THE AXIOM OF INFINITY: A NEW PRESUPPOSITION OF THOUGHT.

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IT so happened that when the first number of the *Hibbert Journal* appeared, containing an article by Professor Royce on the Concept of the Infinite, I had been myself for some time meditating on the logical bearings and philosophical import of that concept, and was actually then engaged in marking out the course which it seemed to me a first discussion of the matter might best follow. The order and scope of his treatment were so like those I had myself decided upon that I should naturally have felt a pardonable pride in the coincidence, had not this feeling been at the same time quite lost in a stronger one, namely, that of the evident superiority of his manner to any which I could have hoped to attain. Indeed, so patient is his exposition of elements, so rich is it in suggestiveness, so intimately and instructively, according to his wont, has he connected the most abstruse and recondite of doctrines with the most obvious and seemingly trivial of things, and so luminous and stimulating is it all, that one must admire the ingenuity it betrays, and cannot but wonder whether after all there really are in science or philosophy any notions too remote and obscure to be rendered intelligible even to common sense, if only a sufficiently cunning pen be engaged in the service.

While his paper is thus replete with inspiring intimations of the "glorious depths" and near-lying interests of the doctrine treated, and is, in point of clearness and vivid portrayal of its central thought, a model beyond the art of most, it is not, I believe, equally happy when judged on the severer ground of its critico-logical estimates. Even on this ground, I do not hesitate, after close examination, to adjudge it the merit of *general* soundness. That, however, it is thoroughly sound, completely mailed against every possible assault of criticism, is a proposition I am by no means prepared to maintain. Quite the contrary, in fact. Nor can the defects be counted as trivial. One of them especially, which it has in common with other both earlier and later discussions of the subject, notably that by Dedekind himself and, more recently, that by Mr Bertrand Russell in his imposing treatise on *The Principles of Mathematics*, is of the most radical nature, concerning as it does no less a question than, I do not say merely that of the validity, but that of the possibility, of existence-proofs of the infinite.

And here I may as well state at once, lest there should be some misapprehension in respect to purpose, that the present writing is not primarily designed to be a review of Professor Royce's or of other recent discussions of the infinite. Reviewed to some extent they will be, but only incidentally, and mainly because they have declared themselves, erroneously as I think, upon that most fundamental of questions, namely, *whether it is possible, by aid of the modern concept, to demonstrate the existence of the infinite.* Argument would seem superfluous to show the immeasurable import of this problem, whether it be viewed solely in its immediate logical bearings, or also mediately, through the latter, in its bearings upon philosophy, upon theology, and, only more remotely, upon religion itself. It is chief among the aims of this essay, to open that problem anew, to appeal from the prevailing doctrine concerning it, in the hope of securing, if possible, a readjudication of the matter which shall be final.

This subject of the infinite, how it baffles approach! How immediate and how remote it seems, how it abides and yet eludes the grasp, how familiar it appears, mingling with the elemental simplicities of the heart, continuously weaving itself into the intimate texture of common life, and yet how austere and immense and majestic, outreaching the sublimest flights of the imagination, transcending the stellar depths, immeasurable by the beginningless, endless chain of the ages! Comprehend the infinite! No wonder we hear that none but the infinite itself is adequate to that. *Du gleichst dem Geist, den du begreifst.* Be it so. Perhaps, then, we are infinite. If not,

"'Wie' fass' ich dich, unendliche Natur?"

Or is it finally a mere illusion? And is there after all no infinite reality to *be* seized upon? Again, if not, what signifies the finite? Is that to be for ever without definition, except as reciprocal of that which fails to be? Is the All really enclosed in some vast ellipsoid, without a beyond, incircumscriptible, devoid alike of tangent plane and outer point? Are we eternally condemned to seek therein for the meaning and end of processes that refuse to terminate? And is, then, this region, too, but a locus of deceptions, "of false alluring jugglery"? Is analysis but the victim of hallucination when it thinks to detect the existence of realms that underlie and overarch and compass about the domain of the countable and measurable? And does the spirit, in its deeper musings, in its pensive moods, only *seem* to feel the tremulous touch of transfinite waves, of vitalising undulations from beyond the farthest shore of the sea of sense?

One fact at once is clear, namely, that, whatever ultimate justification the hypothesis may find, thought has never escaped the necessity of *supposing* the universe of things to be intrinsically somehow cleft asunder into the two Grand Divisions, or figured, if you will, under the two fundamental complementary all-inclusive Forms, which, from motives more or less distinctly felt and also just, as we shall see, though not quite justified, have been, from time immemorial, designated as the Finite and the Infinite. And these great *terms* or their verbal equivalents — for concepts in any strict sense they have not been—though always vague and shifting, for ever promising, but never quite delivering the key to their identities into the hand of Definition, have, nevertheless, in every principal scene, together played the gravest *rôle* in the still unfolding drama of speculation. Or, to change the figure, they have been as Foci, one of them seemingly near, the other apparently remote, neither of them quite itself determinate, but the two conjointly serving always to determine the ever-varying eccentricity of the orbit of thought; and doubtless the vaster lines that serve to bind the differing epochs of speculation into a single continuous system can best be traced by reference to these august terms as co-ordinate poles of interest.

As a simple historical fact, then, philosophy has indeed, with but negligible exception, throughout *assumed* the existence of both the finite and the infinite. That is one thing. Another fact of distinct and equal weight, no matter whether or how we may account for it, is that man, in accord with the deeper meaning of the Protagorean maxim, has always felt himself to have within, or to be somehow, the potential measure of all that is. Is it insignificant that this faith—for that is what it seems to be—as if an indestructible character of the race, as if an invariant defining property of the germ plasm itself whence man springs and derives his continuity, should have survived every vicissitude of human fortune? that it should have been indeed, if not the substance, at least the promise, of things hoped for, the evidence, too, of things not seen, marking and sustaining metaphysical research from the earliest times? And, what is more, the spirit of such research, curiosity, I mean, fit companion and counterpart of that abiding faith, unlike "experience and observation," has known no bounds, but, on the contrary, finding within itself no fatal principle of limitation, it has ever disdained the scale of finite things as competent to take its measure, and boldly asserted claim to the entire realm of being.

These questions, however, have been something more than fascinating. Perhaps their rise, but not their manifold development, much less their profound significance for life and thought, is to be adequately explained on the hypothesis of insatiate curiosity alone. It must be granted that their presence, especially in the arena of dialectic, *has* been often due simply to their intrinsic magical charm for "summit-intellects." And doubtless the play-instinct, deep-dwelling in the constitution of the mind, has often made them serve the higher faculties merely as intricate puzzles, to beguile the time withal. But, in general, the questions have worn a sterner aspect. Philosophy has been not merely allured, it has been constrained, to their consideration; constrained not only because of their inherence in problems of the conscience, especially in that most radical problem of finding the simplest system of postulates that shall be at once both necessary and sufficient to explain the moral feeling; but constrained still more powerfully by the insistent demands that issue from the religious consciousness. But this is yet not all. For man cannot live by these august interests alone. And it is profoundly significant, both as witnessing to the final interblending, the fundamental unity, of all the concerns of the human spirit, and as revealing the ultimate depth and dignity of all its interests, that questions about the infinite quite similar to those that claim so illustrious parentage in Ethics and Philosophy, admit elsewhere of humbler derivation, and readily own to the lowliest of origins. Man, indeed, merely to live, has had to measure and to count, and this homely necessity, fruitful mother of mystery and doubt, *independently* set the problems of the indefinitely small and the indefinitely great; and so it was that needs quite as immediate and austere as those of Morals and Religion—I mean the exigencies of Science, and especially of Mathematics—demanded on their own ground, in the very beginnings of exact knowledge, that the understanding transcend every possible sequence of observations, pass the uttermost limit of "experience," which, refine and enlarge it as you may, remains but finite, and literally lay hold on infinity itself.

To this ancient irrevocable demand, thus urged upon the reason from every cardinal point of human interest, genius has responded as to a challenge from the gods, and I submit that the response, the endeavour of the reason actually to subjugate extra-finite being and compel surrender of its secrets by the organon of thought, constitutes the most sublime and strenuous and inspiring enterprise of the human intellect in every age.

What of it? Long centuries of gigantic striving, age on age of philosophic toil, immeasurable devotion of time and energy and genius to a single end, the intellectual conquest of transfinite being—what has it all availed? What triumphs have been won? I speak, narrowly, of the conquest, and demand to know, not whether it has been accomplished—for that were a foolish query—but whether, strictly speaking, it has been begun. Let not the import of the question be mistaken. No answer is sought in terms of such moral or "spiritual" gains as may be incident even to efforts that miss their aim. Everyone knows that seeking has compensations of its own, which indeed are ofttimes better than any which finding itself can give. And it seems sometimes as if the higher life were chiefly sustained by unsought gains incident to the unselfish pursuit of the unattainable. The circle has not been squared, nor the quintic equation solved, nor perpetual motion invented; neither indeed can be; yet it would show but meagre understanding of the ways of truth to men, did one suppose all the labour devoted to such problems to have been without reward. So, conceivably, it might be with this problem of the infinite. It may be granted that, even supposing no solution to be attainable, the ceaseless search for one, the unwearied high endeavour of the reason through the ages, presents a spectacle ennobling to behold, and of which mankind, it may be, could ill afford to be deprived. It may be granted that incidentally many insights have been won which, though not solutions, have nevertheless permanently enriched the literature of the world and are destined to improve its life. It may be granted than in every time some doctrine of infinity, some philosophy of it, has been at least effective, has helped, that is, for better or worse, to fashion the forms of human institutions and to determine the course of history. Concerning none of these things is there here any question. As to what the question precisely is, there need not be the slightest misapprehension. The fact is that for thousands of years philosophy has recognised the presence of a certain definite Problem, namely, that of *extending the dominion of logic, the reign of exact thought, out beyond the utmost reach of finite things into and over the realm of infinite being*, and this problem, by far the greatest and most impressive of her strictly intellectual concernments, philosophy has, for thousands of years, arduously striven to solve. And now I ask—not, has it been worth while? for that is conceded, but—has she advanced the *solution* in any measure, and, if so, in what respect, and to what extent?

We are here upon the grounds of the *rational* logos. The whole force and charge of the question is directed to matter of concept and inference. Fortunately, the answer is to be as unmistakable as the question. It must be recognised, of course, that the "problem," as stated, is exceedingly, almost frightfully, generic, comprising a host of interdependent problems. One of these, however, is pre-eminent: without its solution *none* other *can* be solved; with its solution, *any* other *may* be eventually. That problem is the problem of conception, of definition in the unmitigated rigour of its severest meaning; it is the problem of discovering a certain principle, of finding, without the slightest possibility of doubt or indetermination, the intrinsic line of cleavage that parts the universe of being into its two grandest divisions, and so of telling finally and once for all precisely what for thought the infinite is and what for thought the finite is.

And now, thanks to the subtle genius of the modern Teutonic mind, this ancient problem, having baffled the thought of all the centuries, has been at last completely solved, and therein our original question finds its answer: *The conquest has been begun.* Bernhard Riemann, profound mathematician and—important fact, of which, strangely enough, too many philosophers seem invincibly unaware—profound metaphysician too, having pointed out, in his famous *Habilitationschrift*,[1] the epoch-making distinction between mere boundlessness and infinitude of manifolds similar to that of space, the greater glory was reserved for three contemporary compatriots of his—Bernard Bolzano,[2] Richard Dedekind,[3] and George Cantor,[4] the first an acute and learned philosopher and theologian, with deep mathematical insight, the other two brilliant mathematicians, with a strong bent for metaphysics—to win independently and about the same time the long-coveted insight into the intrinsic nature of infinity. And thus it is a distinction of our own time that within the memory of living men the *defining mark* of the infinite first failed to elude the grasp, and that that august term, after the most marvellous career of any in the history of speculation, has been finally made to assume the prosaic form of an exact and completely determined concept, and so at length to become available for the purposes of rigorously logical discourse.

Pray, then, what is this concept? Of various equivalent forms of statement, I choose the following: *An assemblage (ensemble, collection, group, manifold) of elements (things, no matter what) is infinite or finite according as it has or has not a* PART *to which the whole is just* EQUIVALENT *in the sense that between the elements composing that part and those composing the whole there subsists a unique and reciprocal (one-to-one) correspondence.*

[1] "Ueber die Hypothesen, welche die Geometrie zu Grunde liegen," *Ges. Werke.* Also in English by W. K. Clifford.

[2] "Paradoxien des Unendlichen."

[3] "Was sind und was sollen die Fahlen."

[4] Memoirs in *Acta Mathematica,* vol. ii., and elsewhere.

If we may trust to intuition in questions about reality, assemblages,[1] infinite as defined, actually abound on every hand. I need not pause to indicate examples. Those pointed out in Professor Royce's mentioned paper may suffice; they will, at all events, furnish the reader with the "clew, which, once familiar to his hand, will lengthen as he goes, and never break." The concept itself I regard as a great achievement, one of the very greatest in the history of thought. Not only does it mark the successful eventuation of a long and toilsome search; it furnishes criticism with a new standard of judgment, it at once creates, and gives the means of meeting, the necessity for a re-examination and a juster evaluation of historic doctrines of infinity; and it is greater still, I believe, as a destined instrument of exploration in that realm which it has opened to the understanding and whose boundary it defines.

Is that judgment not extravagant? For the concept seems so simple, is so apparently independent of difficult presuppositions, that one cannot but wonder why it was not formed long ago. Had the concept in question been early formed, the history and present status of philosophy and theology, and of science too, had doubtless been different. But it was not then conceived. Now that we have it, is it too unbewildering to be impressive? Shall we esteem it lightly just because we can comprehend it, because it does not mystify? Simple it is indeed, almost as simple as the Newtonian law of gravitation, nearly as easy to understand as the geometric interpretation of imaginary quantities, hardly more difficult to grasp than the notion of the conservation of energy, the Mendelian principle of inheritance, or than a score of other central concepts of science. But shallow indeed and foolish is that criticism which values ideas according to their complexity, and confounds the simple with the trivial.

[1] The very simplest possible example of such a manifold is that of the count-numbers. The whole collection can be paired in one-to-one fashion with, for example, half the collection, thus: 1, 2; 2, 4; 3, 6; . . . .; the totality of even and odd being just equivalent to the even.

As an immense city or a vast complex of mountain masses, seen too near, is obscured as a whole by the prominence of its parts, so the larger truth about any great subject is disclosed only as one beholds it at a certain remove which permits the assembling of principal features in a single view, and a proportionate mingling of reflected light from its grander aspects. Accordingly it has seemed desirable, in the foregoing preliminary survey, to hold somewhat aloof, to conduct the movement, in the main, along the path of perspective centres, in order to allow the vision at every point the amplest range. It is now proposed to draw a little closer to the subject and to examine some of its phases more minutely. In respect to the modern concept of infinity, we desire to know more fully what it really signifies, we wish to be informed how it orients itself among cardinal principles and established modes of thought. But recently born to consciousness, it has already been advanced to conspicuous and commanding station among fundamental notions, and we are concerned to know what, if any, transformations of existing doctrine, what readjustments of attitude towards the universe without us or within, what changes in our thought on ultimate problems of knowledge and reality, it seems to demand and may be destined to effect. In a word, and speaking broadly, we wish to know not merely in a narrow sense what the new idea is, but, in the larger meaning of the term, what it "can."

I shall first speak briefly of the so-called "positive" character of the definition, an alleged essential quality of it, a seeming property which criticism is wont to signalise as a radical or intrinsic virtue of the concept itself. Quite independently of the mathematicians Dedekind and Cantor, who, we have seen, were the independent originators of the new formulation, the then old philosopher, Bolzano, bringing to the subject another order of training and of motive, arrived at notions of the finite and infinite, which on critical examination are found to be essentially the same as theirs, though greatly differing in point alike of view and of form. Bolzano's procedure is virtually as follows:—Suppose given a class C of elements, or things, of any kind whatsoever, as the sands of the seashore, or the stars of the firmament, or the points of space, or the instants in a stretch of time, or the numbers with which we count, or the total manifold of truths known to an omniscient God. Out of any such class C, suppose a series formed by taking for first term one of the elements of C, for second term two of them, and so on. Any term so obtainable is itself obviously a class or group of things, and is *defined* to be finite. The indicated process of series formation, if sufficiently prolonged, will either exhaust C or it will not. If it will, C is itself *demonstrably* finite; if it will not, C is, on that account, *defined* to be infinite. Now, say Professor Royce and others, a definition like the latter, being dependent on such a notion as that of inexhaustibility or endlessness or boundlessness, is negative; a certain innate craving of the understanding remains unsatisfied, we are told, because the definition presents the notion, not in a positive way by telling us what the infinite actually *is*, but merely in a negative fashion by telling us what it is *not*. Undoubtedly the claim is plausible, but is it more? Bolzano affirmed and exemplified a certain proposition, in itself of the utmost importance, and throwing half the needed light upon the question in hand. That proposition is: *Any class or assemblage (of elements), if infinite according to his own definition of the term, enjoys the property of being equivalent, in the sense above explained, to some proper part of itself.* Though he did not himself demonstrate the proposition, it readily admits of demonstration, and, since his time, has in fact been repeatedly and rigorously proved. Not only that, but the converse proposition, giving the other half of the needed light, has been established too: *Every assemblage that* HAS *a part "equivalent" to the whole,* IS *infinite in the Bolzano sense of the term.*

It so appears, in the conjoint light of those two theorems, that the property seized upon and pointed out by the ingenious theologian is in all strictness a *characteristic*, though derivative, mark of the infinite as he conceived and defined it. It is sufficiently obvious, therefore, that this derivative property might logically be regarded as *primitive*, made to serve, that is, as a ground of definition. Precisely this fact it is which was independently perceived by Dedekind and Cantor, with the result that, as they have presented the matter, a collection, or manifold, is infinite if it *has* a certain property, and finite if it has it *not*. And now, the critics tell us, it is the infinite which is positive and the finite which is negative.

The distinction appears to me to be entirely devoid of essential merit. It seems rather to be only another interesting example of that verbal legerdemain for which a certain familiar sort of philosophising has long been famous. For what indeed is positive and what negative? Are we to understand that these terms have absolute as distinguished from relative meaning? The distinction, I take it, is without external validity, is entirely subjective, a matter quite at will, being dependent solely on an arbitrary *ordering* of our thought. That which is first put in thought is positive: the opposite, being subsequently put, is negative; but the *sens* of the time-vector joining the two may be reversed at the thinker's will. It is sometimes contended that that which *generally* happens in the world, and so constitutes the *rule*, is intrinsically positive. As a matter of fact a moving body "in general" continuously changes its distance from every object. Such change of distance from *every* other object would accordingly be a positive something. Then it would follow that the classic definition of a sphere-surface as the locus of a moving point which does *not* change its distance from a certain specified point, is really negative. Obviously it avails nothing essential to disguise the negativity by some such seemingly positive phrase as "constant" distance. The trick is an easy one. If, again, it be allowed that, a process being once started, its continuation is positive, its termination negative, then it would result that *in*exhaustibility is positive and exhaustibility negative, whence we should have to own that it is Bolzano's definition which is positive and that by Dedekind and Cantor negative. It hardly admits of doubt that the matter is purely one of an arbitrarily chosen point of view. Every positive is the negative of its negative; every negative, the positive of its positive. Each of these reciprocals is incomplete without, implicit in, determined by, the other. The distinction is here of no importance. What is important is that, no matter which of the definitions be adopted as such, the other then states a derivable property of the thing defined. In either case the *concept* of the infinite remains the same, it is merely its *garb* that is changed. I am very far from intending, however, to assert herewith that, because the definitions are logically equivalent, they must needs be, or indeed are so practically, that is, as instruments of investigation. That is another matter, which, I regret to say, our somewhat pretentious critiques of scientific method furnish no better means of settling than the wasteful way of trial. Everyone will recall from his school-days Euclid's definition of a plane as being a surface such that a line joining any two points of the surface lies wholly in the surface. Logically that is equivalent to saying: A plane is such an assemblage of points that, any three independent points of the assemblage being given, one and only one third point of the assemblage can be found which is equidistant from the given three. But, despite their logical equivalence, who would contend that, for elementary purposes, the latter notion is "practically" as good as the Greek? And so in respect to the infinite, I am free to admit, or rather I affirm, that, on the score of usability, the Dedekind-Cantor definition is greatly superior to its Bolzanoan equivalent. Professor Royce has indeed ingeniously shown how readily it lends itself to philosophic and even to theologic uses.

I turn now to the current assertion by Professor Royce and Mr Russell that the modern concept of the infinite, of which I have given above in italics an exact statement, to which the reader is referred, in fact denies a certain ancient axiom of common sense, namely, the axiom of whole and part. I am not about to submit a brief in behalf of the traditional conception of axioms as self-evident truths. That conception, as is well known, has been once for all abandoned by philosophy and science alike, while to mathematicians in particular no phenomenon is more familiar than that of the coexistence of self-coherent bodies of doctrine constructed on distinct and self-consistent but incompatible systems of postulates. The co-ordination of such incompatible theories is quite legitimate and presents no cause for regret or alarm. The forced recession of the axioms from the high ground of absolute authority, so far from indicating chaos of intellection or ultimate dissolution of knowledge, signifies a corresponding deepening of foundation ; it means an ascension of mind, the proclamation of its creative power, the assertion of its own supremacy. And henceforth the denial of specific axioms, or the deliberate substitution of one set for another, is to be rightly regarded as an inalienable prerogative of a liberated spirit. The question before us, then, is one merely of fact, namely, whether a certain axiom is indeed denied or contradicted by the modern concept of the infinite.

It is in the first place to be observed that the statement itself of that concept avoids the expression, "equality of whole and part," but instead of it deliberately employs the term "equivalence." The word actually used by Dedekind himself is *ähnlichkeit* (similarity). But, says Professor Royce, "equivalence" is just what the axiom really means by equality. It is precisely this statement which I venture to draw in question. If we know that each soldier of a company marching along the street has one and but one gun on his shoulder, then, we are told, even if we do not know *how* many soldiers or guns there are, we do know that there are "*as* many" soldiers as guns. What the definition in question, taken severely, itself affirms in this case, is that the assemblage of guns is "equivalent or similar" to that of the soldiers. Let us now suppose that in place of soldiers we write, for example, "all positive integers," and in place of guns, "all even positive integers"—the integers are plainly susceptible of unique and reciprocal association with the even integers,—then the definition again asserts, as before, "equivalence" of these assemblages. Note that thus far nothing has been said about *number* as an expression of *how many*. If there *be* a number that tells how many things there are in one assemblage, that same number doubtless tells how many there are in any "equivalent" assemblage, and just because the number, if there be one, is the *same* for both, the two are said to be *equal* by axiom. In this view, equality of groups means more than mere "equivalence"; it means, besides, sameness of their numbers, and so applies *only* in case there be numbers. But common sense, whose axiom is here in court, has neither found, nor affirmed the existence of, a number telling, for example, how many integers there are. On the other hand, in case of assemblages for which common sense *has* known a number, the axiom of whole and part is admittedly valid without exception. It thus appears that the axiom supposed, regarded, however unconsciously but nevertheless in intention, as applicable only in case there be a number telling how many, is, in all strictness, not denied by the concept in question. Numbers designed to tell how many elements there are in an assemblage having a part "equivalent" to the whole are of recent invention, and it may be remarked in passing that this invention bears immediate favourable witness to the fruitfulness of the new idea. Such transfinite numbers once created, then undoubtedly, and not before, the question naturally presents itself whether "equivalence" shall be translated "equality," or, what is tantamount, whether the latter term shall be generalised into the former; "generalised," I say, for, though it is true that, as soon as the transfinite numbers are created, there is, in case of an infinite collection and some of its parts, a conjunction of "equivalence" and "sameness of number," yet equality does not of itself deductively attach, for the transfinite numbers are in *genetic* principle,[1] *i.e.*, radically, different from the number notion which the concept of equality has hitherto connoted. The question as to the mentioned translation or generalisation is, therefore, a question, and it is to be decided, not under spur or stress of logic, but solely from motives of economy acting on grounds of pure expedience. If the decision be, as seems likely because of its expedience and economy, favourable to such translation or generalisation, then indeed the old axiom, as above construed, still remains uncontradicted, is yet valid within the domain of its asserted validity. It is merely that a new number-domain has been adjoined which the old verity never contemplated, and in which, therefore, though it does not apply, it never essentially pretended to; but on account of which adjunction, nevertheless, for the sake of good neighbourship, it is constrained, not indeed to retract its ancient claims, but merely to assert them more cautiously and diplomatically, in preciser terms. Even then, in case of quarrel, it is the generaliser who should explain, and not a defender of the generalised.

And now to my final thesis I venture to invite the reader's special attention, and beg to be held with utmost strictness accountable for my words. The question is whether it is possible, by means of the new concept, to demonstrate the existence of the infinite; whether, in other words, it can be proved that there are infinite systems. That such demonstration is possible is affirmed by Bolzano, by Dedekind, by Professor Royce, by Mr Russell, and in fact by a large and swelling chorus of authoritative utterance, scarcely relieved by a dissenting voice. After no little pondering of the matter, I have been forced, and that, too, I must own, against my hope and will, to the opposite conviction. Candour, then, compels me to assert, as I have elsewhere[2] briefly done, not

[1] *Cf.* Couturat, *L'Infini mathématique,* Appendix.

[2] "The Axiom of Infinity and Mathematical Induction," *Bulletin of the American Mathematical Society,* vol. ix., May 1903.

only that the arguments which have been actually adduced are all of them vitiated by circularity, but that, in the very nature of conception and inference, by virtue of the most certain standards of logic itself, every potential argument, every possible attempt to prove the proposition, is foredoomed to failure, destined before its birth to take the fatal figure of the wheel.

The alleged demonstrations are essentially the same, being all of them but variants under a single type. It is needless, therefore, in support of my first contention, to present separate examination of them all. Analysis of one or two specimens will suffice. I will begin with one from Bolzano's offering, both because it marks the beginning of the new era of thought about the subject and because subsequent writers have nearly all of them either cited or quoted it, and that, as far as I am aware, always with approval. Bolzano[1] undertakes to demonstrate, among similar statements, the proposition that *die Menge der Sätze und Wahrheiten an sich* is infinite (*unendlich*), this latter term being understood, of course, in accordance with his own definition above given. The attempt, as anyone may find who is willing to examine it minutely, informally postulates as follows: the proposition, There are such truths (as those contemplated in the proposition), is such a truth, T; T is true, is another such truth, T'; so on; *and*, the indicated process is inexhaustible. Now, these assumptions, which are essential to the argument, and which any careful reader cannot fail to find implicit in it, are, possibly, all of them, correct, but the last is so evident a *petitio principii* as to make one look again and again lest his own thought should have played him a trick.

In case of Dedekind's demonstration, which has been heralded far and wide, the fallacy is less glaring. The argument is far subtler, more complicate, and the *versteckter Zirkel* lies deeper in the folds. But it is undoubtedly there, and its presence may be disclosed by careful explication.

[1] "Paradoxien," sect. 14.

Let the symbol t stand for thought, *any* thought, and denote by $\acute{t}$ the thought that t *is* a thought. For convenience, $\acute{t}$ may be called the image of t. On examination, Dedekind's proof is found to *postulate* as *certainties*: (1) If there be a t, there is a $\acute{t}$, image of t; (2) if there be two distinct t's, the corresponding $\acute{t}$'s are distinct; (3) there is a t; (4) there is a t which is not a $\acute{t}$; (5) every t is *other* than its $\acute{t}$. These being granted, it is easy to see, by supposing each t to be paired with its $\acute{t}$, as object with image, that the assemblage ϑ of all the t's and the assemblage ϑ' of all $\acute{t}$'s are "equivalent." But by (4) there is a t not in ϑ', which latter is, therefore, a *part* of ϑ. Hence ϑ is infinite, by definition of the term.

Let this matter be scrutinised a little. Assuming only the mentioned postulates and, of course, the possibility of reflection, it is obvious that by pairing the t of (4) with its image $\acute{t}$, then the latter with *its* image, and so on, a sequence S of t's is started which, because of (1) and (5), is incapable of termination. This S, too, by Dedekind's proof, is an infinite assemblage. Accordingly, postulate (1), without which, be it observed, the proof is impossible, postulates, in *advance* of the argument, *certainty* which, if the argument's conclusion be true, *transcends* the *finite before* the inference that an *infinite exists* either is or can be *drawn.* The reader may recall how the Russian mathematician Lobatschewsky said, "In the absence of proof of the Euclidian postulate of parallels, I will assume that it is not true"; and how thereupon there arose a new science of space. Suppose that, in like manner, we say here, "In the absence of *proof* that an act once found to be mentally performable is endlessly so performable, we will assume that such is not the case," then, whatever else might result—and of that we shall presently speak—one thing is at once absolutely certain: Dedekind's "argument" would be quite impossible. The fact is that a more beautiful circle than his is hardly to be found in the pages of fallacious speculation, or admits of construction by the subtlest instruments of self-deceiving dialectic, though it must be frankly allowed that Mr Russell's[1] more recent movement about the same centre is equally round and exquisite.

And this disclosure of the fatal circle in the attempted demonstration serves at once to introduce and exemplify the truth of my second contention, which is that all logical discourse, of necessity, *ex vi termini*, presupposes certainty that transcends the finite, where by logical discourse I mean such as consists of completely determined concepts welded into a concatenated system by the ancient hammer of deductive logic. The fact of this presupposition, of course, cannot be *proved*, but, and that is good enough, it can be *exhibited* and *beheld*. To attempt to "prove" it would be to stultify oneself by assuming the possibility of a deductive argument A to prove that the conclusion of A cannot be drawn unless it is assumed in advance. The fact, then, if it be a fact, and of that there need not be the slightest doubt, is to be added to that small group of fundamental simplicities which can at best be *seen*, if the eye be fit.

Consider, for example, this simplest of syllogistic forms: Every element e of the class c is an element $\acute{e}$ of the class $\acute{c}$; every $\acute{e}$ of $\acute{c}$ is an element $\acute{\acute{e}}$ of the class $\acute{\acute{c}}$; ∴ every e of c is an $\acute{\acute{e}}$ of $\acute{\acute{c}}$. I appeal now to the reader's own subjective experience to witness to the following facts: (1) Our *apodictic feeling* is the sole justification of the inference as such; (2) that felt justification is absolute, neither seeking nor admitting of appeal; (3) that sole and absolute justification, namely, the apodictic feeling, is in no slightest degree *contingent* upon the answer to any question whether the multitude of elements e or $\acute{e}$ or $\acute{\acute{e}}$ is or is not, may or may not be found to be, "equivalent" to some part of itself. The feeling of validity here undoubtedly transcends the finite, undoubtedly holds naught in reserve against any possibility of the inference failing as an act should the system of elements turn out to be infinite.

At some risk of excessive clearness and accentuation, for the matter is immeasurably important, I venture to ask the reader

[1] *Principles of Mathematics*, chap. xliii.

to witness how the transcendence or transfiniteness of certainty shows itself in yet another way, not merely in formal deductive *inference*, but also in *conception*. When any concept, as that of Parabola, for example, is formed or defined, it is found that the concept contains implicitly a host of properties not given explicitly in the definition. Properly speaking, the thing defined *is* a certain organic assemblage of properties, of which the totality is implied in a properly selected few of them. Now the fact which it is decisive here to note is that by conception we mean, among other things, that *whenever* the definition may present itself, even though it may be endlessly, a certain invariant assemblage of properties implicitly accompanies the presentation. Without such transfinite certainty of such invariant uncontingent implication, conception would be devoid of its meaning.

The upshot, then, is this: that conception and logical inference alike presuppose absolute certainty that an act which the mind finds itself capable of performing is intrinsically performable endlessly, or, what is the same thing, that the assemblage of possible repetitions of a once mentally performable act is equivalent to some proper part of the assemblage. This certainty I name the *Axiom* of *Infinity*, and this axiom being, as seen, a necessary presupposition of both conception and deductive inference, every attempt to "demonstrate" the existence of the infinite is a predestined begging of the issue.

What follows? Do we, then, *know* by axiom that the infinite is? That depends upon your metaphysic. If you are a radical *a-priorist*, yes; if not, no. If the latter, and I am now speaking as an a-priorist, then you are agnostic in the deepest sense, being capable, in utmost rigour of the terms, of neither conceiving nor inferring. But if we do not *know* the axiom to be true, and so cannot deductively prove the existence of the infinite, what, then, is the *probability* of such existence? The *highest yet attained.* Why? Because the *inductive* test of the axiom, regarded now as a hypothesis, is trying to conceive and trying to infer, and this experiment, which has been world-wide for æons, has seemed to succeed in countless cases, and to fail in none not explainable on grounds consistent with the retention of the hypothesis.

Finally, to make briefest application to a single concrete case. Do the stars constitute an infinite multitude? No one knows. If the number be finite, that fact may some time be ascertained by actual enumeration, and, if and only if there be infinite ensembles of possible repetitions of mental processes, it may also be known by proof. But if the multitude of stars be infinite, that can never be known *except* by proof; this last is possible only if the axiom of infinity be true, and even if this be true, the actual proof may never be achieved.

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