HILBERT`S DIFFERENT AIMS FOR THE FOUNDATIONS OF MATHEMATICS

A dissertation prospectus

by

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December 2006

1. *Introduction*

The foundational ideas of David Hilbert (1862 - 1943) have been generally misunderstood. One of the most damaging misunderstandings is about his consistency program in the nineteen-twenties. It has been taken to have been made impossible by Gödel`s second incompleteness result.[1]

The principal aim of Hilbert`s program was to prove the consistency of analysis. A crucial first step to do it—as was suggested by the arithmetization of analysis in the nineteenth century—was to prove that the usual set of axioms of arithmetic was consistent. Gödel's second incompleteness result showed that if any such set of formal axioms AX is consistent. then the consistency of AX cannot be proved in AX. It implicitly assumes that ordinary first-order logic is used in the axiomatization.

Gödel's result led some logicians to immediately give up hope about Hilbert`s program. Hilbert himself never admitted that this result contradicted his conception of the problem of foundation. It will be shown in the planned dissertation that Hilbert was right in not giving up his foundational aims. One can base AX on a richer (yet elementary) logic, and then a proof of the consistency of arithmetic which is acceptable by Hilbert's standards can be given.

Other misunderstandings of Hilbert's work can be issued under the philosophical interpretations of his foundational views. For example, in many places Hilbert has been called a formalist. Calling him a formalist easily leads to a misleading interpretation of Hilbert's axiomatic approach. An examination of Hilbert's applications of this method to physical sciences strongly suggests a model-theoretical (and hence archetypally non-formalistic) view of mathematics.

Formalism is not the only philosophical view that has been wrongly attributed to Hilbert. Other philosophical interpretations of Hilbert's foundational ideas (by way of finitist and instrumentalist epistemologies in particular) lead to even more seriously misleading reading of Hilbert.[2] For one main reason such interpretations turn one's set of foundational problems into an epistemological problem concerning the certainty of mathematics. Hilbert's foundational standpoint was not an epistemological one. This line of interpretation will be shown to be mistaken. It will be shown to be rooted in the same misunderstandings as those which misled logicians to have taken Gödel's second incompleteness result as to imply the failure of Hilbert's program.[3]

From a wider perspective, one of the reasons for all these wrong interpretations and misunderstandings is a failure to distinguish the different aspects of Hilbert's foundational work and the different aims they seek. It is the purpose of the proposed work to distinguish from each other and investigate the different aspects, and developments that Hilbert's ideas underwent. The main aim is to clarify and to remove several misunderstandings and wrong interpretations of Hilbert's work. This task includes identifying the different specific logical and other conceptual problems that are involved in the different aspects. Often, the progress that has been made on these logical problems serves to put Hilbert's work to sharper perspective.

2. *Hilbert's axiomatic approach*

The foremost aspect of Hilbert's work is his axiomatic approach to the theoretical sciences. Hilbert's view of the nature of mathematics and of the role of logic in mathematics is strictly axiomatic. Mathematics according to Hilbert consists of setting up axiom systems and then deriving theorems from the axioms by purely logical means.

Why was the axiomatization of theories important for Hilbert? And why did he insist that an axiomatization should be logical? The main part of the answer is the following: By organizing everything known in a given field of knowledge into a logical axiom system one can obtain an overview of the entire field. By deriving theorems from axioms in a formal and logical way a mathematician can study the different aspects of this field.

One crucial aspect of Hilbert's axiomatic approach is a model-theoretical view of mathematics and mathematical reasoning. Hilbert's conception of truth and existence in mathematics are model-theoretical conceptions and are envisioned from a model-theoretical viewpoint. He says that his criterion of existence and truth is the consistency of axioms.[4]

On Hilbert's view mathematical axiom systems specify the class of their models. Mathematicians study different aspects of these models by deriving theorems from the axioms. Such conception of axiomatization presupposes the following:

(i) The derivation of the theorems from the axioms must be formal. This does not mean that it has to be completely mechanical. The idea is, rather. that derivations must not introduce any new information.

(ii) The logic used in the derivations of theorems must be complete. Otherwise the axiom system does not capture the intended class of models.

Hilbert recognized (i). This is why he held that all logic is formal. Hilbert seems to have assumed (ii). Indeed, the completeness of propositional logic was first proved in Hilbert's school. Later in 1930 Gödel proved the completeness of the usual first-order logic that had been formulated (to a considerable extent) in Hilbert and Ackerman 1928.

For Hilbert's primary purposes then it suffices that the theorems are all logical (semantic) consequences of the axioms. As was pointed. this does not require that these consequence relations can be implemented by mechanical rules of inference. Thus a second-order axiomatization can serve these primary purposes as well as a first-order one, even though second-order logical truths are not recursively enumerable. For this reason alone it cannot be conclusively said that Hilbert's program was made impossible by Gödel's results.

3. *Hilbert's response to criticisms of set theory*

Hilbert's views on set theory and its uncertainties is one of the central aspects of Hilbert's thoughts on foundational matters. According to Hilbert, paradoxes and uncertainties in mathematics are not problems to be worried about as long as they are remedied by axiomatic treatment. What is required for such treatment is proof of the model-theoretical consistency of the axiom system. On the practical level thereof Hilbert approved Zermelo's axiomatization of set theory. Hilbert seems to have thought that Zermelo's axiomatization relieved us from worries about the handicaps of set theoretical reasoning.

The concrete historical manifestation of the class of problems that are related to axiomatic set theory includes the validity of certain modes of reasoning that are usually taken to be set-theoretical. The best known example is the so-called axiom of choice. Hilbert defended it and hoped that from a suitable point of view it could be seen to be as obvious as 2 + 2 = 4.[5] His epsilon-technique. in which epsilon terms are literally choice terms, was formulated to find out such suitable point of view. It will be argued that Hilbert's aim concerning the axiom of choice can be achieved.

4. *Hilbert's response to intuitionist criticism of mathematics*

Hilbert faced not only one but two "crises of foundations". They have not always been distinguished from each other sufficiently clearly. The first one was the set-theoretical one. the second is intuitionistic one.

What was central to intuitionist criticism is Brouwer's argument that the unlimited application of the law of excluded middle to infinite domains was inadmissible. Based on his criticism of the law of excluded middle—plus a number of epistemological presuppositions—Brouwer rejected a significant portion of classical mathematics in favor of intuitionistic mathematics. Hilbert, respecting Brouwer's criticism at the meta-level, argued and tried to prove that classical mathematics was on a safe ground. No restriction to our logic was necessary. In this regard the status of the law of excluded middle played a central role in the Hilbert – Brouwer controversy.

5. *Hilbert on mathematical reasoning*

One of Hilbert's aims was to be able to understand all mathematical reasoning taking-place on the first-order level. He blamed all the foundational difficulty on the reliance on higher-order conceptualizations. What Hilbert has emphasized and criticized is that both Frege and Dedekind had been dealing with higher-order entities, such as concepts or their extensions. In their place Hilbert wanted to place discrete individual objects that can be given to us intuitively and in immediate experience.

One of the nominalistic assumptions in the philosophy of mathematics is that only individuals are given to us in immediate experience. And in logical terminology this assumption amounts to permitting only to first-order quantification. Hence in a wider philosophical perspective Hilbert's opposition to Frege and Dedekind is not an opposition of a formalist to a non-formalist but an opposition of a nominalist to conceptual realism.

6. *Hilbert's different aims reconsidered*

6.1 *First-order vs. higher-order axiomatizations*

One part of Hilbert's work in the foundations of mathematics consisted in efforts to recapitulate apparently higher-order modes of reasoning on the first-order level. One of them is the axiom of choice. It will be shown by relying on the work of Hintikka, Väänänen and Sandu on independence-friendly (IF) logic that Hilbert's aims in this respect can be achieved. The axiom of choice for example can be argued to be a first-order principle of inference.

6.2 *The law of excluded middle*

In his proof theory Hilbert wanted to construe mathematical reasoning in terms of concrete manipulation. Not only of the symbols for mathematical objects, but of those entities themselves. Intuitionists realized that not even all reasoning in the usual first-order logic can be interpreted in terms of such concrete manipulations. (This insight will be explained by clarifying the role of the law of excluded middle in the *tableau* method.) As a consequence intuitionists tried to formulate formal rules for the so-called intuitionistic logic.

The key role of the law of excluded middle was recognized both by the intuitionists and by Hilbert. However, its precise role was not fully understood. It is not clear that the precise limitations that Brouwer intended have been captured by any explicit logical formulation. On the other hand, Hilbert did not give a full satisfactory justification for the unproblematic uses of the law in mathematics. This whole issue has been put to a new light by the development of IF logic. It dispenses with the law of excluded middle. It is therefore deductively weaker than the ordinary first-order logic. In the proposed section, it will be shown how IF logic can be considered the true intuitionistic logic.

6.3 *Elementary and non-elementary methods*

In order to determine convincing standards for (meta-mathematical) proof methods a clear-cut distinction between elementary and non-elementary methods is needed. Now, IF logic allows the representation of a number of mathematical concepts and principles that cannot be captured by ordinary first-order logic. Among them there is equicardinality, infinity, topological continuity and König's lemma. In this respect it will be shown that the line between what is elementary (also, intuitionistically acceptable) and what is non-elementary in mathematics is (and should be) drawn differently from the way Hilbert seems to have thought.

It is important to know precisely where and how the law of excluded middle enters in the logical and mathematical reasoning. In ordinary first-order logic, the law of excluded middle enters primarily in the use of the cut rules and equivalent rules. Hence it is important to pinpoint the role of the non-cut-free rules in mathematical reasoning. This role can be studied more naturally by means of IF logic than of ordinary first-order logic because cut rules and their equivalents are not eliminable in IF logic.

7. *Conclusion*

By removing the misunderstandings and wrong interpretations of Hilbert's views out of the way, the developments in IF logic will be argued to facilitate a partial revival of Hilbert's different aims. The logical framework of Gödel's theorems is not necessarily the logic Hilbert had to use for his consistency program. First-order logic (as formulated in Hilbert and Ackermann 1928) can be seen as a mid-stage in what Hilbert calls the simultaneous development of logical and mathematical methods. In fact, when we consider IF logic as an alternative to the conventional first-order logic we observe that Hilbert's program can be carried out in the case of arithmetic. Surely, there is the need of further investigation of IF first-order logic with respect to its model-theoretic and proof-theoretic properties. Their relevance to Hilbert's so-called original intentions must be shown. The question whether Hilbert's program can be carried out in the case of higher mathematics must be reconsidered.

Notes

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[1] Actually the misunderstanding also led mathematicians to consider different aims in proof theory which resulted many fruitful proof-theoretical methods. From a mathematical point of view the so-called misunderstanding here was never damaging.

[2] Here one should distinguish the mathematical contributions of finitist and instrumentalist approaches from the philosophically misleading part.

[3] The point is not that finitist and instrumentalist results in the logical foundations of mathematics are useless or fruitless. Their epistemological interpretation that they attribute to Hilbert is misguided.

[4] This might seem a formalist idea, but it has to be understood against the tacit ontology Hilbert seems to have assumed. Hilbert's criterion of truth presupposes a tacit ontology which has not been studied later. It is partly brought out by Husserl who claimed to agree with Hilbert. This ontology involves a super-universe of potential models for the theory, a "model of all models". Indeed Hilbert's paradoxical sounding claim about truth and existence is true in the "model of all models". The (model-theoretical) consistency of a theory implies the existence of models for it in this super-universe of models.

[5] Hilbert 1922, p. 1117.