Epistemic Modality and Absolute Decidability

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Abstract

This paper aims to contribute to the analysis of the nature of mathematical modality, and to the applications of the latter to unrestricted quantification and absolute decidability. Rather than countenancing the interpretational type of mathematical modality as a primitive, I argue that the interpretational type of mathematical modality is a species of epistemic modality. I argue, then, that the framework of two-dimensional semantics ought to be applied to the mathematical setting. The framework permits of a formally precise account of the priority and relation between epistemic mathematical modality and metaphysical mathematical modality. The discrepancy between the modal systems governing the parameters in the two-dimensional intensional setting provides an explanation of the difference between the metaphysical possibility of absolute decidability and our knowledge thereof. I also advance an epistemic two-dimensional truthmaker semantics, if hyperintenisonal approaches are to be preferred to possible worlds semantics.

1 Introduction

This essay aims to contribute to the analysis of the nature of mathematical modality, and to the applications of the latter to unrestricted quantification and absolute decidability. I argue that mathematical modality falls under at least four types; the interpretational, the metaphysical, the non-maximally objective, and the logical. The interpretational type of mathematical modality has traditionally been taken to concern the interpretation of the quantifiers (cf. Linnebo, 2009, 2010, 2013; Studd, 2013); the possible reinterpretations of the intensions of the concept of set (Uzquiano, 2015,a); and the possibility of reinterpreting the domain over which the quantifiers range, in order to avoid inconsistency (cf. Fine, 2006, 2007). The metaphysical type of modality concerns the ontological profile of abstracta and mathematical truth. Abstracta are thus argued to have metaphysically necessary being, and mathematical truths hold of metaphysical necessity, if at all (cf. Fine, 1981). Metaphysical modality is the maximal objective modality.[1] However, the phenomenon of indefinite extensibility of the ordinals, cardinals, and reals is, I argue, possessed of two modalities whose interaction is captured by a two-dimensional semantics, and which consist of an epistemic modality characterizing reinterpretations of quantifier domains, and a non-maximal, hence non-metaphysical, yet still objective modality characterizing ontological expansion.[2] Instances, finally, of the logical type of mathematical modality might concern the properties of consistency (cf. Field, 1989: 249-250, 257-260; Rayo, 2013: 50; Leng: 2007; 2010: 258), and can perhaps be further witnessed by the logic of provability (cf. Boolos, 1993) and the modal profile of forcing (cf. Kripke 1965; Hamkins and Löwe, 2008).

The significance of the present contribution is as follows. (i) Rather than countenancing the interpretational type of mathematical modality as a primitive, I argue that the interpretational type of mathematical modality is a species of epistemic modality. (ii) I argue, then, that the framework of two-dimensional semantics ought to be applied to the mathematical setting. The framework permits of a formally precise account of the priority and relation between epistemic mathematical modality and metaphysical mathematical modality. I target, in particular, the modal axioms that the respective interpretations of the modal operator ought to satisfy. The discrepancy between the modal systems governing the parameters in the two-dimensional setting provides an explanation of the difference between the metaphysical possibility of absolute decidability and

[1] For endorsements of this contention, see Kripke (1980: 99), Lewis (1986), Stalnaker (2003: 203), and Williamson (2016b: 459-460). For an argument in opposition, see Clarke-Doane (2021).

[2] See Khudairi (ms) for further discussion.

our knowledge thereof. (iii) Finally, I examine the application of the mathematical modalities beyond the issues of unrestricted quantification and indefinite extensibility. As a test case for the two-dimensional approach, I investigate the interaction between epistemic and metaphysical mathematical modalities and both large cardinal axioms and Orey sentences which are undecidable relative to the axioms of ZFC, such as the generalized continuum hypothesis. The two-dimensional framework permits of a formally precise means of demonstrating how the metaphysical possibility of absolute decidability and the continuum hypothesis can be accessed by their epistemic-modal-mathematical profile. The logical mathematical modalities – of consistency, provability, and forcing – provide the means for discerning whether mathematical truths are themselves epistemically possible. I argue that, in the absence of disproof, large cardinal axioms are epistemically possible, and thereby provide a sufficient guide to the metaphysical mathematical possibility of determinacy claims and the continuum hypothesis.

In Section **2**, I define the formal clauses and modal axioms governing the epistemic and metaphysical types of mathematical modality. In Section **3**, I discuss how the properties of the epistemic mathematical modality and metaphysical mathematical modality converge and depart from previous attempts to delineate the contours of similar notions. I also advance an epistemic two-dimensional truthmaker semantics, if hyperintenisonal approaches are to be preferred to possible worlds semantics. Section **4** extends the two-dimensional framework to the issue of mathematical knowledge; in particular, to the modal profile of large cardinal axioms and to the absolute decidability of the continuum hypothesis. Section **5** provides concluding remarks.

2 Mathematical Modality

2.1 Metaphysical Mathematical Modality

A formula is a logical truth if and only if the formula is true in an intended model structure, M = <W, D, R, V>, where W designates a space of metaphysically possible worlds; D designates a domain of entities, constant across worlds; R designates an accessibility relation on worlds; and V is an assignment function mapping elements in D to subsets of W.

Metaphysical Mathematical Possibility

$[\![\Diamond\phi]\!]^{v,w} = 1 \Longleftrightarrow \exists w'[\![\phi]\!]^{v,w'} = 1$

Metaphysical Mathematical Necessity

$[\![\Box\phi]\!]^{v,w} = 1 \Longleftrightarrow \forall w'[\![\phi]\!]^{v,w'} = 1$,

with $\Diamond := \neg\Box\neg$

2.2 Epistemic Mathematical Modality

In order to accommodate the notion of epistemic possibility, we enrich M with the following conditions: M = <C, W, D, R, V>, where C, a set of epistemically possibilities, is constrained as follows:

Let $[\![\phi]\!]^c \subseteq$ C;

(ϕ is a formula encoding a state of information at an epistemically possible world).

Intensions

-`pri`$(x) = \lambda c.[\![x]\!]^{c,c}$;

(the two parameters relative to which x – a propositional variable – obtains its value are epistemically possible worlds).

-`sec`$(x) = \lambda c.[\![x]\!]^{w,w}$

(the two parameters relative to which x obtains its value are metaphysically possible worlds).

Then:

- **Epistemic Mathematical Necessity**

  $[\![\blacksquare\phi]\!]^{c,w} = 1 \iff \forall c'[\![\phi]\!]^{c,c'} = 1$

  (ϕ is true at all points in epistemic modal space).

- **Epistemic Mathematical Possibility**

  $[\![\blacklozenge\phi]\!] \neq \emptyset \iff [\![\neg\blacksquare\neg\phi]\!] = 1$

  (ϕ might be true if and only if it is not epistemically necessary for ϕ to be false).

Epistemic mathematical modality can be constrained by consistency, and the formal techniques of provability and forcing. A mathematical formula is false, and therefore metaphysically impossible, if it can be disproved or induces inconsistency in a model.

2.3 Interaction

- **Convergence**

  $\forall$c$\exists$w$[\![\phi]\!]^{c,w} = 1$

  (the value of x is relative to a parameter for the space of epistemically possible worlds. The value of x relative to the first parameter determines the value of x relative to the second parameter for the space of metaphysical possibility).

- **Super-rigidity (2D-Intension):**

$[\![\phi]\!]^{c,w} = 1 \iff \forall$w',c'$[\![\phi]\!]^{c',w'} = 1$

(the intension of ϕ is rigid in all points in metaphysical and epistemic modal space).

2.4 Modal Axioms

- Metaphysical mathematical modality is governed by the modal system KTE, as augmented by the Barcan formula and its Converse (cf. Fine, 1981).

K: $\Box[\phi \to \psi] \to [\Box\phi \to \Box\psi]$

T: $\Box\phi \to \phi$

E: $\neg\Box\phi \to \Box\neg\Box\phi$

Barcan: $\Diamond\exists xFx \to \exists x\Diamond Fx$

Converse Barcan: $\exists x\Diamond Fx \to \Diamond\exists xFx$

- Epistemic mathematical modality is governed by the modal system, KT4, as augmented by the Barcan formula and the Converse Barcan formula.[3]

K: $\blacksquare[\phi \to \psi] \to [\blacksquare\phi \to \blacksquare\psi]$

T: $\blacksquare\phi \to \phi$

4: $\blacksquare\phi \to \blacksquare\blacksquare\phi$

[3] Reasons adducing against including the Smiley-Gödel-Löb provability formula among the axioms of epistemic mathematical modality are examined in Section 5. GL states that '$\blacksquare[\blacksquare\phi \to \phi] \to \blacksquare\phi$'. For further discussion of the properties of GL, see Löb (1955); Smiley (1963); Kripke (1965); and Boolos (1993). Löb's provability formula was formulated in response to Henkin's (1952) problem concerning whether a sentence which ascribes the property of being provable to itself is provable. (Cf. Halbach and Visser, 2014, for further discussion.) For an anticipation of the provability formula, see Wittgenstein (1933-1937/2005: 378). Wittgenstein writes: 'If we prove that a problem can be solved, the concept 'solution' must somehow occur in the proof. (There must be something in the mechanism of the proof that corresponds to this concept.) But the concept mustn't be represented by an external description; it must really be demonstrated. / The proof of the provability of a proposition is the proof of the proposition itself' (op. cit.). Wittgenstein contrasts the foregoing type of proof with 'proofs of relevance' which are akin to the mathematical, rather than empirical, propositions, discussed in Wittgenstein (2001: IV, 4-13, 30-31).

Barcan: $\blacklozenge\exists x Fx \to \exists x \blacklozenge Fx$

Converse Barcan: $\exists x \blacklozenge Fx \to \blacklozenge\exists x Fx$

2.5 Hyperintensional Truthmaker Semantics

If one prefers hyperintensional semantics to possible worlds semantics – in order e.g. to avoid the situation in intensional semantics according to which all necessary formulas express the same proposition because they are true at all possible worlds – one can avail of the following epistemic two-dimensional truthmaker semantics, which specifies a notion of exact verification in a state space and where states are parts of whole worlds (Fine 2017a,b; Hawke and Özgün, forthcoming). According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple $\langle$S, P, $\leq$, v$\rangle$, where S is a non-empty set of states, i.e. parts of the elements in A in the foregoing epistemic modal algebra U, P is the subspace of possible states where states s and t comprise a fusion when s $\sqcap$ t$\in$P, $\leq$ is a partial order, and v: Prop $\to$ (2^S x 2^S) assigns a bilateral proposition $\langle$p$^+$, p$^-\rangle$ to each atom p$\in$Prop with p$^+$ and p$^-$ incompatible (Hawke and Özgün, forthcoming: 10-11). Exact verification ($\vdash$) and exact falsification ($\dashv$) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, forthcoming: 11):

s $\vdash$ p if s$\in[\![$p$]\!]^+$

(s verifies p, if s is a truthmaker for p i.e. if s is in p's extension);

s $\dashv$ p if s$\in[\![$p$]\!]^-$

(s falsifies p, if s is a falsifier for p i.e. if s is in p's anti-extension);

s $\vdash$ $\neg$p if s $\dashv$ p

(s verifies not p, if s falsifies p);

s $\dashv$ $\neg$p if s $\vdash$ p

(s falsifies not p, if s verifies p);

$s \vdash p \wedge q$ if $\exists t,u$, $t \vdash p$, $u \vdash q$, and $s = t \sqcap u$

(s verifies p and q, if s is the fusion of states, t and u, t verifies p, and u verifies q);

$s \dashv p \wedge q$ if $s \dashv p$ or $s \dashv q$

(s falsifies p and q, if s falsifies p or s falsifies q);

$s \vdash p \vee q$ if $s \vdash p$ or $s \vdash q$

(s verifies p or q, if s verifies p or s verifies q);

$s \dashv p \vee q$ if $\exists t,u$, $t \dashv p$, $u \dashv q$, and $s = t \sqcap u$

(s falsifies p or q, if s is the state overlapping the states, t and u, t falsifies p, and u falsifies q);

$s \vdash \forall x\phi(x)$ if $\exists s_1, \ldots, s_n$, with $s_1 \vdash \phi(a_1), \ldots, s_n \vdash \phi(a_n)$, and $s = s_1 \sqcap \ldots \sqcap s_n$

[s verifies $\forall x\phi(x)$ "if it is the fusion of verifiers of its instances $\phi(a_1), \ldots, \phi(a_n)$" (Fine, 2017c)];

$s \dashv \forall x\phi(x)$ if $s \dashv \phi(a)$ for some individual a in a domain of individuals (op. cit.)

[s falsifies $\forall x\phi(x)$ "if it falsifies one of its instances" (op. cit.)];

$s \vdash \exists x\phi(x)$ if $s \vdash \phi(a)$ for some individual a in a domain of individuals (op. cit.)

[s verifies $\exists x\phi(x)$ "if it verifies one of its instances $\phi(a_1), \ldots, \phi(a_n$" (op. cit.)];

$s \dashv \exists x\phi(x)$ if $\exists s_1, \ldots, s_n$, with $s_1 \dashv \phi(a_1), \ldots, s_n \dashv \phi(a_n)$, and $s = s_1 \sqcap \ldots \sqcap s_n$ (op. cit.)

[s falsifies $\exists x\phi(x)$ "if it is the fusion of falsifiers of its instances" (op. cit.)];

s exactly verifies p if and only if $s \vdash p$ if $s \in [\![p]\!]$;

s inexactly verifies p if and only if $s \triangleright p$ if $\exists s' \sqsubset S$, $s' \vdash p$; and

s loosely verifies p if and only if, $\forall t$, s.t. $s \sqcup t$, $s \sqcup t \vdash p$, where $\sqcup$ is the relation of compatibility (35-36);

s $\vdash$ Aϕ if and only if for all t$\in$P there is a t'$\in$P such that t' $\sqcap$ t $\in$P and t' $\vdash$ ϕ;

s $\dashv$ Aϕ if and only if there is a t$\in$P such that for all u$\in$P either t $\sqcap$ u$\notin$P or u $\dashv$ ϕ, where Aϕ denotes the apriority of ϕ.

In order to account for two-dimensional indexing, we augment the model, M, with a second state space, S*, on which we define both a new parthood relation, $\leq$*, and partial function, V*, which serves to map propositions in a domain, D, to pairs of subsets of S*, {1,0}, i.e. the verifier and falsifier of p, such that $[\![P]\!]^{+} = 1$ and $[\![p]\!]^{-} = 0$. Thus, M = $\langle$S, S*, D, $\leq$, $\leq$*, V, V*$\rangle$. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c,i: c ranges over subsets of S, and i ranges over subsets of S*.

(*) M,s$\in$S,s*$\in$S* $\vdash$ p iff:

(i) $\exists$c$_s[\![p]\!]^{c,c} = 1$ if s$\in[\![p]\!]^{+}$; and

(ii) $\exists$i$_{s*}[\![p]\!]^{c,i} = 1$ if s*$\in[\![p]\!]^{+}$

(Distinct states, s,s*, from distinct state spaces, S,S*, provide a multi-dimensional verification for a proposition, p, if the value of p is provided a truthmaker by s. The value of p as verified by s determines the value of p as verified by s*).

We say that p is hyper-rigid iff:

(*) M,s$\in$S,s*$\in$S* $\vdash$ p iff:

(i) $\forall$c'$_s[\![p]\!]^{c,c'} = 1$ if s$\in[\![p]\!]^{+}$; and

(ii) $\forall$i$_{s*}[\![p]\!]^{c,i} = 1$ if s*$\in[\![p]\!]^{+}$

The foregoing provides a two-dimensional hyperintensional semantic framework within which to interpret the values of a proposition. In order to account for partial contents, we define the values of subpropositional entities relative again to tuples of states from the distinct state spaces in our model:

s is a two-dimensional exact truthmaker of p if and only if (*);

s is a two-dimensional inexact truthmaker of p if and only if ∃s'⊏S, s→ s', s' ⊢ p and such that

∃$c_{s'}$⟦p⟧^{c,c} = 1 if s'∈⟦p⟧$^{+}$, and

∃i_{s*}⟦p⟧^{c,i} = 1 if s*∈⟦p⟧$^{+}$;

s is a two-dimensional loose truthmaker of p if and only if, ∃t, s.t. s ⊔ t, s ⊔ t ⊢ p:

∃$c_{s\sqcup t}$⟦p⟧^{c,c} = 1 if s'∈⟦p⟧$^{+}$, and

∃i_{s*}⟦p⟧^{c,i} = 1 if s*∈⟦p⟧$^{+}$.

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions:

- Epistemic Hyperintension:

  $\texttt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$, with s a state in the state space defined over the foregoing epistemic modal algebra, U;

- Subjunctive Hyperintension:

  $\texttt{sec}_{v_{@}}(x) = \lambda i.[\![x]\!]^{v_{@},i}$, with i a state in metaphysical state space I;

- 2D-Hyperintension:

  $\texttt{2D}(x) = \lambda s \lambda w [\![\mathrm{x}]\!]^{s,i} = 1$.

3 Departures from Precedent

The approach to mathematical modality, according to which it yields a representation of the cumulative universe of sets, has been examined by Fine (2005; 2006) and Uzquiano (2015). Fine argues that the mathematical modality should be interpretational; and thus taken to concern the reinterpretation of the domain over which the quantifiers range, in order to avoid inconsistency. Uzquiano argues similarly for an interpretational construal of mathematical modality, where the cumulative hierarchy of sets is fixed, yet what is possibly reinterpreted is the non-logical vocabulary of the language, in particular the membership relation.[4]

On Fine's approach, the interpretational modality is both postulational, and 'prescriptive' or imperatival. The prescriptive element consists in the rule:

'Introduction: !x.C(x)',

such that one is enjoined to postulate, i.e. to 'introduce an object x conforming to the condition C(x)' (2005: 91; 2006: 38).

In the setting of unrestricted quantification, suppose, e.g., that there is an interpretation for the domain over which a quantifier ranges. Fine writes that an interpretation 'I is exten[s]ible – in symbols, E(I) – if possibly some interpretation extends it, i.e. $\Diamond\exists$J(I$\subset$J)' (2006: 30). Then, the interpretation of the domain over which the quantifier ranges is *extensible*, if '$\forall$I.E(I)'. The interpretation of the domain over which the quantifier ranges is *indefinitely extensible*, if '$\Box\forall$I.E(I)' iff '$\Box\forall$I$\Diamond\exists$J(I$\subset$J)', where the reinterpretation is induced via the prescriptive imperative to postulate the existence of a new object by the foregoing 'Introduction' rule (2006: 30-31; 38). Fine clarifies that the interpretational approach is consistent with a 'realist ontology' of the set of reals. He refers to the imperative to postulate new objects, and thereby reinterpret the domain for the

[4]Compare Gödel, 1947; Williamson, 1998; and Fine, 2005.

quantifier, as the 'mechanism' by which epistemically to track the cumulative hierarchy of sets (2007: 124-125).

In accord with Fine's approach, the epistemic mathematical modality defined in the previous section was taken to have a similarly representational interpretation, and perhaps the postulational property is an optimal means of inducing a reinterpretation of the domain of the quantifier. However, the present approach avoids a potential issue with Fine's account, with regard to the the introduction of deontic modal properties of the prescriptive and imperatival rules that he mentions.[5] It is sufficient that the interpretational modalities are a species of epistemic modality, i.e. possibilities that are relative to agents' spaces of states of information.

Developing Parsons' (1983) program, Linnebo (2013) outlines a modalized version of ZF. [6] Similarly to the modal axioms for the epistemic mathematical modality specified in the previous section, Linnebo argues that his modal set theory ought to be governed by the system S4.2, the Converse Barcan formula, and (at least a restricted version of) the Barcan formula. However – rather than being either interpretational or epistemic – Linnebo deploys the mathematical modality in order to account for the notion of 'potential infinity', as anticipated by Aristotle.[7] The mathematical modality is thereby intended to provide a formally precise answer to the inquiry into the extent of the cumulative set-

[5] For an analysis of the precise interaction between the semantic values of epistemic and deontic modal operators, see Author (ms).

[6] See footnote 1. Linnebo (2018) discusses the differences between Putnam's and Parsons' accounts of the role of modality in mathematics. Berry (forthcoming) also discusses the differences between the foregoing. Linnebo (op. cit.: 265-266) avails of two-dimensional indexing for the relation between interpretational and circumstantial modalities. The appeal to epistemic two-dimensional semantics in order to account for interpretational as epistemic and circumstantial as metaphysical modalities and their interaction in this essay was written in 2015 and pursued prior to knowledge of Linnebo's account. My approach differs, as well, by countenancing a hyperintensional, epistemic two-dimensional truthmaker semantics and applying it to the epistemology of mathematics.

[7] Cf. Aristotle, *Physics*, Book III, Ch. 6.

theoretic hierarchy; i.e., in order to precisify the answer that the hierarchy extends 'as far as possible' (2013: 205).[8]

Thus, Linnebo takes the modality to be constitutive of the actual ontology of sets; and the quantifiers ranging over the actual ontology of sets are claimed to have an 'implicitly modal' profile (2010: 146; 2013: 225). He suggests, e.g., that: 'As science progresses, we formulate set theories that characterize larger and larger initial segments of the universe of sets. At any one time, precisely those sets are actual whose existence follows from our strongest, well-established set theory' (2010: 159n21). However – despite his claim that the modality is constitutive of the actual ontology of sets – Linnebo concedes that the mathematical modality at issue cannot be interpreted metaphysically, because sets exist of metaphysical necessity if at all (2010: 158; 2013: 207). In order partly to allay the tension, Linnebo remarks, then, that set theorists 'do not regard themselves as located at some particular stage of the process of forming sets' (2010: 159); and this might provide evidence that the inquiry – concerning at which stage in the process of set-individuation we happen to be, at present – can be avoided.

Another distinction to note is that both Linnebo (op. cit.) and Uzquiano (op. cit.) avail of second-order plural quantification, in developing their primitivist and interpretational accounts of mathematical modality. By contrast to their approaches, the epistemic and metaphysical modalities defined in the previous section are defined with second-order singular quantification over sets.

[8]Precursors to the view that modal operators can be availed of in order to countenance the potential hierarchy of sets include Hodes (1984). Intensional constructions of set theory are further developed by Reinhardt (1974); Parsons (op. cit.); Myhill (1985); Scedrov (1985); Flagg (1985); Goodman (1985); Hellman (1990); Nolan (2002); and Studd (2013). (See Shapiro (1985) for an intensional construction of arithmetic.) Chihara (2004: 171-198) argues that 'broadly logical' conceptual possibilities can be used to represent imaginary situations relevant to the construction of open-sentence tokens. The open-sentences can then be used to define the properties of natural and cardinal numbers and the axioms of Peano arithmetic.

Linnebo and Uzquiano both suggest that their mathematical modalities ought to be governed by the G axiom; i.e. $\diamond\Box\phi \rightarrow \Box\diamond\phi$. The present approach eschews, however, of the G axiom, in virtue of the following. Williamson (2009) demonstrates that – because KT4G is a sublogic of S5 – an epistemic operator which validates the conjunction of the 4 axiom of positive introspection and the E axiom of negative introspection will be inconsistent with the condition of 'recursively enumerable conservativeness' (30). "[I]f a [modal logic] is r.e. (quasi-)conservative then every (consistent) r.e. theory in the language without $\boxdot$ [interpreted as "I know that..."] is conservatively extended by an r.e. theory in the language with $\boxdot$ such that it is consistent in the modal logic for [a recursively enumerable theory] R to be exactly what the agent cognizes in the language without $\boxdot$ while what the agent cognizes in the language with $\boxdot$ constitutes an r.e. theory" (12). As axioms of an agent's consistent, recursively axiomatizable theorizing about the theory of its own states of knowledge and belief, the conjunction of 4 and E would entail that the agent's theory is both consistent and decidable, in conflict with Gödel's (1931) second incompleteness theorem. The modal system, KT4, avoids the foregoing result. In the present setting, the circumvention is innocuous, because the undecidability – yet recursively enumerable quasi-conservativeness – of an epistemic agent's consistent theorizing about its epistemic states is consistent with the epistemic mathematical possibility that large cardinal axioms are absolutely decidable.

My application of epistemic two-dimensional semantics to the epistemology of mathematics departs from full-blooded platonism, as well. According to full-blooded platonism, whatever mathematical objects can exist, do exist, and every consistent mathematical theory describes either a different part of the mathematical universe or distinct mathematical universes altogether (Balaguer, 1998). Thus, ZFC+CH and ZFC+¬CH both "truly describe collections of mathematical objects", holding in distinct albeit equally real mathematical universes (Balaguer, 2001: 97: see also Hamkins, 2012).

Epistemic two-dimensionalism and full-blooded platonism differ on both the nature of their target possibilities and on the status of the actuality of the possibilities. Epistemic two-dimensionalism avails of epistemic possibilities, whereas full-blooded platonism avails of logical possibilities. Further, not all epistemic possibilities are actual according to epistemic two-dimensionalism, whereas the objects of any logically consistent theory actually exist according to full-blooded platonism. One reason to prefer epistemic two-dimensionalism to full-blooded platonism is that the former can be formalized, whereas Restall (2003) has shown that there are significant challenges to formalizing the latter. Another reason to prefer epistemic two-dimensionalism is that – unlike full-blooded platonism – it avoids commitment to the existence of inconsistent universes of sets where e.g. both ZFC+V=L and ZFC+V≠L would obtain.

Finally, Waxman (ms) endeavors to account for the interaction between the imagination and mathematics. Whereas I avail in this paper of conceivability as defined in epistemic two-dimensional semantics – which I refer to in the mathematical setting as epistemic mathematical modality – in order to account for how the epistemic possibility of abstraction principles and large cardinal axioms relates to their metaphysical possibility, Waxman's aim is to account for how imagining a model of a mathematical theory entrains justification to believe its consistency (op. cit.). Unlike Waxman, epistemic mathematical modality is ideal, whereas imagination is, on his account, non-ideal (Waxman, op. cit.: 18; Chalmers, 2002), where ideal conceivability means true at the limit of apriori reflection unconstrained by finite limitations. Unlike Waxman, I believe, further, that imaginative contents are sensitive to hyperintensional subject-matters or topics (cf. Berto, 2018; Canavotto, Berto, and Giordani, 2020).

4 Knowledge of Absolute Decidability

Williamson (2016a) examines the extension of the metaphysically modal profile of mathematical truths to the question of absolute decidability. A statement is decidable if and only if there is a mechanical procedure for deciding it or its negation. Statements are absolutely undecidable if and only if they are "undecidable relative to any set of axioms that are justified" rather than just relative to a system (Koellner, 2006: 153), and they are absolute decidable if and only if they are not absolutely undecidable. In this section, I aim to extend Williamson's analysis to the notion of epistemic mathematical modality that has been developed in the foregoing sections. The extension provides a crucial means of witnessing the significance of the two-dimensional intensional approach for the epistemology of mathematics.

Williamson proceeds by suggesting the following line of thought. Suppose that A is a true interpreted mathematical formula which eludes present human techniques of provability; e.g. the continuum hypothesis (op. cit.). Williamson argues that mathematical truths are metaphysically necessary (op. cit.). Williamson then enjoins one to consider the following scenario: It is metaphysically possible that there is a species which can prove that A. Therefore, A is absolutely provable; that is, A 'can in principle be known by a normal mathematical process' such as derivation in an axiomatizable formal system with quantification and identity. He proposes a safety condition on knowledge. He writes: '"In current epistemological terms, their knowledge of A meets the condition of safety: they could not easily have been wrong in a relevantly similar case. Here the relevantly similar cases include cases in which the creatures are presented with sentences that are similar to, but still discriminably different from, A, and express different and false propositions; by hypothesis, the creatures refuse to accept such other sentences, although they may also refuse to accept their negations'" (11). Williamson writes then that: "The claim is not just that A *would* be absolutely provable *if* there were such creatures. The point is the stronger one that A is absolutely *provable* because there *could* in principle be such creatures."

Williamson's scenario evinces one issue for the 'back-tracking' approach to modal epistemology, at least as it might be applied to the issue of possible mathematical knowledge. On the back-tracking approach, the method of modal epistemology is taken to proceed by first discerning the metaphysical modal truths – normally by natural-scientific means – and then working backward to the exigent incompleteness of an individual's epistemic states concerning such truths (cf. Stalnaker, 2003; Vetter, 2013).

The issue for the back-tracking method that Williamson's scenario illuminates is that the metaphysical mathematical possibility that CH is absolutely decidable must in some way converge with the epistemic possibility thereof. The normal mathematical techniques that Williamson specifies – i.e. proof and forcing – have both an epistemic and a metaphysical dimension. Thus, whether CH is metaphysically necessary – and thus, as Williamson claims, metaphysically possible and absolutely decidable thereby – can only be witnessed by the epistemic means of demonstrating that its absolute decidability is not impossible. Nevertheless, the epistemic mathematical possibility of the decidability of CH is a guide to its metaphysical mathematical possibility.

The significance of the two-dimensional intensional framework outlined in the foregoing is that it provides an explanation of the discrepancy between metaphysical mathematical modality and epistemic mathematical modality. Metaphysical mathematical modality is governed by the system S5, the Barcan formula, and its Converse, whereas epistemic mathematical modality is governed by KT4, the Barcan formula, and its Converse. Thus, epistemic mathematical modality figures as the mechanism, which enables the tracking of metaphysically possible mathematical truth.[9]

Leitgeb (2009) endeavors similarly to argue for the convergence between the notion of informal provability – countenanced as an epistemic modal operator, K – and mathematical truth. Availing of Hilbert's (1923/1996: ¶18-42) epsilon terms for propositions, such that, for an arbitrary predicate, $\mathbf{C}$(x), with x a propositional variable, the term 'ϵp.$\mathbf{C}$(p)' is intuitively interpreted as stating that 'there is a proposition, x(/p), s.t. the formula, that p satisfies $\mathbf{C}$, obtains' (op. cit.: 290). Leitgeb purports to demonstrate that $\forall$p(p $\rightarrow$ Kp), i.e. that informal provability is absolute; i.e. truth and provability are co-extensive.[10] He argues as follows. Let A(p) abbreviate the formula 'p $\wedge \neg$K(p)', i.e., that the proposition, p, is true while yet being unprovable. Let K be the informal provability operator reflecting knowability or epistemic necessity, with $\langle$K$\rangle$ its dual.[11] Then:

1. $\exists$p(p $\wedge \neg$Kp) $\iff$ ϵp.A(p) $\wedge \neg$Kϵp.A(p).

[9]A provisional definition of large cardinal axioms is as follows.
$\exists$xΦ is a large cardinal axiom, because:
(i) Φx is a Σ_2-formula;
(ii) if κ is a cardinal, such that V $\models \Phi(\kappa)$, then κ is strongly inaccessible, where a cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ, and a strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^\lambda < \kappa$ (Cf. Kanamori, 2012: 360); and
(iii) for all generic partial orders $\mathbb{P} \in$V$_\kappa$, and all V-generics G $\subseteq$P, $V[G] \models \Phi$x (Koellner, 2006: 180).

[10]The formula is referred to as the Principle of Knowability, and discussed in further detail in Section 5, below.

[11]See Section 5, for further discussion of the duality of knowledge, and its relation to doxastic operators.

By necessitation,

2. $K[\exists p(p \wedge \neg Kp)] \iff K[\epsilon p.A(p) \wedge \neg K\epsilon p.A(p)]$.

Applying modal axioms, KT, to (1), however,

3. $\neg K[\epsilon p.A(p) \wedge \neg K\epsilon p.A(p)]$.

Thus,

4. $\neg K\exists p(p \wedge \neg Kp)$.

Leitgeb suggests that (4) be rewritten

5. $\langle K\rangle\forall p(p \rightarrow Kp)$.

Abbreviate (5) by $\mathtt{B}$. By existential introduction and modal axiom K, both

6. $\mathtt{B} \rightarrow \exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg\mathtt{B}) \wedge p]$, and

7. $\neg\mathtt{B} \rightarrow \exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg\mathtt{B}) \wedge p]$.

Thus,

8. $\exists p[K(p \rightarrow \mathtt{B}) \vee K(p \rightarrow \neg\mathtt{B}) \wedge p]$.

Abbreviate (8) by $\mathtt{C}(p)$. Introducing epsilon notation,

9. $[K(\epsilon p.\mathtt{C}(p) \rightarrow \mathtt{B}) \vee K(\epsilon p.\mathtt{C}(p) \rightarrow \neg\mathtt{B})] \wedge \epsilon p.\mathtt{C}(p)$.

By K,

10. $[K(\epsilon p.\mathtt{C}(p) \rightarrow K\mathtt{B}) \vee K(\epsilon p.\mathtt{C}(p) \rightarrow K\neg\mathtt{B})]$.

From (9) and necessitation, one can further derive

11. $K\epsilon p.\mathtt{C}(p)$.

By (10) and (11),

12. $K\mathtt{B} \vee K\neg\mathtt{B}$.

From (5), (12), and K, Leitgeb derives

13. $K\mathtt{B}$.

By, then, the T axiom,

14. $\forall p(p \rightarrow Kp)$ (291-292).

Rather than accounting for the coextensiveness of epistemic provability and truth, Leitgeb interprets the foregoing result as cause for pessimism with regard to whether the formulas countenanced in epistemic logic and via epsilon terms are genuinely logical truths if true at all (292).

In response to the attending pressure on the status of epistemic logic as concerning truths of logic, one can challenge the derivation, in the above proof, from lines (12) to (13). The inference depends on line (5), i.e., the epistemic possibility of completeness: $\langle K \rangle \forall p(p \rightarrow Kp)$. Assume that line (5) is valid. Then, the validity of the inference from (12) to (13) can be challenged by the restriction on the quantifier on worlds in the Knowability Principle expressed by (5). The epistemic operator in lines (12) and (13) records, by contrast, the epistemic necessity, rather than the possibility, of the truth of the formulas and subformulas therein. Thus, from (12) either the provability of the provability of propositions or the provability of the unprovability of propositions, one cannot derive (13) the provability of the provability of propositions, because – by (5) – it is only epistemically possible that all true propositions are provable.

A final question is whether Orey sentences such as the Continuum Hypothesis (CH) have a determinate epistemic intension given that there are currently models in which CH is true and models in which CH is false, such that it isn't determinate which epistemic possibility is actual. In response to this worry, the epistemic intension is arguably indeterminate for non-ideal reasoners, yet determinate for ideal ones, such that the epistemic mathematical modality at issue can be divided into non-ideal and ideal varieties.[12]

[12] For the distinction between ideal and prima facie (i.e. non-ideal) conceivability, see Chalmers (2002).

5 Concluding Remarks

In this paper, I have endeavored to delineate the types of mathematical modality, and to argue that the epistemic interpretation of two-dimensional semantics can be applied in order to explain, in part, the epistemic status of large cardinal axioms and the decidability of Orey sentences. The formal constraints on mathematical conceivability adumbrated in the foregoing can therefore be considered a guide to our possible knowledge of unknown mathematical truth.

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