Hume against the Geometers: Extension and Geometry in Hume's Treatise

In the initial sections of Book I, Part II of the *Treatise of Human Nature*[1], David Hume mounts a spirited assault on the doctrine of the infinite divisibility of extension, and defends in its place the contrary position that extension is everywhere only finitely divisible. Later, in Treatise 1.2.4, Hume considers several potential objections to his views on divisibility. The third group of objections is, as he says, "drawn from the mathematics", and in the process of defending himself against these objections Hume presents the reader with an intricate philosophical appraisal of the science of geometry, the branch of mathematics that is the source of the objections.

Here are two reasonable initial questions one might ask about Hume's defense of finitely divisible extension against the opposed geometrical arguments for infinite divisibility:

1. *Is Hume's account of the spatial composition of extension viable?* Do Hume's own positive assertions about points, contiguity, distance, divisibility and related matters form a logically consistent and materially adequate whole?
2. *Does Hume have an effective response to the traditional geometers?* Does Hume's philosophical and methodological appraisal of geometry provide him with adequate reasons for dismissing the geometers' arguments for infinite divisibility?

By a "materially adequate" account of the composition of extension I mean an account that respects those ordinary observations about extension and its measurement that we cannot

[1] All paragraph references are to David Fate Norton & Mary Norton (eds.) David Hume: *A Treatise of Human Nature: Volume 1: Texts*. Clarendon Press (2007). Accompanying page references are given for Nidditch, P. H. & Selby-Bigge, (eds.) (1978). *David Hume, A Treatise of Human Nature*. Oxford. (SBN)

reasonably reject. So, a *viable* account of the composition of extension would be one that deserves to be taken seriously as an account of the world of extension as we experience it: a geometry that is both internally consistent and true to the facts of experience.

One might think that a positive answer to the first question would go at least part of the way toward explaining how Hume proposes to answer to the second question. For suppose Hume has a logically consistent account of the spatial composition of extended objects that is compatible with undeniable facts about extension as it is experienced. Then this account could provide the basis for an alternative geometry. Appealing to that alternative geometry, Hume could simply reject those axioms and derived theorems of traditional geometry that are not consistent with the revised geometry. Since Hume's positive views on space and extension in the Treatise have struck most readers as a very radical departure from the more conventional conceptions of space that are embodied in Euclidean geometry, a Humean replacement geometry would seem to require a deep and thoroughgoing revision of the traditional Euclidean framework.

But Hume does not appear to contemplate any radical, revisionary strategy for dealing with traditional geometry. This point has been emphasized in an article by Emil Badici.[2] Badici notes that Hume espouses a more conservative approach to geometry. While mounting a series of global arguments aimed at countering the geometric arguments for infinite divisibility as a group, Hume suggests that geometry as standardly understood fails only "in this single point" – that is, in its purported proofs of infinite divisibility – while "all of its other arguments" remain

[2] Emil Badici, "On the Compatibility between Euclidean Geometry and Hume's Denial of Infinite Divisibility", *Hume Studies* 34 (2010): 231-44.

intact, and “command our fullest assent and approbation.” (T 1.2.4.32, SBN 52). We may call this position *Geometric Conservatism*:

> *Geometric Conservatism*: The traditional geometric proofs of infinite divisibility are unsound. All other proofs in traditional geometry – those that are not proofs of infinite divisibility – are sound, and they justify our full assent.

I agree with Badici that this is the approach Hume attempts to develop in the Treatise. But I will argue that the attempt fails. Hume’s views about space pose a far more radical and extensive challenge to traditional Euclidean geometry than he seems prepared to acknowledge. If Hume is right about the finite divisibility of extension, then there are many proofs in traditional Euclidean geometry that are not proofs of infinite divisibility but yet are unsound, and do not command assent.

In the first section of this paper, I will develop the case for thinking that Hume’s positive account of the composition of extension in the Treatise poses a radical challenge to traditional geometry. We will see that it is unclear that Hume has provided us with an internally coherent positive account of extension as finitely divisible, but that it is also very likely that if such an account exists, it must diverge significantly from traditional geometry. In the remaining sections of the paper, I will consider five strategies for interpreting Hume’s appraisal of geometry and the traditional geometrical arguments for infinite divisibility. All five strategies have some claim to be found in Hume’s complex account of geometry in the text of the Treatise, although for none of them is the textual case decisive. We will see that each of these interpretive strategies, whether it truly captures Hume’s intentions or not, suffers from serious substantive problems, and so none of them delivers an interpretation of Hume's account that provides him with a way of blocking the geometric arguments for infinite divisibility while maintaining Geometric Conservatism.

1. The *Prima Facie* Case for a Radical Challenge to Euclidean Geometry

Hume's account of the nature of space in the Treatise is presented as part of a more general system concerning space and time, and concerning also our ideas of space and time.[3] For Hume, the ideas we form of extension are complex ideas that consist of other ideas that are their parts. Some of the parts of an idea of extension will themselves be ideas of extension. But no idea of extension, according to Hume, consists of an infinite number of such parts.[4] Nor, consequently, is it the case that *every* part of an idea of extension is composed of further proper parts. Rather every idea of extension is composed of a finite number of simple and indivisible ultimate parts. These ultimate parts, since they are indivisible and consist of no parts themselves, are not extended. They are instead unextended points or minima.[5]

Hume argues that it is at least *possible* for space itself to exist in conformity to our ideas of space (T 1.2.2.8-9, SBN 32 and T1.2.4.1, SBN 39). It is thus possible, he holds, that every real

[3] For more complete, and excellent, accounts of Hume's system than I will present here, see Donald L. M. Baxter, "Hume's theory of space and time in its sceptical context", in David Fate Norton & Jacqueline Anne Taylor (eds.), *The Cambridge Companion to Hume*, Cambridge University Press (2009), and Lorne Falkenstein "Space and time", in Saul Traiger (ed.), *The Blackwell Guide to Hume's Treatise*; Blackwell (2006).

[4] The parts of extension and of ideas of extension are not, for Hume, even "potentially infinite". Hume holds that if an extension A can be divided into extensions B and C, then B and C exist and are already parts of A before the division. For discussion, see Thomas Holden, "Infinite Divisibility and Actual Parts in Hume's Treatise", *Hume Studies*, 28 (2002): 3-25.

[5] Throughout his discussion in T 1.2, Hume assumes that the commitment to the infinite divisibility of extension entails the denial of the existence of indivisible points. This assumption is clearly at work in both T 1.2.2.9, SBN 32 and T 1.2.2.10, SBN 33. The assumption runs counter to the standard contemporary account of extended continua. A classic defense of this contemporary view is found in Adolf Grunbaum, "A Consistent Conception of the Extended Linear Continuum as an Aggregate of Unextended Elements", *Philosophy of Science*, 19 (1952): 288 - 306.

extension consists of a finite number of indivisible parts. But he also argues that the contrary supposition that space consists of infinitely many parts is “utterly impossible and contradictory” (T 1.2.4.1, SBN 39). So, he concludes, it is “certain” space is only finitely divisible. This last part of Hume’s thinking about space is often overlooked, but is worth setting out explicitly:

> *Apodictic Finitism*: The supposition that space is infinitely divisible is impossible and contradictory, and the contrary supposition that space is only finitely divisible can be known with certainty.

As we will see in the second half of this paper, Apodictic Finitism places an additional burden on Hume’s attempt to defend Geometric Conservatism.

Hume’s views about space seem, on their face, to conflict directly and in several obvious ways with the view of space that is embodied in Euclidean geometry, and it is therefore natural to read Hume’s discussion as a defense of the rudiments of some alternative geometry. Lorne Falkenstein and Donald Baxter have both characterized Hume as defending a *discrete geometry*.[6] Falkenstein has also described Hume as endorsing a *finite* or *finitistic* geometry[7]. But clearly, Hume does not explicitly advance any alternative geometry in the Treatise in the form of a rigorous and comprehensive deductive science of space or extension, reduced to some limited set of axioms, postulates and common notions. To any extent, then, that Hume is defending an alternative geometry in the Treatise, that alternative is only implicit in the text.

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[6] See Falkenstein (2006), pp. 62-4 and Baxter (2009), p.123, n. 31.

[7] Lorne Falkenstein, “Hume's Finite Geometry: A Reply to Mark Pressman”, *Hume Studies* 26 (2000): 183-185. Falkenstein uses both terms, “finite” and “finitistic”, in that paper.

But it is by no means obvious what such an alternative geometry would look like when fully developed. Falkenstein has done the most to advance some explicit suggestions along these lines. He proposes the beginnings of one type of model of a discrete geometry of dimension n: the set of n-tuples of integers; that is, the sets Z^n.[8] However, such a set would not itself be a model of a geometry, but only the domain of such a model. The set does not become a model until explicit suggestions are made about how to interpret fundamental geometrical predicate expressions such as “line”, “line segment” “between”, “circle”, “angle”, etc. within that domain. Also, a Humean discrete geometry must include an interpretation of the additional notion of *contiguity* for points, a concept that has no application to the points in Euclidean geometry but is central to Hume’s outlook.[9]

One fundamental geometrical relation is *length*, and conflicting accounts have been offered of how length is to be construed in a Humean discrete geometry. Consider the line consisting of six consecutive and contiguous points: ABCDEF. On one account – call it the *direct enumeration account* - the length of a line goes by the number of points it contains, so the line ABCDEF cleanly decomposes into two parts, ABC and DEF, that are each half the length of the whole line. Donald Baxter, however, interprets Hume as holding that the length of a line goes by summing the number of consecutive minimal extensions in the line.[10] Since, for Hume, no single point is extended, and the smallest possible extension must contain at least two points, then according to Baxter “one must resist the temptation to think that two minimal ideas are twice as long as one.” Instead, Baxter says, when we put two points together contiguously, we

[8] Falkenstein (2006), p. 63
[9] Deciding how Hume understands “contiguity” is essential to interpreting the argument in T 1.2.4.6, SBN 41.
[10] Baxter (2009), p. 109.

get the minimum possible length, and when we add a third point, “the original length doubles”. So, returning to the line ABCDEF, each of the parts ABC and DEF is only 2/5th the length of the whole. Thus, if the physical line ABCDEF is separated cleanly into two parts, without loss of parts, by moving C and D further apart so that they are no longer contiguous, then the sum of the lengths of the resulting parts is only 4/5ths the length of the original. That seems like an unattractive and mysterious consequence, and it would play havoc with any application of geometry to the computation of lengths, areas and volumes, where the computation often depends on decomposing the object to be measured into a finite number of smaller parts, obtaining measures for the parts, and then summing those measures into a measure for the whole object. The direct enumeration account avoids this consequence, but it seems to have the result that the size of A alone is 1/6th the size of the whole, and this in turn seems incompatible with Hume’s rejection of “physical points” on the grounds that an indivisible extension is impossible. (T 1.2.4.3, SBN 40)

We can try to get around this objection by following Rolf George[11] in holding that, for Hume, the point A has *size*, but not *extension*. But this seems odd, since to say that X is 1/Nth the size of Y seems to mean neither more nor less than that X has 1/Nth the extent of Y. We might say instead that A has neither size *nor* extension, and that while AB has a length that is one third the extension of ABCDEF, and AB is itself cleanly divisible into two parts, AB is not divisible into two parts that each have size. But now consider separating ABCDEF into the parts A and BCDEF. This latter approach would have the result that ABCDEF has been cleanly divided

[11] Rolf George, “James Jurin Awakens Hume from His Dogmatic Slumber with a Short Tract on Visual Acuity.” *Hume Studies* 32 (2006): 141-166.

into two parts, one of which has a size that is 5/6ths of the original, the other of which has no size. Again, we seem to have an unattractive consequence of the clean decomposition into parts, without remainder or re-ordering, of an extended whole whose size does not sum to the size of a whole.[12]

Turning from one-dimensional lines to the even more complicated two-dimensional case, we must make some decisions about how to interpret "between", "contiguous" and "straight line" [13] in the space Z^2. A natural first approach to understanding an alternative Humean geometry with a contiguity relation is to say that a straight line is some plurality of points such that, for any three of those points, one of the points lies between the other two, and for any two points in the plurality there is a path of pairwise contiguous points in the plurality running from the first to the second. But how are we to make sense of the crucial notions of *betweenness* and *contiguity* that are used in these definitions?

Suppose we define "between" in the discrete plane Z^2 in the most natural way by appropriating the betweenness relation that Z^2 inherits from the space R^2 of pairs of real numbers, a space in which Z^2 is usually thought to be embedded. So, we say A is *between* B and C in the *Humean Plane* if A is between B and C in R^2 and A, B, and C are all points in Z^2. Should we then say that two points A and B are contiguous in Z^2 just so long as there is no other point

[12] The conception of a whole being composed of parts whose magnitudes do not sum to the magnitude of the whole is not a problem for more recent philosophers such as Grunbaum, cited previously. But it is a problem for Hume, who seems to spurn this notion, and employ its contradictory in the argument at T 1.2.2.2, SBN 29-30. Also, the rejection of the principle that the size of the whole is the sum of the sizes of its parts, in modern accounts of the geometric continuum, is restricted to uncountable decompositions into parts. Geometry is still finitely and countably additive, results used to compute areas by finite decomposition, or by employing limit processes.

[13] When Hume talks of straight lines, he typically uses the term "right line". See, for example, T 1.2.4.26-8, SBN 49-50. He does not think of lines as infinitely extended in two directions, but as what modern geometry books would call a "line segment".

between A and B in Z^2? Or should we adopt a more stringent notion of contiguity, and say instead that A and B are contiguous in Z^2 just in case A and B are at what we would usually think of as a unit distance from each other, where such unit distances only occur in two privileged orthogonal directions?

Either approach gives strange results when considering right triangles. Consider the line segment whose points are (0,0), (0,1), (0,2), (0,3) and another line segment whose points are (0,0), (1,0), etc. up to the point (1000,0). It can easily be seen that there are no points in Z^2 lying exactly between (0,3) and (1000,0).[14] So, on the first proposal for understanding contiguity, those two points are contiguous and form a very short line. The length of that line, since it contains only two points, is clearly shorter than sum of the lengths of the legs of the triangle, no matter whether we use the direct enumeration account of length or Baxter's account of length. The second proposal for understanding contiguity says that these two points are not the endpoints of any line segment, assuming a line segment must be made up of sequences of consecutive, pairwise contiguous points, and so the two lines are not the legs of any right triangle at all, since the would-be triangle lacks a hypotenuse!

An alternative approach would be to begin with contiguity, as defined in the second sense above – that is, as the relation of points that are at unit distance from each other in the two privileged orthogonal directions - and define "betweenness" and "line" in terms of contiguity. We could then say that a line running from a to b is a path of pairwise contiguous points running from a to b that is at least as short as (contains no more points than) any other

[14] The equation of the line in R^2 connecting (0,3) and (1000,0) is $3x + 1000y = 3000$, and any point in Z^2 lying on that line between (0,3) and (1000,0) would have to have a y-coordinate of 1 or 2. But there is no integer value of x satisfying the equation for either of these two values of y.

pairwise contiguous path from a to b. But this approach gives us extravagant lengths for the hypotenuses of right triangles. Right triangles with legs consisting of x points and y points respectively will have zig-zagging hypotenuses of length x + y - 1. Such triangles will also have *multiple* hypotenuses, since there are two distinct minimal paths running from the endpoint of one leg to the endpoint of another. If Hume is proposing a geometry along these lines, its results are very different from the results of Euclidean geometry, where the quantitative measure of the sides of right triangles is governed by the Pythagorean theorem, and where right triangles possess only a single hypotenuse.

Falkenstein suggests, in Falkenstein (2000), that at least *some* right triangles would exist in a Humean discrete geometry, and that the Pythagorean Theorem could still be true of whatever right triangles *do* exist. He argues that for right triangles with commensurable sides, such as 3-4-5 triangles, the Pythagorean theorem would still hold, but we could at the same time regard the lengths of the sides of triangles as being given finitistically by the number of points they contain. But, for the reasons already stated, I don't see how we can get this result with the sort of model he proposes later in Falkenstein (2006), which takes the points in the finitistic space as the points of Z^2. In this space, a line consisting of three points running in the "vertical" direction from A to B, and four points running in the horizontal "direction" from B to C, does not determine a triangle with a hypotenuse of five points running from A to C.[15]

[15] There are other possible models, however, that might help get the desired result. For example, we could model Humean points as unit circles in a Euclidean plane (or unit spheres in 3-space), and then define a Humean figure to be any collection of these unit balls for which no two of them overlap. Then lines can be defined to be pairwise contiguous (tangent) sequences of unit balls running from some ball A to a second ball B where the number of balls in the sequence is less than the number of balls in any other sequence running from A to B. These sequences would have centers that are collinear in the background Euclidean space R^2. This approach might get the desired result, depending on how the unit balls are arranged or *packed* in the space. There might at least some right triangles for which the Pythagorean theorem holds, but whether there are such triangles in any given region of

Perhaps we can get better results by returning to R^2 and looking for other entities in that space with which to model the points in Humean geometry. A natural candidate would be the unit square regions, such as the square region with the four vertex points (0, 0), (1,0), (0, 1) and (1, 1). We can then treat contiguity between two points as *tangency* – sharing either a side or a single vertex – and treat lines, as before, as minimal paths of pairwise contiguous points running from one point to another. But such proposals run into an important problem know as the *Weyl tile problem*.[16] Consider a right triangle whose legs are one vertical path of ten unit squares and one horizontal path of ten unit squares, where the two paths share a common endpoint square. The shortest path between the other two endpoint squares including those squares, will itself be a path of only ten squares – a result radically different from the Euclidean result. This problem is altered somewhat, but not solved, by looking at non-rectangular configurations of points, such as unit equilateral triangles or unit hexagons.

The contemporary philosopher Peter Forrest has proposed a theory of discrete space that is designed to address the challenge of combining finite divisibility with an account of the quantitative relationships holding among the lengths of the sides of triangles and other figures, and his proposal more closely approximates Euclidean results.[17] The technique is a bit more complicated than the proposals we have examined so far, so I will only sketch its general

space would be contingent on the arrangement of balls that happens to obtain in that region. And the approach faces many other challenges, such as how to deal with the relationships between perimeters and areas, and how to arrange balls in such a way as to make the space not too anisotropic for Euclidean geometry to be even a useful approximation.

[16] For a general discussion of this problem see, J.P. Van Bendegem, "Zeno's Paradoxes and the Weyl Tile Argument", *Philosophy of Science*, 54(2) (1987): 295–302.

[17] Forrest, Peter (1995). "Is space-time discrete or continuous? — An empirical question." *Synthese* 103 (3):327--354.

outlines, and modify Forrest's terminology just a bit to make it more immediately applicable to Hume's conceptions.

Consider, once again, the space Z^n of n-tuples of integers and the background space R^n of n-tuples of real numbers in which Z^n is embedded. Define distances and the other standard Euclidean geometrical concepts in the usual way in R^n. Now choose an arbitrary real number *m*, and say that two points A and B are *contiguous* just in case the Euclidean distance from A to B is no greater than *m*. Forrest denotes this space $\mathbf{E}_{n,m}$. (I will call the number *m* the *contiguity diameter*.) Now define the discrete distance from A to B in $\mathbf{E}_{n,m}$ just as we did before: it is the smallest number of points that are the points of a path of pairwise contiguous points running from A to B.

Now, consider some right triangle ABC, with A the vertex of the right angle and whose legs contain x and y number of points respectively, and consider what the above construction gives for the length of its discrete hypotenuse BC. If *m* is small relative to the unit distances in Z^2, the shortest path connecting B and C is still likely to zigzag quite a bit over the Euclidean hypotenuse, and the number of points in that path will differ quite a bit from the result given by the Pythagorean theorem. For example, suppose we look at the triangle with a vertical leg consisting of the points (0,0), (0,1), (0,2), (0,3) and (0,4) and a horizontal leg consisting of (0,0), (1,0), (2,0), (3,0) and (4,0). And suppose we let *m* = 2. Then the lengths of the legs will 5, but the shortest path connecting B and C will be a zigzag line consisting of 9 points – quite far from the Pythagorean distance of roughly 7.07.

But now consider a similar triangle whose legs each consist of not five points, but a million points, and let the contiguity diameter *m* be equal to 100. It should be intuitively clear that, for any given point in the space, we now have many more options for selecting contiguous points, and, starting with B, can now select pairwise contiguous points along a sequential path from B to C that lie much closer to the Euclidean hypotenuse in the background space than was the case in our first example. As a result, the ratio between the number of points in the minimal path from B to C and the number of points in each of the legs will be much closer to the predicted Pythagorean ratio of approximately 1.41. We can then say that for sufficiently large scales, discrete distance relations approximate Euclidean distance relations quite well, although at small scales the two methods might diverge quite radically.

Forrest's ingenious model will make another appearance later in the paper, but for now we can note what it does *not* do. Forrest does not axiomatize his discrete geometry, but we can already see that under such an axiomatization many of the fundamental principles of Euclidean geometry would no longer be true. There would, for example, still exist points between which there lie no other points. Also, non-parallel lines in a single plane might have no intersection point. His model is only meant to give a set of mathematical structures in which, at sufficiently large scales, the quantitative relationships among the lines composing figures in those structures are closely approximated by corresponding quantitative relationships in Euclidean geometry. It remains the case that any geometrical theory that fully spells out the truths about the figures in such a structure would differ quite a bit from Euclidean geometry.

More proposals along these lines could be offered. But I think we have seen enough to realize how deep is the *prima facie* challenge Hume's positive account of the finite divisibility of extension poses for geometry, as traditionally understood. There is an initial problem with the overall coherence of Hume's approach. Any viable geometry, one that is usefully applicable to the measure of extended curves, regions and volumes, will have to respect the principle that the decomposition of extensions into finitely many components gives us pieces whose sizes sum to the size of the whole. But Hume also seems committed to the further principles that (i) every finite extension contains only finitely many ultimate parts, (ii) no ultimate part is extended and (iii) nothing without extension has a magnitude. We have struggled to find a way to harmonize these commitments. But we have also seen that if a coherent discrete geometry can be recovered from Hume's positive account, it is very likely its set of theorems is going to be quite different from the theorems of Euclidean geometry.

2. Hume and the Geometrical Arguments for Infinite Divisibility

But Hume resists this radical direction. Hume's actual approach to geometry is developed in Treatise 1.2.4 in the context of his defense of the finite divisibility of extension against geometrical arguments for the contrary thesis of infinite divisibility. His procedure in addressing these arguments is frustrating for the interpreter. For one thing, he does not explicitly present any of these arguments, but only adverts to them. If Hume had presented the arguments, we might then expect him to explain where precisely they go wrong by pointing to the mistaken premises or inferences they contain. But rather than identifying any specific flaws in the arguments for infinite divisibility, Hume instead seeks to dismiss them as a group based on global considerations about the science of geometry.

This global attack on the arguments for infinite divisibility comes in two main parts. The first part accuses the geometers of self-contradiction in presenting proofs of infinite divisibility that rely on fundamental concepts such as *point*, *line* and *plane*, concepts whose very definitions are, Hume says, inconsistent with the infinite divisibility theorems the geometers purport to prove by arguments relying on these concepts.[18] These *ad hominem* counterarguments are not very compelling, however, because those definitions do not play any essential deductive role in the logical development of Euclidean geometry. They serve only to help the reader grasp the primitive ideas that are employed in the deductions.[19] The second part of Hume's attack depends on his critique of the standard of equality used in geometry, and that is the part of his attack that will be our focus.

Now there are a variety of tactics that one might employ, drawn from the history of philosophical thinking about geometry, to hold onto Euclidean geometry while at the same time defending Humean views about the finitary and discrete character of actually existing extensions in the world of experience. For example, one might take a broadly Platonic approach and claim that geometry describes a realm of ideal mathematical entities that, while in some sense real, are not present in the manifest, experienced world of material bodies and physical space, or even in the world of sensory impressions and ideas. To accept Euclidean geometry, then, would be to accept it as a correct description of that ideal and perfect spatial realm, but not the lower world of matter, sensuous perception and becoming in which we live. Somewhat along the same lines, one might claim that Euclidean geometry describes a merely *possible* world of entities that don't really exist, not even in some ultimate and perfect mathematical realm, but which the actual world of finitely divisible spatial objects

[18] This part of Hume's argument is found in T 1.2.4.8-16, SBN 42-44.

[19] That Euclid's preliminary definitions are only designed to help guide the reader by clarifying the meaning of the fundamental terms of Euclidean geometry, and play no subsequent role in the proofs that follow them, has long been appreciated. See Ian Mueller, "Euclid's Elements and the Axiomatic Method", *The British Journal for the Philosophy of Science*, Vol. 20, No. 4 (Dec. 1969), pp. 289-309.

approximates in some way. To “accept” Euclidean geometry would then only be to accept its broadly logical possibility, and its utility as an approximation of spatial actuality. Another tactic would be to draw on one of the many varieties of deductivist or formalist approaches to Euclidean geometry that have been offered in more recent times.[20]

But we can dispatch these approaches quickly. There is no textual support in the Treatise for any of them. There is no trace of formalist or deductivist thinking in the Treatise, and the other approaches clearly fly in the face of Hume’s explicit animadversions against intellectualism or rationalism in mathematics. Hume takes a different approach, treating geometry as a generally successful description of the actual world of spatially disposed objects, but arguing that it fails in some limited way when it is extended into the realm of the very small. Hume does not seek to avoid the geometers’ challenge to his views by removing their subject from the world of bodies and sensory perceptions. He treats geometry as a theory of the spatial properties of the extended things we experience, but then argues for limitations on the universal applicability and correctness of that theory.

Hume’s strategy for blocking the geometrical arguments for infinite divisibility is clearly intended to turn upon an important distinction he draws between two standards of equality for lines, surfaces, and other extended geometrical objects. The first is a precise standard based upon the actual number of indivisible points contained in those figures, while the second is an

[20] For example, Euclidean geometry could be seen as a pure formalism reducible to some complete axiomatic basis A, and to accept a theorem T, in a broad sense, would be to accept in a stricter sense the conditional *if A, then T*. For a survey of formalist and deductivist approaches to mathematics, see Michael Detlefsen, “Formalism”, in S. Shapiro (ed.), *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford University Press, 2005): 236-317, Ch. 6 in Shapiro, Stewart (2000). *Thinking About Mathematics: The Philosophy of Mathematics*. Oxford University Press, and Linnebo, Øystein (2017). *Philosophy of Mathematics*. Princeton, NJ: Princeton University Press.

imprecise standard based on the general appearances of those figures. Hume says the precise standard is "just", but also "useless".[21] Any actual spatial object does indeed, according to Hume, contain some precise, finite number of indivisible points. If our sense of vision or touch were perfect enough to resolve all the points that make up the extended objects we see or touch, we could precisely count those points, and base our judgments of equality and inequality on these counts.[22] In that sense, the proposed standard is *just*: it is grounded in what Hume takes to be a correct account of the composition of spatial objects and the varying degrees of extension they possess. But, as a matter of fact, our senses are quite imperfect, and therefore the proposed standard is *useless*. (T 1.2.4.19)

So, the precise standard of equality, however well-founded it is in principle, could not be the standard we actually employ when we judge the relative sizes of geometrical figures, and therefore cannot be the standard upon which our experiential knowledge of the proportions of bodies and figures, and of the general maxims of geometry, is based. Instead we base our judgments on the way these objects appear to us. If we are looking at two extended objects, then either one of them will look to be larger than the other, or they will look to be equal in size. But we have also learned from experience that things that look the same size initially might not be concluded to be the same size subsequently, after we have made use of more precise instruments and measuring techniques. So, most of our mature, ordinary perceptual

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[21] First, at T 1.2.4.19, SBN 45.

[22] An instructive passage to read in connection with the just standard, based on the exact numeration of points, occurs at *Treatise* 1.2.1.4-5, where Hume briefly discusses the function of telescopes and microscopes, and the way in which our senses present us with "disproportioned images of things." A *just notion* of a very small object would require an idea containing distinct parts representing each and every part of the object. Hume's view is that the possession of such an idea is indeed possible, but "is extremely difficult, by reason of the vast number and multiplicity of these parts."

judgments of equality and inequality are always tainted by some doubt: the doubt occasioned by our realization that the use of more precise techniques might deliver different results.[23]

Even the use of these more precise techniques, however, ultimately rests on the standard derived from general appearances. If we have a device that measures millimeter lengths, for example, then we might find that two drawn lines that look to be the same length to an unaided view, when viewed from a certain distance, turn out to differ in length by one millimeter when measured carefully with the device. But to arrive at this corrected judgment, we must measure portions of the line piece by piece, and at each step, our judgment that the piece we are measuring is equal in length to the relevant portion of the device we are using is itself based on the perceptual standard.

That Hume's strategy for countering the geometrical arguments for infinite divisibility is supposed to depend on his distinction between the precise and imprecise standard is beyond doubt. When Hume turns to the extended discussion of these arguments beginning at Treatise 1.2.4.17, he begins his analysis with a long discussion of the standard of equality. But determining how exactly these considerations on the distinction between the two standards might provide Hume with the resources to block the infinite divisibility arguments while maintaining Geometric Conservatism is now our task. I will proceed to consider and evaluate five strategies for rebutting these arguments, for each of which there is some basis in Hume's text: the *skeptical strategy*, the *fallibilist strategy*, the *subtraction strategy*, the *quantitative approximation strategy*, and the *mere maxims strategy*. All of these interpretive strategies

[23] Hume's argument for the residual doubt that is always present in our comparison of magnitudes is presented in T 1.2.4.23-4, SBN 47-9.

suffer from serious substantive problems, and so, whether or not any of the five strategies captures Hume's intentions, none of them provides Hume with a way of blocking the geometric arguments for infinite divisibility while maintaining Geometric Conservatism.

2.1. The Skeptical Strategy

One approach to interpreting Hume's account of the failure of the geometric proofs of infinite divisibility emphasizes Hume's skepticism about our beliefs in the existence of mind-independent body. Skeptical readings of Hume's account of our space and our ideas of space have been heavily emphasized in recent reading by Donald Baxter and Donald Ainslie. Baxter and Ainslie have both argued that Hume does not mean to advance any conclusions about extension as it is in itself, but only about extension as it appears to us. Baxter, for example, says that Hume's "concern is simply with how space and time appear to inner and outer sense."[24] And Ainslie argues that Hume "does not mean to be characterizing space as it exists wholly apart from our powers of conception. Instead, in an adumbration of Kant, his claim is restricted to space as it appears to us."[25]

But, in his arguments for finite divisibility, Hume has not adopted a skeptical stance on whether extension is infinitely divisible, but has instead argued quite definitively, and even stridently, that the infinite divisibility of any kind of extension is "impossible and contradictory". He also says that "'tis certain" that space and time do actually exist in conformity with our finitely divisible ideas of them. Even granting that the certainty Hume has in mind here is only a

[24] Baxter, Donald L. M. (2014). Hume on Space and Time. In Paul Russell (ed.), *The Oxford Handbook of Hume*. Oxford University Press USA. Although Baxter highlights Hume's skepticism in this 2014 article, his account of Hume on geometry in Baxter (2009) gives, I think, a more nuanced account.
[25] Ainslie, Donald C. (2010). Adequate ideas and modest scepticism in Hume's metaphysics of space. *Archiv für Geschichte der Philosophie* 92 (1):39-67.

moral one, one cannot defend a stance of certainty of any kind toward some claim by pleading skepticism about the subject matter of that claim. Hume holds that we can know that traditional geometry is *wrong*, not that we are merely in doubt as to whether its ostensible subject matter exists.

Also, if one takes traditional geometry as a theory of the external, perception-independent world of extension, a theory whose theorems entail the existence of either mind-independent space or a world of extended material things whose geometrical properties are just as the theorems describe them, then a skeptic about the existence of mind-independent extension or matter is going to regard geometry as erroneously dogmatic about *all of its claims*, not just about infinite divisibility. But Hume wants to defend the view that the claims of traditional geometry are predominantly sound and fully warranted. Geometry only "fails of evidence" in its proofs of infinite divisibility, "while all its other reasonings command our fullest assent and approbation." (T 1.2.4.32, SBN 52) So, a purely skeptical strategy for interpreting Hume's approach to the geometric proofs for infinite divisibility undermines both his Geometric Conservatism and his Apodictic Finitism.

Nevertheless, these observations leave us with a difficult interpretive problem. How are we to square Hume's confident positive account of the finite divisibility of extension in Treatise 1.2 with the skeptical philosophy that emerges later in Treatise 1.4? That latter part of Book One of the Treatise is in part about something Hume was satisfied to call "the sceptical system," and there is good reason to hold that the skeptical system should be regarded as Hume's own system. Relieving the apparent tension between these two parts of the Treatise requires some attention the main results of Treatise 1.4, and how those results bear on the arguments in

Treatise 1.2. Although a full interpretation of Treatise 1.4 is impossible within the scope of this paper, I believe we can say enough here to solve our reconciliation problem.[26]

I will assume, for the sake of argument, that Hume reaches a genuinely skeptical conclusion in Treatise 1.4, which renders doubtful the existence of extended bodies external to our perceptions. So, are we then to understand the skeptical arguments of Treatise 1.4 as superseding Treatise 1.2 and nullifying or rendering doubtful the earlier positive argument for the finite divisibility of extension? I believe not. To see why not, we need to be more careful than we have been so far about the structure of Hume's argument.

As noted earlier, Hume's argument for the finite divisibility of extension in Treatise 1.2 is based on two main premises:

1. It is possible that space exists and is not infinitely divisible.
2. It is impossible that space exists and is infinitely divisible.

Strictly speaking, all that follows from the above two assertions is that it is possible that space exists, and it is necessarily the case that, if space exists at all, it is only finitely divisible. But Hume says that space certainly *does* exist conformable to the idea of finite divisibility. Hume is implicitly disregarding the option of holding that space doesn't exist at all, and so this appears to be his conclusion:

3. Space exists and is not infinitely divisible.

But when Hume fist presents the argument, he formulates the conclusion like this:

4. No finite extension is infinitely divisible. (T 1.2.2.2, SBN 29-30)

[26] For a deep, comprehensive and sustained reading of Book I, Part 4 of the Treatise, see Ainslie, Donald C. (2015). *Hume's True Scepticism*. Oxford University Press UK.

For Hume, space and extension are the same thing. I think, then, we can restate Hume's conclusion as:

5. Extended things exist, and no extended things are infinitely divisible.

But what is extension? To understand Hume's answer to this question, we can begin by considering his account of our *ideas* of extension. For Hume, ideas of extended things are copies of impressions of extended things, or else they constructed out of such copies by a kind of mental separation and recombination. These impressions of extended things are impressions of colored points disposed in a certain manner. (T 1.2.3.4) So, our ideas of extended things are in turn ideas of colored points disposed in a certain manner. We can then conclude that extended things, for Hume, if they exist at all, and in whatever manner they exist, consist of colored points disposed in some certain manner.

The claim that extended things exist, and that all such extended things are only finitely divisible, doesn't yet tell us where, and in what way, extended things exist. What kinds of extended things are there for Hume? Hume holds that *among* those extended things are the ideas and impressions of extension themselves that exist in our minds. These complex perceptions are extended in exactly the same way as their external objects or "archetypes", *if* those objects exist.[27] But *are* there external extended objects? In Treatise 1.2, we have what appears to be a straightforward *representational realist* framework at work.[28] We are

[27] Support for this interpretation can be found in Treatise 1.4.5, where Hume's assessment of arguments for the immateriality of the soul relies on the view that our perceptions of extended objects are themselves extended. See especially T 1.4.5.15-16. The interpretation I have just given of Hume's views on the nature and spatiality of perceptions of extension is, I believe, consistent with Baxter's interpretation of Hume on the general idea of space in Baxter (2014): 180-183.

[28] I am using this term in the sense of Fogelin, Robert J. (2009), *Hume's Skeptical Crisis*, Oxford University Press, pp. 78-82.

introduced to such things as mites, impressions of mites and ideas of mites. The impressions are said to be caused by rays of light that flow from the objects of perception. (T 1.2.1.4-5, SBN 27-8) This three-level system of external objects, sensory impressions and ideas all seems quite anti-skeptical. But this simple and relatively unproblematic picture is rendered quite problematic later in Treatise 1.4.

However, the skeptical results of Treatise 1.4 do not mean that we cannot attribute statement (5) above to Hume. This is an acceptable conclusion of Hume's argument, even considering the later skeptical complexities developed in Treatise 1.4, because among whatever extended things there might be in the universe, there are, at a minimum, extended *perceptions*. We *cannot*, however, unreservedly attribute to Hume this stronger claim:

6. Extended things external to our perceptions exist, and no extended things are infinitely divisible.

This latter statement expresses a belief Hume holds whenever he is believing in body according to what he later calls *the philosophical system* of belief in body. (T 1.4.2.46-53, SBN 211-16) And this is a belief system Hume presents himself adopting only intermittently, not all the time. (T 1.4.2.57, SBN 218) Still, I think we *can* unreservedly attribute to Hume:

7. *If* extended things external to our perceptions exist, they are not infinitely divisible.

This statement is a straightforward logical consequence of statement (5), because whenever it is true that no Fs are G, it follows logically that if there are any F's that are H, they are not G.

So, we now have a more precise way of understanding Hume's Apodictic Finitism, consistent with the arguments of Treatise 1.2 and the skeptical system developed in Treatise 1.4. Recall the original statement of Apodictic Finitism:

> *Apodictic Finitism*: The supposition that space is infinitely divisible is impossible and contradictory, and the contrary supposition that space is only finitely divisible can be known with certainty.

We now have a clearer interpretation of the second clause in this statement. Hume thinks we can know with certainty that our *perceptions* of extended things are finitely divisible, and we can also know with certainly that, *if* there are *external* things external to our perceptions, they are only finitely divisible as well.

2.2. The Fallibilist Strategy

The skeptical strategy targets the epistemic status of geometry, and attributes to Hume the view that the claims made by geometry are wholly unwarranted, insofar as they are taken to be claims that go beyond the way extension appears to us. We have seen that the skeptical strategy goes too far. But there are numerous passages in Treatise 1.2 and 1.3 that suggest Hume's strategy for blocking the arguments for infinite divisibility does rest on a reappraisal of the epistemic status of geometry, and on his defense of the position that geometrical knowledge is not demonstrative, but fallible and uncertain. The theorems of geometry consist primarily of a collection of universal generalizations. These general propositions are, for the empirically-minded Hume, arrived at by induction from observation of the spatial properties and relations of observable things. Because the foundation for geometry lies in the fallible judgments of the senses, based on the imprecise visual standard of equality, those maxims must possess an epistemic status less than certainty. Hume argues that "geometry falls short of that perfect precision and certainty, which are peculiar to arithmetic and algebra." (T 1.3.1.6, SBN 71-72) And he also points out the *fallibility* of our judgments of equality and inequality:

> *Not only we are incapable of telling, if the case be in any degree doubtful, when such particular figures are equal; when such a line is a right one, and such a surface a plain one; but we can form no idea of that proportion, or of these figures, which is firm and invariable. Our appeal is still to the weak and fallible judgment, which we make from the appearance of the objects, and correct by a compass or common measure; and if we join the supposition of any farther correction, 'tis of such-a-one as is either useless or imaginary.* (T 1.2.4.29, SBN 50-51)

Some interpreters, then, have held that the upshot of Hume's survey of the status of geometry lies in his conclusion that geometry is a fallible and uncertain inductive science, attaining only probability and not demonstrative certainty, and that the uncertain status of geometry destroys the geometers' case for infinite divisibility.[29]

But if this is Hume's main point about geometry, it is clearly inadequate to his purpose. It is not enough for Hume to point out that traditional geometrical knowledge is uncertain, fallible or merely probable. Such is true of any empirical theory, but that fact alone doesn't destroy, or even undermine, the case for whatever conclusions the theory entails. A sound argument can be built on less than certain empirical premises, and its conclusion can still have a degree of warrant proportional to the strength of its premises. But Hume has argued that the infinite divisibility of extension is not just doubtful, but "utterly impossible and contradictory."

[29] An emphasis on the importance of Hume's conception of geometry as a fallible and inductive a posteriori science for his discussion of the geometers' arguments for infinite divisibility can be found in Marina Frasca-Spada (1998); *Space and the Self in Hume's Treatise*; Cambridge University Press: 123-35. De Pierris also develops this point in Graciela De Pierris "Hume's Skepticism and Inductivism Concerning Space and Geometry" in V. de Risi (ed.), *Mathematizing Space*, Trends in the History of Science, Switzerland (2015).

He is thus committed outright to the falsity of the doctrine of infinite divisibility, not its mere uncertainty, and has presented arguments for the contrary supposition that, from all appearances, he regards as completely decisive. Since he knows there are geometric arguments for infinite divisibility, Hume is thus committed to the position that some of the maxims used in these derivations must be false, and that these fundamental principles can be seen to contain at least one falsehood since their falsity follows from decisive arguments for a contrary position.

There is, of course, a long philosophical tradition of treating geometry as a demonstrative and certain science, consisting of propositions that can be known *a priori* via demonstrative proofs from first principles that are both necessary and intuitively certain. It is natural to think that geometric rationalists who maintained this position are Hume's chief opponent, and so defenders of the fallibilism strategy sometimes confuse the task of blocking the arguments for infinite divisibility altogether with the more limited task of showing them to be non-demonstrative.[30] But, as we have seen, Hume's dialectical burden is heavier since he has defended Apodictic Finitism.

Nevertheless, it is worth considering whether we can at least partially vindicate Hume's response to the traditional geometers by allowing him to retreat to a more modest position that drops Apodictic Finitism. Could Hume say that he has shown that his finitistic conception of space is, if not certainly true, at least a viable alternative to the traditional Euclidean conception, and that he has therefore succeeded in showing that the Euclidean conception of

[30] Hume himself might have fallen into this confusion at T 1.2.4.17, SBN 44-5, where he stresses that geometrical proofs of infinite divisibility are "not properly demonstrations." But his subsequent discussion makes it clear that a lack of is not their central failing. He thinks they are simply unsound.

space cannot be justified *a priori*? To develop a fuller understanding of the epistemological and logical issues involved in this topic, I will begin by considering an important discussion of Hume's views on infinite divisibility by James Franklin.[31]

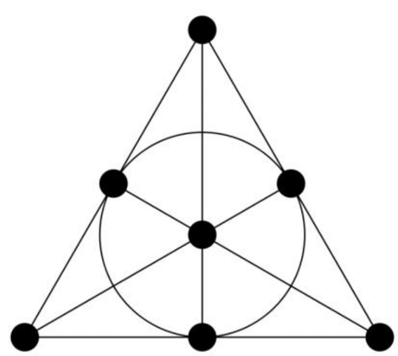
Franklin attempts to provide a partial vindication of Hume, recognizing that "there might be something correct among the errors." And he argues that to understand what is right about Hume we must begin with "what is now known to be the correct answer on the question of infinite divisibility." And what is it that is now known? According to Franklin we now know that both the infinite divisibility and finite divisibility of space are possible:

> "The infinite divisibility of space and time is possible. (This is because there exists a consistent model which incorporates infinite divisibility, namely the set of infinite decimals.) It follows that all supposed proofs of the impossibility of infinite divisibility, whether mathematical or philosophical, are invalid."
>
> …
>
> "It is also possible that space should be discrete or atomic, that is, composed of units and only finitely divisible. (This is because there is a consistent model in this case too: in one dimension, the integers, and m higher dimensions, the lattice of points with integer coordinates.) It follows that all supposed proofs of infinite divisibility are invalid."[32]

[31] Franklin, James (1994). "Achievements and fallacies in Hume's account of infinite divisibility", ***Hume Studies*** 20 (1):85-101.
[32] Franklin, p. 87.

What kind of possibility is Franklin speaking of here? It seems likely he has in mind either logical, broadly logical, or conceptual possibility. His view is that there is a conception of space as discrete and finitely divisible that is logically and conceptually coherent, along with another and equally coherent conception of space as infinitely divisible. And if both conceptions are possible, then if there is any evidential basis for deciding which conception is correct, that evidence would have to be empirical.

But I think there are problems with this approach to vindicating Hume, even if only partially. First, we must be careful about drawing conclusions from the existence of a coherent model of a geometrical space *of some kind* to the viability of that model as a possible representation of the spatial features of the world of our actual experience, or as a possible interpretation of some adequate theory of that space. Contemporary geometers use the term “space” or “geometry” for a large class of abstract structures, some of which are not intended as even remotely plausible models of the geometry of the word we live in, and so do not capture all of even the most elementary observable features of that world. For example, there is a structure called the *Fano Plane* that can be graphically depicted like this:



We are to think up the “lines” in this structure as certain three element subsets of the whole structure, those connected by the three sides, the three medians and the inscribed circle respectively. The Fano Plane is a simple model of the *finite projective plane*, a finite geometrical space that satisfies the following axioms:

1. Every pair of points is connected by exactly one line.
2. Every line intersects every other line in exactly one point.
3. There are four points such that no line contains more than two of them.

Suppose we begin with those three axioms and add the axiom:

4. There are exactly seven points.

This theory is logically consistent, since the Fano Plane is model of it. But no one would regard the Fano Plane an approximation of the world of spatial experience.[33]

So, the mere fact that the space Z^n of integer pairs is a model of *something* doesn't tell us that it is a model of anything that could be plausibly regarded as the geometry of extension as we encounter it in experience. We have seen how challenging it is to build a plausible and useful geometry out of such spaces, so I don't think we are yet able to say that the existence of such abstract mathematical spaces shows that it is even possible that the space of experience is only finitely divisible.

2.3. The Subtraction Strategy: Axiom Excision and Domain Restriction

A different, more logically direct approach to countering the arguments for infinite divisibility while defending Geometric Conservatism and Apodictic Finitism attempts to save the body of Euclidean geometry by the careful but very limited cutting away of a few diseased and

[33] Or background on projective geometry and finite projective planes, see Casse, Rey (2006), *Projective Geometry: An Introduction*, Oxford: Oxford University Press

offending parts in that body. As noted earlier, Hume holds that geometry fails only "in this single point" – that is, in its proofs of infinite divisibility – while "all of its other reasonings" merit respect and approval. (T 1.2.4.32, SBN 52) One way to understand this claim would be to say that, for Hume, most of the proofs and theorems of Euclidean geometry are fine just as they are, but there are a few bad arguments and theorems that need to be rejected. If we simply excise these theorems, and whatever fundamental axiomatic principles they rely on, what is left behind will be sound.[34] Perhaps Hume encourages a revision of geometry, then, but not a radical revision.

But it is not clear that there is any plausible way of doing this that does not leave Euclidean geometry in ruins. The problem is that the postulates, common notions and theorems of Euclidean geometry do not distinguish between the small and the large, and are stated in such a way as to apply to all magnitude scales concurrently and indifferently.[35] To consider one telling example of just how unavailing the theorem excision strategy is, note that many of the theorems and constructions of traditional geometry depend on the assertion that two lines or two curves of some kind will intersect in some single point. It has long been understood that proving these assertions requires making explicit certain assumptions that Euclid left unstated. Subsequent attempts to make the axiomatization of geometry more

[34] This approach has been developed recently by Emil Badici in Badici (2010).

[35] This point can be stated more precisely: Consider any axiomatization of Euclidean geometry EA, and any structure S_{EA} that is a model of that geometry under the interpretation I. Consider a function f: $S_{EA} \rightarrow S_{EA}$ that simply dilates or contracts distances uniformly, and the interpretation I' that interprets all of the formulas ϕ of EA according to the rule $I'(\phi) = I(f(\phi))$. The function I' will also be a model of EA.

rigorous have made it clear that what is required here are postulates entailing the continuity, and thus infinite divisibility, of curves and lines.[36]

One more exacting way of developing the theorem excision strategy is to refrain from jettisoning whole axioms of Euclidean geometry, but to restrict their range of application instead. Perhaps we can fix a domain of the "extremely small", and hold that geometric axioms and theorems fail only inside that domain, but hold everywhere outside that domain.[37] So, for example, while standard Euclidean geometry will contain an axiom such as:

> A: For any two points A and B, there is a point C, not identical to either A or B, that lies between A and B.

the revised geometry would contain only something like this:

> A*: For any two points A and B, if the distance between A and B is greater than the minimal distance k, there is a point C, not identical to either A or B, that lies between A and B.

Similar restrictions might be placed on axioms dealing with angles and angle measure, as opposed to lines and line measure.

The problem with this approach is that breakdowns of geometry in the small can still have very significant consequences on the theorems that can be deduced for geometry in the large, and that even the desired restricted theorems will turn out not to be deducible. Consider that whenever lines and curves intersect, they do so within an arbitrarily small region. If lines

[36] For mathematical background on this lacuna in Euclid, see Marvin Jay Greenberg, "Old and New Results in the Foundations of Elementary Plane Euclidean and Non-Euclidean Geometries" *The American Mathematical Monthly*, vol. 117, no. 3, March 2010, pp. 198-219.

[37] Badici hints at this approach in Badici (2008), p. 240.

and curves can't be guaranteed to intersect within that small region, because that diameter of that region is smaller than the minimal diameter k, then any proofs about areas and volumes based on the existence of such an intersection point will fail, whether the proof deals with a structure smaller than a grain of sand or larger than the Great Pyramid, and even if the theorem is stated with the desired restriction. If Hume's aim was to argue for a straightforward subtraction strategy to block the proofs of infinite divisibility, then our response must be that the principles leading to infinite divisibility are more deeply implicated in traditional geometry than he appears to realize.

We have two more strategies left to consider. These remaining proposals, I believe, more closely fit Hume's discussion of the geometric proofs of infinite divisibility in Treatise 1.2, and offer more promising strategies for blocking those arguments while maintaining Geometric Conservatism. But I believe they are both liable to the same criticism: their conservatism is more apparent than real, and only sweeps the challenging radicalism of Hume's finitistic conception of the composition of extension under an obscuring rug. Far from upholding the soundness of traditional geometric reasoning that is unrelated to infinite divisibility, they give us a traditional geometry that is permeated by unsoundness, and that is filled with arguments that by no means "command our fullest assent and approbation."

3.4 The Quantitative Approximation Strategy

Axiom excision and domain restriction are two ways of trying to make good on the idea that traditional geometry is at least roughly or approximately true. These subtraction strategies attempt to give substance to this general idea by holding that traditional geometry approximates the *logical content* of the true, finitary theory of extension. They propose removing or modifying theorems in the body of traditional geometric results, paring away the parts that are incorrect from a correct internal core. The statements that remain were already implied by Euclidean geometry, but they are only a proper subset of the full set of Euclidean theorems. So, the approximate truth of Euclidean geometry is measured, so to speak, by the proportion of its theorems that are correct, or the proportion of true instances of its lofty general statements.

But another approach to the project of treating Euclidean geometry as an approximation of the true Humean geometry is to focus on the *quantitative* theorems of Euclidean geometry, and to take those theorems as approximately true in a quantitative sense. For example, here are three important quantitative results of Euclidean geometry.

> *The Pythagorean Theorem*: For any triangle T, the square of the length of the hypotenuse of T is equal to the sum of the squares of the lengths of the legs of T.

> *Heron's Theorem*: For any triangle T with sides of lengths a, b and c, and with perimeter $s = a + b + c$, the square of the area of T is equal to $s(s - a)(s - b)(s - c)$.

> *The Circumference Theorem*: For any circle C, the ratio of the length of the circumference of C to the length of the diameter of C is equal to the number π.

The subtraction strategy would either eliminate these theorems altogether from the reformed Humean geometry, or replace them with restricted alternatives, perhaps as follows:

> *The Restricted Pythagorean Theorem*: For any triangle T, if the legs and hypotenuse of T are all of length greater than k, then the square of the length of the hypotenuse of T is equal to the sum of the squares of the length of the legs of T.

But we have seen earlier that it is very hard to recover these kinds of quantitative Euclidean results in a finite geometric space, even if we restrict their range of purported correctness to very large scales. Even with Forrest's model, the Pythagorean theorem is only *roughly* true at such large scales. The quantitative ratio described by the Pythagorean theorem between the squared lengths of hypotenuses of right triangles and the sum of the squared lengths of their legs is only approached as a limit as the sizes of such right triangles considered are increased without bound. So, it could turn out that that in any viable Humean finitary space, the instances of these restricted generalizations are rarely, if ever, precisely true. In that case, we cannot even say that *most* instances of the Pythagorean theorem are true. Given how many such quantitative theorems permeate Euclidean geometry, it looks like the resort to the subtraction strategy is not going to vindicate Hume's Geometric Conservatism.

But does that mean that Euclidean geometry fails to approximate Humean finite geometry? Not necessarily. Perhaps we can salvage the approximation claim by looking not at the number of instances of general claims that are *precisely* true in quantitative terms, but at the number of instances that describe quantitative relationships that are themselves only *approximately* true, according to some appropriate quantitative standard. We might then be able to prove general claims such as the following:

> *The Approximate Pythagorean Theorem*: There is a positive real number x such that, for any triangle T, if the legs and hypotenuse of T are all of length greater than k, the ratio of the square of the length of the hypotenuse of T to the sum of the squares of the lengths of the legs of T is always between 1 - x and 1 + x.

This is another way of saying that, for any triangle T, the ratio of the square of the length of the hypotenuse of T to the sum of the squares of the lengths of the legs of T is always *roughly* equal to 1 – at least if the real number x seems suitably "small" for most instances of concern. Similar modifications could be made to the other quantitative theorems.[38]

There is some evidence in the text of Treatise 1.2.4 that suggests this quantitative approximation strategy best captures Hume's thinking:

> But I go farther, and maintain, that none of these demonstrations can have sufficient weight to establish such a principle, as this of infinite divisibility; and that because with regard to such minute objects, they are not properly demonstrations, being built on ideas, which are not exact, and maxims, which are not precisely true. When geometry decides any thing concerning the

[38] This is the strategy Badici ultimately attributes to Hume in Badici (2008), p. 239.

> proportions of quantity, we ought not to look for the utmost precision and exactness. None of its proofs extend so far. It takes the dimensions and proportions of figures justly; but roughly, and with some liberty. Its errors are never considerable; nor wou'd it err at all, did it not aspire to such an absolute perfection. (T 1.2.4.17, SBN 44-5)

But does this approach really justify Hume's Geometric Conservatism? Given the lack of an account of what "roughly" means in this context, and given also Hume's failure to develop his finitary positive account of the composition of extension, and of the manner of disposition of the parts of extension, beyond a few remarks, it is hard to say how much of the quantitative content of traditional geometry will turn out to be even roughly true. What kind of standard are we to apply? How do we measure the degree of approximation?

There is another major stumbling block in relying on the quantitative approximation strategy to defend Geometric Conservatism: many of the theorems of Euclidean geometry are qualitative existence theorems, not quantitative theorems, and have nothing to do with ratios between lengths, areas and volumes. For example, a full development of Euclidean geometry will prove such propositions as that two non-parallel lines in a plane intersect in one point, and that a line containing a point from the interior of a circle intersects the circle at two points. Are these claims still true, according to Hume? And if not, how much of traditional geometry has been retained?

But the most important problem with the quantitative approximation strategy is that it reinterprets the fundamental *arithmetical* terms of traditional geometry - that is, terms such "equals", "twice" and "one-half" - in such a way that the reinterpreted proofs can by no means still “command our fullest assent and approbation.” Proofs in geometry often involve long chains of reasoning in which two quantities are shown to be equal, or to stand in a certain ratio, by reasoning through a series of intermediate steps. Suppose for example, one has a proof in which one successively proves, for certain geometric quantities, a, b, c, d, e and f, that:

i. a is equal to b

ii. b is equal to c

iii. c is 1/3 of d

iv. d is twice e

v. e is equal to f.

As a result, we conclude that the magnitude of a is 2/3rds the magnitude of f. Now suppose we replace this proof with a reinterpreted proof in which we successively prove:

vi. a is roughly equal to b

vii. b is roughly equal to c

viii. c is roughly 1/3 of d

ix. d is roughly twice e

x. e is roughly equal to f.

Can we now be confident in concluding that the magnitude of a is *roughly* 2/3rds the magnitude of f? Of course not. No matter what degree of precision we assign to “roughly” in the argument, as we multiply the number of *rough* claims of equality in the argument, we will rationally lose confidence that the derived conclusion of a *rough* ratio falls within the same degree of precision. It is hard to see, then, how this flocculent travesty of Euclidean geometry still “commands our full respect and approbation.”

3.5 The Mere Maxims Strategy

The failure of the previous two strategies for defending both Apodictic Finitism and Geometric Conservatism suggest one final alternative, a strategy that does not attempt to separate the bad theorems from the good theorems, or to reinterpret the quantitative terms in those theorems, but instead retains geometry just as it is, while adopting a cautiously judicious and pragmatic attitude about its application, and while refraining from making unqualified truth claims based on its theorems.

Consider, as a first step to motivating this approach, Hume's famous missing shade of blue. In his discussion of the missing shade, Hume seems to have hit upon a counterexample to his previously established general claim that all simple ideas are copied from impressions. And he seems to *accept* that the case of the missing shade *is* a counterexample to the general claim. But he concludes by stating that since the counterexample is so "particular and singular", we should not on the basis of that instance alone "alter our general maxim". (T 1.1.1.10, SBN 5-6)

So, one might conclude that Hume believes that the aim in philosophy and science, as in common life, is only to "adopt" maxims or general rules on the basis of experience, and that the permissibility of the relevant attitude of *adoption* is based on considerations of general usefulness and approximate correctness, and is not necessarily undermined by a single, known counter-instance, or even by a small number of "singular" counter-instances. Hume might hold that the propositions of geometry are merely useful maxims, generally but not universally true, and that the existence of known counterexamples to these propositions should not prevent us from establishing and adopting this useful body of maxims. Traditional geometry should be adopted as is, without modification. It should be developed logically and thoroughly from the traditional principles, and used in its traditional form in application to practical and scientific problems of spatial measurement. But all the while we should bear in mind that it is not exactly and precisely true, and that we are liable to go wrong in using it.

The relevant propositional attitude of *adoption* must not be interpreted as a degree of *belief*. If one has confirmed beyond doubt that there exists a counterexample to the general claim that *every simple idea is copied from an impression*, then one has come to know that general claim is false, and so the appropriate degree of belief in the proposition is zero. So, to continue to adopt the general claim in the face of the counterexample, and treat it as an established maxim, is to continue to use the claim as a guide to life and the process of inquiry, despite knowing that it is not, strictly speaking, true.

I think there is much to recommend the mere maxims approach as an interpretation of Hume's stance in the Treatise, both on geometry and other matters. But if this is indeed the best interpretation of what Hume is saying about the status and content of geometry, then some important reservations must be lodged about the outcome his discussion has reached. The mere maxims strategy is similar in spirit to the quantitative approximation strategy, and it suffers from the same defect.

To employ the mere maxims account as a strategy for saving geometry from Hume's earlier critique of infinite divisibility, and from his positive views on the nature of space and composition of extension, only seems to obscure just how radical those views are. It is one thing to employ an *individual* general rule or maxim as a pragmatic tool for common life, but it is quite another thing to regard a deductive science based on a *large number* of such general rules as retaining the utility of the individual rules of which it is composed. As we know, the propositions *most F and G* and *most G are H* do not jointly entail *most F are H*. If the seemingly universal "all" principles of geometry are disbelieved, and yet adopted and used as only a careless substitute for some body "most" statements, then the utility and accuracy of the resulting theory should be expected to degrade very rapidly, and a scrupulous mind will have low confidence even in the general utility of the principles. If this is Hume's strategy for defending a conservative approach toward geometry, then he is only sweeping his genuine radicalism under the rug.

In a well-known review of Hume's theory of space in the *Treatise of Human Nature*, C. D. Broad concluded that "there seems to me to be nothing whatever in Hume's doctrine of Space except a great deal of ingenuity wasted in recommending and defending palpable nonsense."[39] This assessment has struck some later commentators as too harsh. Donald Baxter, focusing on the connections between Hume's views on space and time and the skeptical approach he develops in the rest of the Treatise, argues that the widespread neglect of this part of Hume's philosophy is unfortunate, and that when those connections are better understood, "the force of Hume's arguments concerning space and time can be appreciated, and the influential criticisms of them can be seen to miss the mark."[40] And Lorne Falkenstein aptly characterizes much of the more recent commentary on Hume's views of space and time as both more sympathetic and more wide-ranging.[41]

[39] Broad, C. D. (1961). Hume's Theory of Space,'. Proceedings of the British Academy 47:161-76.
[40] Baxter (2009): pp. 105-6
[41] Falkenstein (2006): p. 59

Also, although it is certainly the case that the geometries of space and spacetime that are used in contemporary science overwhelmingly retain the infinitely divisible topological and metric structure with which we are all familiar, some serious contemporary work is being done on developing finitistic alternatives to these geometries.[42] So the mere fact that Hume takes space to be finitely divisible should not, by itself, be taken as evidence that Hume's views are nonsense. And many philosophers of a generally empiricist bent might remain convinced that, given the finite capacities of our minds, and the natural limits on our abilities to perceive and imagine the very small, it is hard to explain how we could *know* space to be infinitely divisible, and so there must be some viable alternative theory of space as only finitely divisible – even if that alternative has not yet been successfully formulated.

Nevertheless, there does seem to be something correct in Broad's earlier harsh assessment. James Franklin remarks that classic philosophical problems such as that of the infinite divisibility of space: "attract the attention of two kinds of philosopher: the technical expert, who follows the scientists into the intricacies, and the Young Turk, eager to rush in where others fear to tread and cut the Gordian knot with his brilliant new insight." [43] And Hume is no technical expert. Geometry was a foundational science in Hume's time, as it remains in our own, and the principles entailing the infinite divisibility of space were presupposed by that science, and employed in its calculations in mechanics, technology and daily life. In rushing in with a radical alternative conception of space as only finitely divisible, Hume surely took on a

[42] Some of this work is described in Van Bendegem, Jean Paul, "Finitism in Geometry", The Stanford Encyclopedia of Philosophy (Spring 2017 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/spr2017/entries/geometry-finitism/>

[43] Franklin (1994): p. 86.

heavy burden, and owed his intellectual community something more substantial than he delivered.

Hume thought the doctrine of infinite divisibility was impossible and contradictory, but he underestimated how deeply that doctrine is embedded in all of geometry, not just the purported proofs of the doctrine itself, and so he shied away from accepting the destructive implications of his radical finitistic alternative for the whole of traditional geometry. His attempt to blunt the force of these implications, while it is indeed ingenious, and contains an insightful and thought-provoking investigation of the way we measure spatial magnitudes, does not succeed in its aim.