NOTES ON AXIOMATIC REASONING

1.

Axiomatic reasoning has a long history. It has been used in many different ways. Aristotle saw it as part of what he called first philosophy. Euclid applied it to geometry. Archimedes used it in solving statics problems. Ptolemy studied astronomical models by it. Spinoza used it in his conceptual demonstrations, Newton in mechanics and optics, Peano in number theory, Hilbert in geometry and physics, Einstein in gravitation theory, Whitehead and Russell in logic. Many other scientists and philosophers used it in different fields.

There was a common purpose in all the different uses. It was to provide a systematic investigation of the structures that are intended to be characterized by axioms. The structures in point are physical systems in physics. They are sometimes seen as systematic overviews of deductive structures. Obvious enough, inferences from axioms to theorems follow deductive patterns. A closer look at axiomatic reasoning however suggests that the structures in question are not limited to deductive structures.

2.

For the purpose of providing a systematic study of relevant structures by axiomatizations, mainly two requirements must be satisfied. First, as complete as possible characterization of the structures must be at hand. Second, some non-informative consequence relations must be in the investigator's logical tool box. The two requirements help in obtaining what twentieth century logicians would call the “tautological” character of statements from premises to conclusions in the systems axiomatized. Complete characterizations would determine the reaches and boundaries of reasoning in the axiom systems. Non-informative consequence relations would determine the ways in which one could possibly find out results in those reaches and boundaries.

It must be noted at this point that completeness may mean different things for different purposes. For purely structural purposes, completeness is a goal to achieve in order to determine what there is to be studied by the means of logical reasoning, without leaving any question open. Such determination presupposes that the structures in question (as well as the objects they involve) in some sense exist. Therefore, the consistency of the axioms is presupposed from the beginning of the axiomatization process.

There are different ways to prove the consistency of the axioms of a theory. This is due to the fact that the choice of the axioms can be motivated in different ways. Since truths and other non-falsities in a theory are all (intended to be) determined by the axioms, they have to be in the reaches of logical consequences drawn from the axioms. If the structures that are intended to be studied by axiomatizations do not exist and the axiom systems in question are not consistent, the whole process becomes nothing but a meaningless manipulation of figures.

Consistency is understood in two ways, accordingly. One is where the axioms are presupposed to hold true in the structures they determine. Another is where they are presupposed to be not-false in the structures. In both ways the ultimate presupposition of axiomatization is the existence of the structures and the objects involved. Hence consistency is presupposed in axiomatizations from the very beginning.

3.

Determination of structures both in physics and in mathematics (and possibly in other fields) specify the subject matter that is intended to be studied. Usually these structures are called physical systems or mathematical models. Originally, the study of physical systems and mathematical models on the basis of general principles of reasoning was seen as part of philosophical investigations. For instance, Aristotle summarized the general character of such study in the following way:

> We must state whether it belongs to one or to different sciences to inquire into the truths which are in mathematics called axioms, and into substance. Evidently, the inquiry into these also belongs to one science, and that the science of the philosopher; for these truths hold good for everything that is, and not for some special genus apart from others. And all men use them, because they are true of being qua being and each genus has being. But men use them just so far as to satisfy their purposes; that is, as far as the genus to which their demonstrations refer extends. Therefore since these truths clearly hold good for all things qua being (for this is what is common to them), to him who studies being qua being belongs the inquiry into these as

well. And for this reason no one who is conducting a special inquiry tries to say anything about their truth or falsity,-neither the geometer nor the arithmetician. Some natural philosophers indeed have done so, and their procedure was intelligible enough; for they thought that they alone were inquiring about the whole of nature and about being. But since there is one kind of thinker who is above even the natural philosopher (for nature is only one particular genus of being), the discussion of these truths also will belong to him whose inquiry is universal and deals with primary substance. Physics also is a kind of Wisdom, but it is not the first kind.-And the attempts of some of those who discuss the terms on which truth should be accepted, are due to a want of training in logic; for they should know these things already when they come to a special study, and not be inquiring into them while they are listening to lectures on it. (*Metaphysics*, Gamma 3)

Such conception of the study of axioms is quite puzzling from the perspectives of some received conceptions about axiomatic reasoning. The idea of axiomatic reasoning is entangled conceptually and historically. It is entangled with such ideas of logic, logical system, mathematical theorizing and scientific theorizing, among others. In his claim about the study of axioms in first philosophy Aristotle seems to have been aware of this entanglement. Nevertheless, systematically and historically there have been (and still are) different conceptions of axiomatic reasoning.

4.

One can distinguish two main types of ideas here. In the historical material they are hard to disentangle from each other. A tension between them is in evidence already at early stages of the history of the method. On the one hand, there is a deduction-oriented conception. On the other hand, there is a model-oriented conception. Both terms (i.e. deduction- vs. model-oriented) are historically inaccurate. In the early stages there was no explicit concept of a model. In addition, the so-called deductive structures have not been always deductive in the strict sense of the term. For example, Euclid's axiomatization has been criticized to have deduction gaps. From a model-oriented point of view however the so-called gaps are filled by continuity assumptions. Even the status of a single point can be problematic thereof, from the historical inaccuracy of the terms in question.

On the model-oriented conception what happens is this: A scientist or mathematician wants to study a certain class of structures. She or he captures them as models of an explicit axiom system. If this succeeds, the class of structures can be studied purely logically. That is, they can be studied without the input of any new information. On the deduction-oriented conception, axiomatization amounts spelling out the cognitive dependencies between different items of information one possesses. Typically the axioms express the knowledge one already possesses. Other truths are reached and/or justified by showing that they are consequences of the axioms.

5.

Aristotle's conception of the axiomatic method can be seen as deduction-oriented in the following way: According to Aristotle, axioms speak about real entities which have cognitive dependencies with further truths. That is to say the axioms are taken to be true as well. That is, their logical consequences belong to the same system of truths as the axioms. In such system, we have some given knowledge in terms of the axioms. Starting from the axioms we seek further knowledge by asking further questions. The cognitive meaning of the axioms (as well as of the terms they involve) is either obvious or definable by what is obvious. So that axioms do not require further proof. Unlike the axioms, further truths are deducible from the axioms by syllogisms. (See *Posterior Analytics*)

On both deduction-oriented and model-oriented conceptions one has to distinguish questions about choice of the axioms from questions about the rules one can follow in drawing inferences from them. With such distinction the following question is brought about: Can such inferences bring actual new information about the structures into the theory? According to the deduction-oriented view they can. On the model-oriented way they cannot.

The structures in question can be mathematical, physical (material), social, legal, pure structures qua structures or what not. The distinction between pure structures and structures actualized in some medium was originally unclear. It is best to take as paradigm example, a theory about the actual physical world. One example illustrating the two different orientations is the totality of Maxwell's equations for electromagnetism. From the point of view of the model-oriented conception of the axiomatic method we know how it was possible to build electronic devices and components in the same way as building pure structures on the basis of a tautological logic which does not bring into the argument any new substantial information. In a sense the model of an electronic component for example is a tautological consequence of the totality of Maxwell equations. Its possible existence lurked there in the depths of Maxwell's theory from the very beginning. Similar examples can be produced for other axiomatized theories as well.

What is especially interesting here are the different consequences of the two main conceptions. For one thing, what can be said of the method of deriving consequences from axioms? Here the model-oriented conception has a strikingly clear logic. This logic must be tautological. That is, it must not introduce any new substantial information into the reasoning. This is because the whole idea is to be able to study the structures in question. In this study the deduction of theorems from axioms is only a small part. It can reveal only what all the models of the axioms have in common.

From the tautological axiomatic deductions it follows that on the model-oriented view all structural differences disappear. Pure and interpreted axiom systems becomes on a par. The logical structure of an axiomatic theory is then independent of any meaning (interpretation) of the non-logical constants of the theory.

6.

On the model-oriented view the axioms of a system do not occupy any special position cognitively. All that they do is delineating the class of their models. The axioms of a system need not be obvious. They need not be certain. They need not be more reliable than the theorems. In so far as the adoption of some axioms is suggested by purely structural intuitions, axioms and theorems have the same role. Their role is to specify models. Eventually, being derivable from the axioms (even purely deductively) does not *per se* enhance the cognitive status of a proposition.

In all these respects the implications of the deduction-oriented picture are significantly different from the model-oriented conception. Historical examples belie the related terms. When Newton formulated his experiment based optical theory he organized it in a form that might first look like a Euclid-style axiom system with axioms, theorems, propositions, problems and proofs. But after having presented a proposition Newton repeatedly says instead of presented a deductive argument, "proof by experiment". That is due to his awareness of the need for correlating the axiom system with observation. According to Newton, physical properties of objects are considered as universal within the reaches of the experimental setup. That makes approximations inevitable until further accuracies with new axiomatizations and new experimental setups are found. In that sense axioms can always be revived by additional assumptions in the light of further evidence.

As can be seen from the differences mentioned above, two kinds of explorations of a class of structures axiomatically that do not fit into the deduction-oriented conception are largely neglected by contemporary philosophical discussions. These are (1) "how possible" arguments and (2) exploration of the different structures captured by the axioms (plus suitable additional assumptions). For one thing, a purely general theory (with only universal quantifiers) does not have empirical consequences, except in conjunction with suitable boundary conditions, including initial conditions. Newton, for example, tried to bridge what he called absolute magnitudes with sensible ones through proofs by experiment in and around his axiomatizations. (See *Opticks*)

7.

One of the most fundamental differences between the deduction-oriented and model-oriented conceptions is that the consequence relation according to the model-oriented conception is semantical (model-theoretical). In contrast, in the deduction-oriented conception it is encapsulated in the notion of logical system. This idea involved the assumption that the logic needed is captured by a finite number of formal rules and inferences. In view of the various incompleteness results it is clear that this notion is nearly useless. Few nontrivial "logical systems" are axiom systems in the model-oriented sense and the construction of sundry "logical systems" has little to do with actual mathematical practice.

8.

Our thesis is something like this: In actual mathematical practice, any attempt to axiomatize a theory involves explicitly or implicitly the overview of the structures to be studied. If the characterization of structures by the axioms is readable from the cognitive meaning of the axioms, then the model theoretical work at the backstage is brought more to the front in the shape of a deductive method. Otherwise, uses of the method are possibly misled by deduction-oriented restrictions on the structures to be studied.

According to the deduction-oriented conception, a theory is axiomatized by reference to a division of labor between axioms and rules of inference. In this division, the setting up of axioms is in order to express the information content of a theory. That is done mainly by representation of cognitive meaning of certain facts of the theory, by writing their information content with the help of well-formed formulas. The inference rules are applied then to the information content in order to derive theorems from the axioms. Admittedly, in such conception the logic applied through the uses of inference rules is preferred to have some preconditioned proof-theoretical properties, such as the requirements of what logicians call a logical system.

In the model-oriented conception, a theory is axiomatized by reference to division of labor not between axioms and rules of inference, but between axioms and a much wider domain of logical construction principles. The wider domain in question can be called principles of thought experimentation. For the purpose of applying them, what is required is only the following: no new information must be introduced into the arguments in model constructions. Clear enough, this requirement does not mean to forbid the use of inference rules. They can be explained in terms of the logic of thought experimentation. Therefore, the logic applied in the model-oriented conception is a logic of model construction. It has no preconditions about the proof constructions other than what the general experimental rationale of axiomatic reasoning requires.

9.

There has been an apprehension of syntactic precision in the studies of some logicians in the past two centuries. Frege and Quine are two of the foremost influential representative names among the logicians in point. Their achievements of precision, like similar achievements of other thinkers, seem to have been in book pages for paradoxical reasons. The deductive purposes of logical theorizing in their works presupposed that the material studied is like a precious abacus which should be protected against falling into pieces by a mistake. The thought mechanism underlying such works seems to be a defensive one. It is based on a misleading conception of axiomatic reasoning and its possible uses. One can find how infectious such conception can be, by observing that even the most model-oriented logicians had fallen into misleading phrases concerning the general conception of logic and the axiomatic method. For example, in Tarski's Introduction to Logic, first paragraph says this:

Every scientific theory is a system of sentences which are accepted as true and which may be called LAWS or ASSERTED STATEMENTS, or, for short, simply STATEMENTS. In mathematics, these statements follow one another in a definite order according to certain principles which will be discussed in detail in Chapter VI [under the title ON THE DEDUCTIVE METHOD], AND THEY ARE, AS A RULE, ACCOMPANIED BY CONSIDERATIONS INTENDED TO ESTABLISH THEIR VALIDITY.

Considerations of this kind are referred to as PROOFS, and the statements established by them are called THEOREMS.

That is not the flexibility of research that Hilbert intended in axiomatic reasoning at all. For example he wrote in a letter to Frege:

Nor can one be too exacting in examining the propositions, for after all they are only propositions of the theory. However, the more developed a theory is and the more ramified its structure, the more self-evident will be the manner of its application to the world of appearances; and it would require a large measure of bad intentions indeed if one wanted to apply the more precise propositions of plane geometry or Maxwell's theory of electricity to appearances other than the ones for which they were intended….

The idea of deductive systematization in the sense of the study of the relations between axioms and their possible results, as well as the assumption that axioms must be true statements from the very beginning, create the main misconception about the axiomatic method.

In the way Tarski defines the axiomatic method, number theory, for example, can be axiomatized as a deductive system. Due to incompleteness results, however, the axiomatization will exclude certain true statements of the system as being not deducible, or not provable from the axioms of the axiom system. In order to capture them as well, the deductive system in question can be extended by adding new axioms for each step of incompleteness phenomena in the initial system. Such a step by step (conservative) extensions of a deductive system, no matter how carefully studied in tandem with its models, show nothing but the existence of a deductive hierarchy of true but not provable statements. That kind of hierarchy is considered as a characteristic feature of the axiomatic method in the twentieth century philosophy of mathematics. The reason why it is considered so is simply that the axiomatic method has been generally understood by reference to the deduction-oriented conception. Even if the models of the deductive hierarchy are studied, for example by set theoretical or predicative tools, those studies would not suffice to meet the general thought experimental conceptualizations about mathematical or otherwise scientific truths. Such insufficiency has been criticized by various critics. Their target was sometimes the axiomatic method itself.

10.

Axiomatic method in the wrong hands creates nothing but proof hierarchies. Those hierarchies might be found insightful in some specialized proof-theoretical searches. In general philosophical sense they provide no substantial insight about the notions of truth, consistency, existence etc..

Still the truth or proof hierarchies (as their analogues in recursive hierarchies) are being used in programmatic aims in philosophy. For example, there are the so-called axiomatic theories of truth. Such theories are duplications of the Tarski's undefinability theorem. There are Gödel hierarchies in meta-mathematics, and there are Tarski hierarchies in semantics. A more recent source of such hierarches is found in Feferman's so-called transfinite recursive "progressions" of axiomatic systems. All are rooted from the same source: the deduction-oriented conception of axiomatization.

11.

Tarski, although his work on deductive method is occasionally misleading in logic about the real model-oriented features of axiomatic reasoning, he himself and logicians who were also mathematicians were not misled most of the time. Simply due to appeals to abstract algebraic methods, in tandem with first-order axiomatizations, Tarski's own methods were on safe ground, model theoretically. They were safe in the sense that model theory comes very close to algebraic geometry, when one tries to define it in algebraic terms. For example, Wilfrid Hodges defines it as the sum of algebraic geometry and the theory of fields. Similarly Chen Chung Chang and Howard Jerome Keisler define it as Universal Algebra plus Logic. However safe ground Tarski's model theoretical methods were on, its logic was not so. The deduction-oriented conception of logic and axiomatic reasoning with their truth and proof hierarchies cannot be taken as realistic solutions to basic problems of methodology.

12.

Deductive methods and model-theoretical methods, both being formal methods, have a confusing record of history. They mainly belong to twentieth century philosophy of mathematics. The study of deductive methods can be classified under the development of the so-called natural deduction methods. Systematization of model-theoretical methods can be classified as a development of the work in the theory of truth. Both deduction-oriented and model-oriented methods have further developments in different branches of mathematics, computer science, cognitive science and other fields. Sometimes they are misleadingly combined as in the case of what are known as axiomatic theories of truth.

The confusions are perhaps due to Tarski's conceptualizations about the so-called methodology of deductive sciences, and his construction of the deductive method for the purposes of axiomatizing theories. Further back the same confusions are rooted in Frege's misconceptions about the axiomatic method. The historical roots of both the deduction and model-oriented conceptions of axiomatic method go back to earlier times than the beginning of the twentieth century.

13.

Tarski's conception of the axiomatic method as an example of the deduction-oriented conception may be found unfair. It is true that Tarski's work is some kind of a starter for model theoretical investigations in the twentieth century. However, it is a borderline case. It is a borderline case with respect to formulation and definition of an axiomatic theory. The reason for it is that it presupposes the completeness of the underlying quantification theory of axiomatizations. That is completeness in the sense of recursive enumerability of all the theorems in a so-called deductive theory. Tarski puts such restriction in the following way:

Every theorem of a given deductive theory is satisfied by any model of the axiom system of this theory; and moreover, to every theorem there corresponds a general statement which can be formulated and proved within the framework of logic and which establishes the fact that the theorem in question is satisfied by any such model.

We have here a general law from the domain of the methodology of deductive sciences which, when formulated in a slightly more precise way, it is known as the LAW OF DEDUCTION (or the DEDUCTION THEOREM).

14.

In Tarski's view the deductive method is "justifiably considered the most perfect of all methods employed in the construction of sciences". There lie all the borderline case confusions about theories and their models. For one reason, the most perfect—if we want to use the word perfect—of all methods is not deductive but thought experimental.

The historical background of the deductivist conception of the axiomatic method lies in Frege's confusions about Hilbert's axiomatic work on geometry. One quotation makes it sufficiently clear that Frege completely misunderstood the model theoretical conception of the axiomatic method. Frege wrote,

> Axioms do not contradict one another, since they are true; this does not stand in need of proof. … The usage of words "axiom" and "definition" as presented in this paper is, I think, the traditional and the most expedient one.

Axioms do not contradict one another if they are shown to be not contradicting each other. That is shown by building a model for the axioms. That is clearly so from the point of the model-oriented conception of axiomatic method.

Tarski's borderline considerations about deductive methodology were in the 1940s. In the 1950s the method of semantic tableaus was discovered and developed. It showed a way to cancel out the borderlines and build the models and logical consequences of axioms (or of any set of statements) with the help of some definitory and strategic rules of model building. In that light the bridge construction between model-oriented and deduction-oriented conceptions of axiomatizations has been shown to be completable. However, the whole gist of semantic tableaus is their rules of model construction. It is not their rules of inference. In that sense the method of semantic tableaus has been a continuation of the model-oriented tradition in the same sense as Hilbert's axiomatic work was.

15.

Right from the beginning of his career Hilbert programmed a set of meta-theoretical problems to investigate from a meta-logical perspective. The separation of the historical cases of deduction-oriented and model-oriented methods has taken place right after that, partly as a consequence of his meta-mathematical work in the nineteen twenties. No such separation was questioned in that time. Even today, the distinction is not sufficiently clear to many philosophers.

It would be crude perhaps if the whole tradition of the axiomatic method in the twentieth century is reduced to Hilbert's discoveries. However, it is also true that popular remembrance of Hilbert's fame is under the heading of an unsuccessful formalism. That is a big misconception of the twentieth century philosophy of mathematics. Still philosophers are calling Hilbert a formalist. (Using a mask for a guilty secret will not make the misunderstanding disappear.) In reality, Hilbert's philosophy of mathematics is at least potentially (but history teaches that it was also actually) the offspring of almost every kind of genuine investigation in the foundations of science in the twentieth century. The label formalism cannot be more than a result of sacrificing Hilbert on the altar of misconceptions about the truth of the matter.

16.

Hilbert was a philosopher of science. Arguably he was one of the greatest. He was more interested in "solution sets" rather than singled out solutions to problems. This does not make him a higher-order logician or a computer scientist. The sets of solutions he was after can be seen in his work in invariant theory, or in the foundations of geometry. Thereof the idea of solution sets can be used to explain his axiomatic thoughts. His sixth problem, which is a call to axiomatize physical theories, is a suggestion to investigate possible sets of solutions to problems of physical theories by way of axiomatic analysis, by developing a model-oriented conception of it.

What was model-oriented in axiomatic reasoning, according to Hilbert? Or did he also have a deduction-oriented conception? From an abstract algebraic point of view, the answer is simple. He had a very sensible model-oriented conception of the axiomatic method. This can be seen as follows: Following Wilfrid Hodges if we define model theory as algebraic geometry + field, we may ask: Who build the camp fire of the foundations of algebraic geometry? Hilbert, mainly by proving the Basis

Theorem.

[Some historical notes about the development of model theory may be helpful in seeing the general interrelations between different traditions in the history of logic. We must separate the subject matter into two: (1) Development of model theory and model-oriented techniques by way of axiomatic or otherwise algebraic methods in the general development of logic(s); (2) Development of the axiomatic method and its model-oriented and deduction-or conceptions.]

17.

Model theory was born with the theorem of Löwenheim about the relation between models and countable models. Löwenheim's work was a continuation of what is called these days the algebraic tradition in logic. It starts with Boole’s work and continues with Schröder’s and Peirce’s. Logicians like Skolem and Tarski can also be seen as to have partly contributed to the same line of thought. Their work in clarifications of the model theoretical thought in logic can be studied as part of (1). However, they can also be studied as falling under (2) with respect to their critical positions in the historical development of elementary logic.

As a matter of fact, the so-called algebraic tradition is a neglected part of the history of logic. In the 1890s Hilbert was shaping his new ideas about the model-oriented conception of the axiomatic method, in algebraic varieties, projective geometry, and general geometry. The philosophy of mathematics of algebraic logicians, who applied algebraic ideas to mathematical logic were in the same neighborhood as Hilbert's or Husserl's philosophies of mathematics. Partly because of the reasons why such neighborhood of philosophies is studied today as a neglected part of the history of logic, the model-oriented conception of the axiomatic method, which was common to all that has been neglected, has been poorly understood by philosophers and the mathematicians. One of the grand mistaken results of the twentieth century philosophy of mathematics is the unfortunate label of Hilbert's philosophy as "formalism".

18.

Hilbert's philosophy of mathematics, as has been slowly recognized in recent studies, can be seen as a synthesis between three or four different traditions in the history of logic, viz. algebraist, logicist, intuitionist, and even cognitivist. Partly from that synthesis, in the 1930s, Kurt Gödel extracted completeness and incompleteness results, Tarski extracted definability and decidability results, Church extracted computational results, Turing extracted the general form of "computors", and Gentzen extracted consistency results. How come then Hilbert's program was an unsuccessful formalism, and on the other hand Gödel was a successful Platonist, or Tarski was a successful model theorist, or Church and Turing were successful computation theorists, or (most importantly) Gentzen was a successful (Hilbertian) proof theorist? The answer seems to lie in the misconceptions about the model-oriented conception of axiomatic method in the twentieth century.

19.

In the year 1899 Hilbert's Grundlagen der Geometrie was published. Poincaré's review of Hilbert's book summarizes the model theoretical import (as well as its deductive import) of Hilbert's accomplishment. Poincare mainly refers to the proof of consistency of the axioms and the completeness of the axioms in a descriptive sense, from the viewpoint of the mathematician. Frege however, was confused about what Hilbert calls axioms, definitions and other things. He thought those words were not used in their traditional sense as he understood the so-called traditional sense. And he thought they were not "expedient" enough. He wrote a letter to Hilbert and asked questions about the points he did not quite understand. In Hilbert's reply to Frege we find the most characteristic features of a model-oriented conception of axiomatic method:

In other words, each and every theory can be applied to infinitely many systems of basic elements. For one merely has to apply a univocal and reversible one-to-one transformation and stipulate that the axioms for the transformed things be correspondingly similar. Indeed, this is frequently applied, for example in the principle of duality, etc.; I also apply it in my independence proofs. The totality of assertions of a theory of electricity does of course hold of every other system of things substituted in place of the concepts magnetism, electricity, … just as the required axioms are fulfilled. However, the state of affairs just indicated can never be a short coming of a theory, and in any case is unavoidable.

Later Weyl called Hilbert's work a "move to the meta-geometric level". His reason was understandable enough. Hilbert's axiomatization and investigation of the dependency relations between the axioms was not only about Euclidean geometry or some other singled out specific geometry. It was about, in general, how to build Cartesian, Archimedean, non-Archimedean, non-Euclidean and other possible geometries on the basis of different sets of axioms characterizing different spaces and different classes of geometric models.

Hilbert's proof of the consistency of the five groups of axioms he introduced is a model theoretical proof. It shows how the axioms can be satisfied by an arithmetical model. That is, by using numbers as the objects of the constructed models of geometry. When Hilbert's grouping of the axioms is observed, it is plain to the eye that notions like incidence (or projection), order, and congruence of the objects investigated, as well as their continuity properties and parallelism as giving the shape of the

Euclidean models are put together in a model theoretically (descriptively) complete form, as an organic whole, which can be identified as an object by itself from the perspective of model-oriented foundations. That was sufficiently meta-theoretical systematization for model theoretical purposes around the end of nineteenth century. There had been no mathematical logic or axiomatizations of set theory yet. (Hilbert had not contributed to those fields yet.) The pointed out descriptions of geometries were model identifications, not only the identification of some domains of deductions from axioms to theorems.

20.

In the model-oriented conception of axiomatic reasoning, objects of a theory are taken as being given. So that whenever we say that something exists, it really has to exist in our models. The purpose of the axioms is to define how things behave in our models, i.e. how they are interrelated; dependent on, independent of each other etc. When the models are defined by the axioms, one has to prove that no contradiction arises from the whole characterization. That is, we show that the models really exist.

21.

In 1934, in their joint work *Grundlagen der Mathematik* Hilbert and Bernays took axiomatization of geometry still of the central importance for their model-oriented conceptualizations. In the first chapter of the two volumes, axiomatic method in refined form was perfected in accordance with the latest developments in different fields up to date. Whether two relations R and S satisfy a given condition in the system, say whether for any given R and S, A(R, S) or not A(R, S) is true became one of the most central questions. Hilbert and Bernays introduce such question as a decision problem. The introduction of the decision problem by no means presupposed that the decision problem at hand was restricted to a problem of mechanical derivability or deducibility etc. On the basis of the simple formulation of the decision problem, Hilbert and Bernays formulated problems of consistency and independence of the axioms, by asking whether a set of axioms G and a given axiom A are satisfiable by a domain, or whether A is independent in the sense that G and not A were consistent. These tasks can be accomplished, as was stated in the *Grundlagen*, by formulating logical inferences and showing that a given axiom system AX is satisfied by a model M of things and relations. Whether the formulation of the logical inferences was to be restricted to mechanical rules of provability was the question. It didn't have an answer then. The incompleteness results were fresh. However, the main point was not stated to formulate a deductive system but a system of things and relations, viz. the existence of models, as in the days of meta-geometric investigations of the 1890s. What Hilbert had shown in his earlier work by proving the consistency of geometry axioms was the existence of models for different geometries (and how to build them) in the same sense as the satisfiability relation considered in the *Grundlagen der Mathematik* in 1934.

In his 1900 address Hilbert had mentioned similarity of devices and relationship of the ideas in mathematics. He was aware then that for the purposes of working out the model-oriented study of axioms, development of an underlying logical theory was essential. The model-oriented and deductive features of axiomatic method necessitated an uninformative inferential net to study consequence relations in an axiomatized theory. That didn't necessarily mean to locate the procedures of the underlying logic under the title of logical inference rules only.

22.

Hilbert worked out the difference between axiomatization of logic and axiomatization of non-logical theories. Hilbert's practicing of the axiomatic method in several fields in tandem shows that the logical constants vs. non-logical constants distinction was not essential for him in the axiomatizations in so far as the intended models are characterized completely. However, he mentions the necessity of reaching a clear-cut distinction someday. As has been considered since then there have been different senses of completeness. Truth in an axiom system meant truth in the intended and only in the intended models. In axiomatizations of (parts of) logic, the goal was to capture truth in all possible models. There we observe a distinction between two different kinds of information.

With its meta-theoretical levels Hilbert's work in logic was unique as in the case of his work in geometry. That uniqueness plays a central role in the development of the sciences in so far as the rationale of the axiomatic method and Hilbert's contributions to it understood correctly.

23.

Why was axiomatization so important for Hilbert? The main part of the answer has already been delivered: As soon as an axiom system is set up, assuming that it has models, it offers an overview on the class of its models. Thereby an axiomatization is a means to develop an structural investigation on the information codified by the resulting axiom system. By solving problems, the theoretical investigator or the practical inventor puts the information to use by exploring what there is to be found out about those models. This is done by thought experiments on the model constructions, on the basis of axioms and the model construction rules.

Surely the deepening of an axiom system can as well be studied by reflecting on the different particular aspects of a mathematical theory without using any axiomatic approach. There is no unbreakable law that no one can obtain a better systematic overview of a mathematical theory without using the axiomatic method.

It would be an oversimplification perhaps to say at this point that axiomatic research was for Hilbert an end in itself without external philosophical justifications. No matter which philosophical justifications have been sowed for it, we have to find them grow as a separate branching of thoughts from the internal working of axiomatic thought experiments. Axiomatic method had always been a means to achieve a clearer understanding of mathematical theories, according to Hilbert. He emphasized that the method forced itself upon his research, rather than it flourished as a branch of personal foundational preferences.

In the historical development of Hilbert's work as an axiomatist, it is plain to the eye that his different applications (as well as the approval of others axiomatizations) match with different periods of heated dispute on the foundations of different fields in logic, mathematics, and physics. Hilbert's axiomatization of geometry corresponded, for example, to the end of that period of epistemological disputes on the Euclidean and non-Euclidean geometries. His encouragement and approval of the axiomatization of set theory corresponded to the end of the period of ontological disputes on infinity partly as a result of the set theoretical paradoxes. His call for the axiomatization of physical theories corresponded exactly to those dates when theories of special and general relativity as well as the discovery of the quantum phenomena were about to shake the grounds. Hilbert's affirmation of the Russell and Whitehead's axiomatization of the predicate logic, and taking over some parts of the unfinished project from a meta-logical perspective corresponded to the days when intuitionistic motivations in epistemology were about to cut the tree of quantificational logic in its young age. These examples are enough to show Hilbert's quickness to respond different crisis periods.

What was common to different crisis periods in geometry, set theory and physics is that in each case there appeared epistemological and ontological issues which were taken to be reasons as to admit some of the theories as correct (true) and some of them incorrect (not true). This whole issue, it seems, according to Hilbert's viewpoint, was ill-formed. The main source of the ill-conceived issues (especially in mathematics, but also in physics) was due to the lack of appreciation of the model-oriented conception of axiomatizations and of the absence of epistemological and ontological concerns in such a conception. Expanded briefly, the absence of epistemological and ontological (or otherwise empirical) concerns is due to a distinction we have to make—and Hilbert assumed such distinction implicitly—between uninterpreted axiom systems and interpreted axiom systems. The distinction perhaps can best be explained by means of Einstein's observation in his "Geometry and Experience" paper: "As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality." What Einstein means here is simply that we separate "the logical-formal" from its "objective or intuitive content". Thereby we separate the uninterpreted axiomatizations from interpreted axiomatizations. Hence, by doing so, the applications of the axiomatic method (in its uninterpreted sense) provides the possibility of various foundational investigations which are freed from epistemological or ontological concerns; and hence from crises in the sciences.

To take one example with a closer look we can go back to Poincaré's criticism of Hilbert's early consistency proof sketch for the axioms of number theory. In order to prove such consistency for the general case, as Poincaré quickly realized, Hilbert needed to use the principle of mathematical induction. The principle of mathematical induction, for the complete description of Hilbert's axiomatization, had to be considered also as one of the axioms of number theory of which consistency was in question. Poincaré in his 1906 paper criticized Hilbert's argument on this point. He pointed out that Hilbert's appeal to mathematical induction in his proof was circular reasoning:

> …the point at issue is reasoning by recurrence and the question of knowing whether a system of postulates is not contradictory
> …
> A demonstration is necessary.
> The only demonstration possible is the proof by recurrence.
> This is legitimate only if we regard it not as a definition but as a synthetic judgment.

Poincaré also took Hilbert's line of thought to have assumed the principle of mathematical induction as a synthetic a priori principle. However, assuming its implicit model-oriented background, Hilbert's argument did not involve any epistemological concerns. In fact it was Hilbert's aim to eliminate epistemological presuppositions from the formulation of foundational problems; in this case, from the meta-mathematical problem concerning mathematical induction. Hilbert's later response to Poincaré makes this point sufficiently clear. The reason why Hilbert's argument appeared to have involved epistemic elements is presumably the then missing (then forthcoming) developments in logical theory, which Hilbert was searching for. Hilbert's own later remark on Poincaré's challenge is that it was a mistake on Poincaré's part that he rejected Hilbert's theory in its "inadequate early stages". The source of Poincaré's mistaken view, according to Hilbert, was that Poincaré did not distinguish between two different methods of induction:

> Poincaré…denied from the outset the possibility of a consistency proof for the arithmetic axioms, maintaining that the consistency of the method of mathematical induction could never be proved except through the inductive method itself. But as my proof theory shows, two distinct methods that proceed recursively come into play when the foundations of arithmetic are

established, namely, on the one hand, the intuitive construction of the integers as numeral (to which there also corresponds, in reverse the decomposition of any numeral, or the decomposition of any concretely given array constructed just as an array is), that is, contentual induction, and on the other hand, formal induction proper, which is based on the induction axiom and through which alone the mathematical variable can begin to play its role in the formal system.

It could still be questioned at this point whether or to what extent Poincaré was right in his general epistemological criticism based on the synthetic a priori character of mathematical induction. However, the epistemological force of Poincaré's criticism made such a discussion uninteresting to run the programmatic aims of Hilbert for the purposes of developing an axiomatic approach as a tool for model constructions, as they have been presented in his other reactions to different possible crises in the sciences. On similar lines there was no need for an appeal to any basic intuition in the sense that intuitionist philosophy of mathematics suggested in our foundational theorizing, according to Hilbert. Foundations could be studied mathematically by improving the logical methods on a meta-theoretical level, i.e. meta-logic and meta-mathemaics were to be developed in tandem, as is known to be the case in the actual historical development of the subjects. What this meant was that the exclusion of certain principles like the axiom of infinity, the axiom of reducibility and the axiom of completeness from meta-mathematics was for the sake of showing that a model-oriented conception of axiomatic foundations did not need such assumptions. It could be developed without making existential assumptions about mathematical infinity or similar notions. Such claim immediately implied that Hilbert's preference was a first-order level theorizing in meta-mathematical theory. It could be applied to different mathematical domains without making actual assumptions about infinite totalities etc. On this point, epistemological interpretations of Hilbert's views were and are based on patent misconceptions about on the one hand Hilbert's philosophy of mathematics and on the other, the model oriented conceptions about the axiomatic thought. All that was (and still is) needed for foundational purposes is first the determination of models by axiomatic analysis, and then, second, model-theoretical consistency proofs for the axiomatizations. This was Hilbert's original project.

In addition to his axiomatization of geometry, Hilbert worked on the axiomatization of different mathematical and physical theories. He started teaching physics from the 1890s on. He continued teaching foundational courses in physics until the end of his career. A typical Hilbert course is told to have been including his latest axiomatizations of almost all the physical theories with the addition of axiomatizations of arithmetic and logic. This is told to be taking place in the one semester courses. It is not probable that he taught all the theories moving from one to another by spending time on details or deductive precision. Even that should suffice to see Hilbert's model-oriented conception as a background of his foundational theorizing.

As is well known, Hilbert's sixth Paris problem was about axiomatization of physical theories. He claimed there that the investigations on the foundations of geometry suggested that physical sciences must be axiomatized. Especially those, in Hilbert's opinion, in which mathematics has a crucial role, were to be axiomatized. What he meant by that, as he mentioned, were probability and mechanics in the first place. As a later continuation of these programmatic aims to axiomatize physical theories, Hilbert's 1917 address to the Swiss Mathematical Society entitled

“Axiomatische Denken” was devoted to emphasizing the importance of different examples of the application of the axiomatic method to different mathematical and physical theories. Hilbert’s conception of the axiomatic method there was again far from being based on a deduction-oriented conception. The content of the paper makes it very clear how Hilbert related different fields that he carried out in his investigations under the roof of axiomatic thought, with an archetypel model-oriented conceptualization.

24.

Geometry always had a central role in Hilbert’s use of the axiomatic method. His lectures can be portrayed as a stream of geometric transformations from theories to theories. It is told that one could find in his one semester course most of the theories of physics more or less axiomatized and in addition geometry, algebra and mathematical logic axiomatized. It is obvious that he was not a crazy proof theorist or crazy set theorist. He was a crazy model-theorist. In one semester all axiomatizations could be surveyed without getting into detailed proofs. Like an album, to put in Wittgenstein's wording, of theory pictures in axiomatic form. Weyl describes his feeling about those courses as follows:

> Most of it went straight over my head. But the doors of a new world swung open for me, and I had not sat long at Hilbert's feet before the resolution formed itself in my young heart that I must by all means read and study what this man had written.

Hilbert himself describes the method as methodologically the most flexible. Why to insert deductive-oriented restrictions or formalist flavor then into Hilbert's philosophy of mathematics?

Unfortunately, almost no recognition seems to have taken place on what Hilbert meant in the following as an exemplary of his model theoretical conceptualizations:

> The edifice of science is not raised like a dwelling, in which the foundations are first firmly laid and only then one proceeds to construct and to enlarge the rooms. Science prefers to secure as soon as possible comfortable spaces to wander around and only subsequently, when signs appear here and there that the loose foundations are not able to sustain the expansion of the rooms, it sets about supporting and fortifying them. This is not a weakness, but rather the right and healthy path of development.

What seems to be characteristic of Hilbert's lectures is (1) Unified view of all the theories, and (2) Interconnections between theories. Such characteristics may take us back to Hilbert's starting point with his work in algebraic varieties. The model theoretical character of the abstract algebraic motivations of Hilbert there were preliminary to his later use of the axiomatic method. It is part of a general historical outlook in the nineteenth century meta-theoretical searches for possible solution sets to the most central problems of the sciences where mathematics played a leading role. The outlooks in question developed in different fields in tandem. Hilbert's axiomatic thoughts provided the most perceptive conceptions about the overall interrelatedness of mathematical theories. From a practical perspective, history teaches that many studies in the twentieth century flourished from Hilbert's axiomatic thoughts. As follow ups to Hilbert's axiomatic thoughts, the development of abstract algebra, meta-mathematics, and computer science are only parts of the general development of the model-oriented investigations into the depths of mathematical and physical theories.