# The repugnant conclusion can be avoided with moral intuitions intact: A lesson in order.

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**Abstract**

The repugnant conclusion poses a conundrum in population ethics that has evaded satisfactory solution for more than four decades. In this article, I show that the repugnant conclusion can be avoided without sacrificing any moral intuitions. This is achieved using non-Archimedean orders, which admit the possibility of pairs of goods for which no amount of one is better than a single unit of the other. I show with minimal assumptions, not only are such goods sensible, they are compulsory. I show that utilitarianism and expected utility theory in their canonical forms are not in general suitable in this setting, and using these tools naively can lead to ethical errors that are arbitrarily serious. Multi-dimensional lexicographic expected utility representations are required. I use fuzzy sets to show there needn't be a sharp transition between non-Archimedean equivalent goods, and the lexicographic ordering may only manifest asymptotically. This might be unavoidable due to intrinsic physical limits on the ability to discriminate between different goods arising from e.g. quantum mechanics. Limited discriminatory power causes preferences to be non-transitive, which can resolve problems related to so-called 'fanaticism' and 'recklessness' wherein 'rational' decision-makers counterintuitively prefer arbitrary long shot bets to a guaranteed lesser outcome.

# 1 Introduction

The repugnant conclusion is the idea that for any finite population of lives of arbitrarily high quality, there is a larger population consisting of lives that are 'barely worth living' that is ethically preferable. It has been argued that this conclusion follows if one accepts that a small reduction in the quality of life for a given population may be 'compensated' by adding more people. This reasoning can be applied recursively to make ethical improvements at each step, ending with a larger population of lives that are each 'barely worth living'.[1, 2, 3]

Many attempts have been made to avoid the repugnant conclusion, but they all appear to require accepting ethical positions that are often perceived to be as counterintuitive or undesirable as the repugnant conclusion itself. For a summary of such attempts and their issues, see [3]. This state of affairs has inspired a number of 'impossibility theorems' that purport to prove there is no theory of population ethics satisfying an intuitively desirable set of axioms that includes avoiding the repugnant conclusion.[4, 5, 6, 7, 8]

Lexicographic orders have been suggested in many places as a potential resolution of the repugnant conclusion.[1, 7, 9, 10, 11, 12, 13, 14] This entails the existence of a pair of goods $x, y$ such that $y$ is better than any number of $x$. While lexicographic orders are sufficient to avoid the repugnant conclusion, they may have other implications that are unpalatable. The question of whether lexicographic orders are sensible and necessary in population ethics remains open. One of the main contributions developed in this article is an affirmative answer to this question.

A pre-requisite for lexicographic orders in population ethics is that welfare is represented by a multi-dimensional vector as opposed to a scalar. This idea has appeared in many places, perhaps most notably advocated by Sen.[15, 16] Arguments in favour of this position are often qualitative and appeal to our intuitions and lived experiences. For example, certain aspects of well-being may intuitively appear to be non-fungible with, or irreducible to, others. These approaches more or less ask us to take it as axiomatic that welfare is represented by a vector.

In this article, I show that one needn't rely on such arguments. A complete description of a population is given by a vector whose components represent the number of life years at each possible distinct life quality. Addition corresponds to combining two populations, and scalar multiplication corresponds to scaling a population up or down. The vector space is multi-dimensional so long as there are at least two distinct qualities of life. This formalism parallels standard mathematical treatments of individual choice in economics, in particular general equilibrium theory.[17] However, instead of a choice over different commodity bundles, here the choice is over different populations.

In the literature, the lexicographic order is often introduced in an ad-hoc fashion that can appear contrived to avoid the repugnant conclusion. Here, I show that the lexicographic order follows as a consequence of ethical preferences between two populations being unchanged by scaling both up by the same factor, or by adding another identical population to both. This may be considered a mathematical expression of totalism. The resulting mathematical structure is a preordered vector space. It has long been known that preordered vector spaces are lexicographically ordered, and the proof I give is not new.[18, 19, 20, 13]

This result can be understood intuitively by imagining a year of life worth living denoted by $q_1$, and a year of life worse than non-existence denoted by $q_2$. It is reasonable to think that $n$ life years at quality $q_1$ plus $m$ life years at quality $q_2$ is indifferent to non-existence for some $n, m > 0$. If ethical preferences are preserved by scaling, then $\lambda(nq_1 + mq_2) \simeq 0 \; \forall \lambda$, and $nq_1 + mq_2$ must therefore belong to a lower Archimedean class than either $q_1$ or $q_2$. Denying the possibility of non-Archimedean equivalent goods requires finding error in this reasoning. My conclusion is that the lexicographic order is an inevitable consequence of totalism.

It has been suggested that lexicographic orders nevertheless lead to counterintuitive conclusions that may make them unsuitable in population ethics. In particular, every finite sequence that starts and ends with non-Archimedean equivalent goods must contain a pair of successive elements that are in different Archimedean classes.[21, 22] This seems unacceptable because the difference between successive elements can be made arbitrarily 'small'.

Several works have suggested this conclusion might be avoided or defused by appealing to some combination of vagueness, incommensurability, indeterminacy, or fuzziness.[11, 9, 23, 12, 24] In this paper, I give a novel approach to this issue that does not rely on any of these concepts. The key innovation compared to earlier work is generalising to transfinite sequences that are indexed by intervals in the ordinal numbers, as opposed to sequences that are indexed by the natural numbers. This allows for the possibility of elements that do not have a predecessor. It is possible to construct a monotone transfinite sequence with arbitrarily small increments spanning multiple Archimedean classes, with no two adjacent elements in different Archimedean classes.

Of course one can always take a finite subsequence that features successive elements where one element is preferred to any number of its successor. However, this is also possible in the case where all goods are Archimedean equivalent. For example, consider a sequence that starts with $n > 0$ years of life worth living and decreases in duration by a small discrete quantity until reaching zero years. The only substantive difference in the case of non-Archimedean goods is that one can have a population consisting of a non-zero number of life years that 'behaves like 0', because it is indifferent to 0. I conclude from this that the existence of non-Archimedean goods poses no additional theoretical difficulty compared to the case where all goods are Archimedean equivalent.

I show that preordered vector spaces admit a natural fuzzification of their positive cone, allowing for the possibility that there may not be a precise apparent border between non-Archimedean equivalent goods. I argue that such limitations are unavoidable if there are intrinsic bounds on the precision with which physical quantities can be measured stemming from e.g. quantum mechanics. This formalism may be thought of as an effective theory that uses fuzzy sets as a simple way to account for the fact that the model cannot be completely physically accurate at all scales. I introduce a fuzzy expected utility representation, and suggest that it is equivalent to an appropriate fuzzification of the von Neumann Morgenstern axioms.

The fuzzy set construction presented here is a generalisation of the 'lexical threshold' view in [12]. I present a novel example showing that fuzziness allows for the possibility that the transition between non-Archimedean goods occurs smoothly, and only becomes fully apparent in the limit of large populations. This limit can be thought of as 'viewing populations under a magnifying glass', allowing discrimination to occur at a higher resolution. Alternatively, it can be intuitively understood by imagining empirically measuring the quality of different lives and performing a statistical test to determine whether they are in different Archimedean classes. Certainty that they are in distinct Archimedean classes only appears asymptotically, in the limit of infinite populations.

In a multi-dimensional context, fuzziness causes preferences to be non-transitive. This leads to a new resolution of problems related to so-called 'fanaticism' and 'recklessness', in which a decision-maker acting in accordance with a seemingly reasonable set of axioms including transitivity may counterintuitively prefer bets that give a sufficiently large payoff with arbitrarily small probability to a lesser, sure payoff.[25, 26, 27] The resolution works by rejecting transitivity at all scales as a normative requirement because it would require perfect discriminatory power, which may be inconsistent with physical laws.

This paper is structured as follows. In section 1, I introduce preordered vector spaces as a natural mathematical structure for totalist population ethics. I show that any non-trivial totally preordered vector space of dimension $> 1$ is non-Archimedean. The conclusion is that not only are non-Archimedean goods sensible, they are compulsory. I elucidate the structure of lexicographically ordered vector spaces, showing that it is possible that there are only two Archimedean classes, and all goods in the lower class are indifferent to zero. However, in principle there can be as many Archimedean classes as there are levels of life quality.

In sections 2, I give a brief introduction to ordinal numbers. This prepares for section 3, where I use a novel argument based on transfinite sequences to demonstrate that there is always an arbitrarily fine-grained, monotone well ordering connecting any pair of non-Archimedean equivalent elements with no two successive elements in distinct Achimedean classes. This reasoning makes no appeal to vagueness, incommensurability or fuzziness. I argue that non-Archimedean equivalent goods pose no additional conceptual difficulty compared to the case where all goods are Archimedean equivalent.

Finally, in section 4, I use a fuzzy set construction to parsimoniously model vagueness/uncertainty. I introduce a fuzzy expected utility representation, and suggest its equivalence to a fuzzification of the von Neumann Morgenstern axioms. Fuzziness leads to non-transitivity, and I put this forward as a new resolution of the problems of 'fanaticism' and 'recklessness'. Using this approach, I demonstrate the possibility that the distinction between non-Archimedean equivalent goods only manifests asymptotically.

# 2 The mathematics of population ethics

## 2.1 Totalist population ethics is a preordered vector space

I begin by showing that vector spaces are the natural mathematical framework for population ethics, and preordered vector spaces are the natural mathematical framework for totalist population ethics.

Let $V$ be a vector space with basis $B$, where each $b \in B$ corresponds to a life year at a different level of quality. $V$ may be finite or infinite dimensional. Each element of $V$ is a finite linear combination of basis vectors where the components indicate the number of life years at each quality. Thus each element of $V$ describes a finite population with a particular profile of life qualities. Addition in $V$ corresponds to combining two populations, while multiplication by a positive number represents scaling a population up or down.

**Definition 1. Preorder.** Consider the following properties of a binary relation $\geq$ on a set $S$ that hold $\forall x, y, z \in S$,

$$x \geq x \qquad \text{Reflexive} \tag{2.1}$$

$$z \geq y \text{ and } y \geq x \Rightarrow z \geq x \qquad \text{Transitive} \tag{2.2}$$

$$x \geq y \text{ or } y \geq x \qquad \text{Complete.} \tag{2.3}$$

The relation $\geq$ is called a preorder if it satisfies 2.1-2.2, and a total preorder if it satisfies 2.1-2.3.

**Definition 2. Preordered vector space.** A preordered vector space is a pair $(V, \geq)$ where $V$ is a real vector space and $\geq$ is a preorder on $V$, such that $\forall x, y, z \in V, \lambda > 0$,

$$y \geq x \Rightarrow y + z \geq x + z \tag{2.4}$$

$$y \geq x \Rightarrow \lambda y \geq \lambda x. \tag{2.5}$$

$(V, \geq)$ will be called a totally preordered vector space if $\geq$ is a total preorder.

Preordered vector spaces capture the idea for any two populations, adding some other population or $\lambda$-fold replication does not affect the preordering. This may be considered a mathematical expression of totalism. To see this, note that if $\geq$ can be represented by a utility function $u : V \mapsto \mathbb{R}$, then the preordered vector space axioms are equivalent to linearity of $u$.

Note that the structure of a preordered vector space would not be expedient in e.g. consumer choice theory. One can imagine a pair of goods $x$ and $z$ (tea bags and milk) that complement each other, along with another pair of goods $y$ and $z$ (orange juice and milk) that don't, in a way that could lead to violations of 2.4. In the population ethics setting, however, it is useful for any inter-personal, inter-population or inter-life-year complementarities to be subsumed under quality of life. This can be achieved by e.g. interpreting a basis vector as a year of life of a given quality spent in a virtual reality machine that is indistinguishable from reality. Then, for example, a year of life enhanced by the company of one's family needn't require vectors representing the lives of family members. In this way, any complementarities can be captured through an appropriate interpretation of the basis vectors. Further, this interpretation maps any possible population with any profile of life qualities onto an element of the vector space. Putting a total preorder on this space can then allow any population ethics question in principle to be answered.

2.5 implies scale invariance. In the consumer choice setting, one can imagine two goods $y$ and $x$ (chocolate and bread) such that one unit of $y$ is preferred to $x$, but there is some $\lambda > 1$ such that $\lambda$ units of $x$ is preferred to $\lambda$ units of $y$. In the population ethics setting, however, it is reasonable to take it as axiomatic that if population $v_2$ is better than population $v_1$, then $v_2$ scaled by $\lambda$ is better than $v_1$ scaled by $\lambda$.

Translation invariance 2.4 allows us to interpret a positive number of lives as a gain, and a negative number of lives as a loss, relative to some status quo. Informally speaking, for any prospective loss we wish to consider, we may imagine a population large enough that such a loss is possible, and assign it the element 0. This is possible because the preordering is preserved by translations. More formally, we may consider the affine space that $V$ is associated to.

## 2.2 Preordered vector spaces are lexicographically ordered

Next, I prove that every preordered vector space is lexicographically ordered. This mathematical result has been known for some time, and is typically used in proofs of non-Archimedean generalisations of the von Neumann Morgenstern theorem.[18, 19, 20, 13] The fresh insight here comes from applying this mathematical result in the setting of population ethics, with the implication that totalism leads to the lexicographic order.

**Definition 3. Convex cone.** A convex cone $C$ in a real vector space $V$ is a subset $C \subseteq V$ such that,

$$C + C \subseteq C \tag{2.6}$$

$$\lambda C \subseteq C \; \forall \lambda \geq 0. \tag{2.7}$$

Here, addition and multiplication for subsets of a vector space are defined by $S_1 + S_2 = \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}$ and $\lambda S = \{\lambda s \mid s \in S\}$. Examples of non-convex and convex cones in $\mathbb{R}^3$ are shown in Figure 1.

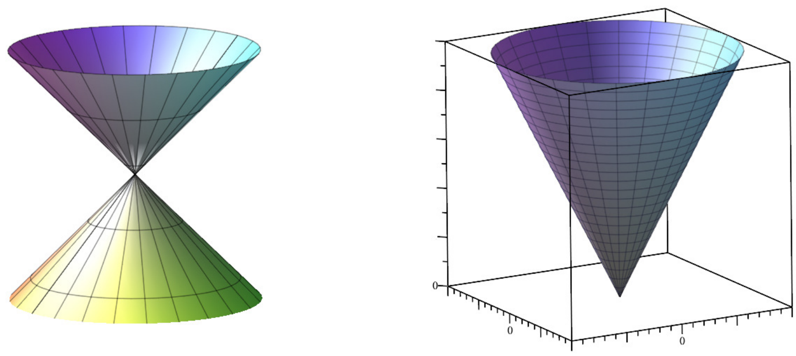

Figure 1: **Left:** A non-convex cone in $\mathbb{R}^3$. **Right:** A convex cone in $\mathbb{R}^3$.

**Theorem 1.** There is a $1-1$ correspondence between convex cones $C$ in a real vector space $V$, and preordered vector spaces $(V, \geq)$, given by defining

$y \geq x$ iff $(y - x) \in C$. $\geq$ is a total preorder iff $C \cup (-C) = V$.

*Proof.* Let $x, y, z \in V$. $C \cup (-C) = V$ implies every element of $V$ is in $C$ or $-C$ or both. This in turn implies that $y \geq x$ or $x \geq y$ $\forall x, y \in V$, which is completeness 2.3.

I will show that every convex cone in $V$ defines a totally preordered vector space. Assume $y \geq x$. Then

$$(y - x) \in C \Rightarrow (y + z) - (x + z) \in C \tag{2.8}$$

$$\Rightarrow (y + z) \geq (x + z), \tag{2.9}$$

which is translation invariance 2.4. Let $\lambda \geq 0$.

$$(y - x) \in C \Rightarrow \lambda(y - x) \in C \tag{2.10}$$

$$\Rightarrow \lambda y \geq \lambda x, \tag{2.11}$$

where I have used 2.7. This is scale invariance 2.5.

Reflexivity 2.1 is trivially true since $x - x = 0$ and every convex cone contains 0, giving $x \geq x$.

Assume that $z \geq y$. Then $(z - y), (y - x) \in C$, and by 2.6,

$$(z - y) + (y - x) = (z - x) \in C \tag{2.12}$$

$$\Rightarrow z \geq x, \tag{2.13}$$

which is transitivity 2.2.

Finally I will show that in every preordered vector space, the set $C := \{v \in V | v \geq 0\}$ is a convex cone. Let $x, y \in C$ and $\lambda \geq 0$. Then $(x + y) \geq y \geq 0$ using 2.4, and $\lambda x \geq 0$ by 2.5. But these are the defining properties of a convex cone 2.6, 2.7. ■

$C$ will be called the positive cone of $(V, \geq)$, and elements in $C$ will be called positive.

**Definition 4. Archimedean equivalence.** Two non-zero elements $x, y \in V$ of a preordered vector space $(V, \geq)$ will be said to be Archimedean equivalent if there exists $n, m \in \mathbb{N}$ such that

$$n|x| \geq |y| \tag{2.14}$$

$$m|y| \geq |x|, \tag{2.15}$$

where $|x| := \max(x, -x)$. I will use the notation $y \gg x$ if $y \geq x$ and $x, y$ are not Archimedean equivalent. $(V, \geq)$ will be called Archimedean if all its non-zero elements are Archimedean equivalent.

**Proposition 1.** Let $x, y \in V$ be elements of a preordered vector space $(V, \geq)$. If $y \gg x$, then $ny \gg x \; \forall n > 0$.

*Proof.* From the definition 4 of Archimedean equivalence, we have $|y| \geq m|x| \; \forall m \in \mathbb{N}$. Assume that there exists $n > 0, m \in \mathbb{N}$ such that $m|x| \geq |ny|$. Then $\frac{m}{n}|x| \geq |y|$, which is a contradiction. ■

This implies that if one unit of $y$ is better than $x$ and $y$ is not Archimedean equivalent to $x$, then any strictly positive number of units of $y$ is better than $x$.

**Theorem 2. Supporting hyperplane theorem.** For any convex subset $S$ of $\mathbb{R}^n$ and any point $x_0$ in its boundary, there exists a supporting hyperplane $H_0$ that contains $x_0$, such that S is a subset of one of the closed half spaces defined by $H_0$.

*Proof.* The proof is an application of the separating hyperplane theorem. See section 2.5 of [28] ■

**Theorem 3.** Every finite-dimensional subspace of a totally preordered vector space is lexicographically ordered.

*Proof.* Let $V_n$ be an n-dimensional subspace of $V$. Then $V_n$ is also a totally preordered vector space, with an induced order given by the positive cone $C_n = C \cap V_n$. That $C_n$ is a convex cone can be seen by taking the intersection of 2.6, 2.7 with $V_n$,

$$(C + C) \cap V_n = V \cap V_n \quad \Rightarrow \quad C_n + C_n = V_n \tag{2.16}$$

$$\lambda C \cap V_n \subseteq C \cap V_n \quad \Rightarrow \quad \lambda C_n \subseteq C_n. \tag{2.17}$$

We also have $C \cap (-C) = V \Rightarrow C_n \cap (-C_n) = V_n$.

If $C_n = V_n$, then

$$x \geq 0 \geq x \quad \Rightarrow \quad x \simeq 0 \; \forall x \in V_n, \tag{2.18}$$

which is the trivial case. If $C_n \neq V_n$, then $C_n$ must have a non-empty boundary that includes 0. Using Theorem 2, the convex cone $C_n$ of $(V_n, \geq)$ has a supporting hyperplane $H_0$ containing 0, defined by $a.x = 0$. Assume $C_n$ has another distinct supporting hyperplane. This would imply there is a non-zero vector $x$ that is in the upper half space defined by one hyperplane, and the lower half space defined by the other. Since $C_n \cap (-C_n) = V_n$ and any supporting hyperplane separates $C_n$ and $-C_n$, this would imply $x = -x$, which is a contradiction. Therefore $H_0$ is the unique supporting hyperplane that contains all of the points on the boundary of $C_n$.

Let $e_1$ be a unit vector pointing in the direction of $a$, which is a normal vector at right angles to $H_0$. Then $e_1$ is not Archimedean equivalent to any of the vectors in $H_0$ that are perpendicular to it. This is because subtracting a vector in $H_0$ from $e_1$ just translates $e_1$ in a perpendicular direction, which always stays in the positive cone $C_n$.

We can repeat the same reasoning inductively starting with the subspace given by the hyperplane $H_0$ to find a subset of basis vectors that lexicographically order $V_n$. ■

It is instructive to look at the case where $V$ is two-dimensional. Ignoring the trivial case where $C = V$, we can choose basis vectors $e_1, e_2$ for $V$ such that $C$ is either $H_{\geq}$ or $(H_{\geq} \setminus L_{<})$ . Here, $H_{\geq}$ is the upper half space of vectors whose vertical component is $\geq 0$, and $L_{<}$ is the strictly negative half-line along the horizontal axis. These positive cones can be visualised in Figure 2.

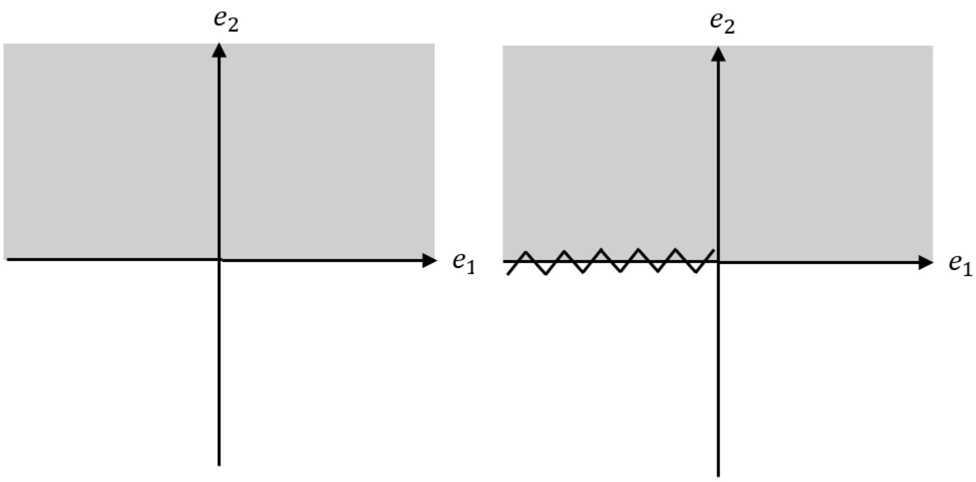


Figure 2: **Left:** The upper half space in $\mathbb{R}^2$. **Right:** The upper half space in $\mathbb{R}^2$ with the strictly negative half-line in the $e_1$ direction removed.

It should be clear that if the vertical component of vector $y$ is greater than or equal to the vertical component of vector $x$, then $y \geq x$. Therefore $e_2$ is the greatest unit vector, and $e_1$ is the least unit vector that is $\geq 0$. In the $(e_1, e_2)$ basis, we have that $(a_1, a_2) \geq (b_1, b_2)$ iff $a_2 > b_2$ or ($a_2 = b_2$ and $a_1 \geq b_1$), which is just the lexicographic order. For any positive vector $v$ that is not a scalar multiple of $e_1$, $v \gg e_1$. All other vectors excluding multiples of $e_1$ and 0 are Archimedean equivalent to each other.

There are vectors that are arbitrarily close in the sense that their difference can be as 'near to zero' as one desires. This provides an intuition for the necessity of non-Archimedean equivalent goods. Given any two linearly independent vectors $v_1, v_2$ corresponding to life years of different qualities, we can always find a linear combination $d$ whose vertical component is zero.

Then $d$ is not Archimedean equivalent to at least one of $v_1$ and $v_2$. For example, if $v_1 > 0$ and $v_2 < 0$ correspond to life qualities that are in the same Archimedean class, then $d = a_1 v_1 + a_2 v_2$ represents $a_1$ life years at quality $v_1$, and $a_2$ life years at quality $v_2$. This can be as 'close' to zero as one wants, leading to a good that is in a lower Archimedean class.

If the positive cone is $(H_{\geq} \setminus L_{<})$, then we have $e_1 \geq 0$ and $-e_1 \not\geq 0$, which implies that $e_1$ is strictly greater than zero, $e_1 > 0$. In this case, among all vectors that are greater than zero and whose components are positive and sum to 1, $e_1$ is the smallest. In other words, $e_1$ corresponds to a year of life that is barely worth living.

On the other hand, if the positive cone is the upper half space $H_{\geq}$, then we have $e_1 \geq 0$ and $e_1 \leq 0$, which implies that $e_1$ is indifferent to zero, $e_1 \simeq 0$. In this case, the set of vectors greater than zero whose components are positive and sum to 1 does not have a minimum, and there is no element that clearly corresponds to a life that is barely worth living.

More generally, let $B_{>}$ denote the subset of basis vectors that are not indifferent to zero. Now assume that there is a 'fixed exchange rate' between these life qualities, i.e there exists $n_i$ such that $b_0 \simeq n_i b_i \; \forall i \in I$, where $I$ is an indexing set for $B_{>}$. Then every vector in the span of $B_0 := \{(b_0 - n_i b_i)\}_{i \in I}$ is indifferent to zero. On the other hand, $b_i \gg b \; \forall b \in B_0, i \in I$. We can use $\simeq$ as an equivalence relation to collapse the space spanned by $B_0$ to a line. The result is a two-dimensional space with a convex cone as in the left hand side of Figure 2.

It is also possible that there are more than two distinct Archimedean classes. In principle, there can be as many distinct Archimedean classes as there are levels of life quality. The set of life years that are strictly greater than zero still need not have a minimal element, similarly to how e.g. there is no smallest strictly positive real number. If it does have a minimal element, this corresponds to the notion of a life year barely worth living. Otherwise, there is no clear concept of a life barely worth living.

## 2.3 Implications of lexicographic ordering in population ethics

In this section, I examine some implications of the lexicographic order in population ethics. In particular, I present the well-known result that the lexicographic order cannot be represented on the real numbers, and I discuss the multidimensional expected utility representation that results from dropping the Archimedean axiom in the von Neumann Morgenstern theorem.

**Proposition 2.** The lexicographic order on $\mathbb{R}^2$ cannot be represented on $\mathbb{R}$.

*Proof.* Let $\geq$ denote the lexicographic order on $\mathbb{R}^2$. Assume there is a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that $u(y) \geq u(x) \Leftrightarrow y \geq x$. Consider the map

$$f : \alpha \to [u((\alpha, 0)), u((\alpha, 1))]. \tag{2.19}$$

The Archimedean property of the real numbers can be used to show that every non-empty interval in the real numbers contains a rational number. I will use $\phi$ to denote a function that selects a rational number from a non-empty interval given as its argument. The function $\phi \circ f : \mathbb{R} \to \mathbb{Q}$ is an injection, since for $\alpha \neq \beta$ we have that either $u((\alpha, 0)) > u((\beta, 1))$, or $u((\beta, 0)) > u((\alpha, 1))$. This in turn implies that $f(\alpha) \cap f(\beta) = \emptyset$, and $\phi(f(\alpha)) \neq \phi(f(\beta))$ for $\alpha \neq \beta$. Then the cardinality of the rationals must be greater than or equal to the cardinality of the reals, which is a contradiction. ■

Proposition 2 has the consequence that there does not exist a real-valued function of the expected utility form that can represent a preordering of probability measures over a non-trivial preordered vector space $(V, \geq)$ of dimension $> 1$. This is because there doesn't even exist a real-valued function that can represent a preordering on the subset of such probability measures that are Dirac measures (i.e. sure outcomes in $V$). The axioms of the von Neumann Morgenstern utility theorem are not satisfied.[29] Note that this is not due to any 'irrationality' on our part regarding our evaluation of uncertain outcomes - the failure occurs when only considering outcomes that happen with probability 1.

This has profound implications for the application of utilitarianism and expected utility theory in population ethics, and any setting involving non-Archimedean preordered vector spaces or lexicographic orders. Any line of reasoning that implicitly or explicitly attempts to represent such orders using real numbers is unsound. If we proceeded ignoring the above, it would be possible to come to ethical conclusions that are arbitrarily wrong. This can occur, for example, by mistakenly taking two goods to be Archimedean equivalent when they are not.

The conflict is with the continuity/Archimedean axiom of the von Neumann Morgenstern utility theorem. If this axiom is weakened, a preordering of probability measures may be represented with multi-dimensional lexicographic expected utilities.[18, 20] Let $P, Q$ be probability measures defined on a sigma algebra over $V$. A lexicographic expected utility representation consists of real-valued functions $u_i$ on each distinct Archimedean class in $V$ such that

$$P \geq Q \Leftrightarrow \mathbb{E}_P[u] \geq \mathbb{E}_Q[u]. \tag{2.20}$$

Here, $\mathbb{E}_P[u]$ is the expectation with respect to the measure $P$ of the vector $u$ whose components are $u_i$. The comparison of $P$ and $Q$ is accomplished using the lexicographic order $\geq$ on a finite dimensional subspace of $V$, since $\mathbb{E}_P[u], \mathbb{E}_Q[u]$ are elements of $V$ that can be written as finite linear combinations of basis vectors. Uncertain prospects are thus lexicographically ordered according to expected utility within Archimedean classes. For more detail on the conditions for such a representation to exist, see Chapter 1-4 of [20].

As an example of multi-dimensional lexicographic expected utility, consider the bets in Table 1 over quantities of two non-Archimedean equivalent goods denoted by $H$ ('higher') and $L$ ('lower').

Table 1: Multi-dimensional expected utility

| | **Bet 1** | **Bet 2** | **Bet 3** |
|---|---|---|---|
| **Probability distribution** | P(H=10) = $\frac{1}{2}$<br>P(H=0) = $\frac{1}{2}$<br>P(L = 2) = 1 | P(H=1) = 1<br>P(L = 3) = 1 | P(H=2) = $\frac{1}{2}$<br>P(H=0) = $\frac{1}{2}$<br>P(L = 7) = 1 |
| **Expected utility** | $\begin{pmatrix}5 & 2\end{pmatrix}$ | $\begin{pmatrix}1 & 3\end{pmatrix}$ | $\begin{pmatrix}1 & 7\end{pmatrix}$ |

Bet 1 is preferred to the other two bets because the first component of its expected utility vector is largest. Bet 3 is preferred to Bet 2 because the first components of their expected utility are equal, but the second component for Bet 3 exceeds that of Bet 2.

Our only assumptions so far are the axioms for a preordered vector space 2. However, we are forced to conclude that there are pairs of goods that are not Archimedean equivalent. Although non-Archimedean life qualities have previously been suggested as a way of avoiding the repugnant conclusion, it has been claimed that if such a pair of life qualities exists, then other counterintuitive results follow. In particular, every finite, monotone sequence beginning and ending with non-Archimedean equivalent goods must contain a successive pair of elements that are in distinct Archimedean classes. I address this issue in section 4 after first introducing some preliminaries on ordinal numbers.

## 3 An informal introduction to ordinal numbers

In this section, I give a brief introduction to ordinal numbers. An understanding of ordinal numbers is necessary for the material in section 4.

Alice likes apples and oranges. She always prefers a larger number of apples/oranges to a smaller number. However, she likes apples more than oranges, to the extent that there is no number of oranges that she would prefer to even a single apple.

Let us try to represent her preferences using numbers. We will assign the number 1 to the bundle consisting of 1 orange, the number 2 to 2 oranges, etc. Then the $\geq$ operator on the natural numbers represents Alice's preferences over oranges.

However, if we limit ourselves to the natural numbers, we immediately run into a problem. The bundle consisting of a single apple cannot be assigned a natural number in a way that respects Alice's preferences. Whichever number it is assigned, there is always a larger natural number available, which would incorrectly imply that there is some number of oranges Alice would prefer to one apple.

In a sense, we have 'run out' of numbers that we can use to order the available bundles. The theory of ordinal numbers was constructed more than a century ago by Cantor, and developed further by von Neumann and others.[30, 31] We give a brief whistlestop tour of this theory here.

There is nothing stopping us from simply defining an abstract number, which we will call $\omega$, and extending the relation $\geq$ by assigning $\omega$ to be greater than any natural number. $\omega$ can then be used to represent the bundle consisting of one apple for Alice.

We can take this further and consider bundles consisting of some number of both apples and oranges. Remembering that Alice always prefers more fruit to less, we may define a new element denoted by $\omega + 1$ that is greater than $\omega$, corresponding to 1 apple and 1 orange. This can be repeated for $\omega + 2$, $\omega + 3$ etc. For two apples, we symbolically assign the element $\omega 2$.

$\omega$ is called the first infinite ordinal number. An arithmetic of ordinal numbers can be constructed recursively using disjoint unions of sets. A well-ordered set is a totally-ordered set in in which every subset has a least element. Given two well-ordered sets $X, Y$, we may define $X + Y$ as the set obtained by taking their disjoint union, and assigning every element of $Y$ to be greater than every element of $X$, but otherwise preserving the ordering within $X$ and $Y$. For example,

$$\{0, 1, 2\} + \{0', 1'\} := \{0, 1, 2, 0', 1'\}, \tag{3.1}$$

where sets are written so that the elements increase from left to right. We can map the set $\{0, 1, 2, 0', 1'\}$ onto $\{0, 1, 2, 3, 4\}$ while preserving the ordering by

$$0 \to 0, \tag{3.2}$$

$$1 \to 1, \tag{3.3}$$

$$2 \to 2, \tag{3.4}$$

$$0' \to 3, \tag{3.5}$$

$$1' \to 4. \tag{3.6}$$

We can go further and identify $\{0, 1, 2, 3, 4\}$ as the ordinal number 5. This is because every element of a well-ordering is uniquely determined by the set of elements that it is larger than. Similarly, $\omega$ can be identified with the set of natural numbers. In this scheme, ordinal numbers are simply canonical representatives of a set of well-orderings that are all equivalent to each other, with the different ordinal numbers providing labels for every possible distinct well-ordered set. The ordinal number associated with a well-ordered set is called its order type.

Note that the above addition operation is commutative for finite ordinal numbers, i.e. $X + Y = Y + X \; \forall X, Y < \omega$. This however, ceases to be the case when considering infinite ordinals. For example,

$$2 + \omega = \{0', 1', 0, 1, 2...\}. \tag{3.7}$$

This ordering is equivalent to $\omega$, since we can simply relabel as follows

$$0' \to 0, \tag{3.8}$$

$$1' \to 1, \tag{3.9}$$

$$0 \to 2, \tag{3.10}$$

$$1 \to 3, \tag{3.11}$$

$$\vdots \tag{3.12}$$

On the other hand,

$$\omega + 2 = \{0, 1, 2... \omega, \omega + 1\}. \tag{3.13}$$

This is not equivalent to $\omega$, since there are two elements in $\omega + 2$ that are greater than all the natural numbers, as opposed to none in $\omega$. Thus $\omega + 2 \neq$

$2 + \omega$.

Multiplication and exponentiation operations can also be defined for ordinal numbers, but I will not develop this here. I refer the reader to some of the many excellent texts on set theory.[32, 33, 34]

# 4 Ordinal numbers and the repugnant conclusion

In this section, I address sequence arguments against the existence of non-Archimedean equivalent goods. Other proposed resolutions in the literature use vagueness, incommensurability, indeterminacy, or fuzziness.[11, 9, 23, 12, 24] The approach here does rely on any of these concepts.

Results in [21, 22] imply that in any finite increasing sequence for which the first and last elements are not Archimedean equivalent to each other, there must be a pair of successive elements that are not Archimedean equivalent. This can used as an argument against the existence of goods that are not Archimedean equivalent, based on the idea that one should be able to construct such a sequence where each element is marginally better than its predecessor. This would then imply that there are a pair of life qualities that are near-identical, but there is no number of the marginally worse life years that is better than just one of the marginally better life years. The argument can be extended to infinite sequences, using transitivity of Archimedean equivalence to obtain the result that in any sequence where every two successive elements are Archimedean equivalent, all elements are Archimedean equivalent.

The problem can be approached by generalising to transfinite sequences. A transfinite sequence is a collection of set elements indexed by an interval in the ordinal numbers (rather than the natural numbers). We can construct an increasing transfinite sequence of vectors $v_1, v_2, \ldots v_\omega$ where every pair of successive elements is Archimedean equivalent, but $v_\omega$ is not Archimedean equivalent to any $v_n$ for $n \in \mathbb{N}$. This is possible because $v_\omega$ does not have a predecessor. That is, there is no natural number that has $\omega$ as its successor, because one can always find a larger natural number. Thus it is perfectly possible to have vectors $v_1$, $v_\omega$ such that $v_\omega$ cannot be 'reached' from $v_1$ by a finite number of marginal increments. If that is not enough, there are ordinal numbers that are arbitrarily large, so one can construct an increasing transfinite sequence of vectors containing pairs of elements that cannot be reached from each other by a countable or uncountable infinite number of marginal increments. If one wishes to have a set of vectors that contains arbitrarily large ascending and descending chains, one can use indexing sets that takes values in e.g. the surreal numbers.

As an example, consider the following sequence in $\mathbb{R}^2$ with the lexicographic order,

$$\begin{pmatrix} 100 \\ 0 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.1}$$

For concreteness, we can think of the top component as the number of life years at a life quality that is worth living, and the bottom component as the number of life years at a life quality that is indifferent to nonexistence.

The final pair of elements in 4.1 are not Archimedean equivalent. We can further subdivide the interval between their top components so that the difference becomes arbitrarily small. One might think that continuing to subdivide in this fashion is unlikely to improve the situation. However, consider

$$\begin{pmatrix} 10^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 10^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 10^0 \\ 0 \end{pmatrix}, \begin{pmatrix} 10^{-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 10^{-2} \\ 0 \end{pmatrix} \dots \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.2}$$

4.2 is clearly more fine-grained than 4.1., since it contains all elements of the form $\begin{pmatrix} 10^{-n} \\ 0 \end{pmatrix}$, $n \in \mathbb{N}$. By contrast, it does not contain any two successive elements that are not Archimedean equivalent. 4.2 is a decreasing transfinite sequence whose order type is $\omega + 1$, where the final element does not have a predecessor.

Finite subsequences like 4.1 whose initial and final elements are not Archimedean equivalent pose no additional conceptual difficulty compared to the case where all goods are Archimedean equivalent. In both cases, one can construct a decreasing sequence with 'small' increments such that there is an element that is preferred to any number of its successor. For example,

$$10^2, 10^1, 10^0, 10^{-1} \dots 10^{-n}, 0. \tag{4.3}$$

The penultimate element is greater than any multiple of 0, despite the fact that the difference between them can be made arbitrarily small. The only substantive difference in the non-Archimedean case is that one can have e.g. a population consisting of a non-zero number of life years in a lower Archimedean class that 'behaves like 0' because it is indifferent to 0. The existence of finite sequences whose initial and final elements are not Archimedean equivalent is thus no more (or less) mysterious than a sequence starting with a life worth living, where each successive term reduces its duration by a small discrete quantity until we are left with a life whose duration is 0.

# 5 Uncertainty and vagueness

## 5.1 Fuzzy preordered vector spaces

The formalism presented so far may be viewed as a theoretical framework for population ethics in circumstances of complete information, sharply demarcated boundaries and ethical preferences that are sensitive to arbitrarily small changes. However, these conditions may not exist in reality. For example, it may not be clear what constitutes a life that is barely worth living, or to give the concept a precise definition. A natural way of generalising to accommodate uncertainty/vagueness in the context of preordered vector spaces is to allow the positive cone to be a fuzzy set.

**Definition 5. Fuzzy set.** A fuzzy set is map $m : V \to [0, 1]$, where $V$ is a set. $m$ is called the membership function.

In the current setting, the value of the membership function may be interpreted as a frequentist or Bayesian probability that a given element of the preordered vector space $(V, \geq)$ is in the positive cone, and therefore greater than or equal to 0. That is, there really is a sharp boundary between elements that are $\geq 0$ and elements that are not, but we do not know exactly where it lies. The uncertainty may originate at least partly from our inability to detect differences below a certain threshold.

Alternatively, the membership function may be interpreted as the 'degree' to which a given vector is greater than or equal to zero. In this view, the 'positive cone' need not have a sharp boundary, similar to the way there is no sharp distinction between e.g. a sunny day and a cloudy day. This interpretation is fundamentally distinct from a probabilistic interpretation, in which there is uncertainty about the answer to a well defined question. By contrast, fuzzy sets allow for the possibility of a question that is not well defined, and answers with truth values ranging between 0 (false) and 1 (true).

Whichever interpretation is adopted, there is no longer a precise apparent border between elements that are not Archimedean equivalent to each other. This is consonant with the intuition that there is no clear dividing line between life qualities that are qualitatively different. For example, we might hold the position that one completely blissful life is better than any number of lives that are barely worth living, without having a clear idea of where exactly the boundary lies between lives that are Archimedean equivalent to the single blissful life, and those that are not.

This induces a new preference relation $\succeq$ on $V$ characterised by a decision rule with threshold $\alpha \in [0, 1]$, defined by

$$y \succeq x \quad \text{iff} \quad m(y - x) \geq \alpha \tag{5.1}$$

It is possible to extend $\succeq$ to probability measures by constructing a representation in terms of a utility function $u$,

$$P \succeq Q \;\Leftrightarrow\; m(\mathbb{E}_P[u] - \mathbb{E}_Q[u]) \geq \alpha. \tag{5.2}$$

I suggest this representation may be equivalent to an appropriate fuzzification of the von Neumann Morgenstern axioms.

## 5.2 Transitivity at all scales is inconsistent with physics

The relation $\succeq$ in 5.1 may be non-transitive. For example, $m$ and $\alpha$ may be such that $\succeq$ restricted to a finite-dimensional subspace of $V$ is the lexicographic semiorder.[35] This can be understood as the lexicographic ordering with imperfect discriminatory power, such that only differences above a certain threshold in a given dimension are detectable. Non-transitivity of the strict preference $\succ$ can also occur as in examples given by Ng.[36] Non-transitivity of indifference $\sim$ (the symmetric part of $\succeq$) can occur along the lines of the example given in seminal work by Luce [37] of an individual who strictly prefers a coffee with one sugar over a coffee with five sugars, but is pairwise indifferent between a series of intermediate coffees that differ by tiny increments in the quantity of sugar.

If there are physical limitations on the ability to discriminate between different goods stemming from e.g. quantum mechanics, then non-transitive preferences may be unavoidable. Non-transitivity arising from limited discriminatory power can resolve problems associated with so-called 'recklessness' and 'fanaticism', wherein a decision-maker acting in accordance with a seemingly reasonable set of axioms counterintuitively prefers bets involving arbitrarily small probabilities of sufficiently large or 'infinite' payoffs to any given sure outcome.[25, 26, 27] This is because sufficiently small probabilities in long shot bets may be indistinguishable from zero, regardless of the magnitude of the payoff. The resolution amounts to dropping transitivity at all scales as a normative requirement because it is inconsistent with physical laws.

As an example, consider the problem of determining whether a coin is fair with probability $\frac{1}{2}$ of landing on heads, or slightly biased, with probability $(\frac{1}{2} + \epsilon)$ of landing on heads. In order to do this, one might seek to make sure the coin is completely symmetric about its axis of rotation, and that it starts completely at rest with zero total momentum. Treating the system classically, it may not be physically possible to determine positions and momenta with arbitrary precision because some non-zero disturbance is introduced to the system simply through measurement. An example illustrating this idea is Heisenberg's microscope.[38]

Alternatively, consider an explicitly quantum mechanical system - a fermion whose spin along a given axis may either be up or down. In order to know the probabilities of these two outcomes, the wavefunction must be determined. However, superposition states are never directly observed. One might hope to perform measurements on a number of identically prepared systems and verify that the probability of either outcome is $\frac{1}{2}$ within some margin of error. But if we are limited to a finite number of systems, this margin of error is necessarily non-zero. Even then, no-cloning theorems show that it is not possible in principle to create identical copies of an arbitrary superposition state, and only imperfect copies can be made.[39]

In this view, fuzziness is a simple way of capturing vagueness/uncertainty that arises because our underlying model cannot be physically accurate at all scales, since it does not properly account for e.g. quantum mechanical effects. Fuzzy preordered vector spaces may be thought of an effective theory of population ethics that provides a simplified model focusing on the most relevant length scales while omitting fine-grained details.

## 5.3 Asymptotic lexicographic order

Equation 5.1 is a generalisation of the lexical threshold view advocated in [12]. In this section, I show that the membership function can allow for a smooth transition between non Archimedean equivalent goods, with the distinction only becoming manifest asymptotically.

To illustrate this idea, consider the case where $x, y \in \mathbb{R}^2$ with

$$y - x = \begin{pmatrix} \delta_2 \\ \delta_1 \end{pmatrix}. \tag{5.3}$$

The crisp cone in Figure 2 is a special case where the membership function is a step function that abruptly transitions from 0 below the horizontal axis, to 1 above it. This is plotted on the left hand side of Figure 3.

We will instead consider a membership function of the form

$$m(y-x) = \frac{1}{1+e^{-(\delta_1^2+1)\delta_2}}. \tag{5.4}$$

This may be thought of as the logistic function with argument $\delta_2$, where the growth rate is variable and equal to $(\delta_1^2+1)$. The boundary only becomes sharp in the limit where $\delta_1$ becomes large. A visualisation is provided on the right hand side of Figure 3.

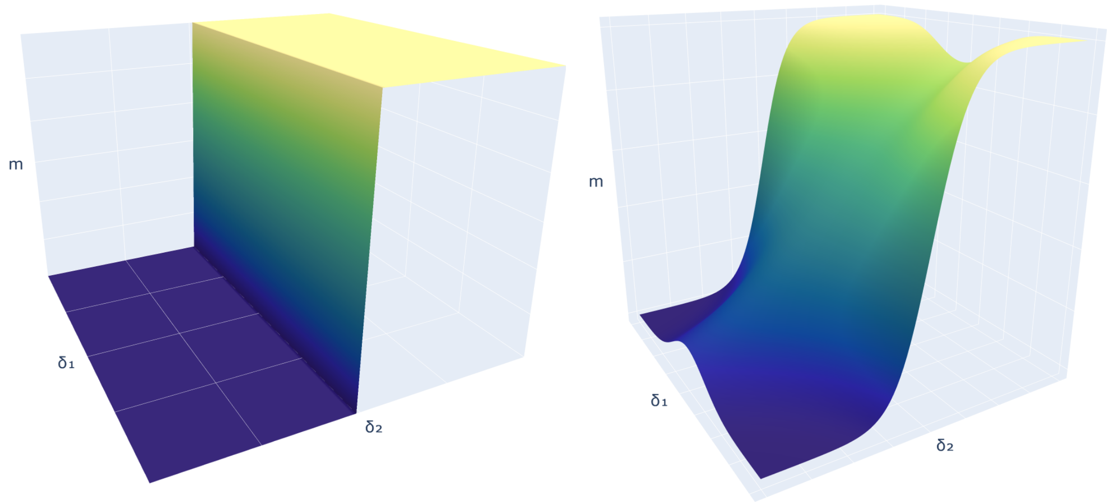


Figure 3: **Left:** Step function. **Right:** Sigmoid function.

This membership function takes value $\frac{1}{2}$ on the $\delta_2 = 0$ plane, which can be thought of as the 'true' boundary of the positive cone as in Figure 2. We have

$$\lim_{\delta_2 \to \infty} m(y - x) = 1 \tag{5.5}$$

$$\lim_{\delta_2 \to -\infty} m(y - x) = 0. \tag{5.6}$$

In words, a sufficiently large difference in in the vertical dimension ensures $y \succeq x$. We also have

$$\lim_{\delta_1 \to \pm\infty} m(y - x) = \begin{cases} 1, & \text{if } \delta_2 > 0, \\ \frac{1}{2}, & \text{if } \delta_2 = 0, \\ 0, & \text{if } \delta_2 < 0. \end{cases} \tag{5.7}$$

This is essentially the lexicographic order - any positive value for $\delta_2$ guarantees $y \succeq x$. However, this is not true for any finite value of $\delta_1$, since

$$y \succeq x \iff \delta_2 \geq \frac{\ln(\frac{\alpha}{1-\alpha})}{\delta_1^2 + 1}. \tag{5.8}$$

This means that in order for $y \succeq x$, $\delta_2$ has to exceed a threshold that varies with the value of $\delta_1$ and that approaches zero as $\delta_1$ becomes large. Thus the lexicographic order only emerges asymptotically.

This membership function has the property that the greater $\delta_1$ is, the smaller $\delta_2$ must be in order for $y \geq x$. This can be interpreted as follows. To the decision-maker, it is not clear whether $e_2$ and $e_1$ are Archimedean equivalent. It is only in the limit where the difference between two populations along the $e_1$ dimension becomes sufficiently large that it becomes apparent they are in different Archimedean classes. The limit $\delta_1 \to \pm\infty$ may be informally thought of as 'looking at $e_1$ through a magnifying glass'. Alternatively, one can think of measuring the welfare of $ne_1, e_2$ and performing a statistical test of whether $e_2 \geq ne_1$. As $n$ becomes larger, we can be increasingly confident that this is the case, with certainty only in the limit.

# 6 Conclusion

In this paper, I have introduced preordered vector spaces, transfinite sequences and fuzzy sets as suitable mathematical tools in population ethics. I have shown that in this setting, not only is the existence of non-Archimedean equivalent goods (i.e. goods for which no amount of one is better than a single unit of the other) sensible, it is compulsory. I note that utilitarianism and expected utility theory in their canonical form fail in this setting, as does any attempt to use real numbers to represent orders in population ethics. This is because non-Archimedean orders cannot generally be represented on Archimedean fields. Generalisations of the von Neumann Morgenstern utility theorem that weaken the continuity/Archimedean axiom lead to multi-dimensional lexicographic expected utility representations. I argue that the existence of non-Archimedean equivalent goods poses no additional theoretical difficulty compared to the case where all goods are Archimedean equivalent. I use a fuzzy set construction to show that there can be a smooth transition between goods that are non-Archimedean equivalent, with the lexicographic order only manifesting asymptotically. This may be a consequence of intrinsic physical limitations on the ability to discriminate between different goods arising from e.g. quantum mechanics. These limitations cause preferences to be non-transitive, which can in turn resolve problems related to 'fanaticism' and 'recklessness'. The main implication of this work is that the repugnant conclusion can be avoided without contradicting any moral intuitions.

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