# Chapter 5.3

# Ranking Theory

*Gabriele Kern-Isberner, Niels Skovgaard-Olsen, Wolfgang Spohn*

**Abstract**: Ranking theory is one of the salient formal representations of doxastic states. It differs from others in being able to represent belief in a proposition (= taking it to be true), to also represent degrees of belief (i.e. beliefs as more or less firm), and thus to generally account for the dynamics of these beliefs. It does so on the basis of fundamental and compelling rationality postulates and is hence one way of explicating the rational structure of doxastic states. Thereby it provides foundations for accounts of defeasible or nonmonotonic reasoning. It has widespread applications in philosophy, it proves to be most useful in Artificial Intelligence, and it has started to find applications as a model of reasoning in psychology.

## 1. Introduction

Epistemic or doxastic attitudes[1] representing how the world is like come in degrees, whether you call them degrees of belief, uncertainty, plausibility, etc. There are various accounts of those degrees, amply presented in this handbook.[2] The interests in those accounts are manifold. Philosophers are concerned with the rational nature of those degrees, AI researchers are interested in their computational feasibility, psychologists deal with their actual manifestations, and all sides argue about how well they are suited to model human reasoning.

However, we also have the notion of belief *simpliciter*. Related notions are those of acceptance or judgment. These are indeed the more basic notions when it comes to truth, to truly representing the world. Beliefs can be true, but degrees of belief cannot. The latter rather relate to action (see chapter 8.2 by Peterson in this volume). Accounts of degrees of belief invariably have great difficulties in doing

1 Strictly speaking, "epistemic" only refers to knowledge, although it is often used more widely. Because we will talk only about belief, we prefer to use "doxastic" throughout. See also chapter 5.1 by van Ditmarsch (in this volume).

2 See chapters 4.1 by Hájek & Staffel, 4.5 by Chater & Oaksford, 4.7 by Dubois & Prade, 8.3 by Glöckner, and 8.4 by Hill (in this volume).

justice to this fundamental point. There is a questionable tendency to take degrees of belief as basic and to belittle those difficulties.

So we need to theoretically account for belief *simpliciter*. The first attempt was doxastic logic (see chapter 5.1 by van Ditmarsch in this volume). However, it is static and misses a dynamic perspective. This has been unfolded in belief revision theory (see chapter 5.2 by Rott in this volume). However, it has problems with iterated belief revision required for a complete dynamic account.

Ranking theory promises both to represent belief *and* degrees of belief and to provide a complete dynamics for both. These features give it a prominent place in the spectrum of possible theories. It was first presented in English in Spohn (1988) and fully developed in Spohn (2012). Easy access is provided in Spohn (2009). Its far-reaching applications in philosophy of science, epistemology, and even to normative reasoning may be found, e.g., in Spohn (2012, 2015, 2019). There is no place here to go into any of them.

Below we present the basics of the theory in Section 2 and its dynamic aspects in Section 3. Section 4 is comparative. Section 5 gives a short introduction to its relevance for Artificial Intelligence, and Section 6 explains how it can be put to use in psychology.

## 2. The Basics of Ranking Theory

Grammatically, "believe" is a transitive verb. In the phrase "*a* believes that *p*", "*a*" refers to a (human) subject and "that *p*" seems to be the object. What does "that *p*" stand for, what are the objects of belief? This is a difficult and most confusing issue extensively discussed in philosophy (under the rubric "propositions"; see, e.g., McGrath 2012). Here, we cut short the issue, as usual in formal epistemology, by saying in a non-committal way that "that *p*" stands for the proposition expressed by "*p*", where that proposition is its truth condition, the set of possibilities or possible worlds in which *p* obtains or "*p*" is true.

Hence, we simply assume a set $W$ of (mutually exclusive and jointly exhaustive) possibilities. These may be coarse-grained and refer only to a few things of interest; they need not consist of entire possible worlds. Each subset of $W$, i.e. each element of the power set $\mathcal{P}(W)$ of $W$, is a *proposition*.

Now, the basic representation of a belief state is simply as the set of propositions believed or taken to be true in that state, its *belief set*. Traditionally, a belief set $\mathcal{B} \subseteq \mathcal{P}(W)$ has to satisfy two rationality requirements: $\mathcal{B}$ must be *consistent*, i.e., $\bigcap\mathcal{B} \neq \emptyset$, and $\mathcal{B}$ must be *deductively closed*, i.e., if $\bigcap\mathcal{B} \subseteq A$, then $A \in \mathcal{B}$.

These two rationality requirements may seem entirely obvious. The rationale of deductive logic is to check what we must not believe and what we are committed to believe. Note, however, that deductive closure is lost when we identify belief with probability above a certain threshold; it easily happens that the probabilities of two propositions is above the threshold, while that of their conjunction is below. Thus, the lottery and the preface paradox and the general desire to stick to a probabilistic representation of belief have led to a contestation of these requirements (see, e.g., Christensen 2005). Here we stick to them as absolutely basic (see chapter 3.1 by Steinberger in this volume). Of course, these requirements can be maintained only under a dispositional understanding of belief; occurrent thought cannot be deductively closed.

The notion of a belief set is static. However, belief sets continuously change, and we must account for how they change (or should rationally change). We cannot do so on a qualitative level. In those changes we often give up old beliefs and replace them by new ones, and then we give up less well entrenched beliefs and keep better entrenched ones (see chapter 5.2 by Rott in this volume). Roughly, this calls for some entrenchment order or, indeed, for some kind of degrees of belief measuring the strength of entrenchment. Here, ranking theory commences. Let us start with some brief formal explanations.

*Definition 1*: $\kappa$ is a *negative ranking function* for $W$ iff $\kappa$ is a function from $\mathcal{P}(W)$ into the set of natural numbers plus infinity $\infty$ such that for all $A, B \subseteq W$:

(1) $\kappa(W) = 0$ and $\kappa(\emptyset) = \infty$,

(2) $\kappa(A \cup B) = \min \{\kappa(A), \kappa(B)\}$ (*the law of disjunction*).

The basic interpretation is that $\kappa$ expresses degrees of *disbelief* (whence the qualification 'negative'). If $\kappa(A) = 0$, $A$ is not disbelieved at all. This allows that $\kappa(\sim A) = 0$ as well (where $\sim A$ is the negation of $A$); then we have indifference or suspension of judgment regarding $A$. If $\kappa(A) > 0$, $A$ is disbelieved or taken to be false, and the more so, the larger $\kappa(A)$. So, positive *belief* in $A$ is expressed by disbelief in $\sim A$, i.e., by $\kappa(\sim A) > 0$ (which implies that $\kappa(A) = 0$).

This interpretation explains axioms (1) and (2). (1) says that the tautology $W$ is not disbelieved, and hence the contradiction $\emptyset$ is not believed. This entails that beliefs are consistent according to $\kappa$. (1) moreover says that the contradiction is indeed maximally disbelieved. And (2) states that you cannot disbelieve a disjunction less strongly than its disjuncts. This entails in particular that if you believe two conjuncts, you also believe their conjunction. Hence, beliefs are deductively closed according to $\kappa$. In other words, the belief set $\mathcal{B} = \{A \mid \kappa(\sim A) >

0} associated with κ satisfies the two basic rationality requirements. Note, moreover, that (1) and (2) entail:

(3) either κ(*A*) = 0 or κ(~*A*) = 0 (or both) (*the law of negation*),

i.e., you cannot (dis)believe *A* and ~*A* at once.

For an illustration, consider Tweetie. Tweetie has, or fails to have, each of three properties: being a bird (*B*), being a penguin (*P*), and being able to fly (*F*). This opens eight possibilities. Suppose you have no idea who or what Tweetie is, but somehow you do not think that it might be a penguin. Then your negative ranks for the eight possibilities (which determine the ranks for all other propositions) may be the following (chosen in some plausible way—but see below how the numbers may be justified):

| κ | $B \cap {\sim}P$ | $B \cap P$ | ${\sim}B \cap {\sim}P$ | ${\sim}B \cap P$ |
|---|---|---|---|---|
| *F* | 0 | 4 | 0 | 11 |
| ~*F* | 2 | 1 | 0 | 8 |

(*Table* 1)

In this case, the strongest proposition you believe is that Tweetie is *either* no penguin and no bird ($\sim B \cap \sim P$) *or* a flying bird and no penguin ($F \cap B \cap \sim P$); all other possibilities are disbelieved. Hence, you neither believe that Tweetie is a bird nor that it is not a bird. You are also indifferent concerning its ability to fly. But you believe, e.g.: if Tweetie is a bird, it is not a penguin and can fly ($\sim B \cup (\sim P \cap F)$); and if Tweetie is a penguin, it can fly ($\sim P \cup F$)—each if-then taken as material implication. Surely, you believe the latter only because you believe that Tweety is not a penguin in the first place. The large ranks in the last column indicate your strong disbelief in penguins *not* being birds. This may suffice as a first illustration.

We will see the reasons for starting with negative ranks. But, of course, we can also introduce the positive counterpart by defining β to be a *positive ranking function* iff there is a negative ranking function κ such that $\beta(A) = \kappa(\sim A)$ for all propositions $A$. β represents *degrees of belief*. Of course, (1) and (2) translate directly into axioms for β.

We may as well represent degrees of belief and degrees of disbelief in a single function by defining τ to be a *two-sided ranking function* iff there is a negative ranking function κ and the corresponding positive ranking function β such that $\tau(A) = \beta(A) - \kappa(A) = \kappa(\sim A) - \kappa(A)$ for all propositions $A$. Thus we have $\tau(A) > 0$, $< 0$, or $= 0$ according to whether $A$ is believed, disbelieved, or neither in τ.

Therefore this is perhaps the most intuitive notion. However, the mathematics is best done in terms of negative ranking functions. It is clear, though, that the three functions are interdefinable.

There is an important interpretational degree of freedom that we have not yet noticed. So far, we said that belief in *A* is represented by $\kappa(\sim A) = \beta(A) = \tau(A) > 0$. However, we may often find it useful to raise the threshold for belief, as we do informally in asking: "Do you *really* believe *A*?" That is, we may as well say that belief in *A* is only represented by $\kappa(\sim A) = \beta(A) = \tau(A) > z$ for some $z \geq 0$. This seems to be a natural move. Belief is vague. Where does it commence, when does it cease? And this vagueness seems well represented by that parameter *z.* This move at the same time enlarges the range of suspension of judgment to the interval from $-z$ to *z*. The remarkable point about axioms (1) and (2) is that they guarantee belief sets to be consistent and deductively closed, however we choose the threshold *z*. They are indeed equivalent to this general guarantee.

## 3. Conditional Ranks, Reasons, and the Dynamics of Ranks

So far, we have sketched only the static part of ranking theory. However, we mentioned that the numeric ranks are essentially used to account for the dynamics of belief; they are not just to represent greater and lesser firmness of (dis)belief. To achieve this, the crucial notion is that of conditional ranks.

*Definition 2*: Let $\kappa$ be a negative ranking function for $W$ and $\kappa(A) < \infty$. Then the *conditional rank* of $B$ *given* $A$ is defined as $\kappa(B \mid A) = \kappa(A \cap B) - \kappa(A)$.

We might rewrite this definition as:

(4) $\kappa(A \cap B) = \kappa(A) + \kappa(B \mid A)$ (the *law of conjunction*).

This is highly intuitive. For, what is your degree of disbelief in $A \cap B$? One way for $A \cap B$ to be false is that $A$ is false; this contributes $\kappa(A)$ to that degree. However, if $A$ is true, $B$ must be false; and this adds $\kappa(B \mid A)$.

It immediately follows for all propositions $A$ and $B$ with $\kappa(A) < \infty$:

(5) $\kappa(B \mid A) = 0$ or $\kappa(\sim B \mid A) = 0$ (*the conditional law of negation*).

This law says that even conditional belief must be consistent. If both, $\kappa(B \mid A)$ and $\kappa(\sim B \mid A)$, were $> 0$, both, $B$ and $\sim B$, would be (dis-)believed given $A$, and this must be excluded, as long as the condition $A$ itself is considered possible. Indeed, given definition 2 and axiom (1), we could axiomatize ranking theory also by (5) instead of (2). Hence, the only substantial assumption written into ranking functions is conditional consistency.

Axioms (1) and (2) did not refer to any cardinal properties of ranking functions. However, the definition of conditional ranks involves arithmetical operations and thus presupposes a cardinal understanding of ranks. We will see below how this may be justified. We hasten to add that one could as well define positive conditional ranks by $\beta(B \mid A) = \kappa(\sim B \mid A)$ and two-sided conditional ranks by $\tau(B \mid A) = \kappa(\sim B \mid A) - \kappa(B \mid A)$.

As an illustration, consider again Table 1 and the conditional beliefs contained therein. We can see that precisely the (material) if-then propositions non-vacuously held true correspond to conditional beliefs. According to the κ specified, you believe, e.g., that Tweetie can fly given it is a bird (since $\kappa(\sim F \mid B) = 1$) and also given it is a bird, but not a penguin (since $\kappa(\sim F \mid B \cap \sim P) = 2$), and that Tweetie cannot fly given it is a penguin (since $\kappa(F \mid P) = 3$). Hence, your vacuous belief in the material implication "if Tweety is a penguin, it can fly" does not amount to a corresponding conditional belief. In other words: "if, then" expresses conditional belief rather than material implication (see also chapter 6.1 by Starr in this volume).

A first fundamental application of conditional ranks lies in the notion of an *epistemic reason*, which is at the center of the entire handbook. It is very natural to say that *A* is a reason for *B* iff *A* speaks in favor of *B* or confirms *B*, if *A* makes *B* more plausible or less implausible, or if *B* is more credible or less incredible given *A* than given ~*A*. This explanation works for any conception of conditional degrees of belief. In a probabilistic interpretation it amounts to Carnap's notion of incremental confirmation or positive relevance (Carnap 1950/62), which is basic for confirmation theory (see chapter 4.3 by Merin in this volume). Ranking-theoretically, it leads to

*Definition 3*: *A* is a *reason for B* relative to the negative ranking function κ or the associated two-sided ranking function τ iff $\tau(B \mid A) > \tau(B \mid {\sim}A)$.

We may show that if *A* is a deductive reason for *B,* i.e., if $A \subseteq B$, then *A* is also a reason for *B* according to definition 3 (given $\kappa(A), \kappa({\sim}B) < \infty$). Clearly, this definition provides only a subjectively relativized notion of a reason entirely depending on the subject's doxastic state. Philosophers strive for a more objective notion of a reason[3], perhaps because they take objective deductive reasons as a paradigm. In our view, the extent to which a more objective notion may be reached is a philosophically fundamental, alas very open issue (see Spohn, 2018).

3 See also chapters 2.1 by Broome, 2.2 by Wedgwood, and 12.2 by Smith (in this volume).

On this account, reasons can take four significant forms, depending on whether $\tau(B \mid A)$ and $\tau(B \mid {\sim}A)$ are positive or negative. E.g., *A* is a *sufficient* reason for *B* iff *B* is believed given *A* and not believed given ~*A*. We suggest that this is indeed the core meaning of the term "sufficient reason", although it is often used differently.

Moreover, we may define that *B* is (doxastically) *irrelevant* to or *independent* of *A* if neither *A* nor ~*A* is a reason for *B.* On this basis, a theory of (conditional) independence can be developed in ranking terms in far-reaching analogy to the probabilistic theory. For instance, the theory of Bayes nets (see chapter 4.2 by Hartmann in this volume) works equally well in ranking theory (see Goldszmidt & Pearl, 1996).

A second fundamental application of conditional ranks lies in the dynamics of beliefs and ranks. As in probability theory, we may say that we should simply move to the degrees of belief conditional on the evidence *E* learned. Thereby, though, the evidence *E* acquires maximal certainty, either probability 1 or positive rank ∞. This seems too restrictive. In general, evidence may be (slightly) uncertain, and our rules for doxastic change through evidence or learning—we do not attend to changes caused in other ways like forgetting—should take account of this. In ranking theory, it is achieved by two principles: first, conditional ranks given the evidence *E* and given its negation ~*E* are not changed by the evidence itself—how could it change them?—and second, the evidence *E* does not become maximally certain, but improves its position by *n* ranks, where *n* is a free parameter characterizing the specific information process.[4] These two assumptions suffice to uniquely determine the kinematics of ranking functions, i.e., ranking-theoretic conditionalization.

In order to see how this works look again at our Tweety example. Suppose you learn in some way and accept with firmness 2 that Tweetie is a bird. Thus you shift up ~*B*-possibilities by 2 and keep constant the rank differences within *B* and within ~*B*. This results in the posterior ranking function κ':

| κ' | $B \cap \sim P$ | $B \cap P$ | $\sim B \cap \sim P$ | $\sim B \cap P$ |
|---|---|---|---|---|
| *F* | 0 | 4 | 2 | 13 |
| ~*F* | 2 | 1 | 2 | 10 |

(*Table* 2)

In κ' you believe that Tweetie is a bird able to fly, but not a penguin; you still neglect this possiblity. So, in κ' you believe more than in κ; in belief revision

4 This is completely analogous to Jeffrey conditionalization in probability theory (see Jeffrey 1983, ch. 11, and chapter 4.1 by Hájek & Staffel in this volume).

theory (cf. Chapter 5.2 of Rott in this volume) this would be called a belief expansion.

Next, to your surprise, you tentatively learn and accept, say with firmness 1, that Tweetie is indeed a penguin. This results in another ranking function κ'', which shifts all *P*-possibilities down by 1 and all ~*P*-possibilities up by 1, so that *P* is indeed believed with firmness 1 (i.e., κ''(~*P*) = 1):

| κ'' | $B \cap {\sim}P$ | $B \cap P$ | ${\sim}B \cap {\sim}P$ | ${\sim}B \cap P$ |
|---|---|---|---|---|
| *F* | 1 | 3 | 3 | 12 |
| ~*F* | 3 | 0 | 3 | 9 |

(*Table* 3)

So, you have changed your mind and believe in κ'' that Tweetie is a penguin bird that cannot fly. In belief revision theory this would be called a belief revision. Obviously, belief contraction (cf. Chapter 5.2 of Rott in this volume), where you simply give up a belief previously held without replacing it by a new one, can also be modeled by ranking-theoretic conditionalization. The example already demonstrates that this rule of belief change can be iteratively applied *ad libitum*.

An important application of ranking-theoretic conditionalization is that it delivers a measurement procedure for ranks that justifies the cardinality of ranks. This procedure refers to iterated belief contraction. Its point is this: if your iterated contractions behave as prescribed by ranking theory[5], then that behavior uniquely determines your ranking function up to a multiplicative constant. That is, your ranks can thereby be measured on a ratio scale (see Hild & Spohn 2008). The consequences of the fact that ranks are measured only on a ratio scale await investigation. They imply, e.g., a problem analogous to the problem of the interpersonal comparison of utilities.

## 4. Comparisons

The formal structure defined by axioms (1) and (2) has been called *Baconian probability* by Cohen (1980). Its first clear appearance is in the functions of potential surprise developed by Shackle (1949). The structure is also hidden in Rescher (1964) and is clearly found in Cohen's own work in Cohen (1970, 1977). The crucial formal advance of ranking theory lies in the definition of conditional ranks, which is nowhere found in these works and which makes the theory a properly cardinal one.

---

5 In fact, we need no more than twofold non-vacuous contractions.

Belief revision theory was precisely about the dynamics of belief. However, they only conceived of entrenchment *orders*. And within their ordinal framework, there was no clear solution of the problem of iterated belief revision (see chapter 5.2 of Rott in this volume).

Possibility theory, building on early work of Zadeh (1978) and developed by Dubois and Prade (see chapter 4.7 by Dubois & Prade in this volume), is in fact equivalent to ranking theory; (2) is the characteristic property of possibility measures. However, the interpretation of those measures was intentionally left open, leaving considerable formal uncertainty as to how conceive of conditional degrees of possibility.

The theory of Dempster-Shafer belief functions, as developed in Shafer (1976), seems to be a far more general theory (see again chapter 4.7 of Dubois & Prade in this volume), which comprises probability theory and also ranking theory as a special case. Shafer (1976, ch. 10) defines so-called consonant belief functions which appear to be equivalent to negative ranking functions. However, their respective dynamic behavior diverges, a fact that prevents reduction of ranking theory to the DS theory (see Spohn 2012, sect. 11.9).

Of course, the largest comparative issue is how ranking theory relates to probability theory. A comparison of their axioms and their form of conditionalization suggests translating the sum of probabilities into the minimum of ranks and the product and the quotient of probabilities into the addition and the subtraction of ranks. This translation works only for negative ranks; that's why the latter provide the formally preferred version of ranking theory. And it explains why very many things that can be done with probability theory also work for ranking theory in a meaningful way.

However, the translation does not justify conceiving ranks in probabilistic terms. As mentioned in section 2, belief in *A* cannot be probabilistically represented by $P(A) = 1 - \varepsilon$, if one sticks to the consistency and deductive closure of belief sets. The relation of probability and belief is hotly debated in philosophy without a clear solution emerging (see, e.g., the proposals of Leitgeb 2017 and Raidl & Skovgaard-Olsen 2016). Therefore, our attitude has always been to independently develop ranking theory as a theory of belief.

## 5. Ranking Functions in Artificial Intelligence

Besides probability theory and logic, ranking functions are among the most popular formalisms used for knowledge representation[6] and reasoning (KRR), and their popularity is still increasing because they provide a very versatile framework for many central operations in KRR, as already sections 2-4 pointed out. Most importantly, ranking functions are a convenient common basic tool for

6 In AI, the distinction between knowledge and belief is usually quite vague.

nonmonotonic reasoning and belief revision. Belief revision has been already explained in more detail in section 3, and nonmonotonic reasoning also deals with belief dynamics in that conclusions may be given up when new information arrives (so, the consequence relation is not monotonic, as in classical logic). Both fields emerged in the 1980's (partly) as a reaction to the incapability of classical logic to handle problems in everyday life that intelligent systems like robots were expected to tackle. Knowledge, or belief about the world is usually uncertain, and the world is always changing. Therefore, AI systems built upon classical logics failed. So-called preferential models (see Makinson 1989) provide an important semantics for nonmonotonic logics, their basic idea is to order worlds according to normality and focus on the minimal, i.e., the most plausible ones for reasoning. Likewise, AGM belief revision theory (see chapter 5.2 by Rott in this volume) needs orderings of worlds to become effective. For both fields, ranking functions offer quite a perfect technical tool that also complies nicely with the intuitions behind the techniques. Moreover, they can also evaluate conditionals and are an attractive qualitative counterpart to probabilities (see section 3).

Judea Pearl was probably the first renowned AI scientist to make use of ranking functions; his famous *system Z* (Pearl 1990) is based on them. He has continuously emphasized the structural commonsense qualities of probabilities and developed ranking functions as an interesting qualitative counterpart to probabilities. He set up his system Z as an "ultimate system of nonmonotonic reasoning" in terms of ranking functions. To date, it is one of the best and most convenient approaches to implement high-quality nonmonotonic reasoning.

Consequently, ranking functions are deeply connected with nonmonotonic and uncertain reasoning and with belief change, which are core topics in KRR. Many researchers make use of them in one way or another even if they rely on more general frameworks. Darwiche & Pearl (1997) presented general postulates for the iterated revision of general epistemic states, but illustrated their account with ranking functions. So did Jin & Thielscher (2007) and Delgrande & Jin (2012) when they devised novel postulates for iterated and multiple revision. Interestingly, the independence properties for advanced belief revision which were proposed in those papers can be related to independence with respect to ranking functions (see Spohn 2012, ch. 7) in analogy with probabilistic independence (see Kern-Isberner & Huvermann, 2017).

Indeed, as suggested in section 3, ranking functions are particularly well suited for iterated belief change because they can easily be changed in accordance with AGM theory, returning new ranking functions which are readily available for a successive change operation. The main AGM operations are revision (adopting a belief) and contraction (giving up a belief), related by Levi and Harper identities (see chapter 5.2). In ranking theory, the connections between these operations are even deeper, since (iterated) contraction is just a special kind of (iterated) ranking conditionalization. Indeed, the results of (Kern-Isberner et al., 2017) show that iterated revision and contractions can be performed by a common methodology.

Continuing on that, and beyond practicality and diversity of ranking functions, it is crucial to understand that they are not just a pragmatically good choice but indeed allow for deep theoretical foundations of approaches to reasoning. It is the ease and naturalness with which they can handle conditionals—very similar to probabilities—that make them an excellent formal tool for modeling reasoning. Given that conditionals are, on the one hand, crucial entities for nonmonotonic and commonsense reasoning and belief change, and, on the other hand, formal entities fully accessible to conditional logics, this capability provides a key feature for logic-based approaches connecting nonmonotonic logics and belief change theories with commonsense and general human reasoning. More precisely, conditional ranks give meaning to differences between degrees in belief when observing *A* vs. *A* & *B* (see the *law of conjunction* in section 3), and the examples of belief change given in section 3 illustrate nicely how it is easily possible to preserve these differences under change when using ranking functions. This property has been elaborated as a *principle of conditional preservation* in Kern-Isberner (2004), giving rise to defining c-representations and c-revisions (all belief changes shown in section 3 are c-revisions). C-representations are c-revisions starting from a uniform ranking function and allowing for reasoning from conditional belief bases. Ranking theory is one of the few formal frameworks that is rich and expressive enough to allow such a precise formalization of conditional preservation which supports both belief change and inductive reasoning as a common methodology, probability theory is another.

## 6. Ranking Theory in Psychology

Potentially, ranking theory has applications for many areas of psychology. To illustrate, psychological research on belief revision has been carried out under the inspiration of AGM theory and probabilistic updating (Baratgin & Politzer 2010; Wolf et al. 2012; Oaksford & Chater 2013; see also chapter 5.4 by Gazzo & Knauff in this volume). Such work could be extended by ranking theory, given that one of its central motivations was to represent a notion of full belief, in contrast to probabilistic update mechanisms, while improving upon AGM theory to allow for iterative revisions. However, as Colombo et al. (2018) note, probabilistic Bayesian approaches are currently enjoying a boom in cognitive science to the neglect of alternative formal frameworks, like ranking theory.

As said, possibility theory is mathematically equivalent to ranking theory, yet differs fundamentally in its intended interpretation. In Da Silva Neves et al. (2002) and Benferhat et al. (2005), possibility theory was subjected to empirical testing. In Da Silva Neves et al. (2002), a direct route was chosen by testing whether participants' possibility judgments satisfy the rationality postulates codified in System P augmented by Rational Monotonicity ($A \mid\!\sim C$, $\neg (A \mid\!\sim \neg B)$; therefore $A \wedge B \mid\!\sim C$),[7] in case their responses violated Monotonicity ($A \mid\!\sim C$; therefore $A \wedge B \mid\!\sim C$). Interestingly, no such direct test of ranking theory based on the participants' judgments of disbelief, or implausibility, has yet been made.

What exists is the following. Isberner & Kern-Isberner (2017) investigated whether belief revision with ranking functions could retrodict findings of temporal delay when processing implausible information in tasks, where plausibility judgments would interfere with the task constraints (a finding known as "the epistemic Stroop effect"). A guiding assumption underlying this work is that ranking theory can be used to represent the situational model that participants construct during language comprehension. Moreover, Eichhorn et al. (2018) proposed a conditional-logical model based on ranking functions, which allows for the elaboration of plausible background knowledge.

In Ragni et al. (2017), it was investigated whether ranking theory could retrodict the suppression effect (Byrne, 1989), where endorsement rates of classically valid *modus ponens* (MP) ($A \rightarrow B$, $A$; therefore $B$) and *modus tollens* (MT) ($A \rightarrow B$, $\neg B$; therefore $\neg A$) are suppressed when further premises indicating possible defeaters are presented. To illustrate, normally inferring “Lisa will study late in the library” from the premises “Lisa has an essay to write” and

---

7 Here, $\mid\!\sim$ represents nonmonotonic or default entailment.

"if Lisa has an essay to write, then Lisa will study late in the library" would be seen as unproblematic. But if the additional premise "if the library is open, then Lisa will study late in the library" becomes available, then participants are much more reluctant to draw this inference. However, as Ragni et al. (2017) show, the inference mechanism exploiting ranking functions does not in itself retrodict the suppression effect. To do so, further assumptions about the underlying knowledge base instantiated in long-term memory need to be made. In addition, Ragni et al. (2017) present experiments that test ranking theory's ability to predict the participants' reasoning with MP and MT once nonmonotonic keywords such as "normally" are inserted. Characteristic of this line of research is that c-representations are used to inductively infer ranking functions that satisfy the constraints set by an assumed knowledge base.

Moreover, the account of conditionals in Spohn (2013, 2015) has inspired a series of experiments. Spohn (2013, 2015) outlines a number of expressive roles of conditionals that go beyond the Ramsey test, which merely takes conditionals to express conditional beliefs. For instance, conditionals may express reason relations as specified in definition 3. In Olsen (2014), a logistic regression model was presented to both formulate predictions for the participants' evaluations of the conclusions of MP, MT, AC ($A \rightarrow B$, $B$; therefore $A$), and DA ($A \rightarrow B$, $\neg A$; therefore $\neg B$) inferences and to suggest a solution to the problem of updating based on conditional information. In Skovgaard-Olsen et al. (2016a), a reason relation reading of the conditional was contrasted experimentally with the Ramsey test in participants' probability and acceptability evaluations of indicative conditionals and support was obtained for the reason relation reading. In Skovgaard-Olsen et al. (2019), it was found that there are patterns of individual variation in these results. In Skovgaard-Olsen et al. (2016b), participants' perceived relevance and reason relation evaluations were investigated and some first evidence for the explications of reason relations and perceived relevance introduced above was obtained. However, characteristic of this latter line of research is that it adopts an indirect route to testing ideas from ranking theory by exploiting its extension to conditionals in Spohn (2013, 2015) and its parallels to probability theory (see also Henrion et al. 1994).

Presently, paradigms operationalizing degrees of beliefs as probabilities are much more well-established than tasks using ranking functions. This is so in spite of the fact that the arithmetical operations of ranking theory require much less computational effort than those of probability theory. For instance, it is well-known that participants exhibit difficulties in properly integrating information about base rates of the rarity of a given disease in evaluating how likely it is that a person has the disease given a positive test result. Yet, interestingly, Juslin et al. (2011) find that the notorious base-rate neglect could be reduced when participants are given the tasks in a logarithmic format. Juslin et al. (2011) therefore conjecture that a linear, additive integration of information is more intuitive in the absence of access to overriding analytic rules that utilize a multiplicative format, like probabilities.

Nevertheless, there is a lack of direct experimental investigations of ranking functions. Perhaps due to the following challenges:

(i) Negative ranking functions are useful for conducting proofs. But it initially presents a conceptual challenge to think in terms of disbelief in negations of propositions as a way of representing full beliefs. Two-sided ranking functions solve this problem. But they come at the cost of having different rules applying to the negative and positive range of the scale.

(ii) An operationalization of ranking functions in terms of iterated contractions exists (see above). But it is not one that has received the same kind of experimental implementation as the operationalization of probabilities in terms of betting quotients.

(iii) Since negative ranks take natural numbers from 0 to infinity, there is no natural way of non-arbitrarily dividing the scale into regions of ascending degrees of disbelief other than a region of zero disbelief, a region of above zero disbelief, and a region of maximal disbelief. In contrast, the crude division of the probability scale of real numbers between 0 and 1 into decimal regions enables participants, and experimenters, to make qualitative differentiations between low degrees of belief (e.g. [0.0,0.3)), middle degrees of belief (e.g. [0,3-0.7]), and strong degrees of belief (e.g.(0.7, 1.0]).

If challenges such as these can be overcome in future work, ranking theory potentially has a lot to offer to psychology, and cognitive science more generally.

**References**

Baratgin, J., & Politzer, G. (2010). Updating: a psychologically basic situation of probability revision. *Thinking & Reasoning, 16,* 253-287.

Benferhat, S., Bonnefon, J. F., & da Silva Neves, R. (2005). An overview of possibilistic handling of default reasoning, with experimental studies. *Synthese, 146,* 53-70.

Byrne, R. M. (1989). Suppressing valid inferences with conditionals. *Cognition, 31,* 61-83.

Carnap, R. (1962). *Logical foundations of probability*. Chicago, IL: University of Chicago Press.

Christensen, D. (2004). *Putting logic in its place: Formal constraints on rational belief*. Oxford: Oxford University Press on Demand.

Cohen, L. J. (1970). *The implications of induction*. London, UK: Methuen.

Cohen, L. J. (1977). *The probable and the provable*. Oxford, UK: Oxford University Press.

Cohen, L. J. (1980). Some historical remarks on the Baconian conception of probability. *Journal of the History of Ideas, 41,* 219-231.

Colombo, M., Elkin, L., & Hartmann, S. (2018). Bayesian cognitive science, monopoly, and neglected frameworks. *The British Journal for the Philosophy of Science*. https://doi.org/10.1093/bjps/axy059

Darwiche, A., & Pearl, J. (1997). On the logic of iterated belief revision. *Artificial Intelligence, 89,* 1-29.

Da Silva Neves, R., Bonnefon, J., & Raufaste, E. (2002). An empirical test of patterns for nonmonotonic inference. *Annals of Mathematics and Artificial Intelligence, 34,* 107–130.

Delgrande, J. P., & Jin, Y. (2012). Parallel belief revision: Revising by sets of formulas. *Artificial Intelligence, 176,* 2223-2245.

Eichhorn, C., Kern-Isberner, G., & Ragni, M. (2018). Rational inference patterns based on conditional logic. In *Thirty-Second AAAI Conference on Artificial Intelligence,* New Orleans, LA.

Henrion, M., Provan, G., Del Favero, B., & Sanders, G. (1994). An experimental comparison of numerical and qualitative probabilistic reasoning. In *Uncertainty Proceedings 1994* (pp. 319-326). Burlington, MA: Morgan Kaufmann.

Hild, M., & Spohn, W. (2008). The measurement of ranks and the laws of iterated contraction. *Artificial Intelligence*, *172*, 1195-1218.

Goldszmidt, M., & Pearl, J. (1996). Qualitative probabilities for default reasoning, belief revision, and causal modeling. *Artificial Intelligence*, *84*, 57-112.

Isberner, M. B., & Kern-Isberner, G. (2017). Plausible reasoning and plausibility monitoring in language comprehension. *International Journal of Approximate Reasoning*, *88*, 53-71.

Jeffrey, R. C. (1983). *The Logic of Decision*. Chicago: University of Chicago Press.

Jin, Y., & Thielscher, M. (2007). Iterated belief revision, revised. *Artificial Intelligence*, *171*, 1-18.

Juslin, P., Nilsson, H., Winman, A., & Lindskog, M. (2011). Reducing cognitive biases in probabilistic reasoning by the use of logarithm formats. *Cognition*, *120*, 248-267.

Kern-Isberner, G. (2004). A thorough axiomatization of a principle of conditional preservation in belief revision. *Annals of Mathematics and Artificial intelligence*, *40*, 127-164.

Kern-Isberner, G., Bock, T., Sauerwald, K., & Beierle, C. (2017). Iterated contraction of propositions and conditionals under the principle of conditional preservation. *EPiC Series in Computing*, 50, 78-92.

Kern-Isberner, G., & Huvermann, D. (2017). What kind of independence do we need for multiple iterated belief change? *Journal of Applied Logic*, *22*, 91-119.

Leitgeb, H. (2017). *The stability of belief: How rational belief coheres with probability*. Oxford, UK: Oxford University Press.

Makinson, D. (1989). General theory of cumulative inference. In M. Reinfrank et al. (Ed.), *Non-monotonic Reasoning* (pp. 1-18). Berlin, Germany: Springer.

McGrath, M. (2012). Propositions, In Zalta E.N. (Ed.), The Stanford Encyclopedia of Philosophy, https://plato.stanford.edu/archives/spr2014/entries/propositions/

Oaksford, M., & Chater, N. (2018). Dynamic inference and everyday conditional reasoning in the new paradigm. In *New Paradigm Psychology of Reasoning* (pp. 111-144). Routledge.

Pearl, J. (1988). *Probabilistic reasoning in intelligent systems: networks of plausible inference*. San Mateo, CA: Morgan Kaufmann.

Pearl, J. (1990). System Z: A natural ordering of defaults with tractable applications to nonmonotonic reasoning. In *Proceedings of the 3rd Conference on Theoretical Aspects of Reasoning about Knowledge* (pp. 121-135). San Francisco, CA. Morgan Kaufmann Publishers Inc.

Ragni, M., Eichhorn, C., Bock, T., Kern-Isberner, G., & Tse, A. P. P. (2017). Formal nonmonotonic theories and properties of human defeasible reasoning. *Minds and Machines*, *27*, 79-117.

Raidl, E., & Skovgaard-Olsen, N. (2017). Bridging ranking theory and the stability theory of belief. *Journal of Philosophical Logic*, *46*, 577-609.

Rescher, N. (1964). *Hypothetical Reasoning.* Amsterdam, Netherlands: North-Holland.

Shackle, G. L. S. (2012). *Expectation in economics*. Cambridge, UK: Cambridge University Press.

Shafer, G. (1976). *A mathematical theory of evidence*. Princeton, NJ: Princeton University Press.

Shenoy, P. P. (1991). On Spohn's rule for revision of beliefs. *International Journal of Approximate Reasoning*, *5*, 149-181.

Skovgaard-Olsen, N. (2014). *Making ranking theory useful for psychology of reasoning*. (Doctoral dissertation, University of Konstanz, Germany). Retrieved from http://nbn-resolving.de/urn:nbn:de:bsz:352-0-262692

Skovgaard-Olsen, N., Singmann, H., & Klauer, K. C. (2016). The relevance effect and conditionals. *Cognition*, *150*, 26-36. https://doi.org/10.1016/j.cognition.2015.12.017

Skovgaard-Olsen, N., Singmann, H., & Klauer, K. C. (2017). Relevance and reason relations. *Cognitive Science*, *41*, 1202-1215. https://doi.org/10.1111/cogs.12462

Skovgaard-Olsen, N., Kellen, D., Hahn, U., & Klauer, K. C. (2019). Norm conflicts and conditionals. *Psychological Review*. https://doi.org/10.1037/rev0000150

Spohn, W. (1988). Ordinal conditional functions: A dynamic theory of epistemic states. In *Causation in decision, belief change, and statistics* (pp. 105-134). Dordrecht: Springer.

Spohn, W. (2009). A survey of ranking theory. In *Degrees of belief* (pp. 185-228). Dordrecht, Netherlands: Springer.

Spohn, W. (2012). *The laws of belief: Ranking theory and its philosophical applications*. Oxford, UK: Oxford University Press.

Spohn, W. (2013). A ranking-theoretic approach to conditionals. *Cognitive Science*, *37*, 1074-1106.

Spohn, W. (2015). Conditionals: A unifying ranking-theoretic perspective. *Philosopher's Imprint*, *15*, 1-30.

Spohn, W. (2018). Epistemic justification: its subjective and its objective ways. *Synthese*, *195*, 3837-3856. https://doi.org/10.1007/s11229-017-1393-0

Spohn, W. (2019). Defeasible normative reasoning. *Synthese, 196*, DOI: 10.1007/s11229-019-02083-2

Wolf, A. G., Rieger, S., & Knauff, M. (2012). The effects of source trustworthiness and inference type on human belief revision. *Thinking & Reasoning*, *18*, 417-440.

Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. *Fuzzy sets and systems*, 1, 3-28.