*This paper was originally submitted as a reply to the paper by Lucas and Redhead in BJPS. The editors, however, decided to end the discussion to the paper of Lucas and Redhead. We can understand their decision, but it is perhaps useful to make our response available at least in the web, in order that no one gets the false impression that we have been overawed by the arguments of Lucas and Redhead.*

# Truth and Provability Again

Jeffrey Ketland and Panu Raatikainen

Lucas and Redhead ([2007]) announce that they will defend the views of Redhead ([2004]) against the argument by Panu Raatikainen ([2005]). They certainly re-state the main claims of Redhead ([2004]), but they do not give any real arguments in their favour, and do not provide anything that would save Redhead's argument from the serious problems pointed out in (Raatikainen [2005]). Instead, Lucas and Redhead make a number of seemingly irrelevant points, perhaps indicating a failure to understand the logico-mathematical points at issue.

What was originally at issue was that Redhead ([2004]) claimed to have established the anti-mechanist conclusion that,

(*) Humanly certifiable truth outruns provability in any formal system, or, that human mind can outrun any machine.

The failings of his argument for this are clearly demonstrated in Raatikainen ([2005]). Nothing Lucas and Redhead now say gives any further support for (*). They only succeed in introducing a number of incorrect claims, and list some facts that are irrelevant for the issue.

The crux of Redhead's original argument concerned the formal system Q, a very weak subsystem of arithmetic without induction, often called Robinson arithmetic. Redhead isolates an arithmetical statement which is not provable in Q. This statement expresses the commutativity of addition of natural numbers. (Redhead does not give the usual model-theoretic argument that commutativity of addition is *not* a theorem of Q.) Redhead makes much ado about the fact that commutativity of addition is not provable in Q, which in fact should not be surprising at all. He proceeds to give a proof, involving a detour through the *meta-theory*, of the commutativity of addition. This proof involves the following three ingredients: the meta-theoretical fact that each instance of commutativity is provable in Q; the *reflection principle* for Q, that each theorem of Q is true; and the *truth-theoretic principle* that truth commutes with universal quantification.

First, note that Redhead doesn't properly explain how *he* certifies that each instance of commutativity is provable in Q. In fact, normal people know it because it can be proved using induction in the meta-theory. He also does not explain how he justifies the reflection principle, except to say that the axioms of Q are 'analytically true'. But since Q only has infinite models, this is very questionable. Is it *analytic* that there are infinitely many objects?

But the central point here is set out in Raatikainen [2005]: Redhead's whole argument here is redundant. Redhead leans, in the course of his argument, on various principles outside Q without a word of explanation how he is justified in holding them. Of course, it is clear that the commutativity of addition can indeed be established assuming all those things. But, as a matter of fact, they involve massive overkill, since ordinary arithmetic induction would suffice! Moreover, this is all *irrelevant*, since none of this gives the faintest support for the bold conclusion (*), that humanly certifiable truths of arithmetic transcend *any* formal system.

In their response, Lucas and Redhead ascribe to us philosophical views that we emphatically do not hold. For example, they claim that we assume that our knowledge of the truth of a meta-theoretic statement about provability in Q ought to guarantee its provability in Q. But we certainly assume no such thing — it would just be absurd. They continue by claiming that we assume what needs establishing, namely, that certifiable truth does *not* outrun provability. But we do not assume that, and we do not even aim to establish that. We only argue that Redhead's argument for the contrary conclusion (*) fails. We do not even claim that the reasoning concerning provability in Q must take a place *in* some formal system; we only submit that it *can* easily be captured in a formal system (for example, in Peano Arithmetic PA; or in a formalized meta-theory containing the truth predicate for arithmetic). But the alleged refutation of mechanism would require one to be able to deny the latter claim.

Lucas and Redhead also talk rather vaguely about "provability" and fail to distinguish clearly between provability in Q, provability in some arbitrary formal system, and absolute provability (i.e., provability by human mind *tout court*), and slide from one to another. Any logician familiar with this material agrees that certifiable truth outruns provability in Q (indeed, we doubt that any rational being denies this). But the anti-mechanist conclusion Lucas and Redhead want to establish is (*), that certifiable truth outruns provability in *any* formalized system (and not just provability in the very restricted and weak system Q). And nothing they say comes even close to confirming that conclusion.

Turning to something of dubious relevance, Lucas and Redhead write: "whereas the first four of Peano's axioms are giving a definition of natural number which works well enough for counting, the least number principle and the induction principle are saying something more, which can be denied without inconsistency, while still leaving us able to count." This is obviously true, but it is hard to see its relevance for any of the issues at stake. In their abstract, Lucas and Redhead also promise to explain the argument for the *a priori* nature of induction. However, nothing of the sort is found in the text. The claim is also puzzling, for Redhead in his [2004] rather calls induction 'notorious' and 'mysterious', and denies at least its analyticity.

Shortly after mentioning certain basic Tarskian principles concerning the notion of truth, Lucas and Redhead say that "sceptics deny that we have such a concept of truth". We have no idea who such sceptics are; they are certainly not *us*, since both of us have written at length about this very notion of truth, and its various properties.

Lucas and Redhead end by stating, "in any formal system, however comprehensive, there would be mathematical truths not provable in that system". This should actually read "in any *consistent* formal system"; but ignoring this error, we of course agree; this is just an informal statement of Gödel's incompleteness theorem. A neater statement is that the set of arithmetic truths is not recursively axiomatizable. But the real question in (*) was about *certifiable* truth, and not about arithmetic truth *simpliciter*. And for that issue, the statement is simply irrelevant. For Gödel's results do *not* imply that the set of *certifiable arithmetic truths* (if there is such a set) is not recursively axiomatizable.

It is important to note that by criticizing Redhead's argument and some other anti-mechanist arguments, we do not – as Lucas and Redhead suggest – thereby want to commit ourselves to the mechanist view that the human mind in practice proceeds purely mechanically. In fact, we are inclined to think that such a mechanist view is false (cf. Feferman [2006]). The point is only that mechanism cannot be refuted by merely referring to mathematical facts such as Gödel's incompleteness theorem, or by Redhead's attempted argument. The issue is more philosophical in character.

## References


Feferman, S. [2006]: 'Are There Absolutely Unsolvable Problems? Gödel's Dichotomy', *Philosophia Mathematica,* Series III vol. 14, 134-152.

Lucas, J. and M. Redhead [2007]: 'Truth and Provability', *British Journa1 for the Philosophy of Science*, **58**, pp. 331–332.

Raatikainen, P. [2005]: 'Truth and Provability – A Comment on Redhead', *British Journa1 for the Philosophy of Science*, **56**, pp. 611–3.

Redhead, M. [2004]: 'Mathematics and the Mind', *British Journal for the Philosophy of Science*, **55**, pp. 731–7.