the semigroup of row matrices with entries from $Z^+ \cup \{0\}$.

Let $S = \{Z^+ \cup \{0\}, +, \times\}$ is a semifield. Clearly V is semivector space over the semifield S.

For every $x \in S$ and $a = (a_1, \ldots, a_9) \in V$ we see

$x \times a = x (a_1, a_2, \ldots, a_9) = (xa_1, xa_2, \ldots, xa_9)$.

For instance if $x = 9 \in S$ and $a = (8, 0, 2, 4, 6, 1, 7, 5, 0) \in V$;

$x \times a = 9 \times (8, 0, 2, 4, 6, 1, 7, 5, 0) = (72, 0, 18, 36, 54, 9, 63, 45, 0) \in V$.

V is a semivector space over the semifield S.

Next we provide another example of a semivector space over a semifield.

Example 1.12: Let $V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} / a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 6\right\}$ be the collection of all 2×3 matrices with entries from $Z^+ \cup \{0\}$. V is a monoid under matrix addition.

Let $S = \{Z^+ \cup \{0\}, +, \times\}$ be the semifield.

Clearly V is a semivector space over the semifield S.

Now we proceed onto define the notion of semilineaar algebra over the semifield [32-8].

Definition 1.4. *Let V be a semivector space defined over the semifield S as in definition.*

We say V is a semilinear algebra over the semifield S if for all v, w ∈ V there is a product operation × defined on V. That is v × w ∈ V and w × v ∈ V.

In short {V, ×} is again a semigroup or infact a monoid.

If {V, ×} is commutative semigroup then the semilinear algebra is commutative otherwise non commutative.

Thus {V, +, ×} is a semiring with multiplicative identity and if V is a semivector space over the semifield S then V is a semilinear algebra over S.

We will provide some examples of the same.

Example 1.13: Let v = { $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$ / $a_i \in Q^+ \cup \{0\}$;

$1 \leq i \leq 9$} be the collection of all 3 × 3 matrices with entries from $Q^+ \cup \{0\}$.

{V, +, ×} is a non commutative semiring with $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ as its multiplicative identity.

V is a non commutative semilinear algebra over the semifield S = $\{Q^+ \cup \{0\}, +, \times\}$.

We now provide an example of a commutative semilinear algebra over the semifield S.

Example 1.14. Let V = $\left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \right.$ / $a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 12\}$

be a collection of 6 × 2 matrices with entries from $Z^+ \cup \{0\}$.

{V, +} is a commutative semigroup with (0) = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ as the identity.

We define natural product $\times_n$ on V (for more about natural product of matrices refer [ ]). That is if A and B ∈ V

where A = $\begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 1 & 1 \\ 9 & 9 \\ 9 & 0 \\ 1 & 9 \end{bmatrix}$ and B = $\begin{bmatrix} 9 & 9 \\ 0 & 9 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 9 \end{bmatrix}$.

Now $A \times_n B = \begin{bmatrix} 2 & 1 \\ 0 & 5 \\ 1 & 1 \\ 9 & 9 \\ 9 & 0 \\ 1 & 9 \end{bmatrix} \times_n \begin{bmatrix} 9 & 9 \\ 0 & 9 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 2\times 9 & 1\times 9 \\ 0\times 0 & 5\times 9 \\ 1\times 1 & 1\times 1 \\ 9\times 2 & 9\times 1 \\ 3\times 9 & 0\times 1 \\ 1\times 4 & 9\times 9 \end{bmatrix} =$

$$\begin{bmatrix} 18 & 9 \\ 0 & 45 \\ 1 & 1 \\ 18 & 9 \\ 27 & 0 \\ 4 & 81 \end{bmatrix} \in V.$$

The identity for $\times_n$ in V is $I_{6\times 1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \in V$ and it is easily verified $I_{6\times 1} \times A = A \times I_{6\times 1}$.

Further for all A, B $\in$ V $A \times_n B = B \times_n A$.

Thus $\{V, +, \times_n\}$ is a commutative semiring.

Clearly $\{V, +, \times_n\}$ is a semilinear algebra which is commutative over the semifield $S = \{Z^+ \cup \{0\}, +, \times\}$.

Now provide an example of a semivector space over a semifield S which is not a semilinear algebra over that semifield S.

Example 1.15. Let V = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix} \right.$

where $a_i \in Q^+ \cup \{0\}$; $1 \le i \le 15\}$ be the collection of all 3 × 5 matrices with entries from $Q^+ \cup \{0\}$.

Now the classical matrix product '×' cannot be defined on V. However {V, +} is a abelian semigroup (monoid).

Thus V is only a semi vector space over the semifield S = $\{Q^+ \cup \{0\}, +, \times\}$ and is not a semilinear algebra under classical matrix product '×' as the usual product cannot be defined for 3 × 5 matrices.

Hence the claim.

Finally we recall definition of the notion of Smarandache semigroup and Smarandache semilinear algebra and S-semi vector space over semifield / S-semiring. For more refer [32-8].

Definition 1.5. *A semigroup {S, *} is said to be a Smarandache semigroup (S-semigroup) if S contains a proper subset G, that G ≤ S such that {G, *} is group under *.*

We will give examples of them.

Example 1.16: Let S = $\{Z_{12}, \times\}$ be a semigroup under $\times$ modulo 12.

Consider G = $\{1, 11\} \subseteq S$; $\{G, \times\}$ is a group by the following table.

$\times$	1	11
1	1	11
11	11	1

Thus S = $\{Z_{12}, \times\}$ is a S-semigroup. For more about S-semigroups refer [32-8].

Now before we proceed onto recall the definition of S-semirings we give examples in the following.

Example 1.17. Let S = $\{[Z^+ \cup \{0\}][x], +, \times\}$ be the semiring of polynomials. Clearly $Z^+ \cup \{0\} \subseteq S$ is a semifield.

We call S a S-semiring.

Definition 1.6. *Let $\{S, +, \times\}$ be a semiring. Let $F \subset S$ be a proper subset of S. If $\{F, +, \times\}$ is a semifield then we define $\{S, +, \times\}$ to be Smarandache semiring (S-semiring).*

We can have several such examples for more refer [32-8].

We now proceed onto define the notion of Smarandache semi vector spaces (S-semivector spaces) in the following.

Definition 1.7. *Let V = {(Z × Z × Z^o)} be a semigrup under +, V is a S-semigroup under +. (Here Z^o = Z^+ ∪ {0}).*

Clearly V is a semivector space over the semifield S = {Z^+ ∪ {0}, +, ×}. Infact V is a S-semivector space over S as V is a S-semigroup.

For more refer [32-8].

Since V is a semigroup under × we define V to be a S-semilinear algebra over S.

V is a S-semigroup under × also. For consider the set.

M = {(1, 1, 1), (–1, 1, 1), (–1, –1, 1), (1, –1, 1)} ⊆ V.

Consider the following Cayley Tahle.

×	(1, 1, 1)	(–1, 1, 1)	(1, –1, 1)	(–1, –1, 1)
(1, 1, 1)	(1, 1, 1)	(–1, 1, 1)	(1, –1, 1)	(–1, –1, 1)
(–1, 1, 1)	(–1, 1, 1)	(1, 1, 1)	(–1, – 1, 1)	(1, – 1, 1)
(1, – 1, 1)	(1, – 1, 1)	(–1, – 1, 1)	(1, 1, 1)	(–1, 1, 1)
(–1, – 1, 1)	(–1, – 1, 1)	(1, – 1, 1)	(–1, 1, 1)	(1, 1, 1)

Clearly M is a group under ×. Hence V is a S-semilinear algebra over S.

Now having seen examples and definitions of these we now define substructure in them. We also define a few properties like linear transformation and so on.

We first define the notion of subsemivector spaces.

Definition 1.8: *Let V be a semivector space over the semifield S. We say W $\subseteq$ V (W a proper subset of V) is subsemivector space of V (or semivector subspace of V) if W itself is a semivector space over the same semifield S.*

We will illustrate this situation by some examples.

Example 1.18. Let V = $\{(a_1, a_2, a_3, a_4) / a_i \in Z^+ \cup \{0\}, 1 \leq i \leq 4\}$ be a semivector space over the semifield

$$S = \{Z^+ \cup \{0\}, +, \times\}.$$

$W_1 = \{(a_1, a_2, 0, 0) / a_1, a_2 \in Z^+ \cup \{0\}\} \subseteq V$ is a subsemivector space of V over the semifield S.

Take $W_2 = \{(a_1, a_2, a_3, a_4) / a_i \in 2Z^+ \cup \{0\}; 1 \leq i \leq 4\} \subseteq V$; W_2 is again a subsemivector space of V over the semifield S.

We proceed onto define subsemilinear algebras or semisublinear algebras of semilinear algebra V over a semifield S.

Definition 1.9.: *Let V be a semilinear algebra over the semifield S. We say a proper subset W $\subseteq$ V is to be a subsemilinear* *algebra of V over S if W itself is a semilinear algebra over the semifield S under the operations of V.*

We will illustrate this situation by some examples.

Example 1.19. Let V = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right.$ / $a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 9\}$ be the collection of all 3 × 3 matrices with entries from $Z^+ \cup \{0\}$.

V is a semilinear algebra over the semifield

$S = \{Z^+ \cup \{0\}, \times, +\}$.

Consider W = $\left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{bmatrix} \right.$ /$a_i \in Z^+ \cup \{0\}; 1 \leq i \leq 6\} \subseteq V$.

Clearly W is a subsemilinear algebra of V over the semifield S with inherited operation from V, on W ⊆ V.

Now having seen example of subsemilienar algebra over a semifield we proceed onto define linear transformation of semivector spaces over semifields.

For more information about S-semigroups, semivector spaces please refer [32-8].

Now we proceed onto define the notion of linear transformation of semivector V and W defined over the same semifield S.

Definition 1.10: *Let V_1 and V_2 be any two semivector spaces defined over the semifield S.*

We say a map / function T: $V_1 \to V_2$ is a linear transformation of semivector spaces if

$$T(\alpha v + u) = \alpha T(v) + T(u)$$

for all u, v $\in V_1$ and $\alpha \in S$.

We illustrate this situation by an example.

Example 1.20. Let $V_1 = \{(a_1, a_2, a_3, a_4) / a_i \in Z^+ \cup \{0\} = Z^o; 1 \leq i \leq 4\}$ be the collection of all 1×4 row matrices with entries from $Z^o = Z^+ \cup \{0\}$.

Let $V_2 = \{ Z_7^o[x]$ be the collection of all polynomials of degree less than or equal to 7}.

Clearly V_1 and V_2 are semivector spaces over the semifield $S = \{Z^+ \cup \{0\} = Z^o, +, \times\}$.

Define $T_1 : V_1 \to V_2$ by $T_1((0, 0, 0, 1)) = x^7 + x^5$

$T_1((0, 0, 1, 0)) = 1 + x^4$

$T_1((0, 1, 0, 0)) = x^2 + x^3$

$T_1((1, 0, 0, 0)) = x^6 + x$.

Now for any $T_1(9(6, 8, 9, 8) + (9, 1, 2, 0))$

(where $9 \in S$, $(6, 8, 9, 8)$ and $(9, 1, 2, 0) \in V_1$)

We have $T_1 (9(6, 8, 9, 8) + (9, 1, 2, 0)) = 9(6(x^6 + x) + 8(x^2 + x^3) + 9 (1 + x^4) + 8(x^7 + x^5)] + 9(x^6 + x) + 1 (x^2 + x^3) + 2(x^4 + 1) + 0(x^7 + x^5) = 54x^6 + 54x + 72x^2 + 72x^3 + 81 + 81x^4 + 72x^7 + 72x^5 + 9x^6 + 9x + x^2 + x^3 + 2x^4 + 2 + 0 + 0 = 72x^7 + 63x^6 + 72x^5 + 83x^4 + 75x^3 + 73x^2 + 63x + 83 \in V_2$.

Thus T_1 is a linear transformation of semivector spaces.

We can have more than one linear transformation.

Define $T_2: V_1 \to V_2$ by $T_2((1, 0, 0, 0)) = x + 1$

$T_2((0, 1, 0, 0)) = x^2 + x^3$

$T_2((0, 0, 1, 0)) = x^4 + x^5$

$T_2((0, 0, 0, 1)) = x^6 + x^7$.

Take $x = (3, 0, 1, 5)$ and $y = (1, 2, 4, 3) \in V_1$ and $9 \in Z^+ \cup \{0\}$.

Now $T_2(9x + y) = T_2 (9(3, 0, 1, 5) + (1, 2, 4, 3))$

$= 9(3(x + 1) + 0 (x^2 + x^3) + 1 (x^4 + x^5) + 5(x^6 + x^7)] + (x + 1) + 2(x^2 + x^3) + 4(x^4 + x^5) + 3 (x^6 + x^7) = 27x + 27 + 0 + 0 + 9x^4 + 9x^5 + 45x^6 + 45x^7 + x + 1 + 2x^2 + 2x^3 + 4x^4 + 4x^5 + 3x^6 + 3x^7 = 48x^7 + 48x^6 + 13x^5 + 13x^4 + 2x^3 + 2x^5 + 28x + 28$.

It is easily verified T_1 and T_2 are distinct transformations.

For more about these concepts refer [37].

Now we can show $(T_1 + T_2)(v) = T_1(v) + T_2(v)$

This is true for all $T_i : V \to V_2$ I an appropriate index.

Next we proceed onto define the notion of linear operator of semivector space V defined over the semifield F.

Definition 1.11. *Let V be a semivector space defined over the semifield F.*

A map or function from V to V is called a linear operator of the semivector space V if $T(\alpha v + u) = \alpha T(v) + T(u)$ for all $\alpha \in S$ and $u, v \in V$.

We will illustrate this by some simple examples.

Example 1.21: Let V = $\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \end{pmatrix} / a_i \in Z^o = Z^+ \cup \{0\}; 1 \leq i \leq 8\}$ be a 2 × 4 matrix with entries from Z^o. V is a semivector space over $S = \{Z^o = Z^+ \cup \{0\}, +, \times\}$ the semifield.

Let $T_1 : V \to V$ given by

$$T\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$T\{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T\{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T\{\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$T\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$T\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$T\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Take $x = \begin{pmatrix} 5 & 6 & 0 & 1 \\ 2 & 0 & 4 & 6 \end{pmatrix}$ and $y \begin{pmatrix} 1 & 2 & 5 & 7 \\ 9 & 0 & 1 & 2 \end{pmatrix} \in V$

Let $\alpha = 9 \in S$.

$$T(\alpha x + y) = 9 + 5\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 6\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ 0 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$+ 0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 6 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \}$$

$$+ 1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 5 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ 7 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + 9 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + 0 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$+ 1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 45 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 54 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 18 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 \end{pmatrix}$$

$$+ \begin{pmatrix} 54 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 56 & 46 & 56 & 5 \\ 16 & 27 & 0 & 37 \end{pmatrix} \in \text{V}.$$

We call T the linear operator from V to V.

However if we have a linear operator I : V → V such that

$$I\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$I\{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so on $I\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ that I(v) = v

for all v ∈ V then we define I to be a identity linear operator on V.

It is easily verified if T_1 and T_2 and two linear operators on V so is $T_1 + T_2$; for $(T_1 + T_2) = T_1 + T_2$ for all v ∈ V.

That $T_1 + T_2$ is again a linear operator. Suppose one is interested in knowing the number of such linear operators from V to V, how to find them.

First we wish to record we define in case of semivector spaces the notion of linear independent and linear dependent as follows.

Suppose V be a semivector space defined over the semifield S we say two vectors v_1, v_2 ∈ V to be linearly dependent if v_1 can be expressed in terms of v_2 as $v_1 = \alpha v_2$ or $v_2 = \beta v_1$, $\alpha, \beta \in S$ otherwise they are linearly in dependent.

For in semivector spaces we do not have a means to say for and v_1 the notion of $-v_1$ for any $v_1 \in V$.

Thus in the example $v_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and

$v_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ are linearly independent, whereas

$v_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ are linearly dependent for $w_2 = 5v$, …

Now take

$v_1 = \begin{pmatrix} 7 & 0 & 6 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$v_3 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $v_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\in V$ in example 1.21 we see $v_1 = 7v_2 + 3v_3 + 5v_4 = 7\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$+ 3\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 7 & 0 & 6 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} = v_1$$

We cannot define this as in case of classical vector spaces; $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$ is linearly independent if and only if $\alpha_i = 0$ for i = 1, …, n, otherwise they are dependent.

This is the marked difference between vector spaces and semivector spaces.

We say a set of elements in a semivector space V is a basis B if every $v \in V$ can be represented uniquely in terms of the base elements in B.

We illustrate this by some examples.

***Example 1.22**:* Let $V = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{pmatrix} / a_i \in Z^o = Z^+ \cup \{0\}$, $1 \leq i \leq 6\}$ be the collection of all 2 × 3 matrices with entries from Z^o. V is a semivector space over the semifield

$$S = \{Z^o, +, \times\}.$$

The basis B for V which is unique (in case of vector spaces the basis is not unique only number of elements is fixed in B) is given as follows.

$$B = \{b_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, b_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$b_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, b_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, b_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}.$$

This B is the basis for V and it is unique in this case and number of elements in B is fixed which is 6.

Any $v = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \end{pmatrix} \in V$ can be represented in terms of elements from B in a unique possible way.

$$v = v_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + v_3 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ v_4 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + v_5 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + v_6 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= v_1b_1 + v_2b_2 + v_3b_3 + v_4b_4 + v_5b_5 + v_6b_6.$$

Hence the claim.

In view of all these we can say if V be the collection of all $n \times m$ matrices with entries from Z^o and if V is a semivector space over the semifield $S = \{Z^o, +, \times\}$ then V has only $n \times m$ number of base elements where $2 \leq m, n < \infty$.

In the next section we proceed onto suggest some problems so that by solving them the reader becomes familiar with the new concept of linguistic matrix semilinear algebras.

Here we provide some problems for the reader. The main reason for doing so is that, this concept is new and if one understands the basics of these concepts thoroughly it would be nice that they can easily follow more difficult concepts on linguistic variables, representation of them and operations on them.

SUGGESTED PROBLEMS

1. Let L be a linguistic variable, intelligence for the linguistic variable intelligence; find the linguistic word w(L) associated with L.

   i) Is w(L) finite or infinite?

   ii) Is w(L) a linguistic continuum?

   iii) Justify your answer.

2. Suppose L be the temperature of water measured in 7 different times on slow heating is given as {3ºC, 5.2ºC, 20ºC, 46.5ºC, 61.3ºC, 78.5ºC, 98ºC}

   i) Give the linguistic word representation w(L) for these values.

   ii) Is w(L) finite? Give the order of w(L)

   iv) If the values are provided what will be w(L)?

iv) Will w(L) be a linguistic continuum?

v) Give the totally ordered chain of w(L) in (i)

3. Suppose performance of students in the classroom is the linguistic variable L.

Suppose w(L) = {good, bad, worst, very good, fair, just good, very fair}

i) What is order w(L)?

ii) Can you give values for w(L)? (Justify your answer).

iii) Is it comparable with problem 2? (Substantiate your claim).

v) What is the difference between linguistic variable temperature in problem 2 and linguistic variable performance of students in problem 3.

vi) Compare the linguistic variable intelligent in problem 1 with the linguistic variable performance of students given in problem 3.

4. Let "performance aspects of students in the classroom" be the linguistic variable L.

i) Describe L by a finite set of words denoted by w(L).

ii) Describe L by a linguistic continuum.

iii) Prove (i) and (ii) are totally ordered sets.

iv) Prove this w(L) cannot be given a numerical representation as in case of the linguistic variable ‘age’ or ‘temperature of water’.

iv) Which tool is better to study concepts like “performance aspects of students”, “Quality of teaching of a teacher”, intelligence of a student in the class room etc?

5. Give examples of a finite semivector space over a semifield.

6. Can V = $\{(Z^o \times Z^o \times Z^o \times Z^o) = \{(a_1, a_2, a_3, a_4) / a_i \in Z^o, 1 \leq i \leq 4\}, +\}$ be a semivector space over the semifield S = $\{Q^o \cup \{0\}, +, \times\}$?

Justify your claim.

7. Prove

$$V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \Bigg/ a_i \in R^o = R^+ \cup \{0\}, 1 \leq i \leq 12\right\} \text{ is a}$$

semivector space over the semifield S = $\{R^o = R^+ \cup \{0\}, \times, +\}$.

i) Find a basis of V over S.

ii) Is the basis B of V of finite order?

iii) Will order of B be 12? Justify your claim.

iv) Find some subsemivector spaces of V over S.

v) Define linear operators from V to V.

vi) Can we say the number of linear operators from V to V is finite? Justify your claim.

8. Let V = {Z^o [x] be a polynomial semiring with coefficients from $Z^o = Z^+ \cup \{0\}$} be a semivector space over the semifield S = {$Z^o = Z^+ \cup \{0\}$, +, ×}.

i) Find a basis B of V over S.

ii) Will $o(B) < \infty$? Justify your claim.

iii) Find all linear operators of V to V over S.

iv) Can the number linear operators from V to V be a finite collection?

v) Find subsemivector subspaces of V over S.

vi) Obtain some special feature associated with this V.

vii) Compare this V with W = {collection of all $n \times m$ matrices with entries from $Z^o = Z^+ \cup \{0\}$; $2 \leq m, n < \infty$} a semivector space defined over the same $S = \{Z^o, +, \times\}$.

viii) Can there be a linear transformation from V to W? Justify your claim.

ix) Can we claim W has a basis B which is of finite order?

x) Let M = { $Z^o_{mn-1}[x]$, all polynomials of degree less than or equal to $mn - 1$ with coefficients from Z^o in the variable x} be a semivector space over the semifield $S = \{Z^o, +, \times\}$.

Can M and W have their basis to be of same order?

xi) Find the collection of all linear transformation from M to W.

xii) Can we say $M \cong W$ that is the two semivector spaces are isomorphic?

9. Let $V = \{(a_1, a_2, \ldots, a_9) / a_i \in Z^o = Z^+ \cup \{0\}; 1 \leq i \leq 9\}$ be the collection of all 1×9 row matrices with entries from Z^o.

i) Prove V is a semivector space over the semifield $S = \{Z^o, +, \times\}$.

ii) Under classical product of matrices prove V is a semilinear algebra over the semifield $S = \{Z^o, +, \times\}$.

iii) Find the basis set of V over S.

iv) Find the collection of all linear operators from V to V.

a) What is the cardinality of that connection?

b) Is it finite or infinite?

c) Does the collection again a semivector space over S?

v) Suppose $W = \{(a_1, a_2, a_3, a_4, a_5, 0, 0, 0, 0) / a_i \in Z^o; 1 \leq i \leq 5\} \subseteq V$ is a subsemivector space over S.

a) Can we define a projection from V to W?

10. Find a basis B for the following semivector space

$$V = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \end{bmatrix} / a_i \in Z^o = Z^+ \cup \{0\} ; 1 \leq i \leq 10 \right\}$$

over the semifield S = $\{Z^o, +, \times\}$.

i) What is the cardinality of B?

ii) Is B unique?

iii) Define at least two linear operators from V to V.

iv) How many linear operators from V to V exist? (Is it finite or infinite?) Justify.

11. Give examples of semivector spaces V over a semifield S which are not semilinear algebras.

12. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} / a_i \in Z^o; 1 \le i \le 12\right\}$ and

W = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix} / a_i \in Z^o; 1 \le i \le 12\right\}$ be two semilinear algebras

under national product $\times_n$ of matrices over the semifield.

S = $\{Z^o = Z^+ \cup \{0\}, +, \times\}$.

i) If $T_1 : V \to W$ is a linear transformation and $M_1 : W \to V$ is a linear transformation of V to W and W to V respectively.

a) Will M_1 and T_1 be related in any way?

b) Which collection B = {all linear transformation from V to W} or C = {all linear transformations from W to V} is a larger set or will o(B) = o(C)? Justify and substantiate your claim!

ii) Can W and V be smilinear algebras under classical product of matrices? (Justify !)

iii) Will V and W be semivector spaces over the semifield $R = \{Q^+ \cup \{0\} = Q^o, +, \times\}$?

iv) Will V and W be semivector spaces over $N = \{2Z^o = 2Z^+ \cup \{0\}$? (Prove your claim).

v) Is N a semifield?

vi) Can we say W_p $\{p\, Z^+ \cup \{0\} = p\, Z^o, +, \times\}$ are semifields for varying primes p?

vi) Are $W_p = \{pZ^o, +, \times\}$ (p a prime) semirings?

13. Let $V = \{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} / a_i \in Z^o = Z^+ \cup \{0\}, 1 \leq i \leq 6\}$

be the semivector space of 2×3 matrices over the semifield $S = \{Z^o, +, \times\}$.

i) Find the subsemivector spaces of V over S.

ii) Show V can be written as a direct sum of subsemivector spaces over S.

iii) Prove $W_1 \oplus \ldots \oplus W_i = V$ then the maximum value i can take is 6 and the minimum value is 2.

iv) Define projection from $P_i : V \to W_i$ where

$W_i = \{\begin{bmatrix} a_i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} / a_i \in Z^o\} \subseteq V.$

Will projection be a linear transformation?

v) Define L_i: $W_i \to V$ by defining

$$L_i\{\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Is L_i a linear transformation of semivector space W_i and V?

14. Develop for a 'real world problem a sagittal linguistic binary map and obtain the corresponding linguistic matrix.

15. Given the linguistic terms of the domain space D and that of the range space R of the problem where D denotes the owners of the bonded labourers working in the textile industry (weavers) and R denotes the cause of those people becoming bonded labourers. The attributes related to the owners of bonded labourers.

O_1 = Globalizaztion / introduction of moder textile machines.

O_2 = Only profit no loss.

O_3 = Availability of raw goods.

O_4 = Demand for finished products.

Then D = {O_1, O_2, O_3, O_4)} and the range space.

R = {B_1, B_2, B_3, B_4, B_5 and B_6}, the description of B_i is as follows ($1 \leq i \leq 6$).

B_1 = No knowledge of other work has not only made them bonded but lead a life of penury.

B_2 = Advent of modern machinery had made them still poor.

B_3 = Salary they earn in a month after reduction for their debt is very low.

B_4 = No savings so they live in debt as with the deducted salary cannot make both ends meet.

B_5 = Government interferes and frees them but they continue to go as bonded labourers as the government does not given them or provide them any alternative livelihood.

B_6 = Hours they work in a day is more than 8 hours.

Transform this into a linguistic matrix for the possible linguistic sagittal diagram that you would draw relating D and R.

i) Give the binary linguistic sagittal diagram.

ii) Obtain the related linguistic matrix of the diagram

iii) If the diagram and the consequent linguistic matrix unique? Justify your claim.

16. Obtain any other interesting property and application of binary linguistic sagittal diagram.

17. If in a real world problem the linguistic terms of the domain space is same as that of the linguistic range space.

    i) Draw the linguistic binary sagittal diagram.

    ii) Draw the linguistic graph.

18. Give a linguistic binary matrix of order say 5×4 given in the following.

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} \text{good} & \phi & \text{bad} & \phi \\ \text{best} & \text{poor} & \phi & \phi \\ \phi & \text{worst} & \text{good} & \text{bad} \\ \phi & \text{bad} & \text{fair} & \phi \\ \text{bad} & \text{bad} & \text{very bad} & \phi \end{bmatrix} \end{array}$$

where 'ϕ' denotes there is no linguistic relation between that x_i and y_j ($1 \le i \le 5$; $1 \le j \le 4$).

Hence or otherwise prove that given a linguistic binary sagittal diagram we can obtain the corresponding linguistic matrix and vice versa.

19. Suppose A and B ($A \neq B$) are two linguistic square matrices with entries from the linguistic set word w(L) related to the linguistic variable L.

i) Find max (A, B}.

ii) Find max {B, A}.

iii) Find min {A, B}.

iv) Find min {B, A}.

v) Find min max {A, B}.

vi) Find max min {A, B}.

vii) Does (v) and (vi) give same linguistic matrix?

viii) Will all the six linguistic matrices give six different linguistic matrices? (Prove)

20. Let A and B two linguistic matrices of order 4 × 5 and 5 × 4 respectively with entries from the same linguistic word / set w(L).

i) Find min {A, B}, max {A, B}, min {B, A} and max {B, A}.

Chapter Two

LINGUISTIC SEMIGROUPS AND THEIR PROPERTIES

In this chapter we define different types of linguistic semigroups and describe their properties. In fact some of them are monoids. We define ideals and subsemigroups of them.

Throughout this chapter S will denote a linguistic set finite or infinite but discrete. I_L will denote a linguistic continuum / interval. I_L is not discrete it is continuous. We will build algebraic structures on them. Let S be any linguistic finite set describing the distance of a star form the earth or the depth of the ocean or the height of a tree or a mountain and so on and so forth. S can describe the performance aspect of a student in studies or the teaching ability of a teacher or a workers performance and so on and so forth.

We will illustrate these situations by some examples.

Example 2.1. Let S = {good, just good, very good, bad, worst, very very good, fair, just fair, very fair, just bad, very bad}

be a linguistic set with eleven linguistic terms or elements.

We see the least element of the set S is worst and the greatest element being very very good.

Now show {S, max} is infact a linguistic semigroup.

We see the linguistic element worst in S is such that max {worst, s}; s any linguistic term in S is s. Thus worst serves as the linguistic identity of S.

Infact max {x, y} = max{y, x} for every x, y $\in$ S. Thus {S, max} is a linguistic monoid with worst as its linguistic identity. We call max as the linguistic operation on S. The linguistic monoid {S, max} is of order 11.

Further max {x, x} = x; so we see {S, max} is an idempotent linguistic semigroup (monoid) which is commutative.

Example 2.2. Let I_L = [fair, best] be a linguistic interval measuring the performance of a worker in an industry.

{I_L, max} is a linguistic semigroup of infinite order. Infact every element in I_L is an idempotent under max operation; for

max {good, good} = good; that is max {s, s} = s for any s $\in$ S.

For any x $\in$ I_L; max {x, x} = x that is why we call {I_L, max} as an idempotent linguistic semigroup. Further the linguistic element fair in I_L is such that max {x, fair} = x for every x $\in$ I_L \ fair. So {I_L, max} is a linguistic commutative monoid as max {x, y} = max {y, x} for any x, y $\in$ I_L.

So {I_L, max} is a linguistic commutative idempotent monoid of infinite order with the fair as its linguistic identity.

Now we proceed onto define this commutative linguistic monoid under max operation.

Definition 2.1. *Let S be the linguistic set (or I_L be a linguistic interval / continuum), {S, max} ({I_L, max}) is a linguistic commutative monoid as the following four conditions are true.*

i) *For any x, y ∈ S (x, y ∈ I_L) we have max {x, y} ∈ S (or I_L). (Closure property)*

ii) *The max operation on S (or I_L) is associative.*

iii) *max {x, y} = max {y, x} for any x, y ∈ S (x, y ∈ I_L) that is max operations is commutative.*

iv) *For any x, y, z ∈ S (or x, y, z I_L) we have max {x, max {y, z}} = max {max {x, y}, z}. That is max operation on S is associative.*

v) *Let l be the least element of S (or I_L) then max {x, l} = x for every x ∈ S (or x ∈ I_L).*

l is defined as the linguistic identity of S (or I_L) for the max operation.

(S, max) (or I_L, max}) is infinite or finite depending on the order of the linguistic set S (always infinite in case of I_L).

vi) *{S, max} is an idempotent linguistic commutative monoid under max as max {x, x} = x for all x ∈ S {or x ∈ I_L}.*

Now we proceed onto describe the min operation on the linguistic set S and the linguistic interval I_L by some examples.

Example 2.3. Let S be a finite linguistic interval / continuum. Consider {S, min} (or {I_L, min}). Let g be the greatest linguistic element of S(or I_L).

It is observed min{x, y} = x or y for x, y $\in$ S.

So min {x, y} $\in$ S and min {g, x} = x for all x $\in$ S.

Further it is left as an exercise for the reader to verify min operation on S in both closed, commutative and associative.

Clearly {S, min} {or {I_L, min}) is a linguistic commutative monoid with g as its linguistic identity.

We observe that this linguistic monoid {S, min} is different from {S, max} leading to two distinct linguistic monoids for which of the linguistic greatest element g of S is the linguistic identity of {S, min} and l the least linguistic element of S is the linguistic identity of {S, max}.

Now for any x, y $\in$ S (or x, y $\in$ I_L) we see min {x, x} = x for all x $\in$ S (or x $\in$ I_L). So {S, min} is also a linguistic commutative idempotent monoid.

As we have defined {S, max} we can also define {S, min} the only difference being here the linguistic identity is g the greatest element of S (or I_L).

Next we proceed onto define the motion of linguistic subsemigroup or submonoid of {S, max} (or {I_L, max}) and

{S, min} (or $\{I_L, \min\}$).

We will first illustrate this situation by some examples.

Example 2.4. Let I_L = {[worst, best] be a linguistic interval / continuum}, $\{I_L, \max\}$ is an infinite linguistic commutative idempotent monoid with worst $\in I_L$ as its linguistic identity.

Consider the linguistic interval J_L = [fair, good] $\subseteq I_L$.

Clearly $\{J_L, \max\}$ is again linguistic commutative idempotent submonoid of the linguistic monoid $\{I_L, \max\}$. However the linguistic identity of $\{J_L, \max\}$ is fair but that of $\{I_L, \max\}$ is worst.

Let P_L = [worst, fair] $\subseteq I_L$ be the linguistic subinterval of I_L. $\{P_L, \max\}$ is again a linguistic commutative idempotent submonoid of $\{I_L, \max\}$.

We see both $\{I_L, \max\}$ and $\{P_L, \max\}$ have the linguistic identity to be the same viz worst.

All of them (linguistic monoid and submonoids) are of infinite order.

Consider M = {good, bad, very bad, just fair, fair, very good} $\subseteq I_L$, a linguistic subset of I_L. We see {M, max} is again a linguistic commutative idempotent submonoid of $\{I_L, \max\}$ with very bad as its linguistic identity.

Clearly M is of finite order and order of M is 6.

Thus $\{I_L, \max\}$ can have infinite number of linguistic idempotent commutative submonids of both finite and infinite order.

The following observations are putforth as results.

Theorem 2.1. *Let {S, max} ({I_L, max}) be a linguistic commutative idempotent monoid with linguistic identity l. Every linguistic commutative idempotent submonoid of {S, max} (or {I_L, max}) need not have l to be its linguistic identity.*

Proof is left as an exercise to the reader.

Now it is important to note the following.

Theorem 2.2. *Let {S, max} (or {I_L, max}) be the linguistic commutative idempotent monoid.*

Every linguistic commutative idempotent subsemigroup of {S, max} (or {I_L, max}) is also a submonoid and has a linguistic identity of its own.

Proof is left as an exercise to the reader.

Now we give examples of linguistic submonoids in case of min operation.

Example 2.5. Let P = {I_L = [shortest, tallest], min} be a linguistic commutative idempotent monoid with respect to min operation on the linguistic interval / continuum [shortest, tallest] = I_L "tallest" is the linguistic identity of P.

Consider

Q = {[short, tall] = $J_L \subseteq I_L$ = {[shortest, tallest], min} $\subseteq$ P is the linguistic commutative idempotent submonid of P of infinite order but its linguistic identity is different from that of P viz. tall.

Let

M = {[shortest, just tall] = $T_L \subseteq I_L$ = [shortest, tallest], min} $\subseteq$ P

is also a linguistic commutative idempotent submonoid of P with the linguistic identity being just tall.

Consider S = {short, very short, tall, very tall, very very short, just tall, medium height} $\subseteq I_L$

be a linguistic discrete subset of I_L . {S, min} is a linguistic commutative idempotent submonoid of I_L with very tall as its linguistic identity.

We see all the linguistic subsemigroups of P are linguistic commutative idempotent submonoids of P.

Now we give linguistic submonoids of a finite linguistic set S under min operation.

Example 2.6. Let S = {very good, good, best, poor, very poor, very bad, bad, just bad, very very fair, fair, just fair, medium} be the linguistic set {S, min} is a linguistic idempotent commutative monoid of order 12 with best as its linguistic identity.

Every linguistic element is a linguistic submonoid of order 1.

Every linguistic element of order two is again a linguistic commutative idempotent monoid.

For instance {{good, very good}, min} is a linguistic submonoid of order two. Infact there are $12C_2 = \frac{12.11}{1.2} = 66$ such linguistic submonoids of order two.

Likewise we see {{poor, very poor, just bad}, min} is again a linguistic submonoid of order three with poor as its linguistic identity.

We have $12C_3 = \frac{12.11.10}{1.2.3} = 220$ such linguistic submonoids of order three.

We have $12C_4$ number of linguistic submonoids of order four exists. There are 495 such linguistic submonoids.

On similar lines we have $12C_5$ linguistic submonoids of order 5 exists and so on.

We have 12 linguistic submonoids of order 11 where 10 of them will have the same linguistic identity best.

Thus there are $12C_1 + 12C_2 + 12C_3 + \ldots + 12C_{11}$ number of linguistic submonoids in S.

The definition of linguistic monoids with min operation can be carried out as in case of linguistic monoids with max operation the only change being the greatest element of the linguistic set or continuum will serve as the linguistic identity contrary to max operation were the least linguistic term will serve as the linguistic identity.

We just give the theorem in case of min operation.

Theorem 2.3. *Let {S, min} (or I_L, min}) be linguistic commutative idempotent monoid on the linguistic set S (or on the linguistic interval / continuum). If g is the greatest element of S (or I_L); g need not in general be the linguistic identity of every linguistic submonoid of {S, min} (or {I_L, min}) and every subset of S (or I_L) is a linguistic submonoid under min operation.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed onto give examples of ideals in linguistic monoids which we choose to call as linguistic ideals of the linguistic monoid.

Example 2.7. Let S = {very big, big, small, very very small, medium, just big, biggest}

be the linguistic set of order 7. {S, max} is a linguistic commutative idempotent monoid.

Let P = {very very big, small} be a linguistic subset of S. {P, max} is again a linguistic commutative idempotent sub monoid of S.

We will find out if P is a linguistic ideal of S and find out whether max{s, p} $\in$ P for all s $\in$ S and p $\in$ P

max{s, very very big} = biggest for s = biggest $\in$ S.

So P is not a linguistic ideal of S as biggest $\notin$ P.

Take M = {biggest, small, big} ⊆ S; M is a linguistic subset of S, {M, max} is a linguistic submonid of {S, max}.

Will check if M is a linguistic ideal of {S, max}.

We find max {s, m} for all s ∈ S and m ∈ M.

If max {s, m} ∈ M for all s ∈ S and m ∈ M then we claim M is a linguistic ideal of S.

Check max {very big {biggest, small, big}} = {biggest, very big} ≠ M.

max {big, {biggest, small, big}} = {biggest, big} ≠ M.

max {small, {biggest, small, big}} = M.

max{very very small, M} = {small, big, biggest} = M.

max{medium, M} = {medium, big, biggest} ≠ M.

max {just big, M} = {just big, big, biggest} ≠ M.

max {biggest, M} = biggest and so on.

Clearly SM is not contained in M so M is not an ideal of S.

Now consider the linguistic set

W = {biggest, max} ⊆ {S, max}; W is a linguistic submonoid of S or order one.

We see for every s ∈ S and this w = {biggest}, max{s, w} = w so SW ⊆ W hence W is a linguistic ideal of S.

But order / cardinality of W is one hence we call W only as a improper or trivial linguistic ideal of S.

Let us take again a linguistic set

V = {big, very big, medium, just big, biggest} $\subseteq$ S.

{V, max} is a linguistic submonoid of S.

Further max{S, V} = V that is SV $\subseteq$ V; so V is also a linguistic ideal of S.

We have given a proper linguistic ideal of S. Since S is a commutative linguistic monoid the question of right or left linguistic ideal does not arise.

Let Y = {big, very big, very very small, medium, just big, biggest} $\subseteq$ S be a linguistic subset of S.

Clearly {Y, min} is a linguistic submonoid of S; but Y is not an ideal as SY $\not\subset$ Y.

For s = smallest $\in$ S and y = very very small be in Y min{s, y} = s = smallest $\notin$ Y (that is s = smallest is not in Y; s $\notin$ Y). So {Y, min} is not a linguistic ideal of S.

Take R = {very big, big, small, very very small, just big, biggest} $\subseteq$ S a linguistic subset of R of order 6.

We see {R, max} is a linguistic submonoid of S.

But R is not a linguistic ideal of S for if we take s = medium $\in$ S and r = small $\in$ R max (s, r) = s = medium but s $\notin$ R; hence our claim.

So we make certain observations in case of finite linguistic sets S under max operation to have proper linguistic ideals.

i) It is observed that any linguistic set is a totally ordered set. For this linguistic set follows the order;

   biggest > very big > big > just big > medium > small > very very small - I

ii) If the elements of the linguistic set is taken from I such that

   i) biggest

   ii) biggest > very big

   iii) biggest > very big > big

   iv) biggest > very big > big > just big

   v) biggest > very big > big > just big > medium

   vi) biggest > very big > big > just big > medium > small

Only these six linguistic subsets of S are linguistic ideals and there are no other linguistic ideals under max operator.

So if we form the linguistic continuum call I as decreasing order continuum then for the linguistic ideal to exist we only take a continuous cut off interval (or continuum) from I.

Any continuous subcontinuum but with the starting the greatest element will form a linguistic ideal of S. So we first make the following definition.

Definition 2.2. *Let S = {x_i / i = 1, 2, 3, …, n} be a linguistic set such that x_n is the largest / biggest / brightest / tallest the greatest linguistic term of S.*

Thus $x_n > x_{n-1} > x_{n-2} > \ldots x_2 > x_1$ *I*

We define I as the continuous linguistic decreasing order chain depicting S.

We say any part of the (continuous) linguistic continuum are of the following types

- *i)* $x_n > x_{n-1} > \ldots > x_r$ *which is called as linguistic greatest (some* x_r*;* $r \neq 1$*) decreasing subcontinuum (or subchain).*
- *ii)* $x_m > x_{m-1} > \ldots > x_1$ *is defined as a linguistic least decreasing continuous subcontinuum subchain which ends with least* ($m \neq n$)*.*
- *iii) Consider the linguistic continuous subcontinuum (subchain) given by* $x_p > x_{p-1} > \ldots > x_t$*,* $p \neq n$ *and* $t \neq 1$*.*

We see (iii) above linguistic continuous subcontinuum is neither least nor greatest decreasing subcontinuum.

So we have 3 types of linguistic subcontinuum or subchains.

These chains are basically introduced to characterize linguistic ideals of infinite linguistic monoids.

We can define the 3 types different when we start from the least linguistic term so it is increasing so that,

$$x_1 < x_2 < \dots < x_n.$$

This is the linguistic increasing continuum which is continuous and is an increasing order chain depicting S.

Note: If any linguistic term is removed then we cannot call it as the linguistic chain depicting S. It may depict only a linguistic subset of S and for it to be a linguistic subchain it is mandatory no element in the original chain of S is left out in that subchain if x_{t_1} x_{t_2} x_{t_3} … x_{t_n} are the subset then it is for some x_s, x_{s+1}, …, x_{s+r} so no in between elements of the subchain is missed if say x_r, x_{r+3}, x_{r+4}, …, x_{r+7} is taken from S then this is not the subchain of the linguistic chain of S as x_{r+1} and x_{r+2} are missed.

Now we define for the same linguistic set S = {x_1, …, x_n} the increasing continuous chain of S, where x_1 is the least element of S and x_n is the largest element of S.

The linguistic continuous increasing chain of S is

$$x_1 < x_2 < x_3 < \dots < x_n.$$

Now we have only three types of linguistic continuous increasing subchains.

i) $x_1 < x_2 < x_3 < \ldots. < x_r$ is called the increasing least term continuous linguistic subchain of the linguistic chain S ($r \neq n$).

ii) $x_t < x_{t+1} < \ldots < x_n$ is called the continuous linguistic increasing subchain with the greatest ($t \neq 1$) term of the chain related to S.

iii) Now $x_r < x_{r+1} < \ldots < x_m$, $m \neq n$; $r \neq 1$ is a continuous increasing subchain of the chain associated with the linguistic set S.

We will describe these situations in the following:

$$x_n > x_{n-1} > \ldots > x_2 > x_1$$

is the linguistic decreasing continuous chain of S.

$$| x_n > x_{n-1} > \ldots > x_r | > x_{r-1} > \ldots > x_1; \quad r \neq 1$$

is the continuous greatest decreasing linguistic subchain of the chain of S denoted by the lines from x_n to x_r.

$$x_n > \ldots > | x_m > x_{m-1} > \ldots > x_1 |, m \neq 1$$

is the continuous least decreasing linguistic subchain of the chain of S denoted by the lines from x_m to x_1.

$$x_n > \ldots > | x_m > x_{m-1} > \ldots > x_{r+1} > x_r | > \ldots > x_1,$$

$m \neq n$; $r \neq 1$ is the continuous decreasing linguistic chain of S denoted by the lines from x_m to x_r.

Now the increasing continuous linguistic chain of the linguistic set S is

$$x_1 < x_2 < \ldots < x_n.$$

Now $| x_1 < x_2 < \ldots < x_r | < x_{r+1} < \ldots < x_n$; $r \neq n$

is the increasing continuous least linguistic subchain of the linguistic chain of S.

Consider

$x_1 < x_2 < \ldots < | x_r < x_{r+1} < \ldots x_n |$; $r \neq 1$

is the increasing continuous greatest linguistic subchain of the linguistic chain of S.

Let $x_1 < x_2 < \ldots < | x_r < x_{r+1} < \ldots < x_m| < x_{m+1} < \ldots < x_n$

is the increasing continuous linguistic subchain of the linguistic chain of S.

Now we give examples of ideals of a finite linguistic set S.

Example 2.8. Let S = {very big, big, small, very very small, medium, just big, biggest} be the linguistic set of order 7. {S, min} be the linguistic commutative monoid and the biggest linguistic term in S is the linguistic unit of {S, min}.

We see the ideals of {S, min} are

$$P = \{\text{very very small}\} \subseteq S$$

is such that {P, min} is a linguistic submonoid which is also an ideal of S. It is a singleton element.

Consider the singleton set {biggest} $\subseteq$ S, {biggest, min} is a linguistic submonoid of {S, min} however {biggest, min} is not a linguistic ideal of S.

Consider P = {small, very very small} ⊆ S; {P, min} is a linguistic submonoid of {S, min}, and is also a linguistic ideal of S.

Take Q = {small, medium} ⊆ S, {P, min} is a linguistic submonoid of S but is not a linguistic ideal of S.

The increasing continuous linguistic chain of S is as follows.

very very small < small < medium < just big < big < very big < biggest (A)

We see that

{{very very small}, min} is a linguistic ideal of S.

{{very very small, small}, min} is again a linguistic ideal of S.

{{small, medium}, min} is not a linguistic ideal of S. It is observed the linguistic least subchain cannot be got from small < medium for very very small; the linguistic least element of S is missing.

Take the linguistic subset

M = {small, very very small, medium} ⊆ S

is such that {M, min} is a linguistic submonoid of S and {M, min} is also a linguistic ideal of S and the elements of M forms a linguistic increasing least subchain of the linguistic chain of S, given by the following

very very small < small < medium.

Now consider

N = {very very small, small, medium, just big, big} $\subseteq$ S;

{N, min} is a linguistic submonoid of {S, min} and is infact a linguistic ideal of S and the set N $\subseteq$ S forms a linguistic increasing least subchain of the linguistic chain of S given by the following

very very small < small < medium < just big < big (a)

If we have to give the linguistic ideal N a least linguistic decreasing subchain structure then we have

big > just big > medium > small > very very small (b)

(b) gives the least linguistic decreasing subchain of N and the ideal of {S, miin} and (a) gives the least linguistic increasing suhchain of N, the ideal of {S, min}.

So with every ideal of {S, min} we can have representation by two subchains both are least increasing (and decreasing) linguistic subchains of the linguistic chain of S.

This is the same case with linguistic ideals of {S, max}. We will first make it clear none of the linguistic ideals of {S, min} can be linguistic ideals of {S, max} and vice versa.

For it is mandatory in case of linguistic ideals of {S, max} they should contain the greatest element as it is clearly evident from the above examples in case of linguistic ideals of {S, min} the least element is to be present in the linguistic subchain that is why least decreasing linguistic subchain and least increasing linguistic subchain.

So in case of linguistic ideals of {S, max} it will be greatest decreasing linguistic subchain and greatest increasing linguistic subchain.

We see both are equivalent for either increasing order or decreasing order.

Now for the same linguistic set S we give some examples of linguistic ideals of {S, max}.

Let A = {biggest, big} $\subseteq$ {S, max} be the linguistic subset of S. {A, max} is only a linguistic monoid of S and {A, max} is not a linguistic ideal of {S, max} for the linguistic set A does not form a linguistic subchain of the increasing or decreasing linguistic subchain of S.

Let B = {biggest, very big, just big, big, medium} $\subseteq$ S

be a linguistic subset of S.

{B, max} is a linguistic submonoid of {S, max}. Further {B, max} is linguistic ideal and the linguistic subchain associated with {B, max} is

biggest > very very big > big > just big

which is the decreasing greatest linguistic subchain of the linguistic chain S and

just big < big < very very big < biggest

is an increasing greatest linguistic subchain of the linguistic subchain S.

Thus we have the following theorem which can characterize an ideal of linguistic monoids {S, max} and {S, min}.

Theorem 2.4. *Let S = {x_1, x_2, …, x_n} be a finite linguistic set with x_1 as its least linguistic element and x_n as its greatest linguistic element.*

Let

$x_1 < x_2 < \ldots. < x_n$ *L and*

$x_n > x_{n-1} > x_{n-2} > \ldots > x_2 > x_1 \ldots$, *G*

where L is the linguistic increasing chain of S and G is the linguistic decreasing chain respectively of S. {S, min} and {S, max} be the linguistic monoids.

i) *I is a linguistic ideal of {S, min} if and only if the linguistic elements I = {x_1, …, x_m} forms a linguistic decreasing least subchain of the chain*

$$x_m > x_{m-1} > \ldots > x_1 \ (m \neq n)$$

of the linguistic decreasing chain of S; and a least linguistic increasing chain

$$x_1 < x_2 < x_3 < \ldots < x_{m-1} < x_m \ (m \neq n)$$

of the linguistic increasing chain of S.

ii) *J is a linguistic ideal of the linguistic monoid {S, max} if and only if J = {x_n, x_{n-1}, …, x_r} and the elements of J satisfies the linguistic greatest decreasing subchain*

$$x_n > x_{n-1} > \ldots > x_r \; (r \neq 1)$$

of the linguistic decreasing chain of S and J satisfies the linguistic greatest increasing subchaining

$$x_r < x_{r+1} < x_{r+2} < \ldots < x_{n-1} < x_n$$

of the linguistic increasing chain of S.

Proof is direct and hence left as an exercise to the reader.

Next we study linguistic ideals in the linguistic monoids of the linguistic continuum $\{[x_1, x_n], \max\}$ and $\{[x_1, x_n], \min\}$ with examples.

Example 2.9. Let I_L = [worst, best] be a linguistic interval / continuum with worst as the linguistic least element and best is the linguistic greatest element.

We see the increasing linguistic continuous chain of infinite length of I_L is

worst < … < just best < best

which is also known as increasing linguistic chain of I_L.

Any linguistic subchain of I_L can only be a closed linguistic subinterval of I_L that contributes to a linguistic continuous increasing subchain of the linguistic continuous increasing chain of I_L.

We will illustrate by this example.

Let J_L = [worst, good] $\subseteq$ [worst, best] $\subseteq$ I_L be the linguistic subinterval of I_L. The linguistic increasing subchain of J_L is as follows.

worst < … < just good < good (a)

Now consider P_L = [fair, best] ⊆ [worst, best] ⊆ I_l

the linguistic subinterval of I_L. The linguistic increasing subchain of P_L is as follows.

fair < … < good < … < very very good < … best (b)

The increasing linguistic subchain (a) is called as the linguistic least element / term increasing subchain of the linguistic chain of S whereas (b) is defined as the greatest linguistic increasing subchain of the linguistic chain associated with S.

Consider the linguistic subinterval

[fair, very good] = Q_L ⊆ I_L = [worst, best].

The increasing linguistic subchain of the subinter Q_L is as follows:

fair < very fair < … < good < … < very good (c)

The linguistic increasing subchain (c) of the linguistic increasing subchain of Q_L is neither the least nor the greatest increasing linguistic subchain of the linguistic chain associated with I_L.

Now the linguistic decreasing chain of I_L is as follows.

best > very very … very good > … > very very … very bad > worst

Now for the above 3 subintervals J_L, P_L and Q_L we give their respective linguistic decreasing subchains.

For J_L = [worst, good] the linguistic decreasing subchain is

good > just good > very very fair > very fair > … > fair >… > very bad > … > worst

is the least linguistic decreasing subchain of the linguistic chain I_L.

For the linguistic subinterval P_L = [fair, best] $\subseteq$ I_L the linguistic decreasing subchain is

best > … > very good > good > just good > … > fair

and it is the linguistic decreasing greatest subchain of the linguistic decreasing chain of I_L.

Q_L = [fair, very good] $\subseteq$ I_L the linguistic decreasing subchain is

very good > good > just good > … > fair

which is neither the least nor the greatest linguistic decreasing subchain of the decreasing linguistic chain of I_L.

The theorem characterizing the linguistic ideals of {I_L, max} and {I_L, min} can be done as in case of linguistic finite set or discrete set S which forms the linguistic monoids under the operations max and min.

Theorem 2.5. *Let I_L = [l, g] be a linguistic interval or continuum where l is the least element and g is the greatest term of I_L.*

Let

$$g > \cdots > \ell \qquad (G)$$

be the decreasing linguistic continuous chain of I_L and

$$\ell < \cdots < g \qquad (L)$$

be the increasing linguistic continuous chain of I_L.

$\{I_L, \max\}$ and $\{I_L, \min\}$ be the linguistic monoids of I_L with max and min operators respectively.

i) *P_L is a linguistic ideal I of $[l, m] = P_L$ $(m \neq g)$ $\{I_L, \min\}$ if and only if P_L forms a least linguistic decreasing subchain of the linguistic decreasing chain of I_L that is*

$$m > \cdots > l \ (m \neq g)$$

and a least linguistic increasing subchain

$$l < \cdots < m \ (m \neq g)$$

of the linguistic increasing chain of I_L.

ii) *Q_L $[t, g] \subseteq I_L$ forms a linguistic ideal of $\{I_L, \max\}$ if and only if Q_L satisfies the linguistic greatest decreasing subchain*

$$g > \cdots > t \quad (t \neq \ell)$$

of the linguistic decreasing chain of I_L and Q_L satisfies the greatest increasing linguistic subchain

$$t < \ldots < g \ (t \neq \ell)$$

of the linguistic increasing chain of I_L.

Proof is direct and left as an exercise to the reader.

Now we observe from the two theorems that any linguistic increasing or decreasing subchain which is neither the least nor the greatest cannot have linguistic ideals associated with them for the linguistic monoids under max and min operations.

We make other important observations and list in the following.

i) S a linguistic set finite or infinite is always a totally ordered set.

ii) S the linguist set has always a greatest element and a least element.

iii) {S, max} and {S, min} are always linguistic monoids.

iv) All linguistic monoids of {S, max} and {S, min} have some linguistic submonoids which are linguistic ideals.

v) The linguistic submonoids M_i of the linguistic monoids {S, max} or {S, min} need not in general have the linguistic identity as that of {S, max} or {S, min}. They can be different. For each linguistic submonoids M_i may have different linguistic identities. This is very different from the classical monoids.

vi) All the four results are true if S is replaced by the linguistic continuum or interval I_L.

Now we use different linguistic semigroups using the linguistic power set of a set S or I_L.

We will first illustrate this situation by some exmaples.

Example 2.10. Let S = {good, best, fair, bad, very bad} be a linguistic set of order 5. P(S) is a power set of S of order 2^5 if ϕ the empty subset is added.

Now P(S) = {ϕ, {best}, {bad}, {very bad} {good}, {fair}, {best, bad}, {fair, good}, … {best, very bad}, {best, bad, fair}… {best, very bad, good}, … {best, very bad, bad, good}, S = {best, very bad, bad, good, fair}} be the linguistic subset of S.

We see P(S) is only a partially ordered set and not a totally ordered set. For {fair} and {best, bad} are not ordered by inclusion relation. However {fair} and {fair, good} are ordered by the inclusion relation {fair} $\subseteq$ {fair, good}.

Thus {P(S), $\subseteq$} the linguistic power set and the set inclusion is only a partially ordered set.

However all the linguistic sets are totally ordered sets unlike the linguistic power sets. We use the operation $\cup$ and $\cap$ in the classical sense only.

Now {P(S), $\cap$} is a linguistic subset semigroup of finite order which is commutative. We can say {S} the linguistic set of P(S) acts as the linguistic subset identity of {P(S), $\cap$}.

For $\{S\} \cap \phi = \phi$, $\{S\} \cap \{\text{good}\} = \{\text{good}\}$,

$\{S\} \cap \{\text{best, very bad, bad}\} = \{\text{best, very bad, bad}\}$ and so on, hence our claim; $\{S\}$ is the linguistic subset identity of $\{P(S), \cap\}$.

So we can say $\{P(S), \cap\}$ is a subset linguistic monoid with $\{S\}$ as its linguistic subset identity. Thus we see for the linguistic power set subset monoid $\{P(S), \cap\}$ the biggest subset of P(S) which is S itself forms the linguistic subset identity of $\{P(S), \cap\}$.

On the contrary consider the linguistic subset monoid $\{P(S), \cup\}$ the least subset of P(S) acts as the linguistic subset identity of $\{P(S), \cup\}$.

Here ϕ the empty subset of P(S) which is the least linguistic term acts as the linguistic subset identity of $\{P(S), \cup\}$. For take $A \in P(S)$ we see $A \cup \{\phi\} = A$ hence the claim. Thus $\{P(S), \cup\}$ is the linguistic subset monoid.

Clearly both the linguistic subset monoids $\{P(S), \cup\}$ and $\{P(S), \cap\}$ are commutative idempotent linguistic subset monoids of order 2^5.

We find subset linguistic submonoids of these linguistic subset monoids $\{P(S), \cup\}$ and $\{P(S), \cap\}$.

Consider the linguistic subset

$$\{T, \cap\} = \{\phi, \{\text{best}\}, \{\text{good}\}\} \subseteq \{P(S), \cap\}.$$

Clearly {T, ∩} is not a linguistic subset submonoid. For this is only a linguistic subset subsemigroup.

The operation ∩ on {T, ∩} is given by the following table

∩	ϕ	{good}	{best}
ϕ	ϕ	ϕ	ϕ
{good}	ϕ	{good}	ϕ
{best}	ϕ	ϕ	{best}

Clearly this has no linguistic subset identity. We also see T is only a linguistic partially ordered subset the linguistic power set P(S) which is linguistic subset subsemigroup of order 3.

For $\phi \subseteq$ {best} and $\phi \subseteq$ {good}.

However {T, ∪} is not even closed for

{best} ∪ {good} = {best, good} $\notin$ T.

∪	ϕ	{good}	{best}
ϕ	ϕ	{good}	{best}
{good}	{good}	{good}	{good, best}
{best}	{best}	{good, best}	{best}

Hence the claim.

Here we make the following observations.

i) Linguistic subsets of the linguistic power P(S) in general need not be a linguistic subset submonoid or linguistic subset subsemigroup.

ii) T $\subseteq$ P(S) may be a linguistic subset subsemigroup under '$\cap$' and T need not be a linguistic subset subsemigroup under $\cup$ or vice versa.

To this effect we have given an example.

Consider the linguistic subsets.

M = {ϕ, {good, best}, good, bad}, {good, best, bad}}.

We see {M, $\cup$} is a linguistic subset submonoid of {P(S), $\cup$} given by the following table.

$\cup$	ϕ	{good, best}	{good, bad}	{good, best, bad}
ϕ	ϕ	{good, best}	{good, bad}	{good, best bad}
{good, best}	{good, best}	{good, best}	{good, best, bad}	{good, best bad}
{good, bad}	{good, bad}	{good, best, bad}	{good, bad}	{good, bad best}
{good, best, bad}	{good, best, bad}	{good, bad, best}	{good, bad, best}	{good, bad, best}

Now the linguistic subset submonoid {M, ∩} is given by the following table.

∩	ϕ	{good, best}	{good, bad}	{good, bad best}
ϕ	ϕ	ϕ	ϕ	ϕ
{good, best}	ϕ	{good, best}	{good}	{good, best}
{good, bad}	ϕ	{good}	{good, bad}	{good, bad}
{good, bad best}	ϕ	{good, best}	{good, bad}	{good, best bad}

Clearly {M, ∩} is not even a closed linguistic subset collection.

So {M, ∩} is not a linguistic subset submonoid or linguistic subset subsemigroup of {P(S), ∩}.

Next we wish to describe linguistic subset ideals (nontrivial) if any in {P(S), ∪} and {P(S), ∩}.

The first question is {T, ∩} an linguistic subset ideal of the linguistic subset monoid {P(S), ∩}. Yes; {T, ∩} is a linguistic subset ideal of the linguistic subset monoid {P(S), ∩}.

So we observe even linguistic subset subsemigroups are linguistic subset ideals of {P(S), ∩}.

Consider the collection of subsets

R = {ϕ, {good}, {bad}, {good, bad best, fair, very bad}}

$\subseteq$ P(S).

We find the table of R under $\cap$ in the following.

$\cap$	ϕ	{good}	{bad}	{good, bad, best, fair}
ϕ	ϕ	ϕ	ϕ	ϕ
{good}	ϕ	{good}	ϕ	{good}
{bad}	ϕ	ϕ	{bad}	{bad}
{good, bad, best, fair}	ϕ	{good}	{bad}	{good, bad, best, fair, very bad}

{R, $\cap$} is a linguistic subset submonoid of {P(S), $\cap$}.

We see {R, $\cap$} is not a linguistic subset ideal of {P(S), $\cap$}.

For take s = {good, bad, fair} $\in$ P(S) and let

t = {good, bad, best, fair} $\in$ R.

s $\cap$ t = {good, bad, fair} $\cap$ {good, bad, best, fair}

= {good, bad, fair} $\notin$ R.

So {R, $\cap$} is not a linguistic subset ideal of {P(S), $\cap$}. Another important observation is that the linguistic subset identity of the linguistic subset submonoid

{R, ∩} is {good, bad, best, fair} which is not the linguistic subset identity of {P(S), ∩}. Only {S} is the linguistic subset identity of {P(S), ∩}.

Now we find the table of {R, ∪}.

∪	ϕ	{good}	{bad}	{good, bad, best, fair}
ϕ	ϕ	{good}	{bad}	{good, bad, best, fair}
{good}	{good}	{good}	{good, bad}	{good, bad, best, fair}
{bad}	{bad}	{bad, good}	{bad}	{good, bad, best, fair}
{good, bad, best, fair}	{good bad, best, fair}	{good, bad, best, fair}	{good, bad, best, fair}	{good, bad, best, fair}

Clearly R is not even closed under ∪ for {bad, good} ∉ R. Thus {R, ∪} is not a linguistic subset subsemigroup or a linguistic subset submonoid of {P(S), ∪}.

Now we try to observe the following. As in the case of linguistic set where {S, ∩} and {S, ∪} are both linguistic subset monoids and every subset T ⊆ S is such that {T, ∪} and {T, ∩} are linguistic subset submonoids of {S, ∪} and {S, ∩} respectively.

However every linguistic subset submonoid of $\{S, \cap\}$ or $\{S, \cup\}$) need not be a linguistic subset ideal. But in case of $\{P(S), \cap\}$ and $\{P(S), \cup\}$ we see every proper linguistic subsets collection of P(S) need not in general be linguistic subset subsemigroups or linguistic subset submonoids.

We need to find the condition for linguistic subset monoids to contain linguistic subset ideals.

Infact it is a difficult task to find a necessary and sufficient condition for linguistic subset submonoid or linguistic subset subsemigroups.

Now we proceed onto define, describe and develop the notion of linguistic subset monoids in case of linguistic interval / continuum.

Let $I_L = [l, g]$ be the linguistic interval / continuum. $P(I_L)$ is the linguistic power set of I_L. Since I_L is of infinite order so is $P(I_L)$. We see $P(I_L)$ has $\{\phi\}$ to be least linguistic element / subset and $\{I_l\}$ to be the greatest linguistic subset of $P(I_L)$. $\{P(I_L), \cap\}$ is the linguistic subset monoid of infinite order. $\{P(I_L), \cup\}$ is also a linguistic subset monoid of infinite order. $\{P(I_L, \cap\}$ has $\{I_L\}$ the largest linguistic subset of $P(I_L)$ to be the linguistic subset identity.

Similarly for $\{P(I_L), \cup\}$ has the linguistic subset $\{\phi\}$ of $P(I_L)$ to be the linguistic subset identity.

For any $A \in \{P(I_L), \cup\}$ we see $A \cup \phi = A$ and for

$$A \in \{P(I_L), \cap\} \text{ we have } A \cap \{I_L\} = A.$$

Thus $\{I_L\}$ is the linguistic subset identity of $\{P(I_L), \cap\}$.

We see $\{P(I_L), \cap\}$ can have finite order linguistic subset subsemigroup even though $\{P(I_L), \cap\}$ is an infinite order linguistic subset monoid.

Consider $T = \{\{\phi\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}\} \subseteq P(I_L)$ where $a_1, a_2, a_3, a_4 \in I_L$ are singleton elements.

The table for T under $\cap$ linguistic operation is as follows.

$\cap$	$\{\phi\}$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_4\}$
$\{\phi\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$
$\{a_1\}$	$\{\phi\}$	$\{a_1\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$
$\{a_2\}$	$\{\phi\}$	$\{\phi\}$	$\{a_2\}$	$\{\phi\}$	$\{\phi\}$
$\{a_3\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$	$\{a_3\}$	$\{\phi\}$
$\{a_4\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$	$\{a_4\}$

Thus $\{T, \cap\}$ is only a linguistic subset subsemigroup of $\{P(I_L), \cap\}$.

Now for the same linguistic subset T of $P(I_L)$ we give the table for the '$\cup$' linguistic operation

$\cup$	$\{\phi\}$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_4\}$
$\{\phi\}$	$\{\phi\}$	$\{a_1\}$	$\{a_2\}$	$\{a_3\}$	$\{a_4\}$
$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_1, a_2\}$	$\{a_1, a_3\}$	$\{a_1, a_4\}$
$\{a_2\}$	$\{a_2\}$	$\{a_2, a_1\}$	$\{a_2\}$	$\{a_2, a_3\}$	$\{a_2, a_4\}$
$\{a_3\}$	$\{a_3\}$	$\{a_3, a_1\}$	$\{a_3, a_2\}$	$\{a_3\}$	$\{a_3, a_4\}$
$\{a_4\}$	$\{a_4\}$	$\{a_4, a_1\}$	$\{a_4, a_2\}$	$\{a_4, a_3\}$	$\{a_4\}$

So T is not even closed under the linguistic operator $\cup$.

Now we give yet another example of a linguistic subset semigroups. Consider R = $\{\{\phi\}, \{a_1\}, \{a_2\}, \{a_1, a_2\}\} \subseteq P(I_L)$ where $\{a_1\}$ and $\{a_2\}$ are singleton linguistic terms of I_L.

We find the table '$\cup$' for the linguistic subset R.

$\cup$	ϕ	$\{a_1\}$	$\{a_2\}$	$\{a_1, a_2\}$
ϕ	ϕ	$\{a_1\}$	$\{a_2\}$	$\{a_1, a_2\}$
$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_1, a_2\}$	$\{a_1, a_2\}$
$\{a_2\}$	$\{a_2\}$	$\{a_1, a_2\}$	$\{a_2\}$	$\{a_1, a_2\}$
$\{a_1, a_2\}$	$\{a_1, a_2\}$	$\{a_1, a_2\}$	$\{a_1, a_2\}$	$\{a_1, a_2\}$

Clearly $\{R, \cup\}$ is a linguistic subset submonoid of $\{P(I_L), \cup\}$ of order four. Further $\{R, \cup\}$ is not a linguistic subset ideal of $\{P(I_L), \cup\}$.

Consider the table of R under $\cap$

$\cap$	$\{\phi\}$	$\{a_1\}$	$\{a_2\}$	$\{a_1, a_2\}$
$\{\phi\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$	$\{\phi\}$
$\{a_1\}$	$\{\phi\}$	$\{a_1\}$	$\{\phi\}$	$\{a_1\}$
$\{a_2\}$	$\{\phi\}$	$\{\phi\}$	$\{a_2\}$	$\{a_2\}$
$\{a_2, a_1\}$	$\{\phi\}$	$\{a_1\}$	$\{a_2\}$	$\{a_1, a_2\}$

We see $\{R, \cap\}$ is the linguistic subset submonoid of this infinite linguistic subset monoid $\{P(I_L), \cap\}$.

Further $\{a_1, a_2\} \in R$ acts as the subset linguistic identity of $\{R, \cap\}$.

In fact $\{R, \cap\}$ is also a linguistic subset ideal of $\{P(I_L), \cap\}$.

We will provide some examples of linguistic subset monoids and the possible existence of linguistic subsets submonoids or subsemigroups or ideals first in case of a finite linguistic set S.

Example 2.11. Let $S = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ be a linguistic set describing the customer satisfaction of some product.

P(S) be the linguistic power set of S which includes the empty subset $\{\phi\}$ and the full linguistic subset of $\{S\}$.

$\{P(S), \cup\}$ and $\{P(S), \cap\}$ are linguistic subset monoids of finite order which is commutative and every element in P(S) is a linguistic subset idempotent both under $\cup$ and $\cap$.

Consider $T = \{x_1, x_2, x_5, x_6, x_9\} \subseteq S$, a linguistic subset of P(T), the linguistic power set of T is a subset of the linguistic power set P(S) of S.

$\{P(T), \cup\} \subseteq \{P(S), \cup\}$ is a linguistic subset submonoid of the linguistic subset monoid.

The linguistic subset identity of P(T) is ϕ as that of P(S). However $\{P(T), \cup\}$ is not a linguistic subset ideal of $\{P(S), \cup\}$ for if $A \in P(S) \setminus P(T)$ then $A \cup x \notin P(T)$ for any $x \in P(T)$.

For the same linguistic subset T $\subseteq$ S consider the linguistic power set P(T) under the operator $\cap$; {P(T), $\cap$} is again a linguistic subset submonoid of {P(S), $\cap$}. {P(T), $\cap$} has its linguistic subset identity to be $\{x_1, x_2, x_5, x_6, x_9\}$ = {T} but the linguistic subset identity of {P(S), $\cap$} is

$$\{S\} = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\},$$

which are different linguistic subsets.

Consider {P(T), $\cap$} for any x $\in$ P(T) and A $\in$ P(S) we have x $\cap$ A $\in$ P(T) hence {P(T), $\cap$} is a linguistic subset ideal of {P(S), $\cap$}.

Consider the linguistic subset {S} $\in$ P(S), {{S}, $\cap$} and {{S}, $\cup$} are linguistic subset monoids. We see W = {{S}, $\cup$} is in fact a linguistic subset ideal of {P(S), $\cup$}. As order of this W is one we call this linguistic subset ideal as trivial one.

{P(S), $\cap$} is a linguistic subset monoid and {{S}, $\cap$} is again a linguistic subset submonoid of {P(S), $\cap$} but is not a linguistic subset ideal of {P(S), $\cap$} as if A $\in$ P(S) \{S} then

A $\cap$ {S} = A $\notin$ W; hence the claim.

So {W, $\cup$} is a linguistic subset trivial ideal whereas {W, $\cap$} is not a linguistic trivial subset ideal of {S, $\cap$}.

Consider V = {ϕ, $\cup$} is a linguistic subset monoid of order one. But V is not a linguistic subset ideal of {P(S), $\cup$}.

But V = {{ϕ}, $\cap$} is a linguistic subset submonoid of {P(S), $\cap$} and infact linguistic subset ideal of {P(S), $\cap$} which we call as trivial linguistic subset ideal of {P(S), $\cap$}.

In view of this we can categorically give certain sufficient condition for the existence of linguistic subset ideals of {P(S), $\cap$} as well as non-existence of linguistic subset ideals of {P(S), $\cup$} in the following.

Theorem 2.6. *Let P(S) be the linguistic power set of the linguistic set S. Every proper subset M of S (M $\subseteq$ S) is such that {P(M), $\cap$} $\subseteq$ {P(S), $\cap$} where {P(S), $\cap$} is a linguistic subset monoid. {P(M), $\cap$} is a linguistic subset submonid of {P(S), $\cap$} and {P(M), $\cap$} is a linguistic subset ideal of {P(S), $\cap$}. For {P(S), $\cup$} the linguistic subset monoid every proper subset M of S is such at {P(M), $\cup$} is a linguistic subset submonid which is not a linguistic subset ideal of {P(S), $\cup$} and for every such linguistic subset submonid only the set {ϕ} is the linguistic subset identity of {P(M), $\cup$}.*

Proof is left as an exercise to the reader.

Next we proceed onto claim for any linguistic set S, P(S) its linguistic power set; {P(S), $\cup$} and {P(S), $\cap$} are linguistic subset semilattices.

Next we proceed onto define different type of operations on P(S) the linguistic power set of the linguistic set S.

We first give examples of them.

Example 2.12. Let S = {good, bad, worst, medium, very bad, very good, just bad, best}

be a linguistic set of 8 elements we have the following total order on S given in the following:

best > very good > good > medium > just bad > bad > very bad > worst

Now P(S) \ {ϕ} be the linguistic power set of S.

We work only with P(S) \ {ϕ}.

For any two set A and B we define min {A, B} as follows: here A = {good, bad, best} and

B = {good, very bad, worst, very good} in P(S) \ {ϕ}

min {A, B}

= min {{good, bad, best} {good, very bad, worst, very good}}

= {min {good, good}, min {good, very bad}, min {good, worst}, min {good, very good}, min {bad, good}, min {bad, very bad} min {bad, worst}, min {bad, very good}, min {best, good}, min {best, very bad}, min {best, worst}, min {best, very good}}

= {good, very bad, worst, bad, very bad, very good}.

We see cardinality of min {A, B} is seven.

Next we find for the same subsets A and B of P(S) the max operation in the following:

max {A, B} = max {{good, bad, best}, {good, very bad, worst very good}}

= {max{good, good}, max {good, very bad}, max {good, worst}, max {good, very good}, max {bad, good} max {bad, very bad}, max {bad, worst}, max {bad, very good}, max {best, good}, max {best very bad}, max {best, very good}, max {best, worst}}

= {good, very good, bad, best}.

We see cardinality of max {A, B} is four. It is clearly seen max {A, B} ≠ min {A, B}.

Example 2.13. Let S = {good, fair, bad, best} be a linguistic set. P(S) be the power set of S.

{P(S) \ {ϕ}, min} and {P(S) \ {ϕ}, max} are both linguistic subset semigroups of finite order.

They are both linguistic subset monoids. For {P(S) \ {ϕ}, min}, {best} is the linguistic subset identity of P(S) \ {ϕ}.

Similarly {P(S) \ {ϕ}, max} has the subset {bad} to be the linguistic subset identity of {P(S) \ {ϕ}, max}.

Thus both {P(S) \{ϕ}, max} and {P(S) \ {ϕ}, min} are linguistic subset monoids of finite order.

We need thus sort of operations on the linguistic power set P(S) \ {ϕ} mainly in problems the expert researcher many a times needs to study of the impact or effect of one linguistic term over the other. This sort of study is not possible when one uses the classical operators ∪ and ∩.

So only we felt it necessary to study the linguistic algebraic structures using {P(S) \ {ϕ}, min} and {P(S) \ {ϕ}, max}; further we did not want any of the dominant or insignificant elements like the linguistic set S itself or the linguistic element ϕ. Now the least and the greatest elements of the linguistic continuum or chain contributed by S has significant influence on the set S when the operators min or max is used.

We see P(S) \ {ϕ} is only a partially ordered linguistic set. Further invariably if A, B ∈ P(S) \ {ϕ}, min {A, B} = C say then C as a set has a larger cardinality of A and B or atleast least the cardinality of A or B.

Similarly for the case of max {A, B}.

We will describe this situation with more examples.

Example 2.14. Let S = {dull, very dull, medium, just medium, little bright, very bright, bright, brightest}

be the linguistic set associated with the performance of a class VIII student.

The linguistic increasing and decreasing continuous chains of S are as follows.

very dull < dull < just medium < medium < little bright < bright < very bright < brightest I and brightest > very bright > bright > little bright > medium > just medium > dull > very dull II

Now P(S) \ {ϕ} be the linguistic power set of the linguistic set S.

Let A = {bright, medium, dull, just medium} and

B = {brightest, very dull, dull} ∈ P(S) \ {ϕ}.

We find min {A, B}.

min {A, B} = {bright, very dull, dull, medium, just medium} 1

max {A, B} = {brightest, bright, medium, dull, just medium} 2

A ∩ B = {dull} 3

A ∪ B = {bright, brightest, medium, dull, just medium, very dull} 4

We see |A| = 4, |B| = 3, |A ∩ B | = 1

|A ∪ B | = 7,

|min {A, B}| = 5 and

|max {A, B}| = 5.

Cardinality of max {A, B} and min {A, B} are 5, the same however max {A, B} ≠ min {A, B}.

Now take A = {brightest, dull, very dull, very bright, bright, medium} $\subseteq$ P(S).

We see A $\cap$ A = A 1

A $\cup$ A = A 2

min{A, A} = {A} = max {A, A}.

So we see all the four operations, $\cap$, $\cup$, max and min are idempotent operators on P(S).

Let A = {dull, very dull, little bright, very bright} and B = {brightest, medium} belong to P(S);

A $\cap$ B = none (undefined).

A $\cup$ B = {dull, very dull, little bright, very bright, brightest, medium},

max {A, B} = {brightest, medium, little bright, very bright}.

min {A, B} = {dull, very dull, little bright, medium, very bright}

|A| = none, |A $\cup$ B | = 6,

|max {A, B}| = 4 and |min {A, B}| = 5.

We see max {A, B} contains the set B.

Similarly min{A, B contains A.

Now we want to find for linguistic subset monoids using min or max operations on linguistic subsets of a power set S, P(S) \ {ϕ}.

First of all from the observations max {A, B} for any A, B ∈ P(S), S a linguistic set is such that max {A, B} is again in P(S) \ {ϕ}.

Similarly min{A, B} for any A, B ∈ P(S) is again in P(S) \ {ϕ}.

The following theorem are left as an exercise for the reader to prove.

Theorem 2.7. *Let S be any linguistic set P(S) \ {ϕ} be the linguistic power set associated with S.*

i) {P(S) \ ϕ, min} is a linguistic subset monoid

ii) {P(S) \ ϕ, max} is a linguistic subset monoid.

iii) The greatest element of S which is a singleton set is the linguistic subset identity of {P(S) \ ϕ, min}.

iv) The least element of S which is singleton set is the linguistic subset identity of {P(S) \ {ϕ}, max}.

We will give some examples them.

Example 2.15. Let S = {low, very low, medium, just medium, lowest, just high, very high, highest}

be a linguistic set measuring the temperature of a day.

The increasing linguistic chain associated with S is as follows:

lowest < very low < low < just medium < medium < just high < very high < highest.

The decreasing continuous linguistic chain associated with S is as follows.

highest > very high > just high > medium > just medium > low > very low > lowest.

However the linguistic power set P(S) \ {ϕ} we cannot have a decreasing or increasing continuous linguistic chain associated with them.

So for P(S) \ {ϕ} we take the least element of S as the least linguistic subset, here {lowest} and the greatest linguistic subset as {highest}.

Only these will serve as identities of the linguistic monoids {P(S) \ {ϕ}, min} and {P(S) \ ϕ, max}.

Now we illustrate this for if we take

A = {very low, low, just medium, just high, highest} ∈ {P(S) \ ϕ, min}.

The linguistic identity of {P(S) \ {ϕ}, min} is {highest} the greatest linguistic term of {P(S) \ {ϕ}, min}.

Consider min {A, {highest}} = {min {very low, highest}, min {low, highest}, min {just medium, highest}, min {just high, highest}, min {highest, highest}}

= {very low, low just medium, just high, highest} = A. So for the linguistic subset identity is {highest} for min operation.

Now consider for the same A in {P(S) \ {ϕ}, max} the subset linguistic identity of {P(S) \ {ϕ}, max} is the linguistic singleton subset {lowest}.

For max {A, {lowest}} is {max {very low, lowest}, max {low, lowest}, max {just medium, lowest}, max {just high, lowest}, max {highest, lowest}}

= {very low, low, just medium, just high, highest} = A.

Thus {P(S) \ {ϕ}, max} is a linguistic subset monoid with the linguistic subset {lowest} as its linguistic identity.

However finding ideals in case of {P(S) \ ϕ, min} or {P(S) \ ϕ, max} happens to be a challenging one.

Even if a linguistic subset B of S is taken and {P(B)\{ϕ}} the linguistic power subset of the linguistic power subset P(S) \ {ϕ} of S. However {P(B) \ {ϕ}, max} is a linguistic subset subset monoid of {P(S) \ {ϕ}, max}.

Similarly the linguistic power subset

B) \ {ϕ} ⊆ P(S) \ {ϕ} is also a linguistic subset submonoid under min of

{P(S) \ {ϕ}}. Clearly {P(B) \ {ϕ}, min} is linguistic not a subset ideal of {P(S) \ {ϕ}, min} and {P(B) \ {ϕ}, max} is the linguistic not a subset ideal of {P(S)\{ϕ}, max}.

So finding linguistic subset ideals of {P(S) \ {ϕ}, max} and {P(S) \ {ϕ}, min} is a difficult problem.

Infact we putforth the following problem.

Let S be a linguistic set P(S) the linguistic power set of S {P(S) \ {ϕ}, min} and {P(S) \ {ϕ}, max} be linguistic subset monoids.

Characterize the linguistic subset ideals of {P(S) \ {ϕ}, min} and {P(S) \ {ϕ}, max}.

Next we define using linguistic interval continuum built linguistic power set and define on them linguistic subset monoid in the following.

Let I_L = [l, g] be the linguistic continuum / interval with l as the least element and g the greatest element.

The linguistic decreasing chain associated with I_L is as following

$$g > \cdots > \ell \qquad \text{(a)}$$

and the linguistic increasing chain associated with I_L is as follows.

$$\ell < \cdots < g \qquad \text{(b)}$$

We define $P(I_L)$ \ {ϕ} as the linguistic power set of I_L which is of infinite order: {$P(I_L)$ \ {ϕ}, min} is a linguistic (power set) subset monoid with {g} the linguistic subset which acts the linguistic identity subset of {$P(I_L)$ \ {ϕ}, min}.

Similarly $\{P(I_L) \setminus \{\phi\}, \max\}$ is a linguistic subset monoid with the linguistic subset identity {1} as the linguistic identity of $\{P(I_L) \setminus \{\phi\}, \max\}$.

We will give by examples linguistic subset submonoids of $\{P(I_L) \setminus \{\phi\}, \max\}$and $\{P(I_L) \setminus \{\phi\}, \min\}$.

Example 2.16. Let I_L = {[worst, best]} be a linguistic interval / continuum. The linguistic increasing and decreasing chains of I_L is as follows:

worst < · · · < fair < · · · < good < · · · < best - I

is the decreasing linguistic continuous chain of I_L.

best > · · · > very good > · · · > fair > bad > · · · > very bad > · · · > worst - II

is the linguistic increasing continuous chain of I_L.

We are mainly giving these two linguistic continuous chains of I_L for basically to characterize the linguistic subset ideals of $\{P(I_L) \setminus \{\phi\}, \min\}$or $\{P(I_L) \setminus \{\phi\}, \max\}$.

Now consider the linguistic subinterval

J_L = [bad, good] $\subseteq$ [worst, best] = I_L and let

$P(J_L) \setminus \{\phi\}$the linguistic subsets of J_L, clearly $P(J_L) \subseteq P(I_L)$.

Now $\{P(J_L) \setminus \{\phi\}, \max\}$ and $\{P(J_L) \setminus \{\phi\}, \min\}$ are linguistic subset submonoid of $\{P(I_L) \setminus \{\phi\}, \min\}$ with the linguistic subset {good} as its linguistic subset identity.

However the linguistic subset identity of {P(I_L) \ {ϕ}, min} is the linguistic subset {best}.

Clearly {P(I_L) \ {ϕ}, min} and {P(J_L) \ {ϕ}, min} have different linguistic subset identities.

Consider {P(J_L) \ {ϕ}, max} ⊆ {P(I_L) \ {ϕ}, max}, clearly {P(J_L) \ {ϕ},max} is a linguistic subset submonoid of the linguistic subset monoid {P(I_L) \ {ϕ}, max}.

The linguistic subset identity of {P(I_L) \ {ϕ}, max} is the linguistic subset {worst} whereas the linguistic subset identity of {P(J_L) \ {ϕ}, max} is the linguistic subset {bad}.

So both have different linguistic subset identity though one is the linguistic subset submonoid of the other.

Thus the important observation about linguistic subset monoids is that their linguistic subset submonoid in general need not have the same identity as that of the subset monoid.

Now we see the linguistic subset submonoid {P(J_L) \ {ϕ}, min} is not a linguistic subset ideal as take the linguistic subset x ={worst} ∈ P(I_L) \ {ϕ}, and any subset {y} ∈ P(J_L) \ {ϕ}, clearly min {{worst}, {y}} = {z} where {worst} ∈ {z} but worst ∉ J_L so the linguistic term worst cannot be in any of the linguistic subsets contributed by P(J_L) \ {ϕ}.

Hence {P(J_L) \ {ϕ}, min} is not a linguistic subset ideal of the linguistic subset monoid {P(I_L) \ {ϕ}, min}.

Now we will find out if $\{P(J_L) \setminus \{\phi\}, \max\}$ is a linguistic subset ideal of the linguistic subset monoid $\{P(I_L) \setminus \{\phi\}, \max\}$. Let us take the linguistic subset {best} $\in P(I_L)\setminus\{\phi\}$ and for any linguistic subset $y \in P(J_L) \setminus \{\phi\}$ we find max {{best}, y} = {w} say.

Claim {w} $\notin P(J_L) \setminus \{\phi\}$ for the linguistic subset {w} must contain the element best but best $\notin J_L$ = [bad, good] so {w} $\notin P(J_L) \setminus \{\phi\}$, hence $\{P(J_L) \setminus \{\phi\}, \max\}$ is not a linguistic subset ideal of $\{P(I_L) \setminus \{\phi\}, \max\}$.

Consider the linguistic interval

R_L = [worst, fair] $\subseteq$ [worst, best] = I_L.

Now $P(R_L) \setminus \{\phi\}$ be the linguistic power set of the linguistic power subset $P(I_L) \setminus \{\phi\}$. $\{P(R_L) \setminus \{\phi\}, \max\}$ and $\{P(R_L), \min\}$ are linguistic subset submonoids of

$\{P(I_L) \setminus \{\phi\}, \max\}$ and $P(I_L)\setminus\{\phi\}, \min\}$ respectively.

Clearly for any x $\in$ [worst, fair] and y = [best] the linguistic subset of $\{P(I_L) \setminus \{\phi\}, \max\}$ max {x, {best}}= {best} and {best} $\notin P(R_L) \setminus \{\phi\}$.

Hence $\{P(R_L) \setminus \{\phi\}, \max\}$ is not a linguistic subset ideal of $\{P(I_L) \setminus \{\phi\}, \max\}$.

Now is $\{P(R_L)\setminus\{\phi\}, \min\}$ a linguistic subset ideal of $\{P(I_L)\setminus\{\phi\},\min\}$.

Consider any linguistic subset

x = {very very good, good, best} $\in P(I_L) \setminus \{\phi\}$

and any {y} ∈ {P(R_L) \ {ϕ}, min}. Clearly min {{y}, {very very good, good, best}} = {y}.

Hence {P(R_L) \ {ϕ}, min} is a linguistic subset ideal of {P(R_L)\{ϕ}, min}.

Let Q_L = [fair, best] ⊆ [worst, best] = I_L be a linguistic subinterval of I_L. P(Q_L) \ {ϕ}, be the linguistic power set of Q_L. P(Q_L) \ {ϕ} ⊆ P(I_L) \ {ϕ}. By default of notation we in some places use just P(Q_L) for P(Q_L) \ {ϕ} or P(I_L) for P(I_L) \ {ϕ}, and so on.

We see {P(Q_L), max} and {P(Q_L), min} are both linguistic subset submonoids of {P(I_L), max} and {P(I_L), min} respectively.

Now we want to find out whether they are linguistic subset ideals of {P(Q_L), max} and {P(Q_L), min}.

Let us take any linguistic subset {x} ∈ {P(Q_L), max} and y = {worst, bad, very bad} ∈ P(I_L) be the linguistic subset.

Clearly max {x, {bad, very bad, worst}} = x so we see {P(Q_L), max} is always a linguistic subset ideal of {P(I_L), max}.

Now for the same {x} in {P(Q_L), min} and for the same y = {bad, very bad, worst} in P(I_L) we find min {{x}, {bad, very bad, worst}}

= {bad, very bad, worst}; for all {x} ∈ P(Q_L) = P([fair, best]).

Thus {P(Q_L), min} is not a linguistic subset ideal of {P(I_L), min}.

In view of all these we put forth the following theorem.

Theorem 2.8. *Let I_L = [l, g] be the linguistic interval / continuum with l the least element and g of the greater element of the interval*

$$\ell < \cdots < g$$

is an increasing linguistic chain of I_L and

$$g > \cdots > \ell$$

is linguistic decreasing chain of I_L. {P(I_L), max} and {P(I_L), min} be the linguistic subset monoids.

i) {P(J_L), min} $\subseteq$ {P(I_L), min} is a linguistic subset ideal of {P(I_L), min} if and only if the linguistic subinterval J_L is of the form [l, f] $\subseteq$ [l, g] where $f \neq g$..

ii) {P(Q_L), max} $\subseteq$ {P(I_L), max} is a linguistic subset ideal of {P(Q_L), max} if and only if the linguistic interval of the form Q_L = [t, g] $\subseteq$ [l, g] and $t \neq l$.

Proof of (i) given J_L = [l, f] $\subseteq$ [l, g] and f $\neq$ g is the linguistic subinterval of I_L. P(J_L) be the linguistic power set of J_L and P(J_L) $\subseteq$ P(I_L).

Clearly {P(J_L), min} is a linguistic subset submonoid of the linguistic subset monoid {P(I_L), min}.

To prove {P(J_L), min} is a linguistic subset ideal of {P(I_L), min} we have to prove for every linguistic subset A in P(I_L) and for any linguistic subset x in P(J_L) we must have min

{A, x} $\in$ P(J_L). This is true only in case of P(J_L) for no element which is common between P(J_L) and P(I_L) is lesser than linguistic terms in P(J_L) hence the claim.

This is so because the lesser part of the continuum which is covered by J_L is continuous linguistic least subchain of the linguistic chain I_L and there is no linguistic term in $I_L \setminus J_L$ which is lesser than any linguistic term in J_L. If such things happen, J_L cannot be a linguistic ideal of {P(I_L), min}.

That is presence of any linguistic term in a linguistic set S $\subset$ I_L or interval J_L contained in I_L is smaller than any of the elements in S or J_L then {P(J_L), min} cannot be a linguistic ideal of {P(I_L), min} for if x $\in$ I_L and x $\notin$ J_L i.e. x $\notin$ P(J_L) and x is smaller than some or one element in P(J_L) say y then min {x, y} = x and x $\notin$ P(J_L) hence the claim.

On similar lines {P(Q_L), max} is a linguistic subset submonoid of {P(I_L), max}.

Now to show {P(Q_L) max} is a linguistic subset ideal we have to show for every linguistic subset x $\in$ P(I_L) and for every linguistic subset y $\in$ P(Q_L), max {x, y} $\in$ P(Q_L).

This is clear from the fact every linguistic subset x $\in$ P(I_L) if it is fully or partly is not in P(Q_L) then every element which not in Q_L i.e. in P(Q_L) is lesser than every element in Q_L hen max {x, y} is always in P(Q_L).

On the contrary if any linguistic term is greater than any element (linguistic term) in Q_L then max {that element, x} for any x is not in Q_L hence not in $P(Q_L)$ so $\{P(Q_L), \max\}$ is not an ideal.

We will illustrate this situation by an example.

Example 2.17. A linguistic interval / continuum related to students IQ is given [least IQ, highest IQ] = I_L.

We see [least IQ, good IQ] = $B_L \subseteq I_L$. $P(I_L)$ be the linguistic power set associated with I_L and $P(B_L)$ be the linguistic power set of B_L; $P(B_L) \subseteq P(I_L)$.

Clearly $\{P(B_L), \min\}$ is a linguistic subset submonoid of the linguistic subset monoid $\{P(I_L), \min\}$. Further $\{P(B_L), \min\}$ is a linguistic subset ideal of $\{P(I_L), \min\}$ for and linguistic subsets x in $P(B_L)$ and y in $P(I_L)$ we have min $\{x, y\} \in P(B_L)$ as all linguistic terms in the continuum $I_L \setminus B_L$ are greater than every linguistic term in B_L.

Clearly $\{P(B_L), \max\}$ is also a linguistic subset submonoid of the linguistic subset monoid $\{P(I_L), \max\}$. However $\{P(B_L), \max\}$ is not a linguistic subset ideal of $\{P(I_L), \max\}$.

For take x = {highest IQ} $\in P(I_L)$, the linguistic subset of $P(I_L)$ and y be any linguistic subset of $P(B_L)$; max {y, {highest IQ}} = {highest IQ} $\notin P(B_L)$ hence $\{P(B_L), \max\}$ is not a linguistic subset ideal of $\{P(I_L), \max\}$.

Now consider Q_L = [average IQ, highest IQ] $\subseteq$ [least IQ, highest IQ] = I_L is a linguistic subinterval of I_L which is continuum linguistic subchain of the linguistic chain of I_L.

We see {P(Q_L), max} is a linguistic subset submonoid of the linguistic monoid {P(I_L), max}. As every element in $I_L \setminus Q_L$ is lesser then every other element in Q_L we see for any linguistic subset x of P(I_L) and for every linguistic subset y of P(Q_L) max {x, y} $\in$ P(Q_L).

Thus {P(Q_L), max} is a linguistic subset ideal of {P(I_L), max}.

Now consider the linguistic interval

R_L = [medium IQ, good IQ] $\subseteq$ I_L = [least IQ, highest IQ].

We see {P(R_L), max} is a linguistic subset submonoid of {P(I_L), max}.

However {P(R_L), max} is not a linguistic subset ideal of {P(I_L), max} for take

{very good IQ, highest IQ, higher IQ} = y $\in$ P(I_L) and

x = {medium IQ, good IQ} $\in$ P(R_L).

We see max {x, y} = y $\notin$ P(R_L) so {P(R_L), max} is not a linguistic subset ideal of {P(I_L), max}.

Consider a linguistic subset of some singleton sets in [least IQ, highest IQ] = S

= {{least IQ}, good IQ} {fair IQ}, {medium IQ}, {poor IQ}, {very poor, IQ}} $\subseteq$ P(I_L).

Now {S, max} is a linguistic subset submonoid of {P(I_L), max}.

Similarly {S, min} is a linguistic subset submonoid of {P(I_L), min}. Clearly {S, min} is not an ideal of {P(I_L), min}, for if x = {good IQ} in S and y = {very very poor IQ} $\in$ P(I_L) then min {x, y} = {very very poor IQ} $\notin$ S so {S, min} is not a linguistic subset ideal of {P(I_L), min}.

Clearly {S, max} is a linguistic subset submonoid of {P(I_L), max}. However {S, max} is not a linguistic subset ideal of {P(I_L), max}.

For take x = {good IQ} $\in$ S and y = {highest IQ} $\in$ P(I_L) we see max {x,y} = y and y $\notin$ S so {S, max} is not a linguistic subset ideal of {P(I_L), max}.

Now consider

T = {[very bad IQ, fair IQ] $\cup$ [good IQ, very very good IQ] be a linguistic subset of I_L. P(T) be the linguistic power subset of T.

P(T) $\subseteq$ P(I_L). {P(T), min} is a subset linguistic submonoid of {P(I_L), min}. However {P(T), min} is not a linguistic subset ideal of {P(I_L), min} for take the linguistic subset

x = {very bad IQ} $\in$ P(T) and y = {least IQ} $\in$ P(I_L);

we see min {very bad IQ, least IQ} = {least IQ} = y $\notin$ P(T). Hence our claim.

Now {P(T), max} is a linguistic subset submonoid of the linguistic subset monoid of {P(I_L), max}.

However {P(T), max} is not a linguistic ideal of {P(I_L), max}. For if we take

y = {highest IQ, very high IQ, very very high IQ} $\in$ P(I_L) and

x = {fair IQ, good IQ} $\in$ P(T).

Then max {x, y} = y $\notin$ P(T). Hence our claim {P(T), max} is not a linguistic ideal of {P(I_L), max}.

Thus the theorem characterizing the linguistic ideals in {P(I_L), max} and {P(I_L), min} are in keeping with these examples.

Thus we have discussed in this chapter four types of linguistic semigroups using either linguistic sets of linguistic intervals using the four operations $\cup$, $\cap$ max and min.

We have proved all of the four operations are distinct but however yield linguistic monoids of both finite and infinite order according as S is finite or infinite. In all cases the linguistic monoids given by the linguistic interval I_L or the continuum is always infinite.

However we can for all these infinite or finite we can have linguistic submonoids which can be of finite order. We have given a necessary and sufficient condition for the these linguistic monoid to be linguistic ideals.

Further it is proved that at all times these are only linguistic monoids.

Finally all these four types of linguistic monoids are idempotent linguistic monoids for

$$\min \{x, x\} = x,$$

$$\max \{x, x\} = x,$$

$$x \cap x = x \text{ and } x \cup x = x$$

for every linguistic term x in the linguistic set S or in the linguistic continuum I_L.

Likewise we have four types of linguistic subset monoids build four P(S) or $P(I_L)$ the linguistic power set of the linguistic set S or the linguistic interval or continuum I_L. Under these four operations P(S) the linguistic power set is a linguistic subset monoid. It is pertinent to record at this juncture by the linguistic power set P(S) we always mean $P(S) \setminus \{\phi\}$ for we do not usually work in general with the empty linguistic set. Mostly when we do so we will specially specifiy it.

Now for $\{P(S), \cup\}$ and $\{P(S), \cap\}$ we include the empty linguistic world $\{\phi\}$. However usually we do not include the linguistic empty word in {P(S), max} and {P(S), min}.

In all the four cases (operations) these linguistic power sets forms a linguistic commutative monoid in which every linguistic subset of P(S) is a linguistic idempotent. These results hold good even in case of $P(I_L)$ the linguistic power set of the linguistic infinite continuum or interval.

In this case also we have characterized the condition for a linguistic subset ideal to exist.

The operations (linguistic) using max or min is very different.

Also both the linguistic power sets P(S) and $P(I_L)$ can be finite linguistic subset submonoids. However the linguistic power set of the linguistic continuum cannot have linguistic subset ideals of finite order. All linguistic subset ideals of $P(I_L)$ under all the four operations are always of infinite order.

We have sometimes mentioned these repeatedly not only for emphasis but also as the fact lies that this study of such algebraic structures using linguistic set or linguistic interval or continuum happens to be very new. So there may not be books or research papers based on these concepts. Finally we propose some exercise problems for the reader to understand this new notion.

Suggested Problems

1. Given a linguistic set S. Is it possible for {S, max} or {S, min} to be only linguistic semigroups and not linguistic monoids? Justify your claim.

2. Prove given a linguistic set S or a linguistic interval / continuum I_L. S and I_L is a totally ordered linguistic sets.

3. Define increasing (decreasing) linguistic chains of a linguistic set S of cardinality 10.

   i) Find all linguistic subchains of the linguistic chain associated with S.

   ii) How many such linguistic subchains exists for |S| = 10?

4. For the linguistic continuum / interval

I_L = [dullest, brightest].

i) Prove we can have infinite number of linguistic increasing or decreasing subchains of the linguistic chain associated with I_L.

ii) Prove we can have infinite number of linguistic ideals of {I_L, max} and {I_L, min}.

iii) Prove we can have infinite number of linguistic subsemigroups (or submonoids) of the linguistic monoids {I_L, min} and {I_L, max} which are not linguistic ideals of {I_L, min} or {I_L, max} respectively.

iv) Obtain any other special feature associated with these linguistic monoids {I_L, max} and {I_L, min}.

5. How many linguistic subset ideals exist for the linguistic monoid {P(S), max} where S = {good, bad, best, very bad, just good, very very good, fair, very fair, just fair, medium worst, very worst, very good}?

   i) Give the linguistic chain of S.

   ii) Study the question for {P(S), min}.

   iii) How many linguistic subset monoids are not linguistic ideals of (a) {P(S), max} and (b) {P(S), min}?

   iv) Find the linguistic monoid {S, max} and its linguistic submonodis which are not linguistic ideals.

v) Study question (iv) in case of {S, min}.

vi) Compare {S, min} with {P(S), min}.

vii) Compare {S, max} with {P(S), max}.

viii) Obtain some interesting properties about these linguistic monoids.

6. Take a linguistic continuum [l, g] = I_L.

i) Find all the four linguistic monoids using I_L.

ii) Give at least three linguistic ideals for these linguistic monoids built in I_L.

iii) For the linguistic subset $P(I_L)$ find all the four ideals and compare them.

v) Find at least 8 linguistic subset collection from $P(I_L)$ and check for them to be linguistic subset ideals of these four ideals.

vi) If a $\{P(J_L), \max\}$ is a linguistic ideal of $\{P(I_L), \max\}$ will $P(J_L)$ be linguistic ideal of $P(I_L)$ under min or $\cap$ or $\cup$ operations?

7. For S = {good, bad, fair, very fair, best, very bad} a linguistic set of order 6;.

i) Find P(S) the linguistic power set of S. Is P(S) a totally ordered set?

ii) Find the linguistic chain of S. Is S a totally ordered set?

iii) Find all ideals in {P(S), max}.

iv) Will the linguistic ideals of {P(S), max} be also a linguistic ideals of {P(S), $\cup$}? Justify your claim.

v) Will the linguistic ideals of {P(S), min} be also the linguistic ideals of {P(S), $\cap$}?

vi) Can there be common linguistic ideals between {P(S), $\cap$} and {P(S), $\cup$}? Justify your claim.

vii) Obtain any other interesting properties about these linguistic ideals.

8. Let I_L = [l, g] be a linguistic infinite continuum. Find all linguistic ideals of {I_L, max}.

i) Show the linguistic ideals form a linguistic ideal of chains.

ii) Can we say the linguistic monoid {I_L, max} has a linguistic chain?

iii) Does {I_L, max} have more number of linguistic subchains? Justify your claim.

iv) Can {I_L, max} the linguistic monoid has a chain of linguistic ideals? Justify your claim.

v) Will {I_L, max} the linguistic monoid have a chain of linguistic submonoid which are not ideals?

vi) Can $\{P(I_L), \max\}$ the linguistic monoid have chain of linguistic ideals? Justify your claim!

vii) Can the linguistic monoid $\{P(I_L), \cap\}$ has linguistic ideals which form a chain?

viii) Can the linguistic monoid $\{P(I_L), \cup\}$ have many chains of linguistic ideals?

9. For the linguistic set S given in the problem 7; let P(S) be the power set of S with the empty linguistic set.

a) Draw the diagram of the linguistic semilattice $\{P(S), \cup\}$ and $\{P(S), \cap\}$.

b) How many linguistic chains are given by $\{P(S), \cup\}$?

c) How many linguistic chains exists for $\{P(S), \cap\}$?

d) Are the numbers in (b) and (c) the same? Justify your claim!

e) Can we have a diagram for $\{P(S), \max\}$ or $\{P(S), \min\}$? Justify your claim.

f) Can they have chains of linguistic subsets?

g) Obtain any other striking feature enjoyed by the four types of linguistic subset monoids and compare them.

10. Let $I_L = [l, g]$ be a linguistic interval.

i) Prove we can have infinite number of finite linguistic submonoids of $\{I_L, \min\}$ and $\{I_L, \max\}$.

ii) Can $\{I_L, \min\}$ and $\{I_L, \max\}$ have finite linguistic nontrival ideals?

iii) Can $\{P(I_L), \min\}$ and $\{P(I_L), \max\}$ have finite order linguistic subset submonoids of finite order? If so give one or two examples of each.

iv) Can $\{P(I_L), \max\}$ and $\{P(I_L), \min\}$ have finite subset ideals? Justify your claim.

v) Prove that both $\{P(I_L), \max\}$ and $\{P(I_L), \min\}$ can have only linguistic subset ideals of infinite order.

vi) Prove $\{I_L, \max\}$ and $\{I_L, \min\}$, the linguistic monoids can have only infinite order linguistic ideals.

vii) Obtain any other special feature associated with $\{I_L, \max\}$ and $\{I_L, \min\}$.

11. a) Prove in case all the four linguistic subset monoids $\{P(I_L), \cap\}$, $\{P(I_L), \max\}$, $\{P(I_L), \min\}$ and $\{P(I_L), \cup\}$ every singleton is a linguistic subset submonoid and not a linguistic subset ideal.

b) Prove these four linguistic subset monoids have no linguistic subset submonoids of order greater than 1.

12. For a finite linguistic set S find the probable construction of finite linguistic automation and finite linguistic semiautomation.

13. Can we say finite linguistic monoids will find applications in all the places where monoids find their applications?

14. Give examples of atleast 10 distinct linguistic interval or continuum associated with 10 different problems (like measuring intelligence, performance aspects of students etc).

15. Can we apply these finite linguistic sets to finite linguistic automation?

16. Give one or two examples of finite linguistic automation using finite linguistic set S (here off will be denoted by 0 and one will be replaced by ON.

17. Obtain any other practical applications of linguistic finite sets.

18. Can linguistic continuum find its applications in practical real world problems?

19. Prove linguistic subset monoids of a power set P(S) of the linguistic set S can have several linguistic chains of same length.

20. Prove $P(I_L)$ for any linguistic interval I_L can have several infinite length linguistic chains.

21. Hence or otherwise prove $P(I_L)$ and P(S) are not totally ordered linguistic subsets collection.

22. Prove every linguistic submonoid of all linguistic monoids $\{I_L, \max\}$ or $\{I_L, \min\}$ are totally ordered.

23. Prove in general every linguistic subset submonoid of the linguistic subset monoid $\{P(I_L), \max\}$ (or $\{P(I_L), \min\}$) not totally ordered linguistic subset submonoids.

24. Can the linguistic subset monoid $\{P(I_L), \max\}$ (or $\{P(I_L), \min\}$) have linguistic subset submonoid which is totally ordered? If one such exists can you characterize them?

25. Does there exist a linguistic subset submonoid of } $\{P(I_L), \max\}$ which has a linguistic subset ideal which is totally ordered?

26. Can we say all linguistic subset ideals of $\{P(I_L), \max\}$ are totally ordered?

27. Does their exist a linguistic subset ideal of $\{P(I_L), \min\}$ which is not a totally ordered? Justify your claim.

28. Prove all linguistic ideals of $\{S, \max\}$ or $\{S, \min\}$ {S a finite linguistic set} the linguistic monoids are totally ordered.

29. Will problem 28 be true if S is replaced by the linguistic continuum I_L?

30. Can we say $\{P(I_L), \cap\}$ the linguistic monoid has linguistic ideals?

31. Will the linguistic ideals mentioned in problem 30 form an ordered chain (that is they (subsets) form a totally ordered collection?

Chapter Three

LINGUISTIC SEMILINEAR ALGEBRAS AND LINGUISTIC SEMIVECTOR SPACES

In this chapter we for the first time introduce the new notion of linguistic semivector spaces and linguistic semi linear algebra using the linguistic set S or linguistic interval I_L. Further we prove several results in this direction.

Finally we prove linguistic matrices are linguistic semi linear algebras over the respective sets on which they are defined.

Recall in chapter 1 we have defined the concept of linguistic semirings and linguistic semifields of both finite and infinite order. We also have defined the new notion of subset linguistic semirings and subset linguistic semifields of both finite and infinite order.

Now we wish to first recall the definition of semivector spaces over semifields.

[20] has defined in the year 1993. In case of linguistic semifield we do not have the concept of characteristic of a

semifield. Further our identity with respect to min corresponds to the maximal linguistic element in the linguistic set S.

For min {g, s} = s for every s $\in$ S, where g is the greatest element in S.

If l is the linguistic least element in S then l serves as the linguistic identity with respect to max; max {l, s} = s for all s $\in$ S.

We first recall definition of classical semivector space from [20].

Definition 3.1. *We call V a semi vector space over a semifield S if the following conditions are true*

i) *(V, +) is an additive commutative monoid with 0 as the additive identity.*

ii) *For all s $\in$ S and v $\in$ V, v · s, s · v $\in$ V where '·' is defined as the product of a scalar s in S and the vector v in V (every element in V will be known as a vector and that of s in S are called scalars).*

iii) *For 0 $\in$ S, $0 \cdot v = v \cdot 0 = 0$.*

iv) *$(a\, b) \cdot \alpha = a\,(b \cdot \alpha)$ for all*

$\alpha \in V$ *and a, b* $\in S$.

$(a + b) \cdot \alpha = a \cdot \alpha + b \cdot \alpha$

$a \cdot (\alpha + \beta) = a \cdot \alpha + a \cdot \beta$

for all α, $\beta \in V$ and a, b $\in S$.

vi) For $1 \in S$; $1 \cdot \alpha = \alpha \cdot 1 = \alpha$ for all $\alpha \in V$.

For more above semivector spaces refer [20, 32-8].

We have given several examples of them in [20].

Before we make the abstract definition we will proceed onto describe this situation by examples.

Example 3.1. Let S be a linguistic set.

S = {worst, bad, good, best, fair, very bad, just good, just fair, very good}.

We see {S, max, min} is a linguistic semifield.

V = {S, max} is a commutative linguistic semigroup with worst as the identity.

Clearly V is a linguistic semivector space over the semifield {S, max, min} for if v $\in$ V we see for all s $\in$ S; min {v, s} is in V.

If worst $\in$ S;

min {worst, best} = worst $\in$ V; for best $\in$ V.

It is easily verified = {S, max} is a linguistic semi vector over the linguistic semifield {S, max, min}.

On similar lines we can have W = {S, min} to be a linguistic semivector space over the linguistic semifield

{S, max, min} where max {w, s} ∈ W for all w ∈ W and s ∈ S; and best as its linguistic identity of W.

Infact for max {best, s} = best ∈ W.

Unlike other semivector spaces (classical one) we see in case of linguistic semivector spaces we always have two semivector spaces V = {S, max} and W = {S, min} defined over the linguistic semifield {S, min, max}.

This is one of the marked differences between the classical semivector spaces and linguistic semivector spaces.

We have given {S, max} or {S, min} only over {S, max, min}.

We will give different ones where the linguistic set basically used for both linguistic semifield and linguistic semi vector spaces is not the same.

Example *3.2.* Let V = {[worst, best], max} be a linguistic semigroup with worst as its identity.

S = {worst, bad, good, very bad, fair, just fair, very good} be a linguistic set such that {S, min, max}, is the linguistic semifield.

For any v ∈ V and s ∈ S we have min{s, v} ∈ V.

Infact V is a linguistic semivector space with infinite cardinality.

Suppose W = {[worst, best], min} be the linguistic monoid with the linguistic identity best.

W is a linguistic semivector space over {S, min, max}.

Here also W is of infinite cardinality.

It is important to note {S, max} is a linguistic semigroup and W = {[worst, best], max, min} be the linguistic semifield.

Clearly {S, max} is not a linguistic semivector space over W.

For very very very bad ∈ W and

min {very very very bad, good} = very very very bad ∉ S.

Hence {S, max} is not a linguistic semivector space over the linguistic semifield W = {[worst, best], max, min}.

Similarly {S, min} be a linguistic semigroup. W be as before a linguistic semifield space W.

S is not a linguistic semivector space over W for best ∈ W and for s = good ∈ S, max {best, good} ∉ S. Hence the claim.

So what can be a criteria that for {S, min} or {S, max} linguistic semigroup to be a linguistic semivector space over the semifield F = {B, min, max}.

It is mandatory the linguistic line (interval) B is a linguistic subset of S. If S ⊆ B and S is a proper subset of B then certainly S cannot be a linguistic semivector space over the linguistic semifield {B, max, min}.

Now we make the abstract definition of the linguistic semivector space V over the linguistic semifield S.

Definition 3.2. *Let V be a linguistic set (or line / interval) with min (or max) operation.*

{V, min} (or {V, max}) is a linguistic monoid {S, max, min} be a (linguistic set or linguistic interval / line) linguistic semifield.

We define {V, min} (or {V, max}) to be a linguistic semivector space over S, if for all v ∈ V and for all s ∈ S, max {v, s} is in V (or min {v, s} is in V).

This is equivalent to saying for {V, max} a linguistic monoid is such that max {V, S} ⊆ V that is

max{V, S} = {max {v, s}, for all v ∈ V and s ∈ S is in V} is a proper subset of V}

(in case of {V, min} min{V, S } ⊆ V that is min{V, S}

= {min {v, s}; for all v ∈ V and s ∈ S} is contained in V, then {V, min} or ({V, max}) is a linguistic semivector space over the semifield {S, min, max}.

We will provide some more examples of them.

Example 3.3. Let V = {tall, short, just tall, very tall, shortest, tallest, medium, just medium, very short, very very tall}

be a linguistic set. {V, max} and {V, min} are linguistic monoids of order 10.

Let S = {tall, just tall, medium, very short, just medium, tallest}

be a linguistic subset of V. {S, min, max} is a linguistic semifield with tallest as the linguistic identity for {S, min} the linguistic monoid and for {S, max} the linguistic monoid very short is the linguistic least identity.

Clearly V is a linguistic semivector space over the linguistic semifield S.

One of the very natural questions is will every linguistic subset of V be a linguistic semi vector subspace of V over the linguistic semifield S.

This question is formulated from the fact that if we have {S, min} (or {S, max}) to be linguistic semigroup or a linguistic monoid then every proper linguistic subset of S (including singleton set) will continue to be a linguistic subsemigroup or a linguistic submonoid.

Likewise if we take any linguistic semiring {P, min, max} or a linguistic semifield then every subset of P say Q will be such that {Q, min, max} will be a linguistic subsemiring or a linguistic subsemifield.

However for every linguistic semivector space {V, min} or {V, max} defined over the linguistic semifield {S, min, max} every subset of V is not in general a linguistic subsemivector space (linguistic semivector subspace) of V.

We will illustrate this situation by an example.

Example 3.4. Let V = {best, worst, good, fair, very good, bad, very very bad, just bad, just fair, just good, very fair, worst}

be a linguistic set, {V, max} is a linguistic monoid and {V, min} is also a linguistic monoid.

Let S = {best, worst, good, fair, very good, bad, very very bad} $\subseteq$ V be a proper linguistic subset of V.

Clearly {S, min, max} is a linguistic semifield. {V, max} is a linguistic semivector space over S.

Now we take W = {bad, just bad, very fair, fair} $\subseteq$ V, {W, min} and {W, max} are linguistic monoids. But {W, min} and {W, max} (by default of notation {W, min, max}) is not a linguistic subsemi vector space over S.

Hence the claim.

Thus we have make a necessary and sufficient condition for a linguistic semivector subspaces to exist over the linguistic semifield over which it is defined.

This putforth in the following theorem.

Theorem 3.1. *Let {V, min} and {V, max} ({V, min, max}) be a linguistic semivector spaces over the linguistic semifield {S, min, max}. Let W $\subseteq$ V be a proper linguistic subset of V. {W, min} and {W, max} (that is ({W, min, max} by default of notation) is a linguistic subsemivector spaces if and only if S $\subseteq$ W.*

Proof: We have V to be a linguistic semi vector space over the linguistic semifield S if and only if S $\subseteq$ V. On the other hand if W $\subseteq$ V, where W is a proper linguistic subset of V, then if W is to be linguistic semivector space over S then S $\subseteq$ W and vice versa. Thus we have this condition to be a mandatory one for the linguistic semivector subspaces to exist.

Now we can define special type of linguistic semivector subspace which is choose to all linguistic strong semi vector subspace or (by default of expression) linguistic semivector strong subspace of the linguistic semivector space.

Definition 3.3. *Let {V, max} or {V, min} ({V, min, max by default of notation represents the two semivector spaces {V, min} and {V, max}) be a linguistic semivector space over the linguistic semifield {S, max, min}.*

Let W $\subseteq$ V be a linguistic proper subset of V and T $\subseteq$ S be a linguistic proper subset of S.

If {W, min} or {W, max} ({W, min, max}) be a linguistic semivector subspaces of V over the linguistic subsemifield T.

Then we define {W, min, max} as the strong linguistic semivector subspace of over the linguistic subsemifield T of S.

We will illustrate this situation by some examples.

Example 3.5. Let {V = [worst, best] min} where I_L = [worst, best] = V; {V, min}, {V, max} (or {V, max, min}) are linguistic semivector spaces over the linguistic semifield

{S = [bad, very good], min, max}.

Consider W = [fair, good] $\subseteq$ [worst, best] = V

be a proper linguistic subset of V.

Clearly {W, max} (or {W, min}, {W, max, min}) are not linguistic subsemivector subspaces of V over the semifield S.

However if we take

T = [just fair, just good] $\subseteq$ [bad, very good] = S;

it can be proved {T, max, min} is a proper linguistic subsemifield of S.

Further {W, max} or {W, min} ({W, max, min}) is a linguistic strong subsemivector subspace over the linguistic subsemifield {T, min, max} of {S, min, max}.

Clearly T $\subseteq$ W.

The following condition is mandatory.

If W $\subseteq$ V is a be strong linguistic subsemivector space over T $\subseteq$ S then it is mandatory T $\subseteq$ W otherwise W will not be a strong linguistic subsemivector space over T.

We can putforth this as the following theorem.

Theorem 3.2. *Let {V, max, min} be a linguistic semivector space over the linguistic semifield {S, max, min}. {W, min, max} $\subseteq$ {V, max, min} is a strong linguistic semivector subspace of V over {T, max, min} $\subseteq$ {S, max, min} if and only if T $\subseteq$ W.*

Proof is left as an exercise to the reader.

Note: If both V and S are only linguistic sets. Now it is not possible to define dimension or generating set of the linguistic semivector spaces; or to be more specific it is not possible to talk about basis or dimensions of exactly in a way in which we talk of classical semivector spaces or classical vector spaces.

Now we cannot speak of linguistic basis or linguistic elements which can generate the linguistic semivector spaces. The only thing we can say is infinite linguistic semivector spaces. The only thing we can say is infinite linguistic semivector space and finite linguistic vector space and nothing more. This is one of the major difference between the classical semivector space and the linguistic semivector space.

Example 3.6. Let us consider P(S) the linguistic power set of the linguistic set

S = {best, bad, good, very bad, very good, worst, fair, medium, very medium, just medium, just fair, very fair}. V = {P(S), min} is a linguistic commutative monoid which is also a linguistic subset semivector space over the linguistic semifield {S, min, max}. We see the cardinality of V over S is 2^{12}.

We just leave it as a simple exercise to prove or disprove V cannot be generated over S by any finite proper subset of V.

Example 3.7. Let S be as in the above example 3.6.

Take M = {best, very good, good, bad, medium, just medium, fair} $\subseteq$ S

to be a linguistic subset of S.

Now we know P(M), linguistic power subset of the linguistic power set S. That is P(M) $\subseteq$ P(S) is a proper linguistic subset {P(M), min} and {P(M), max} are linguistic subset commutative moniods.

Further both {P(M), min} and {P(M), max} are not linguistic subset semivector subspaces of {P(S), min} and {P(S), max} respectively over the linguistic semifield

{S, min, max}.

This is so because the linguistic set S $\not\subset$ M.

Suppose we replace S by M, i.e.

$$\{M, \min, \max\} \subseteq \{S, \min, \max\}$$

the linguistic subset semi subfield of the linguistic semifield S then {P(M), min} and {P(M), max} are linguistic strong subset subsemivector spaces of {P(S), min} and {P(S), max} respectively over the linguistic subsemifield {P(M), max, min}.

There are several such linguistic strong subset subsemivector subspaces. However no proper subset of {S}, P(S) \ {S} is a linguistic subset subsemivector space of {P(S), min, max} over the linguistic semifield {S, min, max}.

If we take the singleton set {S} $\in$ P(S) then {{S}, min} and are trivially linguistic subset semivector subspace of {P(S), min} over the linguistic semifield {S, min, max}.

Study in this direction is both interesting, involving and innovative.

The reader is expected to study them and try to find related properties of subset linguistic semivector spaces using {{S}, max}.

Example 3.8. Let I_L = [dullest, brightest] be a linguistic interval. V = {I_L, min} and W = {I_L, max} are linguistic semivector spaces over the linguistic semifield S = {I_L, min, max}.

Now let P(I_L) be a linguistic power set of I_L; P(I_L) is of infinite order as I_L is of infinite order {P(I_L), max} and {P(I_L), min} are linguistic subset semivector spaces over the linguistic semifield S.

Now let us consider the linguistic subinterval

$$[\text{just dull, very bright}] = J_L \subseteq [\text{dullest, brightest}] = I_L.$$

Now $\{J_L, \max\}$ and $\{J_L, \min\}$ are not linguistic semivector subspaces of $\{I_L, \max\}$ and $\{I_L, \min\}$ respectively over S.

Likewise if $P(J_L)$ is a linguistic power set of the linguistic subinterval $J_L \subseteq I_L$ then also $\{P(J_L), \min\}$ and $\{P(J_L), \max\}$ are not linguistic subset semivector subspaces of $\{P(I_L), \min\}$ $\{P(I_L), \max\}$ respectively over the linguistic semifield S. However $\{P(J_L), \min\}$ and $\{P(J_L), \max\}$ are linguistic strong subset subsemivector spaces over the linguistic subsemifield $\{J_L, \min, \max\}$.

Similarly $\{J_L, \min\}$ and $\{J_L, \max\}$ are linguistic strong subsemivector spaces over the linguistic subsemifield

$$\{J_L, \min, \max\} \subseteq \{I_L, \min, \max\}.$$

We see it is very difficult to get proper linguistic subset subsemivector spaces if we built them over the same set as that of the linguistic set whose power set is considered.

Hence to have many linguistic subset subsemivector subspaces it is mandatory if S is the linguistic set used for the linguistic power set P(S) then we take for the linguistic semifield M and M a proper subset of S $\{P(M), \min\}$ and

{P(M), max} will be a subset subsemivector spaces over the semifield {M, min, max}.

Let us give other types of linguistic semivector spaces using linguistic matrices.

We will first provide some examples of them.

Example 3.9. Let B = {collection of all 1 × 5 linguistic row matrices with elements from the linguistic set

S = {good, bad, best, just good, very good, fair, just bad, very very bad and medium}

We know {B, min} is a linguistic monid with (best, best, best, best, best) as the linguistic identity.

We know {B, max} is a linguistic monoid with

{very very bad, very very bad, very very bad, very very bad, very very bad} as the linguistic identity.

{B, min}, ({B, max} or by default of notation {B, max, min}) is a linguistic semivector space over the linguistic semifield.

{S = {very very bad, best, medium, good, bad} min, max}.

We can say B is a finite order linguistic semivector space over the linguistic semifield S.

Infact order of B is 9^5.

Example 3.10. Let A = {collection of all 1 × 10 linguistic matrices with entries from [shortest, tallest], the linguistic interval measuring height}, {A, min}, ({A, max} or {A, min, max}) is a linguistic semivector space over the linguistic semifield S = {[shortest, tallest], max, max}.

We see the cardinality or the number of elements in A is infinite infact an infinite power.

Now we wish to show by this example that the V which we have constructed is not a linguistic semivector space over the linguistic semifield T.

The example is as follows.

Example 3.11. Let V = {collection of all 1 × 9 linguistic row matrices with entries from the linguistic interval [medium, just tall], {V, min} is a linguistic monoid. {V, max} is a linguistic monoid {V, max, max} by default of notation is not a linguistic semivector space over the linguistic semifield

{S = [shortest, tallest], min, max}.

For consider the linguistic matrix

x = (just tall, just tall, …, just tall) in {V, min}.

Let tallest ∈ S

max {tallest, x} = (tallest, tallest, …, tallest) which is clearly not an element of {V, min}.

Therefore V is not a linguistic semivector space over the linguistic semifield S.

Consider

y = (medium, medium, …, medium) ∈ {V, max}. Our claim {V, max} is not a linguistic semivector space over the linguistic semifield S.

For if we take shortest ∈ S;

min {shortest, y} = (shortest, shortest, …, shortest) ∉ {(V, max}.

Hence {V, max} is not a linguistic semivector space over the linguistic semifield {S, min, max}.

Thus we have to find a condition for a linguistic row matrix collection to be a linguistic semivector space over the linguistic semifield.

Theorem 3.3. *Let V = {collection of all linguistic 1 × n row matrices with entries from the linguistic set S or from the linguistic interval I_L} {V, max} (or {V, min}, {V, min, max}) is linguistic semivector space over the linguistic semifield*

B = {T, the linguistic set with max and min or J_L the linguistic interval with max and min} if and only if $T \subseteq S$ (or $J_L \subseteq I_L$).

Proof is direct and is left as an exercise to the reader.

We can have in this case also both linguistic subsemivector spaces over linguistic semifields and strong linguistic subsemivector spaces over linguistic subsemifields.

However we give illustrative examples of them.

Example 3.12. Let V = {collection of all 1 × 6 row linguistic matrices with entries from the linguistic interval / continuum [dullest brightest]}, {V, min} and {V, max} are special linguistic semivector spaces over the linguistic semifield

{S = [dullest, just bright], min, max}.

Consider P = [dull, medium] ⊆ [dullest, brightest] = V a linguistic subinterval of the linguistic interval [dullest, brightest] = V.

Now let W = {collection of all 1 × 6 row linguistic matrices with entries from P ⊆ V}, {W, min} and {W, max} are not linguistic subsemivector spaces of V over S.

For consider {W, min}; clearly {W, min} is a commutative linguistic monoid.

Now for any w ∈ W and s ∈ S we should have

max {w, s} to be in W if W is to be a linguistic semivector subspace of V over S.

Let w = {dullest, ..., dullest) ∈ W and s = just bright ∈ S.

Now

max (s, w) = max (just bright, just bright, …, just bright)

which is clearly not a linguistic matrix in W.

Hence {W, min} is not a linguistic subsemivector space of V over S.

Now consider {W, max} and {W, min} are commutative linguistic moniods. To prove {W, max} is a linguistic subsemivector space of W over S, we have to prove for any w ∈ W and s ∈ S, min {w, s} ∈ W.

Consider s = dullest ∈ S and w = (dull, dull, …, dull) ∈ W.

We find min {s, w} = (dullest, dullest, …, dullest) ∉ W hence {W, max} is not a linguistic subsemivector space of V over S.

We see {W, max} is not a linguistic subsemivector space of V because the linguistic interval on which W is defined is P = [dull, medium], whereas linguistic interval over which V is defined is [dullest, just bright] and clearly

[dullest, just bright] ⊄ [dull, medium].

Infact [dull, medium] ⊆ [dullest, just bright].

Hence the claim.

Suppose we see if T = {collection of all 1 × 6 linguistic row matrices defined over the interval

Q = [dullest, bright] ⊆ [dullest, brightest]}.

Then clearly {T, max} is a linguistic subsemivector space of V over M.

For M = [dullest, just bright] is properly contained in the linguistic interval [dullest, bright] = Q.

Study in this direction is interesting.

Now we will have to define the notion of linguistic semilinear algebra A over the linguistic semifield S. If A is to be linguistic semilinear algebra over S then {A, min} and {A, max} are linguistic semivector spaces over S and {A, min, max} is a semifield or to be more precise both the operations are defined.

We have seen all the linguistic semivector spaces so far defined are also linguistic semilinear algebras.

That is why in most of the places we use the term {S, min, max} is a linguistic semivector space instead of using the terms separately {S, min} and {S, max} are linguistic semivector spaces defined over a linguistic semifield {P, min, max}; $P \subseteq S$.

Now we will proceed onto give examples of these linguistic structures by examples.

Example 3.13. Let V = {collection of all 5×1 linguistic column matrices with entries from [worst, best], the linguistic interval measuring the services of repair centres to their customers].

{V, max} is a linguistic monoid.

$$\text{Let } A = \begin{bmatrix} \text{bad} \\ \text{fair} \\ \text{good} \\ \text{very bad} \\ \text{medium} \end{bmatrix} \text{ and } B = \begin{bmatrix} \text{good} \\ \text{very bad} \\ \text{fair} \\ \text{medium} \\ \text{good} \end{bmatrix}$$

be two 5×1 linguistic column matrices in V.

Clearly max {A, B} = max $\left\{ \begin{bmatrix} \text{bad} \\ \text{fair} \\ \text{good} \\ \text{very bad} \\ \text{medium} \end{bmatrix}, \begin{bmatrix} \text{good} \\ \text{very bad} \\ \text{fair} \\ \text{medium} \\ \text{good} \end{bmatrix} \right\}$

$$\begin{bmatrix} \max\{\text{bad, good}] \\ \max\{\text{fair, very bad}] \\ \max\{\text{good, fair}\} \\ \max\{\text{very bad, medium}] \\ \max\{\text{medium, good}\} \end{bmatrix} = \begin{bmatrix} \text{good} \\ \text{fair} \\ \text{good} \\ \text{medium} \\ \text{good} \end{bmatrix} \in V.$$

Thus {V, max} is a commutative linguistic monoid of

infinite order with the linguistic identity I = $\begin{bmatrix} \text{worst} \\ \text{worst} \\ \text{worst} \\ \text{worst} \\ \text{worst} \end{bmatrix} \in V.$

We see max {A, I} = max {I, A} = A for all A in V.

On similar lines we can prove {V, min} is a commutative linguistic monoid with the linguistic identity.

$$L = \begin{bmatrix} \text{best} \\ \text{best} \\ \text{best} \\ \text{best} \\ \text{best} \end{bmatrix} \in V.$$

It is verified min {A, L} = min {L, A} = A for all A in V.

Consider S = {[worst, best], max, min}. S is a linguistic semifield of infite order.

Now to show {V, max} is a linguistic semivector space over the linguistic semi semifield S.

For if s = very bad ∈ S and $A = \begin{bmatrix} \text{bad} \\ \text{good} \\ \text{best} \\ \text{very bad} \\ \text{fair} \end{bmatrix} \in \{V, \max\}$

to prove {V, max} is a linguistic semivector space over S we have to prove for any s ∈ S min {s, A} ∈ V

consider s = just bad ∈ S

$$\min \left\{ s, \begin{bmatrix} \text{bad} \\ \text{good} \\ \text{best} \\ \text{very bad} \\ \text{fair} \end{bmatrix} \right\} = \begin{bmatrix} \min\{\text{just bad, bad}\} \\ \min\{\text{just bad, good}\} \\ \min\{\text{just bad, best}\} \\ \min\{\text{just bad, very bad}\} \\ \min\{\text{just bad, fair}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{bad} \\ \text{just bad} \\ \text{just bad} \\ \text{very bad} \\ \text{just bad} \end{bmatrix} \in V.$$

Hence {V, max} is a linguistic semivector space over the linguistic semifield {S, min, max}.

Now to show {V, max, min} is a linguistic semilinear algebra over {V, max, min}.

We see {V, max} and {V, min} are linguistic semi vector spaces over the linguistic semi field {S, max, min}.

So {V, max} will enjoy the special product min so that {V, max, min} becomes a linguistic semilinear algebra over the liniguistic semi field {S, min, max}.

On similar lines we see {V, min} will enjoy a special product max so that {V, min, max} becomes a linguistic semilinear algebra over the linguistic semifield {S, max, min}.

Thus we have got a class of linguistic semilinear algebras using these row and column linguistic matrices.

Now we give examples of linguistic square matrices which are linguistic semilinear algebras.

Example 3.14. Let B = {collection of all 4 × 4 linguistic square matrices with entries from the linguistic interval [very lazy, best] which describes the nature of the worker in an industry}.

Let S = {[medium, just good], min max} be the linguistic semifield of infinite cardinality. {B, max} is a linguistic monoid. To prove this take

$$A = \begin{bmatrix} \text{lazy} & \text{very lazy} & \text{medium} & \text{active} \\ \text{just lazy} & \text{lazy} & \text{active} & \text{lazy} \\ \text{medium} & \text{active} & \text{lazy} & \text{medium} \\ \text{good} & \text{very good} & \text{very lazy} & \text{active} \end{bmatrix} \text{ and}$$

$$D = \begin{bmatrix} \text{good} & \text{medium} & \text{lazy} & \text{good} \\ \text{very lazy} & \text{good} & \text{active} & \text{lazy} \\ \text{active} & \text{lazy} & \text{good} & \text{active} \\ \text{just active} & \text{very lazy} & \text{good} & \text{good} \end{bmatrix}$$

be two 4 × 4 linguistic square matrices in B.

We find max {A, D}

$$= \max \{ \begin{bmatrix} \text{lazy} & \text{very lazy} & \text{medium} & \text{active} \\ \text{just lazy} & \text{lazy} & \text{active} & \text{lazy} \\ \text{medium} & \text{active} & \text{lazy} & \text{medium} \\ \text{good} & \text{very good} & \text{very lazy} & \text{active} \end{bmatrix},$$

$$\begin{bmatrix} \text{good} & \text{medium} & \text{lazy} & \text{good} \\ \text{very lazy} & \text{good} & \text{active} & \text{lazy} \\ \text{active} & \text{lazy} & \text{good} & \text{active} \\ \text{just active} & \text{very lazy} & \text{good} & \text{good} \end{bmatrix} \}$$

$$= \begin{bmatrix} \max\{\text{lazy, good}\} & \max\{\text{very lazy, medium}\} \\ \max\{\text{just lazy, very lazy}\} & \max\{\text{lazy, good}\} \\ \max\{\text{medium, active}\} & \max\{\text{active, lazy}\} \\ \max[\text{good, just active}\} & \max\{\text{very good, very lazy}\} \end{bmatrix.$$

$$\begin{matrix} \max\{\text{medium, lazy}\} & \max\{\text{active, good}\} \\ \max\{\text{active, active}\} & \max\{\text{lazy, lazy}\} \\ \max\{\text{lazy, good}\} & \max\{\text{medium, active}\} \\ \max\{\text{very lazy, good}\} & \max\{\text{active, good}\} \end{bmatrix}$$

$$= \begin{bmatrix} \text{good} & \text{medium} & \text{lazy} & \text{good} \\ \text{very lazy} & \text{good} & \text{active} & \text{lazy} \\ \text{active} & \text{lazy} & \text{good} & \text{active} \\ \text{just active} & \text{very lazy} & \text{good} & \text{good} \end{bmatrix} \text{ is in } \{B, \max\}.$$

Thus {B, max} is a linguistic semigroup of infinite order which is commutative.

Now we prove {B, max} linguistic commutative monoid. For this we have to find the linguistic identity 4 × 4 matrix.

Consider I =

$$\begin{bmatrix} \text{very lazy} & \text{very lazy} & \text{very lazy} & \text{very lazy} \\ \text{very lazy} & \text{very lazy} & \text{very lazy} & \text{very lazy} \\ \text{very lazy} & \text{very lazy} & \text{very lazy} & \text{very lazy} \\ \text{very lazy} & \text{very lazy} & \text{very lazy} & \text{very lazy} \end{bmatrix} \in \{B, \max\}$$

is such that for all x ∈ (B, max, max {x, I} = x.

Thus {B, max} is a linguistic monoid which is commutative and of infinite order.

Now consider {B, min}, to prove {B, min} is a linguistic commutative semigroup we have to show for any A, D ∈ {B, min}, min {A, D} ∈ {B, min}.

For the above linguistic matrices A and D, we see

$$\min\{A, D\} = \begin{bmatrix} \text{lazy} & \text{very lazy} & \text{lazy} & \text{active} \\ \text{very lazy} & \text{lazy} & \text{active} & \text{lazy} \\ \text{medium} & \text{lazy} & \text{lazy} & \text{medium} \\ \text{just active} & \text{very lazy} & \text{very lazy} & \text{active} \end{bmatrix}$$

is in {B, min}.

Further min{A, D} = min {D, A}.

So {B, min} is a linguistic commutative semigroup of infinite cardinality.

We claim {B, min} is a linguistic commutative monoid of infinite cardinality for

$$J = \begin{bmatrix} \text{best worker} & \text{best worker} & \text{best worker} & \text{best worker} \\ \text{best worker} & \text{best worker} & \text{best worker} & \text{best worker} \\ \text{best worker} & \text{best worker} & \text{best worker} & \text{best worker} \\ \text{best worker} & \text{best worker} & \text{best worker} & \text{best worker} \end{bmatrix}$$

in {B, min} will act as the linguistic identity matrix for the linguistic monoid {B, min} as min {A, J} = A for all A in {B, min}.

Now we want to study the following example.

Example 3.15. Let P = {collection of all 3 × 6 linguistic matrices with entries from the linguistic set S}.

{P, min} and {P, max} are linguistic commutative monoids.

Further {S, min} and {S, max} are also linguistic commutative monoids. Infact {P, max, min} and {S, max, min} are linguistic semifields.

We know {P, max, min} is a linguistic semilinear algebra over the linguistic semifield {S, max, min}.

However {S, min, max} is not a linguistic semivector space or semilinear algebra over {P, max, min}. Infact to be more precise {S, min} and {S, max} are not even linguistic semi vector spaces over the linguistic semifield {P, max, min}.

Now we can give the following theorem in case of linguistic matrices.

Theorem 3.4*. Let P = {collection of all m × n linguistic matrices with entries from the linguistic set S or the linguistic continuum / interval I_L}.*

i) {P, min} is a linguistic commutative monoid.

ii) {P, max} is a linguistic commutative monoid.

iii) {S, I_L, min} is a linguistic commutative monoid.

iv) {S, I_L, max} is a linguistic commutative monoid.

v) {S, I_L, min, max} is a linguistic semifield.

vi) *{P, min, max} is a linguistic semifield.*

vii) *{P, min} is a linguistic semivector space over the linguistic semifield {S, min, max}.*

viii) *{P, max} is a linguistic semivector space over the linguistic semifield {S, min, max}.*

ix) *{P, max, min} ({P, min max}) is a linguistic semilinear algebra over the linguistic semifield {S, min, max}.*

Proof is direct and this task is left as exercise to the reader.

Now we are going to define different type of linguistic semilinear algebra over linguistic semifields.

First we illustrate this situation by some examples.

Example 3.16. Let M = {All 5 × 5 linguistic matrices with entries from I_L = [slowest, speediest]}.

{M, min} and {M, max} are linguistic monoids of infinite order which are commutative.

$$I = \begin{bmatrix} \text{speediest} & \text{speediest} & \dots & \text{speediest} \\ \text{speediest} & \text{speediest} & \dots & \text{speediest} \\ \vdots & \vdots & \dots & \vdots \\ \text{speediest} & \text{speediest} & \dots & \text{speediest} \end{bmatrix}$$

is the linguistic identity for the linguistic monoid {M, min}.

Similarly J = $\begin{bmatrix} \text{slowest} & \text{slowest} & \ldots & \text{slowest} \\ \text{slowest} & \text{slowest} & \ldots & \text{slowest} \\ \vdots & \vdots & \ldots & \vdots \\ \text{slowest} & \text{slowest} & \ldots & \text{slowest} \end{bmatrix}$

is the linguistic identity for the linguistic monoid {M, max}.

Now we know {I_L, max, min} is a linguistic semifield of infinite order.

Infact we have two different linguistic semilinear algebra {M, max, min} and {M, min, max} defined over the linguistic semifield {I_L, max, min}.

Let A = $\begin{bmatrix} \text{high} & \text{slow} & \text{high} & \text{medium} & \text{high} \\ \text{medium} & \text{high} & \text{slow} & \text{slow} & \text{slow} \\ \text{slow} & \text{high} & \text{medium} & \text{high} & \text{very high} \\ \text{very slow} & \text{slow} & \text{high} & \text{medium} & \text{slow} \\ \text{high} & \text{slow} & \text{high} & \text{high} & \text{medium} \end{bmatrix}$

and

B = $\begin{bmatrix} \text{slow} & \text{very slow} & \text{slowest} & \text{high} & \text{slow} \\ \text{slow} & \text{high} & \text{slow} & \text{medium} & \text{high} \\ \text{high} & \text{slow} & \text{very slow} & \text{slow} & \text{slow} \\ \text{very high} & \text{slow} & \text{medium} & \text{high} & \text{medium} \\ \text{very slow} & \text{high} & \text{high} & \text{very high} & \text{very slow} \end{bmatrix}$

be any two linguistic matrices from M.

We find min {max {A, B}} =

$$\begin{bmatrix} \text{slow} & \text{medium} & \text{slow} & \text{medium} & \text{medium} \\ \text{slow} & \text{slow} & \text{medium} & \text{slow} & \text{slow} \\ \text{slow} & \text{slow} & \text{slow} & \text{medium} & \text{slow} \\ \text{slow} & \text{very slow} & \text{very slow} & \text{medium} & \text{slow} \\ \text{slow} & \text{high} & \text{slow} & \text{medium} & \text{medium} \end{bmatrix}.$$

Now we find max {min {A, B}} =

$$\begin{bmatrix} \text{high} & \text{high} & \text{high} & \text{high} & \text{medium} \\ \text{slow} & \text{high} & \text{slow} & \text{medium} & \text{high} \\ \text{high} & \text{high} & \text{high} & \text{very high} & \text{high} \\ \text{high} & \text{slow} & \text{medium} & \text{medium} & \text{medium} \\ \text{high} & \text{medium} & \text{medium} & \text{high} & \text{medium} \end{bmatrix}.$$

It is easily verified

$$\min \{\max \{A, B\}\} \neq \max \{\min, \{A, B\}\}.$$

Thus in view of this we get the following linguistic semi linear algebra with the above defined operations.

V_1 = {M, min, min {max {A, B}},

V_2 = {M, min, max {min {A, B}},

V_3 = {M, max, min {max {A, B}},

V_4 = {M, max, max {min {A, B}},

V_5 = {M, max, min} and

V_6 = {M, min, max} are the 6 distinct linguistic semi linear algebras defined over the linguistic field {I_L, min, max}.

Next we show that this result is not true in case of all matrices. They are true only in case of linguistic square matrices.

Thus only in case of linguistic square matrices we get 6 different linguistic semi linear algebras, in all other cases we have only 2 linguistic semi linear algebras for

min {max {A, B}} or max{min {A, B}} are not defined when the linguistic matrices are not square matrices.

Now we define yet another concept of direct product of a linguistic set or linguistic interval / continuum in the following.

Definition 3.4. *Let S be a linguistic set or I_L a linguistic interval / continuum.*

Now the linguistic direct product of S (or I_L) denoted by $S \times \ldots \times S = \{s_1, \ldots, s_n\}$ where $s_i \in S$; $1 \leq i \leq n$; $2 \leq n < \infty\}$ (or $I_L \times \ldots \times I_L = (l_1, \ldots, l_n) / l_i \in I_L$; $1 \leq i \leq n$; $2 \leq n < \infty\}$) is the linguistic set (or interval) producted n-times.

When n = 2 we get the linguistic plane which has been defined in book 1 of linguistic series [22-5]

When n = 3 we will have the three dimensional linguistic space and for any n, $3 < n < \infty$ we have the n-dimensional space.

The very purpose of recalling this is we can say $S \times S \times S \times \ldots \times S = \{(s_1, s_2, \ldots, s_n)$ *with* $s_i \in S;\ 1 \leq i \leq n\}$ *can also be realized as the* $1 \times n$ *linguistic row matrices.*

Hence we can describe following.

Let S be a linguistic set (or I_L a linguistic interval / continuum).

The direct product

$$M = S \times S \times \ldots \times S \text{ (or } N = I_L \times I_L \times \ldots \times I_L)$$

$= \{(s_1, \ldots, s_n) \text{ (or } (l_1, \ldots, l_n)) \,/\, s_i \in S;\ 1 \leq i \leq n\ (l_i \in I_L;\ 1 \leq i \leq n)\}$

is a linguistic commutative monoid under min operation; that is {M, min} is a linguistic monoid which is commutative. On similar lines {M, max} is also a linguistic commutative monoid.

Infact {M, min, max} is a linguistic semifield.

We now give examples of linguistic semilinear algebras over linguistic semifields using linguistic direct products.

Example 3.17. Let $V = S \times S \times S \times S \times S$ be the direct product of a linguistic set S. We know {S, min, max} is a linguistic semifield. {V, max} is a linguistic commutative monoid. Similarly {V, min} is again a linguistic commutative monoid. Further {V, max} is a linguistic semivector space over the linguistic semifield {S, max, min}.

Similarly {V, min} is again a linguistic semivector space over the semifield {S, max, min} {V, min, max} and {V, max, min} are linguistic semilinear algebras over the linguistic semifield {S, min, max}.

We can have several such linguistic semilinear algebras using the linguistic semifield {S, min, max}.

It is not possible to find out the basis or generating elements for these linguistic set or linguistic continuum.

The main reasons for this is in case of linguistic sets or linguistic interval / continuum we have max or min operation on them which are idempotents in nature that is min {x, x} = x and max {x, x} = x for all linguistic elements x in S or I_L.

We call this properly as linguistic idempotent operator.

Finally the important issue being can we have some sort of mapping or function between linguistic semivector spaces V and W defined over the linguistic semifield.

We have defined the notion of linguistic mapping in linguistic book series I [20-5].

However for the sake of completeness we describe the notion of linguistic mapping or function.

Let H = {good, bad, fair, just good, very good, very bad, best}

K = {good, bad, very very bad, worst, best, just bad, fair, very fair, just good}.

Let L_1 be a linguistic map from H to K.

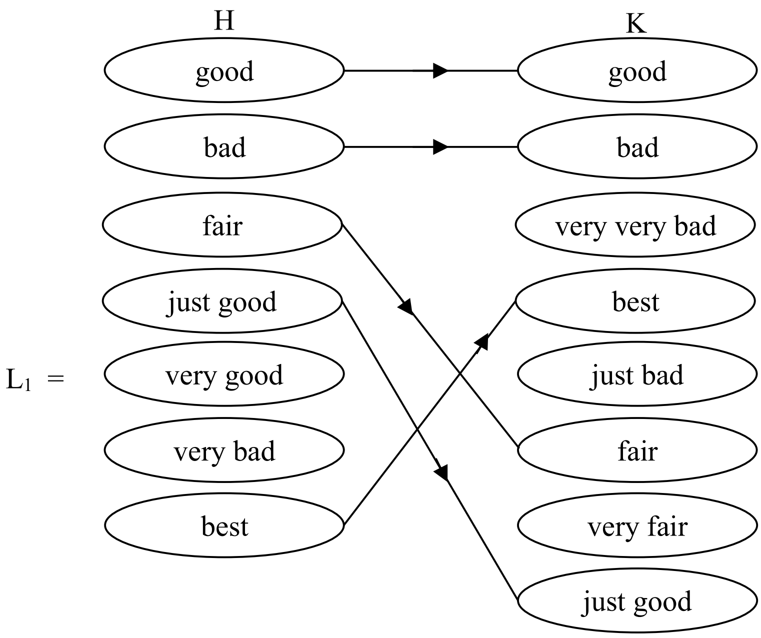

Figure 3.1

We say this linguistic map $L_1 : H \to K$ is an incomplete identical map that is H and K are two sets of experts working on a same problem they wish to map the identical terms onto the other so only we call L_1 as incomplete linguistic mapping or is an incomplete linguistic function [22-5].

For the same linguistic set say H and K we have the following linguistic map say L_2.

L_2 =

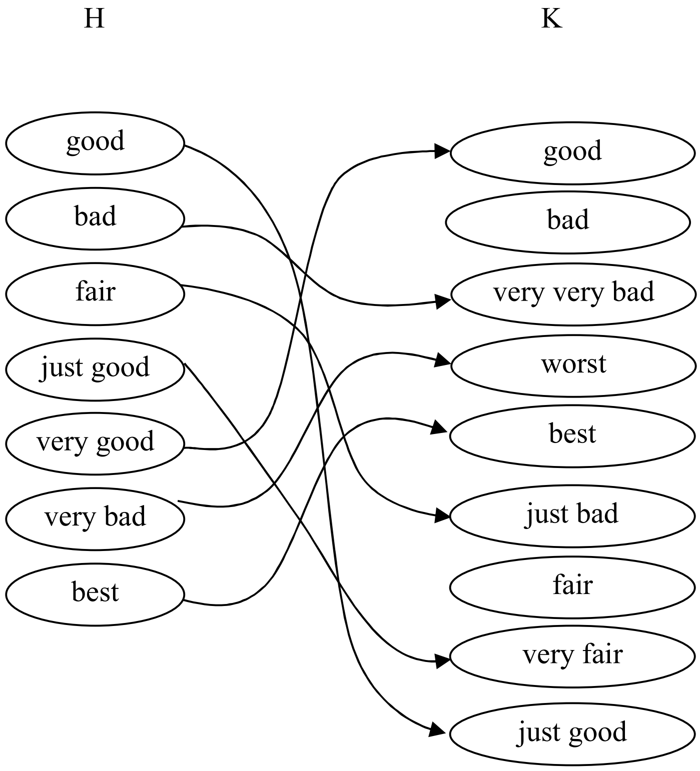

Figure 3.2

When we observe the linguistic mapping L_2 we record the following.

L_2 (good) = just good

L_2 (bad) = very very bad

L_2 (fair) = just bad

L_2 (just good) = very fair

L_2 (very good) = good

L_2 (very bad) = worst

L_2 (best) = best

Bad and fair are left out in the range space.

This linguistic map clearly distinguishes that the expert H is a more lenient towards grading or equivalently the K is more strict than H.

Now this linguistic map L_2 is a not a identical linguistic map we call it as graded linguistic map. However it is complete as in the classical sense for every linguistic term in H is mapped to some term in K.

No linguistic term is left out in H. We can have yet another type of mapping where one may feel the expert H is strict and expert K is very lenient.

In that case the linguistic map is of the following form.

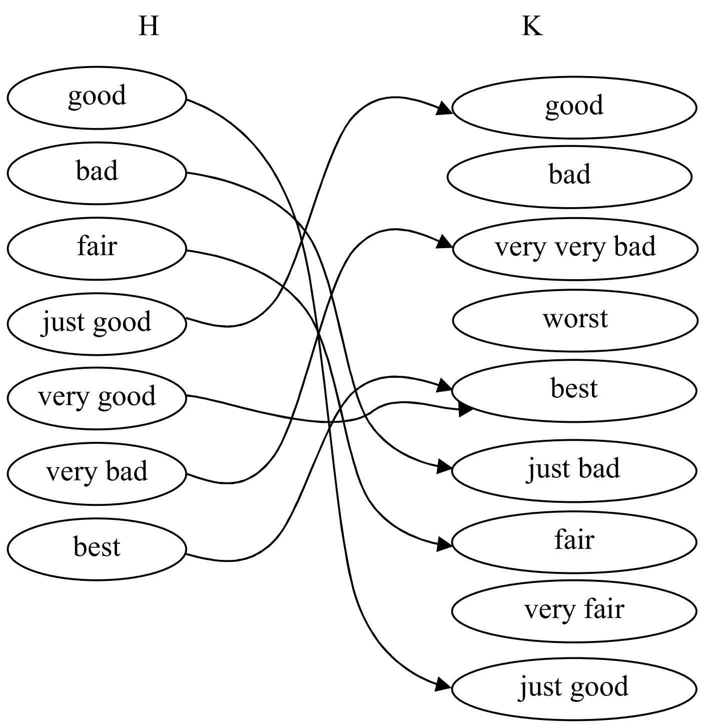

Figure 3.3

Thus is one of the ways the linguistic mapping is made.

One of the famous way of getting the mapping is if P and S are two linguistic sets such that P $\subseteq$ S then we do the linguistic embedding.

That is if $L_3 : P \to S$ then $L_3(p) = p \in S$; yet $L_3(P) = P \subseteq S$.

We call L_3 as the identity linguistic embedding.

We can have 3 types of linguistic and some times we can have a linguistic map which can be very arbitrary which are just a map like the one described by the following.

For the same set of linguistic sets H and K we have the following linguistic map L_4 which we choose to call all such types of linguistic maps are arbitrary linguistic maps.

L_4:

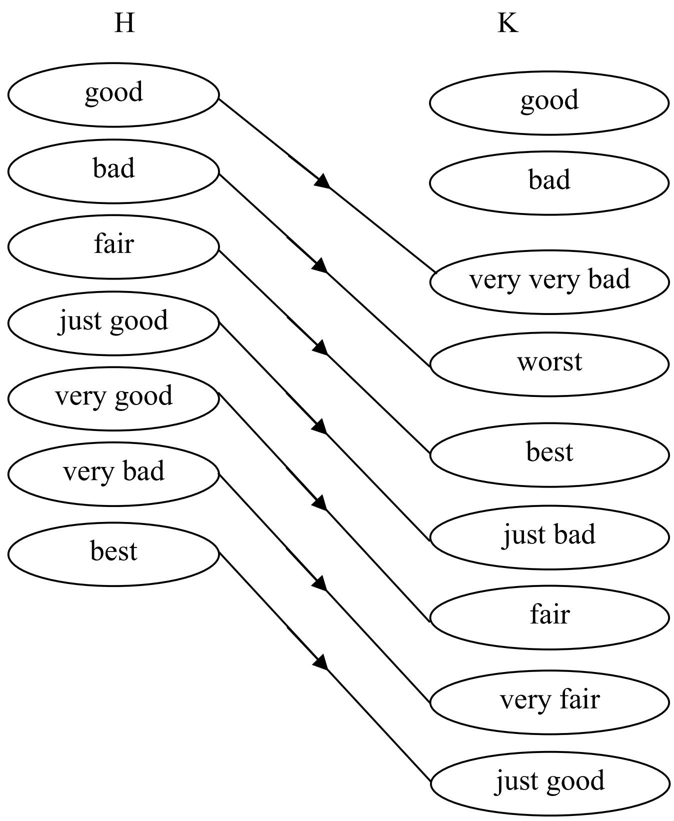

Figure 3.4

This L_4 is one such arbitrary one. In algebraic linguistic structure we have done several such types of linguistic maps [22-5]. No rhyme or reason plays a role while mapping is carried out in such an arbitrary manner. Even a layman will condemn such maps.

On similar lines we can have linguistic maps defined on linguistic intervals I_L. They can also be maps of the form from a linguistic finite set to a linguistic interval or continuum. Such study can be leading to different types of linguistic maps.

One special among them is a projection linguistic map. Its reverse way is called the linguistic map as injection map. One can have a one to one linguistic map if both the domain linguistic space and the range linguistic space are one and the same or is a one to one linguistic map.

Let J_L and I_L be two linguistic intervals / continuum.

Let L be a linguistic map from the two linguistic intervals J_L and I_L.

L: $J_L \to I_L$ can be such that L(s) = s and s $\in I_L$ in this case either $J_L \subseteq I_L$ or $J_L = I_L$.

Any one interested in these concepts can refer [22-5].

Now we proceed onto develop the mappings to the case of linguistic semivector spaces and linguistic semilinear algebras.

We give the abstract definition of linguistic transformation of linguistic semivector spaces defined over the same linguistic semifield S.

Definition 3.5. *Let V and W be two linguistic semivector spaces with {V, max} and {W, max} defined over the same linguistic semifield {S, min, max}.*

We define the map L_{max}: {V, max} → {W, max} to be a linguistic transformation of the two linguistic semivector spaces over the linguistic semifield S if the following conditions are satisfied by L_{max}.

i) *L_{max} is a linguistic map from V → W.*

ii) *L_{max}(max (a, b)) = max (L_{max}(a), L_{max}(b)) that if c = max (a, b) then $L^{(c)}_{max}$ = max ($L^{(a)}_{max}$, $L^{(b)}_{max}$) for all c, a, b ∈ V and L_{max}(a), L_{max}(b), $L^{(c)}_{max}$ ∈ W.*

iii) *For s ∈ S and a ∈ V; L_{max} (min (s, a)) = min (L_{max}(s), L_{max}(a)) that if min (s, a) = d then L_{max} (d) = min (L_{max}(s), L_{max}(a)).*

Then we define L_{max} to be a linguistic transformation of the linguistic semivector spaces {V, max} to {W, max} defined over the linguistic semifield {S, max, min}.

On similar lines we define for any two linguistic semivector spaces {V, min} and {W, max} defined over the linguistic semifield {S, max, min}; L_{min} to be linguistic transformation if the following conditions are true.

i) L_{min} is a linguistic map from V → W.

ii) L_{min} (min (a, b)) = min (L_{min}(a) L_{min}(b)) that if c = min (a, b) then min (L_{min}(a), L_{min}(b)) = L_{min}(c) for all a, b, c ∈ V.

iii) For a ∈ V and s ∈ S we have L_{min} (max (s, a)) = max (L_{min}(s), L_{min}(a)) and if max (s, a) = d then L_{min}(d) = max (L_{min}(s), L_{min}(a)).

Then we define L_{min} to be a linguistic transformation of the linguistic semivector space {V, min} to {W, min} defined over the same linguistic semifield S

We define linguistic transformation L from {V, max, min} to {W, max, min}, the two linguistic semilinear algebras if L satisfies both L_{min} and L_{max}.

We will provide an example of the same.

Example 3.18. Let V = {{best, good, bad, fair, just good, very bad, just bad, very fair just fair}, max} and

W = {{very very good, good, very bad, fair, worst bad, just bad, very fair, just good}, max}

be any two linguistic semivector spaces over the semifield

S = {{good, bad, fair, very fair, very bad, just good}, max, min}}.

Now L: {V, max} → {W, max} is defined as follows

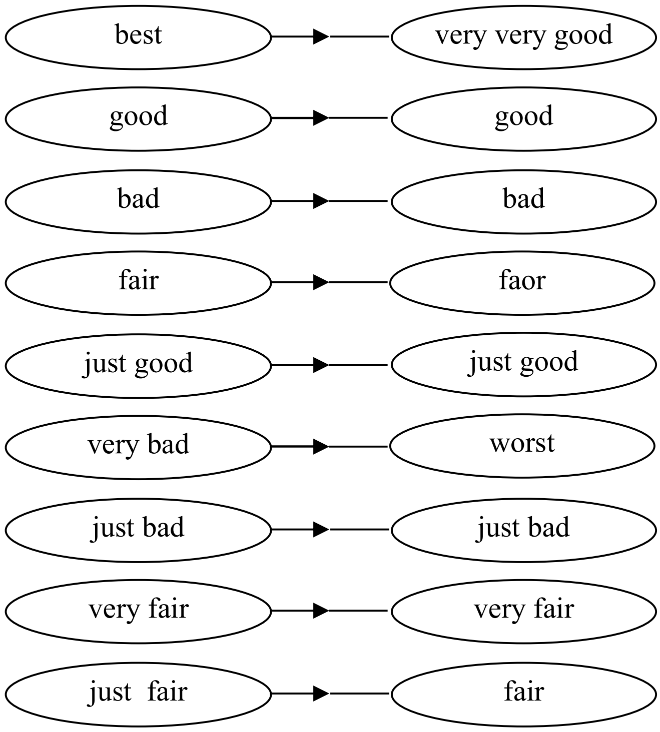

Figure 3.5

It is easily verified L is a linguistic transformation of semivector spaces {V, max} to {W, max}.

We can also define the linguistic transformation from a linguistic semivector subspace {P, max} of a linguistic semivector space {V, max} defined over the linguistic semifield {S, min, max}.

The linguistic map from $l : V \to P$ is called a linguistic projection if

$l(p) = p$ for all $p \in V$

l(v) = least linguistic term in P if $v \notin P$ that is $v \in V \setminus P$.

The following observations are mandatory.

There can be no two linguistic semivector subspaces W_1 and W_2 of the linguistic semivector space {V, max} such that $W_1 \cap W_2 = \{\phi\}$; where {V, max} is defined over the linguistic semivector space {S, min, max}.

It is mandatory $W_1 \cap W_2 = \{S\}$ for W_1 and W_2 to be linguistic semivector subspaces of {V, max} over S.

So the questions of direct sum as in the classical semivector spaces and the mapping (linguistic transformations which are projection) may not in general satisfy any of these properties enjoyed by classical linear transformations and the projections.

Above all we cannot define the concept of kernel of a linguistic linear transformation. But however we will to build an appropriate analogue so we make the following statement.

Let {V, max} be the linguistic semivector space defined over the linguistic semifield {S, max, min}.

Let l be the least linguistic term element of V and g be the great linguistic term / element of V.

Let L_{max} be a linguistic transformation from V to V.

The set of all elements x in (V, max) such that $L_{max}(x) = l$ will be called as linguistic kernel of L_{max} under the max operator on V.

On similar lines all $x \in V$ such that $L_{min}(y) = g$ for those $y \in V$ will be defined as the linguistic kernel of L_{min}:

$$\{V, \min\} \to \{V, \min\}.$$

This is only under the assumption that L_{min} and L_{max} are well defined linguistic transformations from {V, min} to {V, min} and $\{V, \max\} \to \{V, \max\}$ respectively.

It is important record that one can do lots of research in this direction for it is in a dormant state.

Suggested Problems

1. Compare linguistic semivector spaces V and classical semivector spaces W defined over their respective semifield.

2. Let I_L = [shortest, tallest] be the linguistic interval (continuum).

   Prove V = $\{I_L, \max\}$ and W = $\{I_L, \min\}$ are linguistic semivector spaces over the linguistic semifield

   $$S = \{I_L, \min \max\}.$$

   i) Can I_L have linguistic subsemivector space of finite order (has finite number of elements)?

      ii) Can I_L have linguistic subsemivector spaces with infinite cardinality?

iii) Will B = {[shortest, just], max} a linguistic semivector subspace of V? Justify your claim.

3. Let V = {{best, worst, bad, good, very very bad, fair, just fair, just good, just bad, very bad, medium, just medium, very medium, very best}, min max}

   be a linguistic semivector space over the linguistic semifield

   S = {{bad, good, best, very very bad, just good, just bad}, min, max}}.

   i) Find all linguistic subsemivector spaces of V over S.

   ii) How many such linguistic semivector subspaces of V over S? Find the exact number.

4. Let V and S be as in problem (3). Find all linguistic strong subsemivector spaces of V over the linguistic subsemifields T of S.

5. Find any special interesting properties of linguistic semivector spaces {V, min} and {V, max} over the linguistic semifield {S, min, max} (S and V are appropriate linguistic sets or linguistic intervals).

6. Can linguistic semivector spaces over linguistic semifields mentioned in problem (5) have a linguistic basis or a generalizing linguistic set? Justify your claim!

7. Is it in general possible to find a generating proper linguistic subset which can generate a linguistic semivector space over a semifield? (Substantiate your answer with proof).

8. Give a characterization theorem to assert the condition for a linguistic strong semi vector subspaces defined over a linguistic subsemifield to exist.

9. Find all linguistic strong subsemivector spaces of {V, min} and {V, max} (where V = {good, bad, very bad, just bad, very very bad, just good, very good, fair, very fair, best, worst very worst} is the linguistic set) over the linguistic semifield

   F = {S = {good, bad, very good, fair, very bad, worst, best}, min max}.

   i) How many linguistic subsemivector spaces of V over S exist which can never be linguistic strong subsemivector spaces over any linguistic propoer subsemifield of S?

10. Find all special features enjoyed by {P(S), min} = V; S a linguistic set be a linguistic subset semivector space over the linguistic semifield {S, min, max}.

    Can {P(S), min} have proper subset linguistic subsemifields of order greater than two over the linguistic semifield {S, min, max}? Justify your claim!

11. Let M = {collection of all 1 × 6 linguistic matrices with entries from I_L = [most incapable, highly capable]}.

i) Prove {M, min} is a linguistic commutative monoid of infinite cardinality.

ii) Prove {M, max} is a linguistic commutative monoid of infinite cardinality. Give its linguistic identity.

iii) Prove {M, min, max} is a semifield of infinite cardinality.

iv) Prove P = {I_L, max} is a linguistic monoid of infinite cardinality. What is its linguistic identity?

v) Prove Q = {I_L, min} is a linguistic commutative monoid of infinite cardinality. Find Q's linguistic identity.

vi) Prove T = {I_L, min, max} is an infinite dimensional linguistic semifield.

vii) Prove {M, max} is a linguistic semivector space of infinite dimensionality over the linguistic semifield T.

viii) Prove {M, min} is an infinite dimensional linguistic semivector space over the linguistic semifield T.

ix) Prove {M, min, max} (and {M, max, min} are linguistic semi linear algebras of infinite dimension over T.

12. Let P(S) be the linguistic power set of the linguistic set S. Prove

i) {P(S), min} is a linguistic commutative monoid and give the linguistic identity.

ii) {P(S), max} is a linguistic commutative monoid with linguistic identity.

iii) {P(S), min, max} is a linguistic semifield.

iv) Prove {S, min} and {S, max} are linguistic commutative moniods and find their respective linguistic identities.

v) Prove {S, min, max} is a linguistic semifield.

vi) Prove {P(S), min} is a linguistic semivector space over {S, min, max}.

vii) Prove {P(S), max} is a linguistic semivector space over {S, min, max}.

viii) Prove {P(S), min, max} is a linguistic semilinear algebra over {S, min, max}.

ix) Can {S, min} be a linguistic semivector space over {P(S), min, max}? Justify your claim.

x) Can {S, max} be a linguistic semivector space over {P(S), min, max}? Prove your claim.

xi) Will {S, min, max} be a linguistic semilinear algebra over the linguistic semifield {P(S), min, max}? Justify your claim.

13, Distinguish for any linguistic structure S and its linguistic power set P(S) on all the linguistic algebraic structure they enjoy.

14. Let M = {collection of all 1 × 9 linguistic row matrices with entries from a linguistic set S}.

i) Prove {M, min} is a linguistic row matric monoid and give the linguistic identity matrix.

ii) Prove {M, max} is a linguistic row matrix monoid and give the linguistic identity matrix.

iii) Will the linguistic identities in (i) and (ii) be the same? Justify your claim.

iv) Prove {M, min, max} is a linguistic semifield.

v) Prove {M, min} is a linguistic semivector space over the linguistic semifield {S, max, min}.

vi) Prove {M, max} is a linguistic semivector space over the linguistic semifield {S, max, min}.

vii) Prove {M, max, min} is a linguistic semilinear algebra over the semifield {S, max, min}.

viii) Can {S, max, min} be a linguistic semilinear algebra over {M, max, min}?

ix) Can {S, max} be a linguistic semivector space over the linguistic semifield {M, max, min}? Prove your claim!

i) Can {S, min} be a linguistic semivector space over the linguistic semifield {M, max, min}? Justify your claim.

15. Let N = {collection of all 7 × 2 linguistic matrices

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix}$$

where $a_i \in I_L$ = [worst, best]; $1 \leq i \leq 14$}.

Study questions (i) to (x) of problem (14) for this collection.

16. Let A and B be two linguistic matrices given in the following

$$A = \begin{bmatrix} \text{good} & \text{bad} & \text{fair} & \text{worst} & \text{best} \\ \text{bad} & \text{bad} & \text{fair} & \text{good} & \text{good} \\ \text{best} & \text{best} & \text{good} & \text{fair} & \text{fair} \\ \text{good} & \text{medium} & \text{good} & \text{worst} & \text{fair} \\ \text{just good} & \text{bad} & \text{best} & \text{good} & \text{good} \end{bmatrix}$$

and

$$B = \begin{bmatrix} \text{very bad} & \text{bad} & \text{good} & \text{good} \\ \text{good} & \text{very good} & \text{fair} & \text{just fair} \\ \text{bad} & \text{just bad} & \text{best} & \text{fair} \\ \text{good} & \text{bad} & \text{worst} & \text{bad} \\ \text{fair} & \text{best} & \text{fair} & \text{very bad} \end{bmatrix}$$

i) Find min{max {A, B}}.

ii) Is min {max {B, A}} defined, justify your claim?

iii) Can min {A, B} obtained? Prove your claim.

iv) Does max {min {A, B}} defined?

v) Find max {min {A, B}}.

vi) Is max {min {A, B}} = min{max{A, B}?

vii) Does max {B, A} exist?

viii) Will max {min {A, B} = min {max {A, B}}?

17. Define and describe a linguistic transformation from two linguistic semivector spaces {V, max} to {W, max} defined over the linguistic semifield {S, max, min},.

18. Define the notion of linguistic inverse transformation for any given linguistic transformation

L_{max} :(V, max) $\rightarrow$ (W, max)

where (V, max) and (W, max) are linguistic semivector spaces over the same linguistic semifield (S, max, min}.

i) Does the inverse linguistic transformation always exist? Justify your claim.

ii) Can one find linguistic kernel of L_{max}?

iii) Obtain any other special property enjoyed by linguistic transformation L_{max} from $V \to W$.

iv) Obtain the special feature associated with L_{min} $\{V, min\} \to \{W, min\}$.

iv) Will the linguistic kernel of L_{min} and L_{max} be the same? Justify your claim

19. Let {V, max] be a linguistic semivector space over the linguistic semifield {S, min, max}.

    Assuming S and V both finite linguistic sets prove the following.

    i) Find all subset linguistic semivector subspaces W_i of V over linguistic subsemifields of S.

    ii) Define a linear linguistic transformation from

    $L^{i}_{max} : V \to W_i$; for all i.

    What is the linguistic kernel of these linguistic transformation L^{i}_{max} ?

    iii) Are these linguistic kernel of L^{i}_{max} same or different for different W_i's? Justify your claim.

iv) Let L^{i}_{min} : {V, min} → {U_i, min} be a linguistic transformation which is a projection.

Find the linguistic kernel of L^{i}_{min} .

v) Do the kernels L^{i}_{min} vary for each i or for some they can be the same? Justify your claim!

20. Let L^{i}_{min} {V, max} → {W, max} where is a linguistic transformation of semivector spaces defined over the linguistic semifield {S, max, min}

i) What is kernel of L_{max}?

ii) Will kernel of L_{max} be a linguistic semivector subspace of {W, max} or {V, max}? Justify your claim.

iii) Obtain any other special property enjoyed by the linguistic transformation L_{max}.

21. Let T_{max} be a linguistic linear operator from {V, max} to {V, max}, {V, max} semivector space defined over the linguistic semifield over {S, max, min}.

i) What is kernel T_{max}?

ii) Does kernel T_{max} exist?

iii) Is kernel T_{max} will be a linguistic subspace of {V, max}?

iv) Obtain any of the special and distinct features enjoyed by T_{max}.

22. Let T_{min}: {V, min} → {V, min} be a linguistic linear operator study questions (i) to (iv) of problem (21).

23. Let W = {All 1 × 6 linguistic matrices with entries from the linguistic interval I_L = [very bad, best]} and

V = {All 1 × 6 linguistic matrices with entries from the linguistic interval [worst, very very good]}.

Let S = {[very bad, very very good], max, min} be the linguistic semifield. Will {V, max} and {W, max} be linguistic semivector spaces over the semifield S.

i) Build L_{max}: {V, max} → {W, max} the linguistic linear transformation and find ker L_{max}.

ii) Build T_{max}: {V, max} → {V, max} the linguistic linear operator and find ker T_{max}.

iii) If J_{max}: {W, max} → {V, max} be a linguistic linear transformation find ker J_{max}.

iv) Bring out the differences between ker T_{max} and kerJ_{max}.

v) Can we compose J_{max} . I_{max}?

vi) Will J_{max} . I_{max} be a linguistic transformation from {W, max} → {W, max}? Justify your claim.

vii) What will be the composition map $I_{max} . J_{max}$? Does it exist? If it exists find the domain and range space?

viii) Find atleast two proper linguistic semivector subspaces V_1 and V_2 of {V, max} and show the define the linguistic projections

p^1_{max} : {V, max} → {V_1, max} and

p^2_{max} : {V, max} → {V_2, max}.

a) Are these distinct linguistic projections?

b) Find ker p^1_{max} and ker p^2_{max}. Are they distinct or the same?

ix) For {M_1, max} a linguistic strong semivector subspace of {V, max} over a linguistic semi subfield {P, max, min} ⊆ {S, max, min} find a linguistic projection.

q_{max} : {V, max} → {M_1, min} and find the ker q_{max}.

x) Compare this ker q_{max} with that of ker p^1_{max} ?

xi) Obtain any other interesting properties about them.

24. Let M = {collection of all 5 × 5 linguistic matrices with entries from a linguistic set S of finite order}.

{M, max} is linguistic semivector space over the linguistic semifield {S, max, min}.

i) Study questions (i) to (xi) of problem 23 for this M.

ii) Prove {M, max, min} is a linguistic semilinear algebra over the linguistic semifield

{S, min, max}.

iii) Find some linguistic semilinear subalgebras P of M over the linguistic semifield {S, min, max}.

iv) Define in M two strong linguistic semilinear subalgebras P_1 and P_2 over some proper linguistic subsemifields of {S, min, max}.

v) Define a linguistic projective operator P_L: M $\rightarrow$ P.

vi) Define T_L^1 and T_L^2 two linguistic projective strong operations from

$T_L^1 : M \rightarrow P_1$ and $T_L^2 : M \rightarrow P_2$.

vii) Find $T_L^1 . T_L^2$ and $T_L^2 . T_L^1$ and obtain the domain and range of these composition operators.

viii) What is kernel T_L^1 and T_L^2? Are they identical?

ix) Find the kernel P_L.

x) How does a general linguistic operator vary from the linguistic strong operator?

xi) Obtain any interesting properties associated with these 3 types of linguistic operators.

25. Give an example of a linguistic semivector space defined over a linguistic semifield which is not a linguistic semilinear algebra.

26. Can we ever obtain a generating linguistic set for a linguistic semivector space defined over a linguistic semifield?

27. Characterize the condition for a linguistic strong semivector subspace to exist for a linguistic semivector space V defined over a linguistic semifield S.

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About the Authors

Dr.W.B.Vasantha Kandasamy is a Professor in the School of Computer Science and Engg, VIT University, India. In the past decade she has guided 13 Ph.D. scholars in the different fields of non-associative algebras, algebraic coding theory, transportation theory, fuzzy groups, and applications of fuzzy theory of the problems faced in chemical industries and cement industries. She has to her credit 694 research papers. She has guided over 100 M.Sc. and M.Tech. projects. She has worked in collaboration projects with the Indian Space Research Organization and with the Tamil Nadu State AIDS Control Society. She is presently working on a research project funded by the Board of Research in Nuclear Sciences, Government of India. This is her 136th book. On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.
She can be contacted at vasanthakandasamy@gmail.com
Web Site: http://www.wbvasantha.in

Dr. K. Ilanthenral is Associate Professor in the School of Computer Science and Engg, VIT University, India. She can be contacted at ilanthenral@gmail.com

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Dr. Florentin Smarandache is a Professor of mathematics at the University of New Mexico, USA. He got his MSc in Mathematics and Computer Science from the University of Craiova, Romania, PhD in Mathematics from the State University of Kishinev, and Postdoctoral in Applied Mathematics from Okayama University of Sciences, Japan. He is the founder of neutrosophy (generalization of dialectics), neutrosophic set, logic, probability and statistics since 1995 and has published hundreds of papers on neutrosophic physics, superluminal and instantaneous physics, unmatter, quantum paradoxes, absolute theory of relativity, redshift and blueshift due to the medium gradient and refraction index besides the Doppler effect, paradoxism, outerart, neutrosophy as a new branch of philosophy, Law of Included Multiple-Middle, multispace and multistructure, degree of dependence and independence between neutrosophic components, refined neutrosophic set, neutrosophic over-under-offset, plithogenic set, neutrosophic triplet and duplet structures, quadruple neutrosophic structures, DSmT and so on to many peerreviewed international journals and many books and he presented papers and plenary lectures to many international conferences around the world. http://fs.unm.edu/FlorentinSmarandache.htm
He can be contacted at smarand@unm.edu