On Accuracy and Coherence with Infinite Opinion Sets

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Abstract

There is a well-known equivalence between avoiding accuracy dominance and having probabilistically coherent credences (see, e.g., de Finetti 1974, Joyce 2009, Predd et al. 2009, Schervish et al. 2009, Pettigrew 2016). However, this equivalence has been established only when the set of propositions on which credence functions are defined is finite. In this paper, we establish connections between accuracy dominance and coherence when credence functions are defined on an infinite set of propositions. In particular, we establish the necessary results to extend the classic accuracy argument for probabilism originally due to Joyce (1998) to certain classes of infinite sets of propositions including countably infinite partitions.

1 Introduction

A central norm in the epistemology of partial belief is probabilism: a person's degrees of belief—or *credences*—should satisfy the laws of probability.[1] There is a long tradition in the spirit of Savage (1971) and de Finetti (1974) of appealing to the epistemic virtue of accuracy to justify probabilism (also see Rosenkrantz 1981). One particular form of argument is the accuracy dominance argument for probabilism introduced by Joyce (1998). Let a set $\mathcal{F}$ of propositions be an *opinion set* and a function $c : \mathcal{F} \to [0, 1]$ a *credence function on* $\mathcal{F}$. Let a credence function be *coherent* if it satisfies the axioms of probability. A credence function c' on $\mathcal{F}$ *accuracy dominates* a credence function c on $\mathcal{F}$ if c is more inaccurate than c' no matter how the world turns out to be (where inaccuracy is precisified as in Section 2). Then the existing accuracy dominance arguments purport to vindicate probabilism by showing that a credence function is not accuracy dominated if and only if it is coherent.

However, there is a limitation to almost all of the literature on accuracy arguments for probabilism: the opinion set is assumed to be finite.[2] Indeed, de Finetti (1974), Lindley (1987), Joyce

[1] This paper is based on work done in Kelley 2019.

[2] In an unpublished manuscript, Walsh (2020) gives an accuracy dominance argument in the countably infinite context, to which we return in Section 3. In a related but distinct area, Huttegger (2013) and Easwaran (2013) extend to the infinite setting part of the literature on using minimization of expected inaccuracy to vindicate epistemic principles. See, e.g., Greaves and Wallace 2005. Schervish et al. (2014) prove that in certain countably infinite cases, coherence is sufficient to avoid *strong dominance*. Schervish et al. (2009) and Steeger (2019) explore a different way to weaken the assumption that the opinion set is finite. We return to their work in Section 4.

(1998, 2009), Predd et al. (2009), Leitgeb and Pettigrew (2010a,b), and Pettigrew (2016) all establish their dominance results only for finite opinion sets. In this paper, we remove this assumption and prove dominance results that we hope to be useful in evaluating the extent to which accuracy arguments for probabilism succeed when the opinion set is infinite.

We begin in Section 2 by reviewing the mathematical framework and the standard dominance result for finite opinion sets. Sections 3-5 are concerned with accuracy and coherence in the infinite setting. In Sections 3-4, we make headway on characterizing the opinion sets and accuracy measures for which there is an equivalence between coherence and avoiding dominance as in the finite case. Finally, in Section 5, we extend the accuracy framework to the uncountable setting and prove that coherence is necessary to avoid dominance on uncountable opinion sets.

2 The Finite Case

We first set up the framework that will be used throughout the paper. Fix a set W (not necessarily finite) which represents the set of *possible worlds* and, for now, a finite set $\mathcal{F} \subseteq \mathcal{P}(W)$ of *propositions* that represents an *opinion set*—the set of propositions that an agent has beliefs about.

Definition 2.1. An *algebra* over W is a subset $\mathcal{F}^* \subseteq \mathcal{P}(W)$ such that:

1. $W \in \mathcal{F}^*$;
2. if $p, p' \in \mathcal{F}^*$, then $p \cup p' \in \mathcal{F}^*$;
3. if $p \in \mathcal{F}^*$, then $W \setminus p \in \mathcal{F}^*$.

Definition 2.2. i. A *credence function* on an opinion set $\mathcal{F}$ is a function from $\mathcal{F}$ to $[0, 1]$.

ii. A credence function c is *coherent* if it can be extended to a finitely additive probability function on an algebra $\mathcal{F}^*$ over W containing $\mathcal{F}$. That is, there is an algebra $\mathcal{F}^* \supseteq \mathcal{F}$ over W and a function $c^* : \mathcal{F}^* \to [0, 1]$ such that:

   (a) $c^*(p) = c(p)$ for all $p \in \mathcal{F}$;

   (b) $c^*(p \cup p') = c^*(p) + c^*(p')$ for $p, p' \in \mathcal{F}^*$ with $p \cap p' = \varnothing$;

   (c) $c^*(W) = 1$.

iii. A credence function that is not coherent is *incoherent*.

Remark 2.3. If $\mathcal{F} = \{p_1, \ldots, p_n\}$, we identify a credence function c over $\mathcal{F}$ with the vector $(c(p_1), \ldots, c(p_n)) \in [0, 1]^n$. Thus the space of all credence functions over $\mathcal{F}$ can be identified with $[0, 1]^n \subseteq \mathbb{R}^n$. We often simplify notation by setting $c_i := c(p_i)$.

We now introduce an important subclass of the class of all credence functions, namely the (coherent) credence functions that match the truth values of $\mathcal{F}$ at a world w exactly.

Definition 2.4. Fix an opinion set $\mathcal{F}$. For each $w \in W$, let $v_w : \mathcal{F} \to \{0, 1\}$ be defined by $v_w(p) = 1$ if and only if $w \in p$. We call v_w the *omniscient credence function at world* w. We let $\mathcal{V}_\mathcal{F}$ denote the set of all omniscient credence functions on $\mathcal{F}$. Note that $|\mathcal{V}_\mathcal{F}| \leq 2^{|\mathcal{F}|}$.

Next, we specify the inaccuracy measures we will be concerned with in this section. Fix a finite opinion set $\mathcal{F}$, and let $\mathcal{C}$ denote the set of credence functions on $\mathcal{F}$. We define an *inaccuracy measure* to be a function of the form

$$\mathscr{I} : \mathcal{C} \times W \to [0, \infty].$$

The class of inaccuracy measures we consider is a generalization of the class normatively defended by Pettigrew (2016): the inaccuracy measures defined in terms of what we call a *quasi-additive Bregman divergence*. It is a subclass of the inaccuracy measures assumed in Predd et al. 2009.[3]

Definition 2.5. Suppose $\mathfrak{D} : [0,1]^n \times [0,1]^n \to [0, \infty]$.

1. $\mathfrak{D}$ is a *divergence* if $\mathfrak{D}(\mathbf{x}, \mathbf{y}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$ with equality if and only if $\mathbf{x} = \mathbf{y}$.

2. $\mathfrak{D}$ is *quasi-additive* if there exists a function $\mathfrak{d} : [0,1]^2 \to [0, \infty]$ and a sequence of elements $\{a_i\}_{i=1}^n$ from $(0, \infty)$ such that

$$\mathfrak{D}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} a_i \mathfrak{d}(x_i, y_i),$$

in which case we say $\mathfrak{D}$ is *generated* by $\mathfrak{d}$ and $\{a_i\}_{i=1}^n$.

3. $\mathfrak{D}$ is a *quasi-additive Bregman divergence* if $\mathfrak{D}$ is a quasi-additive divergence generated by $\mathfrak{d}$ and $\{a_i\}_{i=1}^n$, and in addition there is a function $\varphi : [0,1] \to \mathbb{R}$ such that:

   (a) φ is continuous, bounded, and strictly convex on $[0, 1]$;

   (b) φ is continuously differentiable on $(0, 1)$ with the formal definition

   $$\varphi'(i) := \lim_{x \to i} \varphi'(x)$$

   for $i \in \{0, 1\}$;[4]

   (c) for all $x, y \in [0, 1]$, we have

   $$\mathfrak{d}(x, y) = \varphi(x) - \varphi(y) - \varphi'(y)(x - y).$$

   We call such a $\mathfrak{d}$ a *one-dimensional Bregman divergence*.

We take the inaccuracy of a credence function c at a world w to be the distance between c and the omniscient credence function v_w, where distance is measured with a quasi-additive Bregman divergence.

Definition 2.6. Let a *legitimate inaccuracy measure* be an inaccuracy measure given by

$$\mathscr{I}(c, w) = \mathfrak{D}(v_w, c),$$

where $\mathfrak{D}$ is a quasi-additive Bregman divergence.

[3] Using terminology from Definition 2.5, Predd et al. consider a more general class in allowing different one-dimensional Bregman divergences for different propositions.

[4] We do not require $\varphi'(i) < \infty$ for $i \in \{0, 1\}$.

By allowing different weights depending on the proposition, we can accommodate the intuition that some propositions are more important to know than others (see, e.g., Levinstein 2019 for further discussion of the varying epistemic importance of propositions). Even if one thinks that inaccuracy measures should be additive, as Pettigrew (2016) does, relaxing this restriction makes our results more widely relevant. A popular example of an additive legitimate inaccuracy measure is the Brier score (see Section 12, "Homage to the Brier Score," of Joyce 2009):

$$\mathscr{I}(c, w) = \sum_{i=1}^{n} (v_w(p_i) - c(p_i))^2.$$

Remark 2.7. The class of additive Bregman divergences is the class of additive and continuous *strictly proper scoring rules.* See Pettigrew 2016, p. 66. Also see, e.g., Banerjee et al. 2005 and Gneiting and Raftery 2007 for more details on Bregman divergences as well as their connection to strictly proper scoring rules.

We now recall the dominance result connecting coherence to accuracy dominance when the opinion set is finite. It was first proved for the Brier score by de Finetti (1974, pp. 87-90) and extended to any legitimate inaccuracy measure by Predd et al. (2009). See Schervish et al. 2009 for further generalizations of the finite result.

Definition 2.8. For each pair of credence functions c, c^* over $\mathcal{F}$:

1. c^* *weakly dominates* c relative to an inaccuracy measure $\mathscr{I}$ if $\mathscr{I}(c, w) \geq \mathscr{I}(c^*, w)$ for all $w \in W$ and $\mathscr{I}(c, w) > \mathscr{I}(c^*, w)$ for some $w \in W$;

2. c^* *strongly dominates* c relative to $\mathscr{I}$ if $\mathscr{I}(c, w) > \mathscr{I}(c^*, w)$ for all $w \in W$.

Theorem 2.9 (de Finetti 1974, Predd et al. 2009)**.** Let $\mathcal{F}$ be a finite opinion set, $\mathscr{I}$ a legitimate inaccuracy measure, and c a credence function on $\mathcal{F}$. Then the following are equivalent:

1. c is not strongly dominated;

2. c is not weakly dominated;

3. c is coherent.

Further, if c is incoherent, then c is strongly dominated by a coherent credence function.

On the basis of Theorem 2.9, authors in the accuracy literature conclude that an incoherent credence function is objectionable because there is an undominated coherent credence function that does strictly better in terms of accuracy, no matter how the world turns out to be, whereas coherent credence functions are not accuracy dominated in this way. Since it is the basis of the accuracy argument for probabilism in the finite case, Theorem 2.9 is the result we would like to extend to infinite opinion sets. We now make progress toward this goal when $\mathcal{F}$ is countably infinite.

3 The Countable Case: Coherence is Necessary

3.1 Generalized Legitimate Inaccuracy Measures

We begin with a discussion of how to measure inaccuracy in the countably infinite setting. Fix a countably infinite opinion set $\mathcal{F}$ over a set W of worlds (of arbitrary cardinality). Let $\mathcal{C}$ be the set of credence functions over $\mathcal{F}$, which can be identified with $[0,1]^\infty$ (see Remark 2.3). An *inaccuracy measure* remains a map from $\mathcal{C} \times W$ to $[0,\infty]$.

The class of inaccuracy measures that we use are defined in terms of generalizations of quasi-additive Bregman divergences.

Definition 3.1. Suppose $\mathfrak{D} : [0,1]^\infty \times [0,1]^\infty \to [0,\infty]$. Then we call $\mathfrak{D}$ a *generalized quasi-additive Bregman divergence* if

$$\mathfrak{D}(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{\infty} a_i \mathfrak{d}(x_i, y_i),$$

where $\mathfrak{d}$ is a bounded one-dimensional Bregman divergence as in Definition 2.5.3 and $\{a_i\}_{i=1}^\infty$ a sequence of elements from $(0,\infty)$ with $\sup_i a_i < \infty$.[5]

Remark 3.2. Note that $\mathfrak{d}$—defined in terms of φ—being bounded is equivalent to φ' being bounded on $[0,1]$. Further, we may assume that $\varphi(0) = \varphi'(0) = 0$ since $\mathfrak{d}_\varphi = \mathfrak{d}_{\bar{\varphi}}$ if φ and $\bar{\varphi}$ differ by a linear function.[6]

In the appendix, we show that generalized quasi-additive Bregman divergences are examples of what Csiszár (1995) calls *Bregman distances*, which are generalizations of quasi-additive Bregman divergences defined on spaces of non-negative functions.

Suggestively, we make the following definition.

Definition 3.3. Given an enumeration of $\mathcal{F}$,[7] let a *generalized legitimate inaccuracy measure* be an inaccuracy measure $\mathscr{I} : \mathcal{C} \times W \to [0,\infty]$ given by

$$\mathscr{I}(c,w) = \mathfrak{D}(v_w, c) \tag{1}$$

for $\mathfrak{D}$ a generalized quasi-additive Bregman divergence.

Notice that the Brier score extends to a generalized legitimate inaccuracy measure, namely the squared $\ell^2(\mathcal{F})$ norm

$$\mathscr{I}(c,w) = ||v_w - c||^2_{\ell^2(\mathcal{F})} = \sum_{i=1}^{\infty} (v_w(p_i) - c(p_i))^2. \tag{2}$$

We call (2) the *generalized Brier score*.

The name "generalized legitimate inaccuracy measure" is motivated by the observation that a generalized legitimate inaccuracy measure naturally restricted to the finite opinion sets is a legitimate inaccuracy measure. This is because 1) for both the generalized and finite legitimate

[5] Recall that $\sup_i a_i = a \in \mathbb{R} \cup \{+\infty, -\infty\}$ such that $a_i \leq a$ for all $i \in \mathbb{N}$ and for any $b < a$, there is some a_i such that $b < a_i \leq a$.

[6] Proof: Let $\bar{\varphi}(x) = \varphi(x) + ax + b$. Then $\mathfrak{d}_{\bar{\varphi}}(x,y) = \varphi(x) + ax + b - \varphi(y) - ay - b - (\varphi'(y) + a)(x-y) = \varphi(x) + ax + b - \varphi(y) - ay - b - \varphi'(y)(x-y) - ax + ay = \varphi(x) - \varphi(y) - \varphi'(y)(x-y) = \mathfrak{d}_\varphi(x,y)$. Further, if φ satisfies the conditions in Definition 2.5.3, then $\bar{\varphi}$ does as well. Thus we may assume that any one-dimensional Bregman divergence is defined by a function φ such that $\varphi(0) = \varphi'(0) = 0$.

[7] The choice of enumeration does not matter since the terms in the infinite sum defining inaccuracy are non-negative. Thus convergence is absolute and independent of order.

inaccuracy measures, the score of an individual proposition is defined by a one-dimensional Bregman divergence, and 2) for both the generalized and finite legitimate inaccuracy measures, the scores of individual propositions are combined in a weighted additive way to give a score for the entire credence function. To use the terminology of Leitgeb and Pettigrew (2010a), in the finite and countably infinite setting, the local scores are the same and the global scores relate to the local scores in the same way. These observations support the view that, insofar as quasi-additive Bregman divergences are the appropriate functions to use for measuring inaccuracy in the finite setting, generalized quasi-additive Bregman divergences are the appropriate functions to use for measuring inaccuracy in the countably infinite setting.

3.2 Coherence is Necessary

We now state one of our main results: coherence is necessary to avoid accuracy dominance in the countably infinite case.

Theorem 3.4. Let $\mathcal{F}$ be a countably infinite opinion set, $\mathscr{I}$ a generalized legitimate inaccuracy measure, and c an incoherent credence function. Then:

1. c is weakly dominated relative to $\mathscr{I}$ by a coherent credence function; and
2. if $\mathscr{I}(c, w) < \infty$ for each $w \in W$, then c is strongly dominated relative to $\mathscr{I}$ by a coherent credence function.

Proof. See the Appendix. □

Remark 3.5. By analyzing the proof of Theorem 3.4, one can see that the most general way to state the theorem is: assume c is incoherent; if $\mathscr{I}(c, w) < \infty$ for some w, then there is a coherent credence function d such that $\mathscr{I}(d, w) < \mathscr{I}(c, w)$ for all w such that $\mathscr{I}(c, w) < \infty$; if $\mathscr{I}(c, w) = \infty$ for all $w \in W$, then any omniscient credence function weakly dominates c.

Remark 3.6. The following is easy to prove from the results of Schervish et al. (2009): any incoherent credence function c over a countably infinite opinion set is weakly dominated but not necessarily by a *coherent* credence function; and if $\mathscr{I}(c, w) < \infty$ for each $w \in W$, then c is strongly dominated but not necessarily by a *coherent* credence function.[8] Thus the value in the proof strategy to come is that the dominating credence function is proven to be coherent, which is analogous to the finite case.[9,10]

We note that one direction of Walsh's (2020) accuracy dominance result follows immediately from Theorem 3.4. We first recall his result.

[8] Proof sketch: If c is incoherent, then there is some finite $\overline{\mathcal{F}} \subseteq \mathcal{F}$ on which c is incoherent. Restrict c to $\overline{c}$ on $\overline{\mathcal{F}}$. Then by Theorem 2.9, there is some $\overline{d}$ that strongly dominates $\overline{c}$. Extend $\overline{d}$ to a credence function d on $\mathcal{F}$ by copying c off of $\overline{\mathcal{F}}$. Then so long as c has finite inaccuracy at some world, d will weakly dominate c.

[9] Thanks to Teddy Seidenfeld for suggesting this connection to the finite case.

[10] Further, it is often argued that not all dominated credence functions are irrational—only those that are dominated by a credence function which is itself not dominated (see discussion of the Undominated Dominance principle in Pettigrew 2016, p. 22). For the opinion sets and inaccuracy measures dicussed in Section 4, the undominated credence functions will be precisely the coherent credence functions, and so the added strength of Theorem 3.4 is normatively important, as well.

Theorem 3.7 (Walsh 2020)**.** Let $\mathcal{F}$ be a countably infinite opinion set. Let

$$\mathscr{I}(c,w) = \sum_{i=1}^{\infty} 2^{-i}(v_w(p_i) - c(p_i))^2. \tag{3}$$

Then:

1. if c is incoherent, then c is strongly dominated relative to $\mathscr{I}$ by a coherent credence function;

2. if c is coherent, then c is not weakly dominated relative to $\mathscr{I}$ by any credence function $d \neq c$.

Part 1 of this result follows from Theorem 3.4 by defining $\mathscr{I}$ in terms of the generalized quasi-additive Bregman divergence generated by $\{2^{-i}\}_{i=1}^{\infty}$ and

$$\mathfrak{d}(x,y) = x^2 - y^2 - 2xy(x-y) = \varphi(x) - \varphi(y) - \varphi'(y)(x-y),$$

where $\varphi(x) = x^2$. Note that $\mathscr{I}(c,w) < \infty$ for all $c \in \mathcal{C}$ and $w \in W$ as $\sum_i 2^{-i} < \infty$.

4 The Countable Case: The Sufficiency of Coherence

Unlike coherent credence functions on finite opinion sets, coherent credence functions on countably infinite opinion sets can be strongly dominated.

Example 4.1. Let $\mathcal{F} = \{\{n \geq N : n \in \mathbb{N}\} : N \in \mathbb{N}\}$ be an opinion set over $\mathbb{N}$ (including zero). Let

$$c(\{n \geq N\}) = \frac{1}{\sqrt{N+1}}.$$

Then c is coherent—in fact, countably coherent (see Definition 4.6)—but $\mathscr{I}(c,w) = \infty$ for all $w \in W$ when $\mathscr{I}$ is the generalized Brier score. So any omniscient credence function strongly dominates c.

In fact, the classic example of a merely finitely additive probability function—the 0-1 function defined on the finite-cofinite algebra over $\mathbb{N}$ taking value 0 on finite sets—restricts to a coherent dominated credence function.

Example 4.2. Let $\mathcal{F} = \{\{n \leq N : n \in \mathbb{N}\} : N \in \mathbb{N}\}$ be an opinion set over $\mathbb{N}$ (including zero). Let

$$c(\{n \leq N\}) = 0.$$

Then c is coherent—as well as finitely supported and not countably coherent—but $\mathscr{I}(c,w) = \infty$ for all $w \in W$ when $\mathscr{I}$ is the generalized Brier score. So any omniscient credence function strongly dominates c.

The goal of this section is to characterize the opinion sets and inaccuracy measures for which some variant of Theorem 2.9 holds. We extend Theorem 2.9 by proving dominance results for *countably coherent* credence functions and using an opinion set compactification construction to transfer these results to merely coherent credence functions. At points, our results will only apply to the generalized Brier score. We conjecture that any such result extends to any generalized legitimate inaccuracy measure. In any case, this is a well motivated restriction since the Brier score has been defended by many—including Horwich (1982), Maher (2002), Joyce (2009), and Leitgeb and Pettigrew (2010a)—as being a particularly appropriate way to measure inaccuracy.

Throughout the rest of this section we assume the opinion set $\mathcal{F}$ is countably infinite.

4.1 Countable Coherence

We begin by introducing the notion of a *countably coherent* credence function and establishing a characterization theorem regarding countable coherence on *countably discriminating* opinion sets which extends a result of de Finetti (1974).

Definition 4.3. For $\mathcal{F} \subseteq \mathcal{P}(W)$, we define an equivalence relation $\sim$ on W such that $w \sim w'$ if and only if $\{p \in \mathcal{F} : w \in p\} = \{p \in \mathcal{F} : w' \in p\}$. We call the set of equivalence classes of W the *quotient of W relative to $\mathcal{F}$*. If the quotient of W relative to $\mathcal{F}$ is countable, then we call $\mathcal{F}$ *countably discriminating.*

Clearly, any countable opinion set over a countable set of worlds is countably discriminating.

The following characterization of the coherent credence functions on finite opinion sets is due to de Finetti (1974). Recall $\mathcal{V}_{\mathcal{F}}$ is the set of omniscient credence functions on $\mathcal{F}$, which is finite when $\mathcal{F}$ is finite.

Theorem 4.4 (de Finetti 1974)**.** *c is a coherent credence function on a finite opinion set $\mathcal{F}$ if and only if there are $\lambda_w \in [0,1]$ with $\sum_{v_w \in \mathcal{V}_{\mathcal{F}}} \lambda_w = 1$ such that*

$$c(p) = \sum_{v_w \in \mathcal{V}_{\mathcal{F}}} \lambda_w v_w(p)$$

for all $p \in \mathcal{F}$.

Theorem 4.4 is integral to Predd et al.'s proof that coherence is sufficient to avoid dominance in Theorem 2.9. We now show de Finetti's characterization of the coherent credence functions on finite opinion sets extends to countably coherent credence functions on countably discriminating opinion sets.

Definition 4.5. A *σ-algebra* over W is a subset $\mathcal{F}^* \subseteq \mathcal{P}(W)$ such that:

1. $W \in \mathcal{F}^*$;
2. if $\{p_i\}_{i=1}^{\infty} \subseteq \mathcal{F}^*$, then $\bigcup_{i=1}^{\infty} p_i \in \mathcal{F}^*$;
3. if $p \in \mathcal{F}^*$, then $W \setminus p \in \mathcal{F}^*$.

Definition 4.6. Let a credence function c be *countably coherent* if c extends to a countably additive probability function on a σ-algebra $\mathcal{F}^*$ containing $\mathcal{F}$.[11] That is, there is a $c^* : \mathcal{F}^* \to [0,1]$ such that:

1. $c^*(p) = c(p)$ for all $p \in \mathcal{F}$;
2. $c^*(\bigcup_{i=1}^{\infty} p_i) = \sum_{i=1}^{\infty} c^*(p_i)$ for $\{p_i\}_{i=1}^{\infty} \subseteq \mathcal{F}^*$ with $p_i \cap p_j = \varnothing$ for $i \neq j$;

[11] Note that if c is countably coherent on $\mathcal{F}$, then c extends to a countably additive probability function on $\sigma(\mathcal{F})$, the σ-algebra generated by $\mathcal{F}$.

3. $c^*(W) = 1$.

Otherwise, a credence function is *countably incoherent*.

Proposition 4.7. Let $\mathcal{F}$ be a countably discriminating opinion set. Then a credence function c is countably coherent if and only if there are $\lambda_w \in [0,1]$ with $\sum_{v_w \in \mathcal{V}_\mathcal{F}} \lambda_w = 1$ such that

$$c(p) = \sum_{v_w \in \mathcal{V}_\mathcal{F}} \lambda_w v_w(p)$$

for all $p \in \mathcal{F}$.

Proof. See the Appendix. □

4.2 Compactification of an Opinion Space

In this section, we introduce the compactification construction of what we call an *opinion space*. The construction will be relevant to transferring dominance results for countably coherent credence functions to merely coherent credence functions.

Definition 4.8. An *opinion space* is a pair $(W, \mathcal{F})$, where W is a nonempty set and $\mathcal{F} \subseteq \mathcal{P}(W)$.

From here on out we will speak in terms of opinion spaces as opposed to opinion sets in order to keep track of the underlying set of worlds.

Borkar et al. (2003) proved that the opinion spaces which satisfy a certain compactness property are precisely those where the set of coherent credence functions and the set of countably coherent credence functions coincide.

Definition 4.9. Let $(W, \mathcal{F})$ be an opinion space. Let $f(n) \in \{0,1\}$ and set $p_n^{f(n)} = p_n$ if $f(n) = 0$ and $p_n^{f(n)} = p_n^c$ if $f(n) = 1$. Then $(W, \mathcal{F})$ is *compact* if for any choice of $\{p_n\}_{n=1}^\infty \subseteq \mathcal{F}$ and $f : \mathbb{N} \to \{0,1\}$, if $\bigcap_{n=1}^N p_n^{f(n)}$ is nonempty for every N, then $\bigcap_{n=1}^\infty p_n^{f(n)}$ is nonempty.

As an example, note that the opinion spaces from Examples 4.1 and 4.2 are not compact. Indeed, for the first example $\bigcap_{n=1}^\infty p_n = \varnothing$ and yet every finite subset of $\mathcal{F}$ has nonempty intersection; for the second example, $\bigcap_{n=1}^\infty p_n^c = \varnothing$ while $\bigcap_{n=1}^N p_n^c \neq \varnothing$ for every N.

Remark 4.10. Assume $\mathcal{F}$ is closed under finite intersections. Let $\mathcal{A}(\mathcal{F})$ denote the algebra generated by $\mathcal{F}$, and let $\mathcal{T}(\mathcal{F})$ denote the topology generated by $\mathcal{F}$. By the Alexander subbase theorem (see, e.g., Kelley 1975, p. 139), $(W, \mathcal{F})$ is compact if and only if $\mathcal{T}(\mathcal{A}(\mathcal{F}))$ is compact.[12]

Theorem 4.11 (Borkar et al. 2003)**.** The following are equivalent:

1. $(W, \mathcal{F})$ is compact;
2. for every credence function c on $(W, \mathcal{F})$, c is coherent if and only if c is countably coherent.

We now show how to associate a compact space to any space and, in light of Theorem 4.11, a countably coherent credence function to any coherent credence function. Let $(W, \mathcal{F})$ be an opinion space. Let S denote the set of sequences of the form $\{p_n^{f(n)}\}$ (as in Definition 4.9) such that

[12]Proof: The set of elements in $\mathcal{F}$ and their complements form a subbase for $\mathcal{T}(\mathcal{A}(\mathcal{F}))$.

$\bigcap_{n=1}^{N} p_n^{f(n)} \neq \varnothing$ for every N but $\bigcap_{n=1}^{\infty} p_n^{f(n)} = \varnothing$. Define $W^* = W \cup \{x_s : s \in S\}$. Define $\mathcal{F}^* \subseteq \mathcal{P}(W^*)$ as follows: for each $p \in \mathcal{F}$, let S_p denote the set of sequences s of the form $\{p_n^{f(n)}\}$ (as in Definition 4.9) such that $s \in S$, $p_n = p$ for some n, and $f(n) = 0$. Then define

$$p^* = p \cup \{x_s : s \in S_p\}.$$

Finally, let $\mathcal{F}^* = \{p^* : p \in \mathcal{F}\}$. We call $(W^*, \mathcal{F}^*)$ the *compactification* of $(W, \mathcal{F})$. We always denote the compactification of $(W, \mathcal{F})$ by $(W^*, \mathcal{F}^*)$. Further, we let Ψ denote the natural bijection from $\mathcal{F}$ to $\mathcal{F}^*$ given by $\Psi(p) = p^*$.

We first note that $(W^*, \mathcal{F}^*)$ is in fact compact.

Lemma 4.12. For any opinion space $(W, \mathcal{F})$, $(W^*, \mathcal{F}^*)$ is compact.

Proof. See the Appendix. □

Next we note that we can naturally identify a coherent credence function on $(W, \mathcal{F})$ with a countably coherent credence function on $(W^*, \mathcal{F}^*)$.

Lemma 4.13. Let $(W, \mathcal{F})$ be an opinion space and c a coherent credence function on $(W, \mathcal{F})$. Let $(W^*, \mathcal{F}^*)$ be the compactification of $(W, \mathcal{F})$ and define $c^*(\Psi(p)) := c(p)$ for each $p \in \mathcal{F}$. Then c^* is a countably coherent credence function on $(W^*, \mathcal{F}^*)$ and $\mathscr{I}(c, w) = \mathscr{I}(c^*, w)$ for $w \in W$.

Proof. See the Appendix. □

For a coherent credence function c defined on an opinion space $(W, \mathcal{F})$, we let c^* denote the countably coherent credence function on $(W^*, \mathcal{F}^*)$ given as in Lemma 4.13.

Example 4.14. As an example, let us compute the compactification of the opinion space from Example 4.2 and show how to identify a coherent credence function on the space with a countably coherent credence function on its compactification. We note that only for $f(n) = 1$ for all $n \in \mathbb{N}$ is $\bigcap_{n=1}^{N} p_n^{f(n)}$ nonempty for every N while $\bigcap_{n=1}^{\infty} p_n^{f(n)} = \varnothing$. Indeed, assume $f(m) = 0$ for some m. If $f(i) = 1$ for some $i \geq m+1$, then since $p_i^c \cap p_m = \varnothing$, we have $\bigcap_{n=1}^{i} p_n^{f(n)} = \varnothing$. So $f(i) = 0$ for all $i \geq m+1$. But then since $\bigcap_{n=1}^{m} p_n^{f(n)} \neq \varnothing$ and $\bigcap_{n=1}^{m} p_n^{f(n)} \subseteq p_i$ for all $i \geq m+1$, it also follows that $\bigcap_{n=1}^{\infty} p_n^{f(n)} \neq \varnothing$, which contradicts our assumption. So S is a single point x, $W^* = W \cup \{x\}$, and $\mathcal{F}^* = \{\{n \leq N\} : N \in \mathbb{N}\}$. $\mathcal{F}^*$ is identical to $\mathcal{F}$, except that there is a point in the complement of every proposition in $\mathcal{F}^*$. For a coherent credence function c on $(W, \mathcal{F})$, c^* on $(W^*, \mathcal{F}^*)$ is identical to c and is a countably coherent credence function on the compact opinion space $(W^*, \mathcal{F}^*)$. For example, for credence function c in Example 4.2, c^* extends to the countably additive omniscient credence function v_x on the σ-algebra generated by $\mathcal{F}^*$.

Using Theorem 4.11, Lemma 4.12 and Lemma 4.13, the proof strategy for extending Theorem 2.9 is more precisely as follows. First, we establish dominance results for countably coherent credence functions. Second, we associate each coherent credence function on $(W, \mathcal{F})$ with a countably coherent credence function on $(W^*, \mathcal{F}^*)$ as in Lemma 4.13. Lastly, we use the dominance results for countably coherent credence functions to establish dominance results for coherent credence functions in certain cases where there is "accuracy dominance stability" in compactifying.

4.3 W-Stable Opinion Spaces

In this section, we establish the equivalence between coherence and avoiding weak dominance for certain opinion spaces (Theorem 4.19), as well as additional results extending Theorem 2.9 (Corollary 4.22 and Theorem 4.23). We first note that under certain circumstances countably coherent credence functions are not weakly dominated (Proposition 4.16 and Proposition 4.17); then we use the compactification construction from the previous section and a property of an opinion space—*W-stability* (Definition 4.18)—to establish that for certain opinion spaces, mere coherence is also sufficient to avoid weak dominance.

We first prove that if a countably coherent credence function c has finite expected inaccuracy, then c is not weakly dominated.

Definition 4.15. For c a countably coherent credence function and $\mathscr{I}$ a generalized legitimate inaccuracy measure, we say that c has *finite expected inaccuracy relative to* $\mathscr{I}$ if c has a countably additive extension $\bar{c}$ defined on the opinion space $(W, \sigma(\mathcal{F}))$ such that $\mathbb{E}_{\bar{c}}\mathscr{I}(c, \cdot) < \infty$. For c a coherent but not countably coherent credence function and $\mathscr{I}$ a generalized legitimate inaccuracy measure, we say that c has *finite expected inaccuracy relative to* $\mathscr{I}$ if c^* has finite expected inaccuracy relative to $\mathscr{I}$.[13]

Note that it follows by Definition 4.15 that any coherent credence function c has finite expected inaccuracy if and only if c^* has finite expected inaccuracy.

Proposition 4.16. Let $(W, \mathcal{F})$ be an opinion space with $\mathcal{F}$ countably infinite and $\mathscr{I}$ a generalized legitimate inaccuracy measure. If c is a countably coherent credence function with finite expected inaccuracy, then c is not weakly dominated.

Proof. See the Appendix. □

Here is another dominance result for countably coherent credence functions where we assume $\mathcal{F}$ is *point-finite* ($|\{p \in \mathcal{F} : w \in p\}| < \infty$ for all $w \in W$) but weaken the assumption that c has finite expected inaccuracy considerably, namely to *somewhere finitely inaccurate* (there is a $w \in W$ such that $\mathscr{I}(c, w) < \infty$). We also restrict to the generalized Brier score $\mathscr{B}$.

Proposition 4.17. Let $(W, \mathcal{F})$ be a point-finite opinion space with $\mathcal{F}$ countably infinite and $\mathscr{I}$ a generalized legitimate inaccuracy measure. If a credence function c is countably coherent and somewhere finitely inaccurate relative to $\mathscr{B}$, then c is not weakly dominated relative to $\mathscr{B}$.

Proof. See the Appendix. □

We now introduce the notion of *W-stability* which will allow us to use Propositions 4.16 and 4.17 to prove extensions of Theorem 2.9.

Definition 4.18. Let $(W, \mathcal{F})$ be *W-stable relative to* $\mathscr{I}$ if for any coherent credence function c on $(W, \mathcal{F})$, if c is weakly dominated relative to $\mathscr{I}$, then c^* on $(W^*, \mathcal{F}^*)$ is weakly dominated relative to $\mathscr{I}$.

[13]Consider the measure space $(W, \sigma(\mathcal{F}), \mu)$. Note $\mathfrak{d}(d_i, v_w(p_i)) = 1_{p_i}(w)\mathfrak{d}(1, d_i) + (1 - 1_{p_i}(w))\mathfrak{d}(0, d_i)$ so that each term in $\mathscr{I}(d, \cdot)$ is measurable for any credence function d, and so the infinite sum is measurable as the finite sum and limit of measurable functions are measurable. Thus we can take the expectation of $\mathscr{I}(d, \cdot)$ with respect to μ for any credence function d.

Using Proposition 4.16, we get the following sufficient and partly necessary conditions on an opinion space for coherence to be equivalent to not being weakly dominated.

Theorem 4.19. Let $\mathscr{I}$ be a generalized legitimate inaccuracy measure and $(W, \mathcal{F})$ a W-stable opinion space relative to $\mathscr{I}$ where all coherent credence functions have finite expected inaccuracy relative to $\mathscr{I}$. Then the following are equivalent:

1. c is coherent;

2. c is not weakly dominated.

Proof. We prove that if c is coherent, then c is not weakly dominated. Let $(W^*, \mathcal{F}^*)$ be the compactification of $(W, \mathcal{F})$. If c is coherent on $(W, \mathcal{F})$, then c^* is countably coherent by Lemma 4.13. Further c^* has finite expected inaccuracy by definition and the assumption that c has finite expected inaccuracy. So by Proposition 4.16, c^* is not weakly dominated. But since $(W, \mathcal{F})$ is W-stable this implies that c is not weakly dominated. The other direction follows from Theorem 3.4. ☐

Remark 4.20. It is trivial to see that W-stability is necessary for the equivalence of coherence and not being weakly dominated. It is open how far finite expected inaccuracy can be weakened.

Remark 4.21. If $\mathscr{I}$ is defined with summable weights, that is, $\{a_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} a_i < \infty$, then there is a $C < \infty$ such that $\mathscr{I}(c, w) < C$ for all credence functions c and $w \in W$. So, in particular, all coherent credence functions have finite expected inaccuracy relative to $\mathscr{I}$.

If we add in an additional finiteness assumption, then we get the full equivalence of Theorem 2.9.

Corollary 4.22. In Theorem 4.19, if in addition all coherent credence functions c have $\mathscr{I}(c, w) < \infty$ for all $w \in W$, then the following are equivalent:

1. c is coherent;

2. c is not weakly dominated;

3. c is not strongly dominated.

We combine W-stability and Proposition 4.17 to get another set of sufficient conditions on $(W, \mathcal{F})$ for Theorem 2.9 to go through for the generalized Brier score.

Theorem 4.23. Let $(W, \mathcal{F})$ be a W-stable opinion space with $(W^*, \mathcal{F}^*)$ point-finite such that all coherent credence functions are somewhere finitely inaccurate relative to $\mathscr{B}$. Then the following are equivalent:

1. c is coherent;

2. c is not weakly dominated relative to $\mathscr{B}$;

3. c is not strongly dominated relative to $\mathscr{B}$.

Proof. If c is coherent, then c^* is countably coherent on a point-finite opinion set. Further, c^* is somewhere finitely inaccurate relative to $\mathscr{B}$, as c is somewhere finitely inaccurate by assumption. Thus by Proposition 4.17, c^* is not weakly dominated relative to $\mathscr{B}$. By W-stability, c is not weakly dominated relative to $\mathscr{B}$. Clearly if c is not weakly dominated then c is not strongly dominated. Finally, we show that if c is incoherent then c is strongly dominated. First, if c is not somewhere finitely inaccurate, then any omniscient credence function strongly dominates c since $\mathscr{I}(v_w, w') < \infty$ for every $w, w' \in W$ by point-finiteness. If c is somewhere finitely inaccurate then $\mathscr{I}(c, w) < \infty$ for all $w \in W$ by point-finiteness. Thus Theorem 3.4 establishes that c is strongly dominated relative to $\mathscr{B}$. □

Remark 4.24. We can drop the assumption that all coherent credence functions are somewhere finitely inaccurate in Theorem 4.23 if we strengthen W-stable to compact so that $(W, \mathcal{F}) = (W^*, \mathcal{F}^*)$. Indeed, compactness alongside point-finiteness implies coherent credence functions on $(W, \mathcal{F}) = (W^*, \mathcal{F}^*)$ are somewhere finitely inaccurate: if there were a coherent (and thus countably coherent) credence function infinitely inaccurate at all worlds, then it would be strongly dominated by an omniscient credence function, contradicting Proposition 4.27 below.

4.3.1 Partitions

As an application of Theorem 4.19, we establish Theorem 2.9 for countably infinite partitions.[14] In parts of the existing literature (e.g., in Joyce 2009), credence functions are assumed to be defined on a (finite) partition of W to begin with, and so such a result might be especially relevant to extending the accuracy argument for probabilism to countably infinite opinion sets.

Lemma 4.25. A partition is W-stable relative to any generalized legitimate inaccuracy measure.

Proof. See the Appendix. □

Theorem 4.26. Let $(W, \mathcal{F})$ be a partition and $\mathscr{I}$ a generalized legitimate inaccuracy measure. Then the following are equivalent:

1. c is coherent;
2. c is not weakly dominated;
3. c is not strongly dominated.

Proof. The result follows from Corollary 4.22, Lemma 4.25, and the fact that $\mathscr{I}(c^*, \cdot)$ is bounded on W^* for each coherent credence function c. To see the latter, note that since c is coherent it follows that $\sum c_i = \sum c_i^* \leq 1$. For $w \in W$ such that $w \in p_i$, recalling that $\varphi(0) = 0$,

$$\mathscr{I}(c^*, w) = a_i\mathfrak{d}(1, c_i^*) + \sum_{j\neq i} a_j\mathfrak{d}(0, c_j^*) = a_i\mathfrak{d}(1, c_i^*) + \sum_{j\neq i} a_j(c_j^*\varphi'(c_j^*) - \varphi(c_j^*)) \leq C + D\sum_j c_j^* \leq C + D$$

for some constants C, D independent of c^* or w. Similarly, as seen in the proof of Lemma 4.25, $W^* \setminus W = \{w^*\}$ where

$$\mathscr{I}(c^*, w^*) = \sum_{j=1}^{\infty} a_j\mathfrak{d}(0, c_j^*) = \sum_{j=1}^{\infty} a_j(c_j^*\varphi'(c_j^*) - \varphi(c_j^*)) \leq C$$

[14] It has been noted that de Finetti's (1974) original proof of Theorem 2.9 assuming the Brier score extends to countably infinite opinion sets. However, the only proof we have seen is a sketch of the necessity of coherence for countably infinite partitions by (Joyce, 1998, footnote 6), and such a claim could not be true for arbitrary countable opinion sets as Examples 4.1 and 4.2 show. Further, we prove the extension for arbitrary generalized legitimate inaccuracy measures.

for some constant C independent of c^* or w. It follows that i) all coherent credence functions have finite expected inaccuracy and ii) $\mathscr{I}(c, w) < \infty$ for $c \in \mathcal{C}$ and $w \in W$. Thus Lemma 4.25 and Corollary 4.22 establish the result. □

4.4 S-Stable Opinion Spaces

In this section, we establish the equivalence between coherence and avoiding strong dominance for certain opinion spaces (Theorem 4.29). The conditions are in terms of the analogous stability condition—*S-stability* (Definition 4.28)—but a different finiteness assumption, and the proof strategy is the same as for Theorem 4.19.

We begin by establishing that on compact opinion spaces, coherent and thus countably coherent credence functions (recall Theorem 4.11) are not strongly dominated.

Proposition 4.27. Let $(W, \mathcal{F})$ be a compact opinion space and $\mathscr{I}$ a generalized legitimate inaccuracy measure. If c is coherent (and thus countably coherent), then c is not strongly dominated relative to $\mathscr{I}$.

Proof. See the Appendix. □

We now introduce S-stability and the main theorem of this section.

Definition 4.28. Let $(W, \mathcal{F})$ be *S-stable relative to* $\mathscr{I}$ if whenever a coherent credence function c defined on $(W, \mathcal{F})$ is strongly dominated relative to $\mathscr{I}$, then c^* defined on $(W^*, \mathcal{F}^*)$ is strongly dominated relative to $\mathscr{I}$.

Theorem 4.29. Let $\mathscr{I}$ be a generalized legitimate inaccuracy measure and $(W, \mathcal{F})$ an S-stable opinion space relative to $\mathscr{I}$. Assume that $\mathscr{I}(c, w) < \infty$ for each coherent credence function c and $w \in W$. Then the following are equivalent:

1. c is coherent;

2. c is not strongly dominated.

Proof. Assume c is coherent. c^* defined on the compact opinion space $(W^*, \mathcal{F}^*)$ is countably coherent by Lemma 4.13. So by Proposition 4.27, c^* is not strongly dominated. But since $\mathcal{F}$ is S-stable this implies that c is not strongly dominated relative to W. The other direction follows from Theorem 3.4. □

Remark 4.30. It is trivial to see that S-stability is necessary for the equivalence of coherence and avoiding strong dominance. It is open how much the assumption that coherent credence functions satisfy $\mathscr{I}(c, w) < \infty$ for all w can be weakened.

Remark 4.31. Schervish et al. (2009) take a different approach to dropping the assumption that the opinion set is finite: they apply weak and strong dominance notions to finite subsets of arbitrarily sized opinion sets. They also explore connections between the two notions of dominance considered here—weak and strong dominance—and what they call *coherence*$_1$, which amounts to avoiding being susceptible to a finite *Dutch book*.[15]

[15]Thanks to Teddy Seidenfeld for pointing me to this work of Schervish et al.. An additional point worth noting about their work is that they further generalize the finite results of Predd et al. (2009) by i) allowing a wider variety

Remark 4.32. Theorem 4.29 is related to Theorem 1 of Schervish et al. 2014. However, 1) their assumptions are in some ways weaker and in some ways stronger than those in Theorem 4.29[16] and 2) while Schervish et al. (2014) establish that coherence is sufficient for avoiding strong dominance in certain cases, unlike Theorems 4.19 and 4.29, their results do not show that coherence is sufficient for avoiding even weak dominance in certain cases or that incoherence always entails being weakly dominated (and sometimes strongly dominated) by a coherent credence function (see Remark 3.6).

4.5 Further Directions

While Theorems 4.19 and 4.29 come close to characterizing the opinion spaces on which not being weakly and strongly dominated, respectively, are equivalent to coherence, it is open how far the finiteness assumptions in the theorems can be weakened. This is a natural next line of inquiry. In addition, it would be useful to determine characterizations of W- and S-stability in terms of the inaccuracy measure that make it relatively easy to check whether an opinion set is W- or S-stable. Also, there are natural ways to generalize the results above to more closely match the finite results: allow different one-dimensional Bregman divergences for different propositions and allow unbounded one-dimensional Bregman divergences.

Another direction one could go in exploring the sufficiency of coherence for avoiding dominance is as follows: instead of characterizing the countable opinion sets on which Theorem 2.9 goes through, one could characterize the kinds of coherent credence functions for which Theorem 2.9 goes through on any countable opinion set.[17] Doing so might show that while coherence is not enough to avoid dominance in all cases, coherence along with additional plausible constraints is sufficient. In particular, while restricting to finitely supported credence functions is not enough to establish the sufficiency of coherence for avoiding strong dominance (due to Example 4.2), it is open whether *countable* coherence is equivalent to avoiding weak or strong dominance on the restricted class.

5 The Uncountable Case

So far we have been concerned with credences defined on countably infinite opinion sets. We now consider what can be said in favor of probabilism when credences are defined on uncountable opinion sets. When extending from the finite to the countably infinite setting, we used inaccuracy measures that naturally restrict to legitimate inaccuracy measures in the finite case. Similarly, in the uncountable case, we allow for measure theoretically defined inaccuracy measures that naturally restrict to generalized legitimate inaccuracy measures in the countable case. However, for the sake of generality, we allow inaccuracy to be defined by integration against any finite measure.[18]

of inaccuracy measures including those which are merely proper as opposed to strictly proper and ii) by scoring conditional probabilities. A natural direction for future work is to use these relaxations in the finite case to relax assumptions made here. Similarly, Steeger (2019) considers the property of avoiding strong dominance with respect to the Brier score for every finite subset of arbitrarily sized opinion sets (see "sufficient coherence" on p. 38).

[16] Schervish et al. require that the *prevision* for the inaccuracy of the credence function be finite and that inaccuracy be pointwise finite, while we only assume the latter. On the other hand, we require the opinion set to be S-stable while they do not.

[17] Thanks to Thomas Icard and Milan Mosse for suggesting this alternative direction of study.

[18] Note that since the counting measure over $\mathbb{N}$ is not a finite measure, the result below does not directly establish Theorem 3.4.

Due to the measure theoretic construction of the inaccuracy measures we consider, we restrict our attention to measurable credence functions and equate credence functions that are equal almost everywhere. In some measure spaces, like the weighted counting measure spaces underlying generalized legitimate inaccuracy measures, we lose nothing since every credence function is measurable and only the empty set is measure zero. However, in other cases, these assumptions are substantive. We begin by extending the accuracy framework to the measure theoretic setting.

Definition 5.1. Let $(\mathcal{F}, \mathcal{A}, \mu)$ be a measure space and $c : \mathcal{F} \to \mathbb{R}^+$. If c is $\mathcal{A}$-measurable and $\mu(\{p : c(p) \notin [0,1]\}) = 0$, we call c a *μ-credence function*. We say a μ-credence function c is *μ-coherent* if there is a coherent (in the usual sense) credence function c' on $\mathcal{F}$ with $c = c'$ μ-a.e. We say a μ-credence function is *μ-incoherent* if there is no coherent credence function c' such that $c = c'$ μ-a.e.

Definition 5.2. Let $\mathcal{F}$ be an opinion set (of arbitrary cardinality) over a set W of worlds. Let $(\mathcal{F}, \mathcal{A}, \mu)$ be a σ-finite measure space over the opinion set $\mathcal{F}$. Let $\mathcal{C}$ be the space of all μ-credence functions. Assume $\mathscr{I} : \mathcal{C} \times W \to [0, \infty]$ is such that, for all $(c, w) \in \mathcal{C} \times W$, we have

$$\mathscr{I}(c, w) = B_{\varphi,\mu}(v_w, c),$$

where $B_{\varphi,\mu}$ is a *Bregman distance* relative to φ[19] and $(\mathcal{F}, \mathcal{A}, \mu)$ (see Definition A.1). In particular, each v_w is a μ-credence function. Then we call $\mathscr{I}$ an *integral inaccuracy measure on* $(\mathcal{F}, \mathcal{A}, \mu)$.

We now state a dominance result about integral inaccuracy measures. The proof is essentially a measure theoretic version of the proof of Theorem 3.4.

Theorem 5.3. Let $\mathscr{I}$ be an integral inaccuracy measure on a finite measure space $(\mathcal{F}, \mathcal{A}, \mu)$. Then for every μ-credence function c, if c is μ-incoherent, then there is a μ-coherent μ-credence function c' that strongly dominates c relative to $\mathscr{I}$.

Proof. See the Appendix. ☐

Here is an example of how Theorem 5.3 can be used to give an accuracy argument in a concrete uncountable setting. Assume we have a coin with unknown bias $\theta \in [0,1]$ and a set of propositions of the form "$a \leq \theta \leq b$" for each $a, b \in [0,1]$ with $a \leq b$. Then a credence function on this uncountable opinion set can be represented by a function

$$c : X \to [0,1],$$

where $X = \{(a,b) : 0 \leq a \leq b \leq 1\} \subseteq [0,1]^2$. We put the Lebesgue measure λ on X to generalize the additive constraint often assumed in the finite case. We let

$$\mathscr{I}(c, w) = \int_X \mathfrak{d}(v_w(\mathbf{x}), c(\mathbf{x}))\lambda(d\mathbf{x})$$

for a bounded one-dimensional Bregman divergence $\mathfrak{d}$. Then the assumptions of Theorem 5.3 hold, so we get the following dominance result: for any λ-credence function c, if c is a λ-incoherent, then there is a λ-coherent λ-credence function that strongly dominates c.

[19] Again, we assume the one-dimensional Bregman divergence $\mathfrak{d}$ generated by φ is bounded.

6 Conclusion

There is plenty of normative work to be done using the results established above. In light of the failure of coherence being sufficient to avoid strong dominance on certain countably infinite opinion sets, the most pressing question seems to be: is there an accuracy-based argument for probabilism on at least all countable opinion sets? If not, what does this mean for the accuracy project as a whole? Can we give some sort of privileged status to certain kinds of opinion sets or inaccuracy measures for which coherence is equivalent to not being dominated, e.g., partitions? What is the normative status of the stronger condition of countable coherence? Further, while the measure theoretic framework introduced in Section 5 to score inaccuracy of credence functions over opinion sets of arbitrary cardinality seems like a natural extension of the finite and countably infinite frameworks, is it well motivated that inaccuracy does not track the behavior of a credence function on measure zero sets? The hope with this paper is to start a conversation about these questions by first establishing relevant mathematical results.

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A Appendix

A.1 Proof of Theorem 3.4

We review the necessary background before proving Theorem 3.4.

A.1.1 Generalized Projections

Csiszár (1995) showed that what he calls *generalized projections* onto convex sets with respect to Bregman distances exist under very general conditions. We review his relevant results here (but assume knowledge of basic measure theory).

Definition A.1. Fix a σ-finite measure space $(X, \mathcal{X}, \mu)$. The *Bregman distance* of non-negative ($\mathcal{X}$-measurable) functions s and t is defined by

$$B_{\varphi,\mu}(s,t) = \int \mathfrak{d}(s(x), t(x))\mu(dx) \in [0, \infty]$$

where $\mathfrak{d}(s(x), t(x)) = \varphi(s(x)) - \varphi(t(x)) - \varphi'(t(x))(s(x) - t(x))$ for some strictly convex, differentiable function φ on $(0, \infty)$.[20] Note that $B_{\varphi,\mu}(s,t) = 0$ iff $s = t$ μ-a.e. See Csiszár 1995, p. 165 for details.

[20]For $B_{\varphi,\mu}$ to be a distance measure, we do not need to assume that $\varphi(1) = \varphi'(1) = 0$ by the remark following (1.9) in Csiszár 1995.

Remark A.2. Notice that a generalized quasi-additive Bregman divergence $\mathfrak{D}$ with weights $\{a_i\}_{i=1}^{\infty}$ whose generating one-dimensional Bregman divergence $\mathfrak{d}$ is given in terms of φ has a corresponding Bregman distance $B_{\bar{\varphi},\mu}$ with

1. the measure space being $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$, where $\mu(A) = \sum_{i \in A} a_i$ for each $A \in \mathcal{P}(\mathbb{N})$, and
2. $\bar{\varphi}$ on $(0, \infty)$ being a strictly convex, differentiable extension of φ on $[0, 1]$.[21]

Thus non-negative ($\mathscr{P}(\mathbb{N})$-measurable) functions are elements of $\mathbb{R}^{+^{\infty}}$. Note, importantly, that the corresponding generalized legitimate inaccuracy measure $\mathscr{I}$ determined by $\mathfrak{D}$ is also given by the corresponding Bregman distance. That is,

$$\mathscr{I}(c, w) = B_{\bar{\varphi},\mu}(v_w, c).$$

To simplify notation, let B denote $B_{\bar{\varphi},\mu}$ a Bregman distance. Let S be the set of non-negative measurable functions. For any $E \subseteq S$ and $t \in S$, we write

$$B(E, t) = \inf_{s \in E} B(s, t).$$

If there exists $s^* \in E$ with $B(s^*, t) = B(E, t)$, then s^* is unique and is called the *B-projection of t onto E* (see Csiszár 1995, Lemma 2). As Csiszár notes, these projections may not exist. However, a weaker kind of projection exists in a large number of cases. To describe them, we need to introduce a kind of convergence called *loose in μ-measure convergence*.

Definition A.3. We say a sequence $\{s_n\}$ of elements from S converges *loosely in μ-measure* to t, denoted by $s_n \rightsquigarrow_\mu t$, if for every $A \in \mathcal{X}$ with $\mu(A) < \infty$, we have

$$\lim_{n \to \infty} \mu(A \cap \{p : |s_n(p) - t(p)| > \epsilon\}) = 0 \text{ for all } \epsilon > 0.$$

Definition A.4.

i. Given $E \subseteq S$ and $t \in S$, we say that a sequence $\{s_n\}$ of elements from E is a *B-minimizing* sequence if $B(s_n, t) \to B(E, t)$.

ii. If there is an $s^* \in S$ such that every B-minimizing sequence converges to s^* loosely in μ-measure, then we call s^* the *generalized B-projection of t onto E*.

The result that is integral to proving Theorem 3.4 is the following (see Csiszár's Theorem 1, Lemma 2, and Corollary of Theorem 1).

Theorem A.5 (Csiszár 1995)**.** Let E be a convex subset of S and $t \in S$. If $B(E, t)$ is finite, then there exists $s^* \in S$ such that

$$B(s, t) \geq B(E, t) + B(s, s^*) \text{ for every } s \in E$$

and $B(E, t) \geq B(s^*, t)$. It follows that the generalized B-projection of t onto E exists and equals s^*.

[21] Using that φ' exists and is finite at $x = 1$ as we assumed $\mathfrak{d}$ is bounded, we extend φ as follows: for $x \in [1, \infty)$, let $\bar{\varphi}(x) = q(x) = x^2 + bx + c$, where b and c are chosen so $\varphi(1) = q(1)$ and $\varphi'(1) = q'(1)$. Then using the fact that $\bar{\varphi}$ is differentiable at 1 by construction and a function is strictly convex if and only if its derivative is strictly increasing, it is easy to see that $\bar{\varphi}$ is differentiable and strictly convex on $(0, \infty)$.

A.1.2 Extending Partial Measures

We also use an extension result of Horn and Tarski (1948) in the proof of Theorem 3.4. Following Horn and Tarski, we introduce *partial measures* and recall that they can be extended to finitely additive probability functions. Recall the definition of a finitely additive probability function in Definition 2.2 (though we drop the assumption that $\mathcal{F}$ is finite).

Remark A.6. It is a simple corollary of the definition of a finitely additive probability function c over an algebra $\mathcal{F}$ that for any $p, p' \in \mathcal{F}$: if $p \subseteq p'$, then $c(p) \leq c(p')$.

Here is another useful fact about finitely additive probability functions.

Proposition A.7. If c is a finitely additive probability function on an algebra $\mathcal{F}$ and $a_0, \ldots, a_{m-1} \in \mathcal{F}$, then

$$\sum_{k=0}^{m-1} c(a_k) = \sum_{k=0}^{m-1} c\Big(\bigcup_{p \in S^{m,k}} \bigcap_{i \leq k} a_{p_i}\Big) \tag{4}$$

where $S^{m,k}$ is the set of all sequences $p = (p_0, \ldots, p_k)$ with $0 \leq p_0 < \ldots < p_k < m$.

To introduce the notion of a partial measure, we need the following definition.

Definition A.8. Let $\varphi_0, \ldots, \varphi_{m-1}$ and $\psi_0, \ldots, \psi_{n-1}$ be elements of $\mathcal{F}$. Then we write

$$(\varphi_0, \ldots, \varphi_{m-1}) \subseteq (\psi_0, \ldots, \psi_{n-1})$$

to mean

$$\bigcup_{p \in S^{m,k}} \bigcap_{i \leq k} \varphi_{p_i} \subseteq \bigcup_{p \in S^{n,k}} \bigcap_{i \leq k} \psi_{p_i} \text{ for every } k < m \tag{5}$$

where $S^{r,k}$ $(r = m, n)$ is as in Proposition A.7.[22]

Definition A.9. A function c, defined on a subset S of an algebra $\mathcal{F}$ over W, that maps to $\mathbb{R}$ is called a *partial measure* if it satisfies the following properties:

1. $c(x) \geq 0$ for $x \in S$;
2. If $\varphi_0, \ldots, \varphi_{m-1}, \psi_0, \ldots, \psi_{n-1} \in S$ and
$$(\varphi_0, \ldots, \varphi_{m-1}) \subseteq (\psi_0, \ldots, \psi_{n-1}),$$
then
$$\sum_{k=0}^{m-1} c(\varphi_k) \leq \sum_{k=0}^{n-1} c(\psi_k);$$
3. $W \in S$ and $c(W) = 1$.

The following result is the point of introducing the above definitions.

Theorem A.10 (Horn and Tarski 1948)**.** Let c be a partial measure on a subset $\mathcal{F}$ of an algebra $\mathcal{A}$. Then there is a finitely additive probability function c^* on $\mathcal{A}$ that extends c.

[22] Note that if $m > n$, this condition implies $\bigcup_{p \in S^{m,k}} \bigcap_{i \leq k} \varphi_{p_i} = \bigcup_{p \in S^{n,k}} \bigcap_{i \leq k} \psi_{p_i} = \varnothing$ for $k \geq n$.

A.1.3 Proof

We now establish the necessity of coherence to avoid dominance.

Theorem 3.4. Let $\mathcal{F}$ be a countably infinite opinion set, $\mathscr{I}$ a generalized legitimate inaccuracy measure, and c an incoherent credence function. Then:

1. c is weakly dominated relative to $\mathscr{I}$ by a coherent credence function; and
2. if $\mathscr{I}(c, w) < \infty$ for each $w \in W$, then c is strongly dominated relative to $\mathscr{I}$ by a coherent credence function.

Proof. Let $\mathscr{I}$ be a generalized legitimate inaccuracy measure and thus defined by a Bregman distance $B_{\bar{\varphi},\mu}$ (see Remark A.2). We write B for $B_{\bar{\varphi},\mu}$. Let S be the set of non-negative functions on $\mathcal{F}$. Let $E \subseteq S$ be the set of coherent credence functions on $\mathcal{F}$. Then clearly E is convex.

Let c be an incoherent credence function.

Case 1: $\mathscr{I}(c, w) = \infty$ for all $w \in W$. Then since $\mathscr{I}(v_w, w) = 0$ for all $w \in W$, any omniscient credence function weakly dominates c.

Case 2: $\mathscr{I}(c, w') < \infty$ for some $w' \in W$. We show that there is a coherent credence function π_c such that

$$\mathscr{I}(c, w) > \mathscr{I}(\pi_c, w) \text{ for any } w \text{ such that } \mathscr{I}(c, w) < \infty.$$

Since $v_{w'} \in E$, we see that

$$B(E, c) \leq B(v_{w'}, c) = \mathscr{I}(c, w') < \infty.$$

Thus we can apply Theorem A.5 to get a $\pi_c \in S$ such that

$$B(s, t) \geq B(E, c) + B(s, \pi_c) \text{ for every } s \in E. \tag{6}$$

In particular, (6) holds when s is the omniscient credence function at world w for any $w \in W$; and so we see that

$$\mathscr{I}(c, w) \geq B(E, c) + \mathscr{I}(\pi_c, w) \tag{7}$$

for all w, where all numbers in (7) are finite whenever $\mathscr{I}(c, w) < \infty$.

Next we show that π_c is in fact coherent. This is due to the following claim: E is closed under loose convergence in μ-measure where μ is a weighted counting measure on $\mathcal{P}(\mathbb{N})$ defined with weights $\{a_i\}_{i=1}^{\infty}$. To see this, let $c_n \in E$ for each n and $c \in S$. Assume $c_n \to c$ loosely in μ-measure. We show $c \in E$, i.e., c is coherent. Note c is coherent on $\mathcal{F}$ if and only if $c' : \mathcal{F} \cup \{W\} \to [0, 1]$ is coherent on $\mathcal{F} \cup \{W\}$, where $c' = c$ on $\mathcal{F}$ and $c'(W) = 1$. Thus it suffices to assume c and c_n for all n are defined on $\mathcal{F} \cup \{W\}$ with $c(W) = c_n(W) = 1$ for all n.

It is easy to see that loose convergence in a weighted counting measure (where all weights are non-zero) implies pointwise convergence on $\mathcal{F}$, so

$$c(p) = \lim_{n\to\infty} c_n(p) \in [0, 1]$$

for each $p \in \mathcal{F} \cup \{W\}$. To show $c \in E$, it suffices to show c can be extended to a finitely additive probability function on $\mathcal{P}(W)$.

We first show c is a partial measure on $\mathcal{F} \cup \{W\}$. Definitions A.9.1 and A.9.3 clearly hold for c so we just need to show Definition A.9.2 holds. Let $\varphi_0, \ldots, \varphi_{m-1}, \psi_0, \ldots, \psi_{m'-1} \in \mathcal{F} \cup \{W\}$ and

$$\bigcup_{p \in S^{m,k}} \bigcap_{i \leq k} \varphi_{p_i} \subseteq \bigcup_{p \in S^{m',k}} \bigcap_{i \leq k} \psi_{p_i}$$

for every $k < m$. Since the c_n are coherent and thus extend to finitely additive probability functions on algebras containing $\mathcal{F}$, we have by Proposition A.7 and Remark A.6 that

$$\sum_{k=0}^{m-1} c_n(\varphi_k) = \sum_{k=0}^{m-1} c_n(\bigcup_{p \in S^{m,k}} \bigcap_{i \leq k} \varphi_{p_i}) \leq \sum_{k=0}^{m'-1} c_n(\bigcup_{p \in S^{m',k}} \bigcap_{i \leq k} \psi_{p_i}) = \sum_{k=0}^{m'-1} c_n(\varphi_k)$$

using that

$$\bigcup_{p \in S^{m,k}} \bigcap_{i \leq k} \psi_{p_i} = \bigcup_{p \in S^{m',k}} \bigcap_{i \leq k} \psi_{p_i} = \varnothing$$

for $k \geq m'$. Sending n to infinity and using the pointwise convergence of c_n to c on $\mathcal{F} \cup \{W\}$ we obtain that

$$\sum_{k=0}^{m-1} c(\varphi_k) \leq \sum_{k=0}^{m'-1} c(\psi_k).$$

Thus c is a partial measure on $\mathcal{F} \cup \{W\}$. By Theorem A.10, it follows that there is a finitely additive probability function c^* on an algebra $\mathcal{F}^* \supseteq \mathcal{F}$ that extends c and so $c \in E$, which concludes the proof that E is closed under loose μ-convergence.

By Theorem A.5, π_c is the generalized B-projection of c onto E. Also, since

$$B(E, c) = \inf_{s \in E}(s, c) < \infty,$$

there is a B-minimizing sequence $\{s_n\} \subseteq E$ such that $B(s_n, c) \to B(E, c)$ by the definition of infimum. By the definition of a generalized projection, $s_n \rightsquigarrow_\mu \pi_c$. Since E is closed under loose convergence, it follows that $\pi_c \in E$. Further, by Theorem A.5,

$$B(E, c) \geq B(\pi_c, c) > 0,$$

since $\pi_c \neq c$ (as c is incoherent) and $B(s, t) = 0$ if and only if $s = t$ (as μ is a weighted counting measure with all non-zero weights). So for every w such that $\mathscr{I}(c, w) < \infty$, we deduce that

$$\mathscr{I}(c, w) \geq B(E, c) + \mathscr{I}(\pi_c, w) > \mathscr{I}(\pi_c, w).$$

This proves that c is weakly dominated by π_c, and c is strongly dominated by π_c if $\mathscr{I}(c, w) < \infty$ for all $w \in W$. □

A.2 Proofs from Section 4

Proposition 4.7. Let $\mathcal{F}$ be a countably discriminating opinion set. Then a credence function c is countably coherent if and only if there are $\lambda_w \in [0,1]$ with $\sum_{v_w \in \mathcal{V}_\mathcal{F}} \lambda_w = 1$ such that

$$c(p) = \sum_{v_w \in \mathcal{V}_\mathcal{F}} \lambda_w v_w(p)$$

for all $p \in \mathcal{F}$.

Proof. We adapt the proof of Proposition 1 in Predd et al. 2009. Let $\mathcal{F} = \{p_1, p_2, \ldots\}$. Let $\mathcal{X}$ be the collection of all nonempty sets of the form $\bigcap_{i=1}^\infty p_i^*$ where p_i^* is either p_i or p_i^c. Then $\mathcal{X}$ partitions W. Also, $\mathcal{X}$ is in bijection with $\mathcal{V}_\mathcal{F}$, the set of omniscient credence functions.

Indeed, let f map v_w to $\bigcap_{i=1}^\infty p_i^*$ where $p_i^* = p_i$ if $v_w(p_i) = 1$ and $p_i^* = p_i^c$ otherwise. Then for each w, $w \in f(v_w)$ and so $f(v_w) \in \mathcal{X}$. Note f is onto. Indeed, let $w \in \bigcap_{i=1}^\infty p_i^*$, where $\bigcap_{i=1}^\infty p_i^* \in \mathcal{X}$. Then $f(v_w) = \bigcap_{i=1}^\infty p_i^*$. Also, f is injective. Indeed, assume $f(v_w) = f(v_{w'})$. Then

$$f(v_w) = \bigcap_{i=1}^\infty p_i^1 = \bigcap_{i=1}^\infty p_i^2 = f(v_{w'})$$

for $p_i^j = p_i$ or $p_i^j = p_i^c$ for all $i \in \mathbb{N}$ and $j \in \{1,2\}$. If $p_i^1 \neq p_i^2$ for some i, then without loss of generality we may assume $p_i^1 = p_i$ and $p_i^2 = p_i^c$. So $w \in p_i^1$ but $w \notin p_i^2$ and thus $w \notin \bigcap_{i=1}^\infty p_i^2$. But $w \in \bigcap_{i=1}^\infty p_i^1$ by definition of f and so $\bigcap_{i=1}^\infty p_i^1 \neq \bigcap_{i=1}^\infty p_i^2$, which is a contradiction. It follows that $p_i^1 = p_i^2$ for all i, but then by definition of f, this implies $v_w(p_i) = 1$ if and only if $v_{w'}(p_i) = 1$ for all i and so $v_w = v_{w'}$.

It is easy to see that since $\mathcal{F}$ is countably discriminating, $\mathcal{V}_\mathcal{F}$ is countable. It follows that $\mathcal{X}$ is countable. Enumerate the elements of $\mathcal{V}_\mathcal{F}$ and $\mathcal{X}$ by $v_{w_1}, v_{w_2}, \ldots$ and $e_1, e_2, \ldots$, respectively, such that $f^{-1}(e_j) = v_{w_j}$. We have that p_i is the disjoint union of e_j such that $e_j \subseteq p_i$, or equivalently the e_j where $f^{-1}(e_j)(p_i) = 1$. Note i) for any countably additive probability function μ on a σ-algebra containing $\mathcal{F}$ (and thus containing $\mathcal{X}$) and any $p_i \in \mathcal{F}$:

$$\mu(p_i) = \sum_{j=1}^\infty \mu(e_j) f^{-1}(e_j)(p_i).$$

Now we prove the equivalence. Assume c is countably coherent. So c extends to a countably additive probability function μ on a σ-algebra containing $\mathcal{F}$. Then by i),

$$c(p_i) = \mu(p_i) = \sum_{j=1}^\infty \mu(e_j) f^{-1}(e_j)(p_i)$$

for all $p_i \in \mathcal{F}$. But since $\mu(e_j)$ are non-negative and sum to 1 (since the e_j's partition W and μ is a countably additive probability function), we have that c has the form stated.

Now assume $c(p_i) = \sum_{j=1}^\infty \lambda_j v_{w_j}(p_i)$ for all i where $\sum_{j=1}^\infty \lambda_j = 1$. Let $\sigma(\mathcal{F})$ be the smallest σ-algebra on W containing $\mathcal{F}$. Then it is easy to check that the function on $\sigma(\mathcal{F})$ defined by $\bar{v}_{w_j}(p) = 1$ if and only if $w_j \in p$ extends v_{w_j} and is a countably additive probability function on $\sigma(\mathcal{F})$. Then $\sum_{j=1}^\infty \lambda_j \bar{v}_{w_j}$ is a countably additive probability function on $\sigma(\mathcal{F})$ since a countable sum of countably additive probability functions with coefficients that sum to 1 is a countably additive probability function. Since

$$c(p_i) = \sum_{j=1}^{\infty} \lambda_j v_{w_j}(p_i) = \sum_{j=1}^{\infty} \lambda_i \bar{v}_{w_j}(p_i)$$

for all i, it follows that c extends to a countably additive probability function on a σ-algebra containing $\mathcal{F}$. □

Lemma 4.12. For any opinion space $(W, \mathcal{F})$, $(W^*, \mathcal{F}^*)$ is compact.

Proof. Let $\{\Psi(p_n)^{f(n)}\}_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{F}^*$ or their complements as in Definition 4.9. Case 1: for each N there is some $w_N \in W$ such that $w_N \in \bigcap_{n=1}^{N} \Psi(p_n)^{f(n)}$. Then since i) $\Psi(p) \cap W = p$ and ii) $\Psi(p)^c \cap W = p^c$ for any $p \in \mathcal{F}$, it follows that $w_N \in \bigcap_{n=1}^{N} p_n^{f(n)}$ for each N. If there is some $w' \in W$ with $w' \in \bigcap_{n=1}^{\infty} p_n^{f(n)}$ then by i) and ii) it follows that $w' \in \bigcap_{n=1}^{\infty} \Psi(p_n)^{f(n)}$. Otherwise, by construction, we defined some x_s to be such that $x_s \in \bigcap_{n=1}^{\infty} \Psi(p_n)^{f(n)}$. In either case, we are done. Case 2: there is some N such that $\bigcap_{n=1}^{N} \Psi(p_n)^{f(n)} \subseteq W^* \backslash W$. I claim this implies that $\bigcap_{n=1}^{N} \Psi(p_n)^{f(n)} = \varnothing$. Indeed, if there were some $w \in W^* \setminus W$ such that $w \in \bigcap_{n=1}^{N} \Psi(p_n)^{f(n)}$, then that is because $\{p_n^{f(n)}\}_{n=1}^{N}$ is an initial sequence of some sequence $\{\bar{p}_n^{\bar{f}(n)}\}_{n=1}^{\infty}$ such that $\bigcap_{n=1}^{l} \bar{p}_n^{\bar{f}(n)} \neq \varnothing$ for each l and thus, in particular, $\bigcap_{n=1}^{N} p_n^{f(n)} \neq \varnothing$. So there is some $w \in W$ such that $w \in \bigcap_{n=1}^{N} \Psi(p_n)^{f(n)}$ by i) and ii), which is a contradiction. Thus we have established that $(W^*, \mathcal{F}^*)$ is compact. □

Lemma 4.13. Let $(W, \mathcal{F})$ be an opinion space and c a coherent credence function on $(W, \mathcal{F})$. Let $(W^*, \mathcal{F}^*)$ be the compactification of $(W, \mathcal{F})$ and define $c^*(\Psi(p)) := c(p)$ for each $p \in \mathcal{F}$. Then c^* is a countably coherent credence function on $(W^*, \mathcal{F}^*)$ and $\mathscr{I}(c, w) = \mathscr{I}(c^*, w)$ for $w \in W$.

Proof. Since $(W^*, \mathcal{F}^*)$ is compact, we only need to show that c^* is coherent by Theorem 4.11. Thus it suffices to show that c^* can be extended to a finitely additive probability function on $\mathcal{A}(\mathcal{F}^*)$. Since c is coherent, there is a finitely additive probability function $\bar{c}$ such that:

1. $\bar{c}(p) = c(p)$ for $p \in \mathcal{F}$;
2. $\bar{c}(p \cup q) = \bar{c}(p) + \bar{c}(q)$ for $p, q \in \mathcal{F}$ with $p \cap q = \varnothing$;
3. $\bar{c}(W) = 1$.

First, define $\Psi(p^c) := \Psi(p)^c$ for each $p \in \mathcal{F}$. Then each element in $\mathcal{A}(\mathcal{F}^*)$ can be represented by $\bigcup_{i=1}^{N} \bigcap_{j=1}^{M} \Psi(q_{ij})$ where q_{ij} or its complement is in $\mathcal{F}$. We define

$$\bar{c^*}(\bigcup_{i=1}^{N} \bigcap_{j=1}^{M} \Psi(q_{ij})) := \bar{c}(\bigcup_{i=1}^{N} \bigcap_{j=1}^{M} q_{ij}).$$

Using that $p = \Psi(p) \cap W$ and $p^c = \Psi(p)^c \cap W$, we show that $\bar{c^*}$ is a well-defined finitely additive probability function on $\mathcal{A}(\mathcal{F}^*)$ extending c^*. We first show $\bar{c^*}$ is well-defined. Assume that

$$\bigcup_{i=1}^{N} \bigcap_{j=1}^{M} \Psi(q_{ij}) = \bigcup_{i=1}^{N'} \bigcap_{j=1}^{M'} \Psi(r_{ij}).$$

Then this clearly implies that

$$\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}\Psi(q_{ij})\cap W=\bigcup_{i=1}^{N'}\bigcap_{j=1}^{M'}\Psi(r_{ij})\cap W$$

which, noting that $p=\Psi(p)\cap W$ and $p^c=\Psi(p)^c\cap W$, establishes that

$$\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}q_{ij}=\bigcup_{i=1}^{N'}\bigcap_{j=1}^{M'}r_{ij},$$

and so

$$\bar{c^*}(\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}\Psi(q_{ij}))=\bar{c}(\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}q_{ij})=\bar{c}(\bigcup_{i=1}^{N'}\bigcap_{j=1}^{M'}r_{ij})=\bar{c^*}(\bigcup_{i=1}^{N'}\bigcap_{j=1}^{M'}\Psi(r_{ij})).$$

Thus $\bar{c^*}$ is well-defined. Clearly, $\bar{c^*}$ extends c^*. Now, since $W\subseteq W^*$, if

$$\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}\Psi(q_{ij})\cap\bigcup_{i=1}^{N'}\bigcap_{j=1}^{M'}\Psi(r_{ij})=\varnothing$$

then

$$\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}\Psi(q_{ij})\cap W\cap\bigcup_{i=1}^{N'}\bigcap_{j=1}^{M'}\Psi(r_{ij})\cap W=\varnothing$$

and so

$$\bar{c}(\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}q_{ij}\cup\bigcup_{i=1}^{N'}\bigcap_{j=1}^{M'}r_{ij})=\bar{c}(\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}q_{ij})+\bar{c}(\bigcup_{i=1}^{N'}\bigcap_{j=1}^{M'}r_{ij}).$$

Then noting the definition of $\bar{c^*}$ in terms of $\bar{c}$, we establish finite additivity. Lastly, if

$$W^*=\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}\Psi(q_{ij}),$$

then

$$W=\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}\Psi(q_{ij})\cap W,$$

and so

$$\bar{c^*}(\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}\Psi(q_{ij}))=\bar{c}(\bigcup_{i=1}^{N}\bigcap_{j=1}^{M}q_{ij})=\bar{c}(W)=1.$$

This establishes that c^* is coherent on $(W^*,\mathcal{F}^*)$, and so since $(W^*,\mathcal{F}^*)$ is compact, c^* is countably coherent. Further, $w\in p$ if and only if $w\in\Psi(p)$ for each $w\in W$, so v_w defined on $\mathcal{F}$ is the same as v_w defined on $\mathcal{F}^*$ for each $w\in W$. Since $c(p)=c^*(\Psi(p))$ for all $p\in\mathcal{F}$, this establishes that $\mathscr{I}(c,w)=\mathscr{I}(c^*,w)$ for each $w\in W$. □

Proposition 4.16. Let $(W,\mathcal{F})$ be an opinion space with $\mathcal{F}$ countably infinite and $\mathscr{I}$ a generalized legitimate inaccuracy measure. If c is a countably coherent credence function with finite expected inaccuracy, then c is not weakly dominated.

Proof. Since c is countably coherent, let $\bar{c}$ be a countably additive probability function on $\sigma(\mathcal{F})$ extending c such that $\mathbb{E}_{\bar{c}}\mathscr{I}(c,\cdot) < \infty$. Note that since $\mathfrak{d}$ is strictly proper (see Remark 2.7), we know that for any $i \in \mathbb{N}$, $\mathbb{E}_{\bar{c}}\mathfrak{d}(v_w, c_i) < \mathbb{E}_{\bar{c}}\mathfrak{d}(v_w, x)$ for $x \neq c_i$. Assume toward a contradiction that there is a credence function d with $d \neq c$ and $\mathscr{I}(d,w) \leq \mathscr{I}(c,w)$ for each w with strict inequality for some w. Then $\mathbb{E}_{\bar{c}}\mathscr{I}(d,\cdot) \leq \mathbb{E}_{\bar{c}}\mathscr{I}(c,\cdot) < \infty$, so both $\mathscr{I}(d,\cdot)$ and $\mathscr{I}(c,\cdot)$ are integrable with respect to the measure space $(W, \sigma(\mathcal{F}), \bar{c})$. Then let i be any index such that $d_i \neq c_i$. There must be at least one since $c \neq d$. Then

$$\mathbb{E}_{\bar{c}}\mathfrak{d}(v_w, c_i) < \mathbb{E}_{\bar{c}}\mathfrak{d}(v_w, d_i).$$

If $d_i = c_i$ then clearly $\mathbb{E}_{\bar{c}}\mathfrak{d}(v_w, c_i) = \mathbb{E}_{\bar{c}}\mathfrak{d}(v_w, d_i)$. So since $\mathbb{E}_{\bar{c}}\mathscr{I}(c,\cdot) < \infty$ and $\mathbb{E}_{\bar{c}}\mathscr{I}(d,\cdot) < \infty$, we have

$$\mathbb{E}_{\bar{c}}\mathscr{I}(c,\cdot) = \sum_{i=1}^{\infty} a_i \mathbb{E}_{\bar{c}}\mathfrak{d}(v_w, c_i) < \sum_{i=1}^{\infty} a_i \mathbb{E}_{\bar{c}}\mathfrak{d}(v_w, d_i) = \mathbb{E}_{\bar{c}}\mathscr{I}(d,\cdot),$$

which implies that $\mathbb{E}_{\bar{c}}(\mathscr{I}(c,\cdot) - \mathscr{I}(d,\cdot)) < 0$. Thus there is some nonempty set $E \in \sigma(\mathcal{F})$ with $\bar{c}(E) > 0$ on which $\mathscr{I}(c,\cdot) - \mathscr{I}(d,\cdot) < 0$ (since the Lebesgue integral is positive). But this contradicts our assumption that d weakly dominates c, and so we are done. $\square$

Proposition 4.17. Let $(W, \mathcal{F})$ be a point-finite opinion space with $\mathcal{F}$ countably infinite and $\mathscr{I}$ a generalized legitimate inaccuracy measure. If a credence function c is countably coherent and somewhere finitely inaccurate relative to $\mathscr{B}$, then c is not weakly dominated relative to $\mathscr{B}$.

Proof. Assume d weakly dominates c. Note i) c is somewhere finitely inaccurate if and only if $\mathscr{B}(c,w) < \infty$ for all $w \in W$ if and only if $\sum_{i=1}^{\infty} c_i^2 < \infty$. It follows by weak dominance that $\mathscr{B}(d,w) < \infty$ for all $w \in W$ and therefore $\sum_{i=1}^{\infty} d_i^2 < \infty$. Let $\mathscr{B}(c,w) = \mathfrak{D}(v_w, c)$ for $\mathfrak{D}$ a generalized quasi-additive Bregman divergence.

Since $(W, \mathcal{F})$ is point-finite, it is also countably discriminating as there are only countably many finite subsets of $\mathcal{F}$. So by Proposition 4.7, $c = \sum_{j=1}^{\infty} \lambda_j v_{w_j}$ for $\lambda_j \in [0,1]$ with $\sum_{j=1}^{\infty} \lambda_j = 1$. First, note that $\mathfrak{D}(\sum_{j=1}^{\infty} \lambda_j v_{w_j}, c) = 0$ and $(\sum_{j=1}^{\infty} \lambda_j v_{w_j}(p_i) - c(p_i))^2 = 0$ for all i, so

$$\mathfrak{D}(\sum_{j=1}^{\infty} \lambda_j v_{w_j}, c) - \mathfrak{D}(\sum_{j=1}^{\infty} \lambda_j v_{w_j}, d) = \sum_{i=1}^{\infty} a_i (\sum_{j=1}^{\infty} \lambda_j v_{w_j}(p_i) - c_i)^2 - (\sum_{j=1}^{\infty} \lambda_j v_{w_j}(p_i) - d_i)^2.$$

Using that

$$(\sum_{j=1}^{\infty} \lambda_j v_{w_j}(p_i) - c_i)^2 - (\sum_{j=1}^{\infty} \lambda_j v_{w_j}(p_i) - d_i)^2 = \sum_{j=1}^{\infty} \lambda_j [(v_{w_j}(p_i) - c_i)^2 - (v_{w_j}(p_i) - d_i)^2]$$

for each i since $\sum_{j=1}^{\infty} \lambda_j = 1$, we have that

$$\mathfrak{D}(\sum_{j=1}^{\infty} \lambda_j v_{w_j}, c) - \mathfrak{D}(\sum_{j=1}^{\infty} \lambda_j v_{w_j}, d) = \sum_{i=1}^{\infty} a_i \sum_{j=1}^{\infty} \lambda_j [(v_{w_j}(p_i) - c_i)^2 - (v_{w_j}(p_i) - d_i)^2] \tag{8}$$

$$= \sum_{i=1}^{\infty} a_i (\sum_{j:w_j \notin p_i} \lambda_j)(c_i^2 - d_i^2) + a_i (\sum_{j:w_j \in p_i} \lambda_j)((1 - c_i)^2 - (1 - d_i)^2)$$

$$= \sum_{i=1}^{\infty} a_i (c_i^2 - d_i^2) + 2a_i (\sum_{j:w_j \in p_i} \lambda_j)(d_i - c_i)$$

$$= \sum_{i=1}^{\infty} a_i (-c_i^2 - d_i^2) + 2a_i (\sum_{j:w_j \in p_i} \lambda_j) d_i$$

since $c_i = \sum_{j:w_j \in p_i} \lambda_j$. We have $\sum_{i=1}^{\infty} c_i^2 + d_i^2 < \infty$ by i). Thus

$$0 \leq \sum_{i=1}^{\infty} a_i 2(\sum_{j:w_j \in p_i} \lambda_j) d_i < \infty \tag{9}$$

because

$$0 \geq \mathfrak{D}(\sum_{j=1}^{\infty} \lambda_j v_{w_j}, c) - \mathfrak{D}(\sum_{j=1}^{\infty} \lambda_j v_{w_j}, d).$$

Having established (9), we claim we can use the dominated convergence theorem (see, e.g., Theorem 1.4.49 in Tao 2011) to switch limits in (8). Indeed,

$$\sum_{i=1}^{\infty} a_i \sum_{j=1}^{N} \lambda_j [(v_{w_j}(p_i) - c_i)^2 - (v_{w_j}(p_i) - d_i)^2] = \sum_{i=1}^{\infty} a_i (\sum_{1 \leq j \leq N} \lambda_j)(c_i^2 - d_i^2) + 2a_i (\sum_{\substack{j:w_j \in p_i \\ 1 \leq j \leq N}} \lambda_j)(d_i - c_i).$$

Letting

$$g_N(i) = a_i (\sum_{1 \leq j \leq N} \lambda_j)(c_i^2 - d_i^2) + 2a_i (\sum_{\substack{j:w_j \in p_i \\ 1 \leq j \leq N}} \lambda_j)(d_i - c_i)$$

and noting that $-(\sum_{\substack{j:w_j \in p_i \\ 1 \leq j \leq N}} \lambda_j) c_i \geq -c_i^2$ since $c_i = \sum_{j:w_j \in p_i} \lambda_j$, we see that

$$|g_N(i)| \leq a_i (2c_i^2 + d_i^2 + 2a_i (\sum_{j:w_j \in p_i} \lambda_j) d_i).$$

Each of c_i, d_i, and $(\sum_{j:w_j \in p_i} \lambda_j) d_i$ is summable in i and $\sup_i a_i < \infty$. So, the dominated convergence theorem applies, and we can switch limits.

Thus we have

$$\begin{aligned}0 &\geq \mathfrak{D}(\sum_{j=1}^{\infty}\lambda_j v_{w_j}, c) - \mathfrak{D}(\sum_{j=1}^{\infty}\lambda_j v_{w_j}, d) \\ &= \sum_{j=1}^{\infty}\lambda_j \sum_{i=1}^{\infty} a_i[(v_{w_j}(p_i) - c(p_i))^2 - (v_{w_j}(p_i) - d(p_i))^2] \\ &= \sum_{j=1}^{\infty}\lambda_j(\mathfrak{D}(v_w, c) - \mathfrak{D}(v_w, d)) \geq 0\end{aligned}$$

where we used that c and d are both finitely inaccurate for each $w \in W$ to break up the summation in the second line. Thus we conclude that $c = d$, as $\mathfrak{D}(c, d) = 0$ if and only if $c = d$. □

Lemma 4.25. A partition is W-stable relative to any generalized legitimate inaccuracy measure.

Proof. Let $\mathcal{F} = \{p_1, p_2, \ldots\}$ be a partition. Assume a coherent credence function c on $\mathcal{F}$ is weakly dominated by some credence function d. We can assume d is coherent by Theorem 3.4, and so $\sum_{m=1}^{\infty} d_m \leq 1$. We show that c^* is weakly dominated by d^*, thereby establishing that a partition is W-stable.

First, $\mathscr{I}(c^*, w) = \mathscr{I}(c, w)$ and $\mathscr{I}(d^*, w) = \mathscr{I}(d, w)$ for all $w \in W$ by Lemma 4.13. Thus by assumption of weak dominance,

$$\mathscr{I}(c^*, w) \geq \mathscr{I}(d^*, w) \text{ for all } w \in W$$

with a strict inequality for some $w \in W$. We therefore need to only check what happens for $w \in W^* \setminus W$. The compactification of a partition consists in adding one point w^* which is in the complement of all $p^* \in \mathcal{F}^*$. So $W^* = W \cup w^*$ and

$$\mathscr{I}(c^*, w^*) - \mathscr{I}(d^*, w^*) = \sum_{m=1}^{\infty} a_m(\varphi(d_m) - \varphi'(d_m)d_m) - \sum_{m=1}^{\infty} a_m(\varphi(c_m) - \varphi'(c_m)c_m),$$

which I claim is greater than or equal to 0. Indeed, assume toward a contradiction that

$$\sum_{m=1}^{\infty} a_m(\varphi(d_m) - \varphi'(d_m)d_m) < \sum_{m=1}^{\infty} a_m(\varphi(c_m) - \varphi'(c_m)c_m).$$

Then since $d_n \to 0$ as $\sum_{n=1}^{\infty} d_n \leq 1$ and $c_n \to 0$ as $\sum_{n=1}^{\infty} c_n \leq 1$, $\varphi'(d_n) - \varphi'(c_n) \to 0$ (since $\varphi'(0) = \lim_{x\to 0}\varphi(x) = 0$) and so we can find a K such that

$$|\varphi'(d_n) - \varphi'(c_n)| < |\sum_{m=1}^{\infty} a_m(\varphi(d_m) - \varphi'(d_m)d_m) - \sum_{m=1}^{\infty} a_m(\varphi(c_m) - \varphi'(c_m)c_m)|$$

for $n \geq K$. Thus for $n \geq K$,

$$\mathscr{I}(c, w_n) - \mathscr{I}(d, w_n) = \sum_{m=1}^{\infty} a_m(\varphi(d_m) - \varphi'(d_m)d_m) - \sum_{m=1}^{\infty} a_m(\varphi(c_m) - \varphi'(c_m)c_m) + \varphi'(d_n) - \varphi'(c_n) < 0,$$

contradicting that d weakly dominates c. So indeed, d^* weakly dominates c^*. □

Proposition 4.27. Let $(W, \mathcal{F})$ be a compact opinion space and $\mathscr{I}$ a generalized legitimate inaccuracy measure. If c is coherent (and thus countably coherent), then c is not strongly dominated relative to $\mathscr{I}$.

Proof. Let $\mathscr{I}_n(c', w) := \sum_{i=1}^{n} a_i \mathfrak{d}(v_w(p_i), c'(p_i))$ for each $n \in \mathbb{N}$, $w \in W$, and credence function c' on $\mathcal{F}$. Consider a credence function $d \neq c$. Define

$$T^n = \{(v_w(p_1), \dots, v_w(p_n)) : \mathscr{I}_k(c, w) < \mathscr{I}_k(d, w) \text{ for some } k \geq n, w \in W\}$$

and $T = \{e\} \cup \bigcap_{n=1}^{\infty} T^n$, where e is the empty sequence. For each $s, t \in T$, we set $s < t$ if and only if s is an initial sequence of t, and we set the height of $t \in T$ to be the length of the tuple. Then T is a binary tree.

We claim T is infinite. Fix $n \in \mathbb{N}$. Then there is a $t \in T$ with height n if and only if $T^n \neq \varnothing$ if and only if $\mathscr{I}_k(c, w) < \mathscr{I}_k(d, w)$ for some $k \geq n$ and $w \in W$. Let k be the maximum of n and the smallest i such that $c(p_i) \neq d(p_i)$. Then since c restricted to any subset of $\mathcal{F}$ is coherent, by Theorem 2.9, $\mathscr{I}_k(c, w') < \mathscr{I}_k(d, w')$ for some $w' \in W$ and so $(v_{w'}(p_1), \dots, v_{w'}(p_n)) \in T^n$.

By Konig's lemma (see, e.g., Hrbacek and Jech 1999, Sec. 12.3), there exists an infinite branch

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \{(v_{w_n}(p_1), \dots, v_{w_n}(p_n))\}$$

through T, where

$$(v_{w_n}(p_1), \dots, v_{w_n}(p_n)) < (v_{w_m}(p_1), \dots, v_{w_m}(p_m))$$

whenever $n < m$. For each i, let $p_i^* = p_i$ if $v_{w_i}(p_i) = 1$ and $p_i^* = p_i^c$ if $v_{w_i}(p_i) = 0$. Then $w_n \in \bigcap_{i=1}^{n} p_i^*$ since $v_{w_i}(p_i) = 1$ if and only if $v_{w_n}(p_i) = 1$ for $i < n$ as $(v_{w_i}(p_1), \dots, v_{w_i}(p_i)) < (v_{w_n}(p_1), \dots, v_{w_n}(p_n))$. Thus $\bigcap_{i=1}^{n} p_i^* \neq \varnothing$ for each n and so by compactness there is some $w \in \bigcap_{i=1}^{\infty} p_i^*$. Then

$$(v_w(p_1), \dots, v_w(p_n)) = (v_{w_n}(p_1), \dots, v_{w_n}(p_n)) \in T^n$$

for each $n \in \mathbb{N}$. By the definition of T^n, for each $n \in \mathbb{N}$ we have

$$\mathscr{I}_{k_n}(c, w) < \mathscr{I}_{k_n}(d, w)$$

for some $k_n \geq n$. Sending n to infinity, $\mathscr{I}(c, w) \leq \mathscr{I}(d, w)$ and thus d does not strongly dominate c. □

A.3 Proof of Theorem 5.3

Theorem 5.3. Let $\mathscr{I}$ be an integral inaccuracy measure on a finite measure space $(\mathcal{F}, \mathcal{A}, \mu)$. Then for every μ-credence function c, if c is μ-incoherent, then there is a μ-coherent μ-credence function c' that strongly dominates c relative to $\mathscr{I}$.

Proof. Let $\mathscr{I}(c, w) = B_{\varphi,\mu}(v_w, c)$. We write B for $B_{\varphi,\mu}$. Let S be the set of non-negative $\mathcal{A}$-measurable functions on $\mathcal{F}$. Let $E \subseteq S$ be the set of μ-coherent μ-credence functions over $\mathcal{F}$. Then E is convex. Let c be a μ-incoherent μ-credence function. Because μ is finite and $\mathfrak{d}$ is bounded,

$$B(E,c) < \infty.$$

Thus we can apply Theorem A.5 to get a $\pi_c \in S$ such that

$$B(s,c) \geq B(E,c) + B(s,\pi_c) \text{ for every } s \in E. \tag{10}$$

In particular, (10) holds when s is the omniscient credence function at world w for each w, so we obtain

$$\mathscr{I}(c,w) \geq B(E,c) + \mathscr{I}(\pi_c,w) \tag{11}$$

for all w, where all numbers in (11) are finite. We show that π_c is in fact a μ-coherent μ-credence function. It suffices to show that π_c is μ-a.e. equal to a coherent credence function on $\mathcal{F}$ (since $\pi_c \in S$, it is $\mathcal{A}$-measurable). To do so, we prove the following claim: E is closed under loose-convergence in μ-measure.

To see this, let $c_n \in E$ for each n and $c \in S$. Assume $c_n \to c$ loosely in μ-measure. The first thing to notice is that, since μ is finite, loose μ-convergence implies μ-a.e. convergence on a subsequence $\{a_n\}_{n=1}^{\infty}$ of $\{n\}_{n=1}^{\infty}$,[23] so that

$$c(p) = \lim_{n\to\infty} c_{a_n}(p) \in [0,1]$$

for each $p \in \mathcal{G}$ with $\mu(\mathcal{G}^c) = 0$. Since the c_{a_n} are μ-coherent, we can change each c_{a_n} on a (measurable) measure zero set $\mathcal{X}_n$ to get coherent μ-credence functions c_{a_n}. Further, we replace $\mathcal{G}$ with $\mathcal{G} \setminus (\cup_{n=1}^{\infty}\mathcal{X}_n)$. Assuming these adjustments have been made, we have that $c_{a_n} \to c$ on $\mathcal{G}$ with $\mu(\mathcal{G}^c) = 0$, and each c_{a_n} is coherent. We now show $c \in E$ by showing it is equal to a coherent credence function on $\mathcal{F}$ when restricting to $\mathcal{G}$.

First, we extend c (resp. c_{a_n}) to $\overline{c}$ (resp. $\overline{c_{a_n}}$), where $\overline{c}$ (resp. $\overline{c_{a_n}}$) is a credence function on $\mathcal{G} \cup \{W\}$ such that $c = \overline{c}$ (resp. $c_{a_n} = \overline{c_{a_n}}$) on $\mathcal{G}$ and $\overline{c}(W) = 1$ (resp. $\overline{c_{a_n}}(W) = 1$). Then notice that c (resp. c_{a_n}) is coherent on $\mathcal{G}$ if and only if $\overline{c}$ (resp. $\overline{c_{a_n}}$) is coherent on $\mathcal{G} \cup \{W\}$. Thus we work with $\overline{c}$ and $\overline{c_{a_n}}$ instead noting that $\overline{c} = \lim_n \overline{c_{a_n}}$ on $\mathcal{G} \cup \{W\}$. To show $\overline{c} \in E$, we first show $\overline{c}$ is a partial measure on $\mathcal{G} \cup \{W\}$.

Definitions A.9.1 and A.9.3 clearly hold for $\overline{c}$ so we just need to show that Definition A.9.2 holds. Let $\varphi_0, \ldots, \varphi_{m-1}, \psi_0, \ldots, \psi_{m'-1} \in \mathcal{G} \cup \{W\}$ and

$$\bigcup_{p\in S^{m,k}} \bigcap_{i\leq k} \varphi_{p_i} \subseteq \bigcup_{p\in S^{m',k}} \bigcap_{i\leq k} \psi_{p_i}$$

for every $k < m$. Since $\overline{c_{a_n}}$ are coherent on $\mathcal{G} \cup \{W\}$ and thus extend to measures on an algebra containing $\mathcal{G} \cup \{W\}$, we have by Corollary A.7 that

$$\sum_{k=0}^{m-1} \overline{c_{a_n}}(\varphi_k) = \sum_{k=0}^{m-1} \overline{c_{a_n}}(\bigcup_{p\in S^{m,k}} \bigcap_{i\leq k} \varphi_{p_i}) \leq \sum_{k=0}^{m'-1} \overline{c_{a_n}}(\bigcup_{p\in S^{m',k}} \bigcap_{i\leq k} \psi_{p_i}) = \sum_{k=0}^{m'-1} \overline{c_{a_n}}(\psi_k)$$

[23] It is a standard fact that convergence in measure implies a.e. convergence on a subsequence. Now notice that loose convergence implies convergence in measure when the measure is finite.

using that

$$\bigcup_{p\in S^{m,k}}\bigcap_{i\leq k}\varphi_{p_i}=\bigcup_{p\in S^{m',k}}\bigcap_{i\leq k}\psi_{p_i}=\varnothing$$

for $k\geq m'$. Sending n to infinity and using the pointwise convergence of $\overline{c_{a_n}}$ to $\overline{c}$ on $\mathcal{G}\cup\{W\}$ we conclude that

$$\sum_{k=0}^{m-1}\overline{c}(\varphi_k)\leq\sum_{k=0}^{m'-1}\overline{c}(\psi_k).$$

Thus $\overline{c}$ is a partial measure on $\mathcal{G}\cup\{W\}$. By Theorem A.10, it follows that there is a finitely additive probability function c^* on $\mathcal{A}(\mathcal{F})$ such that $c^*=\overline{c}$ on $\mathcal{G}\cup\{W\}$. Thus $c^*|_{\mathcal{F}}$ is a coherent credence function on $\mathcal{F}$ and

$$c=\overline{c}|_{\mathcal{F}}=c^*|_{\mathcal{F}}$$

μ-a.e. (specifically off $\mathcal{G}^c$). Further, we already assumed c is $\mathcal{A}$-measurable and $\{p:c(p)\in[0,1]\}\subseteq\mathcal{G}$. Thus c is a μ-coherent μ-credence function.

The proof is finished just as in the proof of Theorem 3.4. By Theorem A.5, π_c is the generalized projection of c onto E. Since

$$B(E,c)=\inf_{s\in E}(s,c)<\infty$$

there is a B-minimizing sequence $\{s_n\}$ of elements in E such that $B(s_n,c)\to B(E,c)$ by the definition of infimum. By the definition of a generalized projection, $s_n\rightsquigarrow_\mu\pi_c$. Since E is closed under loose convergence, it follows that $\pi_c\in E$. Further, since c is μ-incoherent we know $c\neq\pi_c$ (up to μ-a.e. equivalence) so we see $B(E,c)\geq B(\pi_c,c)>0$ since $B(s,t)=0$ if and only if $s=t$ μ-a.e. Since $\mathscr{I}(c,w)<\infty$ for all w, we deduce that

$$\mathscr{I}(c,w)\geq B(E,c)+\mathscr{I}(\pi_c,w)>\mathscr{I}(\pi_c,w)$$

for all $w\in W$. This proves that c is strongly dominated by π_c, and we are done. □

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