A Modal Logic and Hyperintensional Semantics for Gödelian Intuition

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Abstract

This essay aims to provide a modal logic for rational intuition. Similarly to treatments of the property of knowledge in epistemic logic, I argue that rational intuition can be codified by a modal operator governed by the modal μ-calculus. Via correspondence results between fixed point modal propositional logic and the bisimulation-invariant fragment of monadic second-order logic, a precise translation can then be provided between the notion of 'intuition-of', i.e., the cognitive phenomenal properties of thoughts, and the modal operators regimenting the notion of 'intuition-that'. I argue that intuition-that can further be shown to entrain conceptual elucidation, by way of figuring as a dynamic-interpretational modality which induces the reinterpretation of both domains of quantification and the intensions and hyperintensions of mathematical concepts that are formalizable in monadic first- and second-order formal languages. Hyperintensionality is countenanced via a topic-sensitive epistemic two-dimensional truthmaker semantics.

1 Introduction

'The incompleteness results do not rule out the possibility that there is a theorem-proving computer which is in fact equivalent to mathematical intuition' – Gödel, quoted in Wang (1986: 186).[1]

In his remarks on the epistemology of mathematics, Gödel avails of a notion of non-sensory intuition – alternatively, 'consciousness', or 'phenomenology' (cf. Gödel, 1961: 383) – as a fundamental, epistemic conduit into mathematical

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[1] Note however that, in the next subsequent sentence, Gödel records scepticism about the foregoing. He remarks: 'But they imply that, in such a – highly unlikely for other reasons – case, either we do not know the exact specification of the computer or we do not know that it works correctly' [Gödel, quoted in Wang (op. cit.)].

truths.[2] According to Gödel, the defining properties of mathematical intuition include (i) that it either is, or is analogous to, a type of perception (1951: 323; 1953,V: 359; 1964: 268); (ii) that it enables subjects to alight upon new axioms which are possibly true (1953,III: 353,fn.43; 1953,V: 361; 1961: 383, 385; 1964: 268); (iii) that it is associated with modal properties, such as provability and necessity (1933: 301; 1964: 261); and (iv) that the non-sensory intuition of abstracta such as concepts entrains greater conceptual 'clarification' (1953,III: 353,fn.43; 1961: 383). Such intuitions are purported to be both *of* abstracta and formulas, as well as to the effect *that* the formulas are true. The distinction between 'intuition-of' and 'intuition-that' is explicitly delineated in Parsons (1980: 145-146), and will be further discussed in Section **2**.[3]

In this paper, I aim to outline the logical foundations for rational intuition, by examining the nature of property (iii).[4] The primary objection to Gödel's approach to mathematical knowledge is that the very idea of rational intuition is insufficiently constrained.[5] Subsequent research has thus endeavored to expand

[2]Another topic that Gödel suggests as being of epistemological significance is the notion of 'formalism freeness', according to which the concepts of computability, demonstrability, and definability can be specified independently of a background formal language (cf. Gödel 1946, and Kennedy 2013 for further discussion). Kennedy notes however that, in his characterizations of demonstrability and definability, Gödel assumes ZFC as his metatheory (op. cit.: 383). Further examination of the foregoing is beyond the scope of the present essay.

[3]For differences between Gödel's conception of intuition, and that of, respectively, Kant, Brouwer, and Hilbert, see Parsons (2008: ch. 5).

[4]Parsons (1980) suggests that intuition, which he refers to, further, as the imagination, can be captured by a type of mathematical modality, although he does not provide a modal logic for the modality. Parsons writes: 'We can call the possibility in question *mathematical possibility*; this expresses the fact that we are not thinking of the capabilities of the human organism, and it may even be extraneous to think of this "construction" as an act of the *mind*. The latter construal agrees with the viewpoint of Kant and Brouwer. It is very tempting if we want to say that any string of strokes is *perceptible* or *imaginable*. (It is preferable to reserve these words for tokens, but then one can speak of the *intuitability* of the type.) The idea is that no matter how many times the operation of constructing one more stroke in imagination has been repeated, "we" can still construct one more. However, I think there is really a hidden assumption that there is no constraint on what "we" can perceive beyond the open temporality of these experiences, and some very gross aspects of spatial structure. Kant and Brouwer thought these were contributions of our minds to the way we experience the world. Kant of course thought that we could not know these things *a priori* unless our minds had contributed them. I am not persuaded by this, and in any case I do not want my argument to rest on the notion of *a priori* knowledge' (158-159). Parsons suggests that intuition can be of structures, along with formulas and objects (152). Parsons (2008) is, however, sceptical about the interpretation of mathematical modality as an epistemic modality. Parsons interprets informal provability as an epistemic modality (82), and writes that formal provability is 'prima facie epistemic' and epistemic modalities might 'be related to the fact that mathematical knowledge is characteristically obtained by proof' (81-82). Formulas and the modality in modal-structuralism are argued, however, not to be epistemic, because their truth and the existence of mathematical structures are independent of epistemic states (80-83).

[5]See, e.g., Hale and Wright (2002). Wright (2004) provides a vivid articulation of the issue: 'A major — but not the only – problem is that, venerable as the tradition of postulating intuitive knowledge of first principles may be, no-one working within it has succeeded at producing even a moderately plausible account of how the claimed faculty of rational intuition is supposed to work — how exactly it might be constituted so as to be reliably responsive to basic logical validity as, under normal circumstances, vision, say, is reliably responsive to the

upon the notion, and to elaborate on intuition's roles. Chudnoff (2013) suggests, e.g., that intuitions are non-sensory experiences which represent non-sensory entities, and that the justificatory role of intuition is that it enables subjects to be aware of the truth-makers for propositions (p. 3; ch. 7). He argues, further, that intuitions both provide evidence for beliefs as well as serve to guide actions (145).[6] Bengson (2015: 718-723) suggests that rational intuition can be identified with the 'presentational', i.e., phenomenal, properties of representational mental states – namely, cognitions – where the phenomenal properties at issue are similarly non-sensory; are not the product of a subject's mental acts, and so are 'non-voluntary'; are qualitatively gradational; and they both 'dispose or incline assent to their contents' and further 'rationalize' assent thereof.[7] Boghossian countenances intuitions as 'pre-judgmental and pre-doxastic' (2020: 201). He defines intuition as follows: 'An intuition, as I understand it (following many others), is an intellectual *seeming*. An intellectual seeming is similar to a sensory seeming in being a presentation of a proposition's being true; yet dissimilar to it in not having a sensory phenomenology' (200). He suggests that 'intuitive judgments appear to instantiate a type of *three-step process*: you consider a scenario and a question about it; after sufficient reflection, a particular answer to that question comes to seem true to you, either because, as we saw earlier, you work out that it is true, or because, without working it out, it just comes to strike you as true; finally, you endorse this proposition' (201). Nagel (2013) examines an approach to intuitions which construes the latter as a type of cognition. She distinguishes, e.g., between intuition and reflection, on the basis of experimental results which corroborate that there are distinct types of cognitive processing (op. cit.: 226-228). Intuitive and reflective cognitive processing are argued to interact differently with the phenomenal information comprising subjects' working memory stores. Nagel notes that – by contrast to intuitive cognition – reflective cognition 'requires the sequential use of a progression of conscious contents to generate an attitude, as in deliberation' (231). We will here follow Chudnoff (op. cit.), Bengson (op. cit.), Boghossian (op. cit.) in taking rational intuition to be a non-sensory phenomenal property of mental states such as judgment, and a non-sensory intellectual seeming which presents a proposition as being true.

Rather than target objections to the foregoing essays, the present discussion aims to rebut the primary objection to mathematical intuition alluded to above, by providing a logic for its defining properties. The significance of the proposal is thus that it will make the notion of intuition formally tractable, and might thus serve to redress the contention that the notion is mysterious and ad hoc.

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configuration of middle-sized objects in the nearby environment of a normal human perceiver' (op. cit.: 158).

[6] A similar proposal concerning the justificatory import of cognitive phenomenology – i.e., the properties of consciousness unique to non-sensory mental states such as belief – can be found in Smithies (2013a,b). Smithies prescinds, however, from generalizing his approach to the epistemology of mathematics.

[7] Compare Kriegel (2015: 68), who stipulates that 'making a judgment that p involves a feeling of involuntariness' and 'making a judgment always involves the feeling of mobilizing a concept'.

In his (1933) and (1964), Gödel suggests that intuition has a constitutively modal profile. Constructive intuitionistic logic is shown to be translatable into the modal logic, S4, with the rule of necessitation, while the modal operator is interpreted as concerning provability. Mathematical intuition of set-theoretic axioms is, further, purported both to entrain 'intrinsic' justification, and to illuminate the 'intrinsic necessity' thereof. Gödel (1947/1964: 260-261) suggests that intrinsic necessity is a property of axioms which are 'implied' by mathematical concepts, such as that of set. Gödel (1964) writes: 'First of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation 'set of' ... These axioms show clearly, not only that the axiomatic system of set theory as used today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which only unfold the content of the concept of set explained above' (260-261).[8] Extrinsic justifications are associated, by contrast, with both the evidential probability of propositions, and the 'fruitful' consequences of a mathematical theory subsequent to adopting new axioms (op. cit.: 261, 269).

Following Gödel's line of thought, I aim, in this paper, to provide a modal logic for the notion of 'intuition-that', i.e. intuitions that an axiom might be true.[9]Via correspondence results between modal propositional logic and the bisimulation-invariant fragment of first-order logic, and fixed point modal propositional logic and the bisimulation-invariant fragment of monadic second-order logic [see van Benthem (1983; 1984/2003), Janin and Walukiewicz (1996); and Venema (2014, ms)], a precise translation can be provided between the notion of 'intuition-of', i.e., intuitions of objects and properties, and the modal operators regimenting the notion of 'intuition-that'. I argue that intuition-that can thus be codified by an operator in fixed point modal propositional logic, where the logic is given a dynamic-interpretational interpretation. There is thus a formal correspondence between the operators codifying the notion of 'intuition-that' and the predicates of the second-order logic in which the predicates are interpreted so as to concern the properties of 'intuition-of'. This provides a precise

[8]Note that intrinsic necessity and Gödel's notion of analyticity as true in virtue of the concepts involved might, for Gödel, be convergent notions. With regard to analyticity, Gödel (1972a) writes: 'It may be doubted whether evident axioms in such great numbers (or of such great complexity) can exist at all, and therefore the theorem mentioned might be taken as an indication for the existence of mathematical yes or no questions undecidable for the human mind. But what weighs against this interpretation is the fact that there *do* exist unexplored series of axioms which are *analytic* in the sense that they only explicate the content of the concepts occurring in them, e.g., the axioms of infinity in set theory, which assert the existence of sets of greater and greater cardinality or of higher and higher transfinite types and which only explicate the content of the general concept of set' (305-306, my emphasis). The convergence in the notions of intrinsic necessity and analyticity for Gödel might have been inspired by his interactions with the logical positivists of the Vienna Circle, who similarly identified analyticity with necessity. Parsons (2014: 146) argues for the identification of the two notions for Gödel, although doesn't draw on the similarity between the definitions of the two notions in the 1964 and 1972a works quoted above as evidence.

[9]See Parsons (1979-1980; 1983: p. 25, chs.10-11; 2008: 176).

answer to the inquiry advanced by Parsons (1993: 233) with regard to how 'intuition-that' relates to 'intuition-of'.

This result is of relevance to Williamson's paper, "Boghossian, Mueller-Lyer, the Parrot, and the Nazi". Williamson argues that a knowledge-first account of the phenomenology of pre-doxastic states is preferable to one which appeals to intuition. My modal logic for rational intuition might provide a counterexample to Williamson's claim that a knowledge-first account of the phenomenology of pre-doxastic states is preferable to one which appeals to intuition, because epistemic and justification logics don't capture all of the properties which my modal logic and hyperintensional semantics for rational intuition can capture, and such properties are relevant to the phenomenology of pre-doxastic states, namely, the relation between intuition-that and intuition-of. I believe, too, that my modal logic for rational intuition is preferable to Artemov (1995)'s Justification Logic, where justification terms can be applied to mathematical axioms, for similar reasons.[10]

I argue, then, that intuition-that can further be shown to entrain property (iv), i.e. conceptual elucidation, by way of figuring as an interpretational modality which induces the reinterpretation of domains of quantification (cf. Fine, 2005; 2006) and the reinterpretation of the intensions and hyperintensions of mathematical vocabulary (cf. Uzquiano, 2015). Fine (op. cit.) has countenanced both postulational interpretational and postulational dynamic modalities, where the latter are imperatival. I propose to combine interpretational and dynamic modalities. Modalized rational intuition is therefore expressively equivalent to – and can crucially serve as a guide to the interpretation of – mathematical concepts which are formalizable in monadic first- and second-order formal languages.

Gödel writes: 'I said that in math[ematical] reasoning the non-comput[ational] (*i.e.* intuitive) element consists in intuitions of higher & higher infinities. This is quite true but it [sic - D.E.] this situation can be further analysed & then it turns out that they result (as becomes perfectly clear when these things are carried out in detail) from a deeper & deeper self knowledge of reason [to be more precise from a more & more complete rational knowledge of the <u>essence</u> of reason (of which essence the fac[ulty] of self knowledge is itself a constituent part)] [I believe that comput[ational] reason also results from self knowledge of reason but not from essential but factual knowledge] It seems to me that this is a verification (in the field of math[ematics]) of some tenets of idealistic philosophy' (Gödel, 1963, cited in van Atten, 2015: 160). In chapter **5**, I advance a novel account of the iteration of epistemic states based on modal and hyperintensional fixed points in the hyperintensional μ-calculus, instead of modal axiom 4 interpreted epistemically i.e. the KK principle. The modal logic and hyperintensional semantics for Gödelian intuition treated as a mathematical modality relates to epistemic operators interpreted as concerning knowledge, because intuition has dynamic-interpretational modalities and hyperintensional verifiers which effect conceptual elucidation. See §2. In Elohim (2024), interpretational

[10] See Artemov and Fitting (2019), for further discussion.

modalities and hyperintensional verifiers are treated as a species of epistemic modality and hyperintensionality.

In Section **2**, I countenance and motivate a modal logic, which embeds dynamic logic within the modal μ-calculus (see Carreiro and Venema, 2014). I argue that the dynamic interpretational properties of modalized rational intuition provide a precise means of accounting for the manner by which intuition can yield the reinterpretation of quantifier domains and mathematical vocabulary; and thus explain the role of rational intuition in entraining conceptual elucidation. In Section **3**, I examine remaining objections to the viability of rational intuition and provide concluding remarks.

2 Modalized Rational Intuition and Conceptual Elucidation

In this section, I will outline the logic for Gödelian intuition. The motivation for providing a logic for rational intuition will perhaps be familiar from treatments of the property of knowledge in formal epistemology. The analogy between rational intuition and the property of knowledge is striking: Just as knowledge has been argued to be a mental state (Williamson, 2001; Nagel, 2013b); to be propositional (Stanley and Williamson, 2001); to be factive; and to possess modal properties (Hintikka, 1962; Nozick, 1981; Fagin et al., 1995; Meyer and van der Hoek, 1995), so rational intuition can be argued to be a property of mental states; to be propositional, as recorded by the notion of intuition-that; and to possess modal properties amenable to rigorous treatment in systems of modal logic.

I should like to suggest that the modal logic of Gödelian intuition is the modal μ-calculus (see Carreiro and Venema, 2014).

Suppose that there is a language, L, with the following operations: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\Diamond$, $\Box$, μx. (least fixed point), υx. (greatest fixed point); and the following grammar:

$$\phi := \mathrm{T} \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \psi \mid \Diamond\phi \mid \Box\phi \mid \mu\mathrm{x}.\phi \mid \upsilon\mathrm{x}.\phi$$

Let M be a model over the Kripke frame, $\langle$W, R$\rangle$; so, M = $\langle$W, R, V$\rangle$. W is a non-empty set of possible worlds. R is a binary relation on W. R[w] denotes the set $\{$v$\in$W $\mid$ (w,v)$\in$R$\}$. V is a function assigning proposition letters, ϕ, to subsets of W.

$\langle$M,w$\rangle \Vdash \phi$ if and only if w$\in$V(ϕ).
$\langle$M,w$\rangle \Vdash \neg\phi$ iff it is not the case that $\langle$M,w$\rangle \Vdash \phi$
$\langle$M,w$\rangle \Vdash \phi \wedge \psi$ iff $\langle$M,w$\rangle \Vdash \phi$ and $\langle$M,w$\rangle \Vdash \psi$
$\langle$M,w$\rangle \Vdash \phi \vee \psi$ iff $\langle$M,w$\rangle \Vdash \phi$ or $\langle$M,w$\rangle \Vdash \psi$
$\langle$M,w$\rangle \Vdash \phi \to \psi$ iff, if $\langle$M,w$\rangle \Vdash \phi$, then $\langle$M,w$\rangle \Vdash \psi$
$\langle$M,w$\rangle \Vdash \phi \iff \psi$ iff [$\langle$M,w$\rangle \Vdash \phi$ iff $\langle$M,w$\rangle \Vdash \psi$]
$\langle$M,w$\rangle \Vdash \Diamond\phi$ iff $\langle$R$_w\rangle(\phi)$
$\langle$M,w$\rangle \Vdash \Box\phi$ iff [R$_w$](ϕ), with

$\langle R\rangle(\phi) := \{w\in W \mid R[w] \cap \phi \neq \emptyset\}$
$[R](\phi) := \{w\in W \mid R_d[w] \subseteq \phi\}$
$\langle M,w\rangle \Vdash \mu x.\phi$ iff $\bigcap\{U \subseteq W \mid \phi \subseteq U\}$ (Fontaine, 2010: 18)
$\langle M,w\rangle \Vdash \upsilon x.\phi$ iff $\bigcup\{U \subseteq W \mid U \subseteq \phi\}$ (op. cit.; Fontaine and Place, 2010),
$R_A := \bigcap_{a\in A'} R_a$.

This last clause characterizes the intersection of accessibility relations in the modal logic for rational intuition, such that the pooled intuition can be thought of as a type of distributive property among a set of agents. Interpreting the property as knowledge, Baltag and Smets (2020: 3) write: 'One can now introduce, for each group [A'] $\subseteq$ A, a distributed knowledge operator $K_{A'}\phi$ as the Kripke modality $[R_A]$ ... The logic of distributed knowledge LD has as language the set of all formulas built recursively from atomic formulas p $\in$ Prop by using negation $\neg\phi$, conjunction $\phi \wedge \psi$, and distributed knowledge operators $K_{A'}\phi$ (for all groups [A'] $\subseteq$ A). The logic of distributed knowledge and common knowledge LDC is obtained by extending the language of LD with common knowledge modalities $[O]_{A'}$. These logics are known to be decidable and have the finite model property. [The following comprise] complete proof systems LDC and LD for these logics:

(I) Axioms and rules of classical propositional logic
(II) S5 axioms and rules for distributed knowledge
(K-Necessitation) From ϕ, infer $K_{A'}\phi$
(K-Distribution) $K_{A'}(\phi \rightarrow \psi) \rightarrow (K_{A'}\phi \rightarrow K_{A'}\psi)$
(Veracity) $K_{A'}\phi \rightarrow \phi$
(Pos. Introspection) $K_{A'}\phi \rightarrow K_{A'}K_{A'}\phi$
(Neg. Introspection) $\neg K_{A'}\phi \rightarrow K_{A'}\neg K_{A'}\phi$
(III) Special axiom for distributed knowledge:
(Monotonicity) $K_{A'}\phi \rightarrow K_Q\phi$, for all A' $\subseteq$ Q $\subseteq$ A.'

The notion of pooled rational intuition among a set of agents, as formalized by the logic of distributive knowledge, might be one way formally to account for one aspect of the communitarian conditions on practices, techniques, uses, customs, and institutions which subserve the metasemantic determination and normative status of linguistic contents in the work of the later Wittgenstein (2009).

With regard to the axioms which rational intuition satisfies, K states that $\Box(\phi\rightarrow\psi) \rightarrow (\Box\phi\rightarrow\Box\psi)$; i.e., if one has an intuition that ϕ entails ψ, then if one has the intuition that ϕ then one has the intuition that ψ. 4 states that $\Box\phi \rightarrow \Box\Box\phi$; i.e., if one has the intuition that ϕ, then one intuits that one has the intuition that ϕ. Necessitation states that $\vdash\phi \rightarrow \vdash\Box\phi$. Because intuition-that is non-factive, we eschew in our modal system of axiom T, which states that $\Box\phi \rightarrow \phi$; i.e., one has the intuition that ϕ only if ϕ is the case [cf. BonJour (1998: 4.4); Parsons (2008: 141)].

In order to account for the role of rational intuition in entraining conceptual elucidation (cf., Gödel, 1961: 383), I propose to follow Fine (2006) and Uzquiano (2015) in suggesting that there are dynamic interpretational modalities associated with the possibility of reinterpreting both domains of quantification (Fine, op. cit.) and the non-logical vocabulary of mathematical languages, such as the membership relation in ZF set theory (Uzquiano, op. cit.).[11]

Fine (2005) has advanced modalities which are postulational, and prescriptive. He (op. cit.) suggests, further, that the postulational modality might be characterized as a program in dynamic logic, whose operations can take the form of 'simple' and 'complex' postulates which enjoin subjects to reinterpret the domains. Uzquiano's (op. cit.) generalization of the interpretational modality, in order to target the reinterpretation of the intensions of terms such as the membership relation, can similarly be treated.

In propositional dynamic logic (PDL), there are an infinite number of diamonds, with the form $\langle\pi\rangle$.[12] π denotes a non-deterministic program, which in the present setting will correspond to Fine's postulates adumbrated in the following. $\langle\pi\rangle\phi$ abbreviates 'some execution of π from the present state entrains a state bearing information ϕ'. The dual operator is $[\pi]\phi$, which abbreviates 'all executions of π from the present state entrain a state bearing information ϕ'. π^* is a program that executes a distinct program, π, a number of times ≥ 0. This is known as the iteration principle. PDL is similarly closed under finite and infinite unions. This is referred to as the 'choice' principle: If π_1 and π_2 are programs, then so is $\pi_1 \cup \pi_2$. The forth condition is codified by the 'composition' principle: If π_1 and π_2 are programs, then $\pi_1;\pi_2$ is a program (intuitively: the composed program first executes π_1 then π_2). The back condition is codified by Segerberg's induction axiom (Blackburn et al., op. cit: p. 13). Formally, $[\pi^*](\phi \rightarrow [\pi]\phi) \rightarrow (\phi \rightarrow [\pi^*]\phi)$.

Crucially, the iteration principle permits π^* to be interpreted as a fixed point for the equation: $\mathrm{x} \iff \phi \vee \Diamond\mathrm{x}$. The smallest solution to the equation will be the least fixed point, $\mu\mathrm{x}.\phi \vee \Diamond\mathrm{x}$, while the largest solution to the equation, when $\pi^* \vee \infty_\Diamond$, will be the greatest fixed point, $\upsilon\mathrm{x}.\phi \vee \Diamond\mathrm{x}$. Janin and Walukiewicz (op. cit.) have proven that the modal μ-calculus is equivalent to the bisimulation-invariant fragment of monadic second-order logic.

Fine's simple postulational dynamic modality takes, then, the form:

'(i) Introduction. !x.C(x)', which states the imperative to: '[I]ntroduce an object x [to the domain] conforming to the condition C(x)'.

Fine's complex dynamic-postulational modalities are the following:

(ii) 'Composition. Where β and γ are postulates, then so is $\beta;\gamma$. We may read $\beta;\gamma$ as: do β and then do γ; and $\beta;\gamma$ is to be executed by first executing β and then executing γ.

[11] A variant strategy is pursued by Eagle (2008). Eagle suggests that the relation between rational intuition and conceptual elucidation might be witnessed via associating the fundamental properties of the entities at issue with their Ramsey sentences; i.e., existentially generalized formulas, where the theoretical terms therein are replaced by second-order variables bound by the quantifiers. However, the proposal would have to be expanded upon, if it were to accommodate Gödel's claim that mathematical intuitions possess a modal profile.

[12] Cf. Blackburn et al., op. cit.: 12-14. Semantics and proof-theory for PDL are outlined in Hoare (1969); Pratt (1976); Goldblatt (1987: ch. 10; 1993: ch. 7) and van Benthem (2010: 158).

(iii) Conditional. Where β is a postulate and A an indicative sentence, then A $\rightarrow \beta$ is a postulate. We may read A $\rightarrow \beta$ as: if A then do β. How A $\rightarrow \beta$ is executed depends upon whether or not A is true: if A is true, A $\rightarrow \beta$ is executed by executing β; if A is false, then A $\rightarrow \beta$ is executed by doing nothing.
(iv) Universal. Where β(x) is a postulate, then so is $\forall$xβ(x). We may read $\forall$xβ(x) as: do β(x) for each x; and $\forall$xβ(x) is executed by simultaneously executing each of β(x1), β(x2), β(x3), ..., where x1, x2, x3, ... are the values of x (within the current domain). Similarly for the postulate $\forall$Fβ(F), where F is a second-order variable.
(v) Iterative Postulates. Where β is a postulate, then so is β*. We may read β* as: iterate β; and β* is executed by executing β, then executing β again, and so on ad infinitum' (op. cit.: 91-92).

Whereas Fine avails of postulational interpretational modalities in order both to account for the notion of indefinite extensiblity and to demonstrate how relatively unrestricted quantification can be innocuous without foundering upon Russell's paradox (op. cit.: 26-30), the primary interest in adopting modal μ-logic with modal operators interpreted as dynamic interpretational modalities as the logic of rational intuition is its capacity to account for dynamic reinterpretations of mathematical vocabulary and quantifier domains; and thus to illuminate how the precise mechanisms codifying modalized rational intuition might be able to entrain advances in conceptual elucidation.

The computational profile of modalized rational intuition can be outlined as follows. In category theory, a category C is comprised of a class Ob(C) of objects and a family of arrows for each pair of objects C(A,B) (Venema, 2007: 421). An **E**-coalgebra is a pair $\mathbb{A}$ = (A, μ), with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\rightarrow$ **E**(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390). A coalgebraic model of deterministic automata can be thus defined (391). An automaton is a tuple, $\mathbb{A} = \langle$A, a_I, C, Ξ, F$\rangle$, such that A is the state space of the automaton $\mathbb{A}$; $a_I \in$A is the automaton's initial state; C is the coding for the automaton's alphabet, mapping numerals to the natural numbers; Ξ: A X C $\rightarrow$ A is a transition function, and F $\subseteq$ A is the collection of admissible states, where F maps A to {1,0}, such that F: A $\rightarrow$ 1 if a$\in$F and A $\rightarrow$ 0 if a$\notin$F (op. cit.). The modal profile of coalgebraic automata can be witnessed both by construing the transition function as a counterfactual conditional (cf. Stalnaker, 1968; Williamson, 2007), and in virtue of the convergence of coalgebraic automata with coalgebraic models of modal logic (407). Let

$\Diamond\phi \equiv \nabla\{\phi, \mathrm{T}\}$,

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.)

$[\![\nabla\Phi]\!] = \{\mathrm{w}\in\mathrm{W} \mid \mathrm{R}[\mathrm{w}] \subseteq \bigcup\{[\![\phi]\!] \mid \phi\in\Phi\}$ and $\forall\phi\in\Phi, [\![\phi]\!] \cap \mathrm{R}[\mathrm{w}] \neq \emptyset\}$ (Fontaine, 2010: 17).

Let an **E**-coalgebraic modal model, $\mathbb{A} = \langle$S, λ, R[.]$\rangle$, where λ(s) is 'the collection of proposition letters true at s in S, and R[s] is the successor set of s in S', such that $\mathbb{S}$,s $\Vdash \nabla\Phi$ if and only if, for all (some) successors σ of s$\in$S, [Φ, σ(s)$\in$**E**($\Vdash_{\mathbb{A}}$)] (Venema, 2007: 399, 407), with **E**($\Vdash_{\mathbb{A}}$) a relation lifting of the satisfaction relation $\Vdash_{\mathbb{A}} \subseteq$ S x Φ. Let a functor, **K**, be such that there is a relation $\overline{\mathbf{K}} \subseteq \mathbf{K}$(A) x $\mathbf{K}$(A') (Venema, 2012: 17)). Let Z be a binary relation s.t. Z $\subseteq$ A x A' and $\wp\overline{Z} \subseteq \wp$(A) x $\wp$(A'), with

$\wp\overline{Z}$:= {(X,X') | ∀x∈X∃x'∈X' with (x,x')∈Z ∧ ∀x'∈X'∃x∈X with (x,x')∈Z} (op. cit.). Then, we can define the relation lifting, $\overline{\mathbf{K}}$, as follows:

$\overline{\mathbf{K}}$:= {[(π,X), (π',X')] | $\pi = \pi$' and (X,X')∈$\wp\overline{Z}$} (op. cit.), with π a projection mapping of $\overline{\mathbf{K}}$.[13]

Modal automata are defined over a modal one-step language (Venema, 2020: 7.2). With A being a set of propositional variables, the set, `Latt`(X), of lattice terms over X has the following grammar:

$$\phi ::= \bot \mid \top \mid x \mid \phi \wedge \phi \mid \phi \vee \phi,$$

with x∈X and ϕ∈`Latt`(A) (op. cit.).

The set, `1ML`(A), of modal one-step formulas over A has the following grammar:

$$\alpha \in A ::= \bot \mid \top \mid \Diamond\phi \mid \Box\phi \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \text{ (op. cit.)}.$$

A modal P-automaton $\mathbb{A}$ is a quadruple (A, Θ, a_I), with A a non-empty finite set of states, a_I∈A an initial state, and the transition map

Θ: A x $\wp$P $\rightarrow$ `1ML`(A)

maps states to modal one-step formulas, with $\wp$P the powerset of the set of proposition letters, P (op. cit.: 7.3).

The philosophical significance of the foregoing is that the modal logic of rational intuition can be interpreted by modal coalgebraic automata. The foregoing accounts for the distinctively computational nature of the modal profile of rational intuition, while the modalities are interpreted as dynamic-interpretational modalities which effect reinterpretations of (hyper-)intensions and quantifier domains and thus effect conceptual elucidation.

3 Hyperintensionality

The hyperintensionality of rational intuition can be countenanced in two ways. The first way is via truthmaker semantics, which we here present in a two-dimensional guise. The truthmakers are interpreted as states of intuition. The second way is via topic-sensitivity, which countenances 'two-component' contents comprised of worlds and a mereology of topics i.e. subject matters. In this paper, I will render two-dimensional truthmakers topic-sensitive.

Following the presentation of topic models in Berto (2018; 2019), Canavotto et al (2020), and Berto and Hawke (2021), the diamond, box, and least and greatest fixed point operators can be sensitive to topics, i.e. hyperintensional

[13]The projections of a relation R, with R a relation between two sets X and Y such that R $\subseteq$ X x Y, are

X $\longleftarrow$(π_1) R (π_2)$\longrightarrow$ Y such that π_1((x,y)) = x, and π_2((x,y)) = y. See Rutten (2019: 240).

subject matters. Atomic topics comprising a set of topics, T, record the hyperintensional intentional content of atomic formulas, i.e. what the atomic formulas are about at a hyperintensional level. Topic fusion is a binary operation, such that for all x, y, z$\in$T, the following properties are satisfied: idempotence (x $\oplus$ x = x), commutativity (x $\oplus$ y = y $\oplus$ x), and associativity [(x $\oplus$ y) $\oplus$ z = x $\oplus$ (y $\oplus$ z)] (Berto, 2018: 5). Topic parthood is a partial order, $\leq$, defined as $\forall$x,y$\in$T(x $\leq$ y $\iff$ x $\oplus$ y = y) (op. cit.: 5-6). Atomic topics are defined as follows: Atom(x) $\iff$ $\neg\exists$y < x, with < a strict order. Topic parthood is thus a partial ordering such that, for all x, y, z$\in$T, the following properties are satisfied: reflexivity (x $\leq$ x), antisymmetry (x $\leq$ y $\wedge$ y $\leq$ x $\rightarrow$ x = y), and transitivity (x $\leq$ y $\wedge$ y $\leq$ z $\rightarrow$ x $\leq$ z) (6). A topic frame can then be defined as {W, R, T, $\oplus$, t}, with t a function assigning atomic topics to atomic formulas. For formulas, ϕ, atomic formulas, p, q, r (p_1, p_2, ...), and a set of atomic topics, Utϕ = {p_1, ... p_n}, the topic of ϕ, t(ϕ) = $\oplus$Utϕ = t(p_1) $\oplus$ ... $\oplus$ t(p_n) (op. cit.).

The diamond, box, and least and greatest fixed point operators for rational intuition can then be defined relative to topics:

$\langle$M,w$\rangle \Vdash \Diamond^t\phi$ iff $\langle R_{w,t}\rangle(\phi)$

$\langle$M,w$\rangle \Vdash \Box^t\phi$ iff $[R_{w,t}](\phi)$, with

$\langle R_{w,t}\rangle(\phi)$:= {w'$\in$Wt'$\in$T | $R_{w,t}$[w', t'] $\cap\ \phi \neq \emptyset$ and t'(ϕ) $\leq$ t(ϕ)

$[R_{w,t}](\phi)$:= {w'$\in$Wt'$\in$T | $R_{w,t}$[w', t'] $\subseteq \phi$ and t'(ϕ) $\leq$ t(ϕ)

$\langle$M,w$\rangle \Vdash \mu x.\phi^t$ iff $\bigcap${U $\subseteq$ W | $\phi \subseteq$ U}

$\langle$M,w$\rangle \Vdash \upsilon x.\phi^t$ iff $\bigcup${U $\subseteq$ W | U $\subseteq \phi$}.

Turning to hyperintensional truthmaker semantics, by contrast, a modalized state space model is a tuple $\langle$S, P, $\leq$, v$\rangle$, where S is a non-empty set of states, P is the subspace of possible states where states s and t comprise a fusion when s $\sqcup$ t$\in$P, $\leq$ is a partial order, and v: Prop $\rightarrow$ (2^S x 2^S) assigns a bilateral proposition $\langle p^+, p^-\rangle$ to each atom p$\in$Prop with p^+ and p^- incompatible (Fine 2017a-c; Hawke and Özgün, forthcoming: 10-11). The state space comprises hyperintensional states of rational intuition. Exact verification ($\vdash$) and exact falsification ($\dashv$) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, forthcoming: 11):

s $\vdash$ p if s$\in[\![p]\!]^+$

(s verifies p, if s is a truthmaker for p i.e. if s is in p's extension);

s $\dashv$ p if s$\in[\![p]\!]^-$

(s falsifies p, if s is a falsifier for p i.e. if s is in p's anti-extension);

s $\vdash \neg$p if s $\dashv$ p

(s verifies not p, if s falsifies p);

s $\dashv \neg$p if s $\vdash$ p

(s falsifies not p, if s verifies p);

s $\vdash$ p $\wedge$ q if $\exists$v,u, v $\vdash$ p, u $\vdash$ q, and s = v $\sqcup$ u

(s verifies p and q, if s is the fusion of states, v and u, v verifies p, and u verifies q);

s $\dashv$ p $\wedge$ q if s $\dashv$ p or s $\dashv$ q

(s falsifies p and q, if s falsifies p or s falsifies q);

s $\vdash$ p $\vee$ q if s $\vdash$ p or s $\vdash$ q (s verifies p or q, if s verifies p or s verifies q);

s $\dashv$ p $\vee$ q if $\exists$v,u, v $\dashv$ p, u $\dashv$ q, and s = v $\sqcup$ u

(s falsifies p or q, if s is the fusion of the states v and u, v falsifies p, and u falsifies q);

s $\vdash$ $\forall$xϕ(x) if $\exists$s$_1$, ..., s$_n$, with s$_1$ $\vdash$ ϕ(a$_1$), ..., s$_n$ $\vdash$ ϕ(a$_n$), and s = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$

[s verifies $\forall$xϕ(x) "if it is the fusion of verifiers of its instances ϕ(a$_1$), ..., ϕ(a$_n$)" (Fine, 2017c)];

s $\dashv$ $\forall$xϕ(x) if s $\dashv$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s falsifies $\forall$xϕ(x) "if it falsifies one of its instances" (op. cit.)];

s $\vdash$ $\exists$xϕ(x) if s $\vdash$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s verifies $\exists$xϕ(x) "if it verifies one of its instances ϕ(a$_1$), ..., ϕ(a$_n$)" (op. cit.)];

s $\dashv$ $\exists$xϕ(x) if $\exists$s$_1$, ..., s$_n$, with s$_1$ $\dashv$ ϕ(a$_1$), ..., s$_n$ $\dashv$ ϕ(a$_n$), and s = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$ (op. cit.)

[s falsifies $\exists$xϕ(x) "if it is the fusion of falsifiers of its instances" (op. cit.)];

s exactly verifies p if and only if s $\vdash$ p if s$\in$$[\![p]\!]$;

s inexactly verifies p if and only if s $\rhd$ p if $\exists$s'$\leq$S, s' $\vdash$ p; and

s loosely verifies p if and only if, $\forall$v, s.t. s $\sqcup$ v $\vdash$ p (35-36);

s $\vdash$ Aϕ if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u$\in$P and u' $\vdash$ ϕ, where Aϕ denotes that ϕ is a necessary truthmaker; and

s $\dashv$ Aϕ if and only if there is a v$\in$P such that for all u$\in$P either v $\sqcup$ u$\notin$P or u $\dashv$ ϕ;

s $\vdash$ A(Aϕ) if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u $\in$P and u' $\vdash$ ϕ and there is a u"$\in$P such that u' $\sqcup$ u"$\in$P and u" $\vdash$ ϕ;

s $\vdash$ A($\forall$xϕ(x)) if and only if for all u$\in$P there is a u'$\in$P such that u $\vdash$ [u' $\vdash$ $\exists$s$_1$, ..., s$_n$, with s$_1$ $\vdash$ ϕ(a$_1$), ..., s$_n$ $\vdash$ ϕ(a$_n$), and u' = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$];

s $\vdash$ A($\exists$xϕ(x)) if and only if or all u$\in$P there is a u'$\in$P such that u $\vdash$ [u' $\vdash$ ϕ(a)] for some individual a in a domain of individuals (op. cit.).

In order to account for two-dimensional indexing, we augment the model, M, with a second state space, S*, on which we define both a new parthood relation, $\leq$*, and partial function, V*, which serves to map propositions in a domain, D, to pairs of subsets of S*, {1,0}, i.e. the verifier and falsifier of p, such that $[\![$p$]\!]^+$ = 1 and $[\![$p$]\!]^-$ = 0. Thus, M = $\langle$S, S*, D, $\leq$, $\leq$*, V, V*$\rangle$. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c, i: c ranges over subsets of S, and i ranges over subsets of S*.

(*) M, s$\in$S, s*$\in$S* $\vdash$ p iff:
(i) $\exists$c$_s$$[\![p]\!]^{c,c}$ = 1 if s$\in$$[\![p]\!]^+$; and
(ii) $\exists$i$_{s*}$$[\![p]\!]^{c,i}$ = 1 if s*$\in$$[\![p]\!]^+$

(Distinct states, s, s*, from distinct state spaces, S, S*, provide a multi-dimensional verification for a proposition, p, if the value of p is provided a truthmaker by s. The value of p as verified by s determines the value of p as verified by s*).

We say that p is hyper-rigid iff:

(**) M, s∈S, s*∈S* ⊢ p iff:
(i) $\forall$c'$_s$$[\![p]\!]^{c,c'} = 1$ if s∈$[\![p]\!]^+$; and
(ii) $\forall$i$_{s*}$$[\![p]\!]^{c,i} = 1$ if s*∈$[\![p]\!]^+$

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions. Epistemic two-dimensional truthmaker semantics receives substantial motivation by its capacity (i) to model conceivability arguments involving hyperintensional metaphysics, and (ii) to avoid the problem of mathematical omniscience entrained by intensionalism about propositions[14]:

- Epistemic Hyperintension:

  $\texttt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$, with s a state in the epistemic state space S

- Subjunctive Hyperintension:

  $\texttt{sec}_{v@}(x) = \lambda w.[\![x]\!]^{v@,w}$, with w a state in metaphysical state space W

In epistemic two-dimensional semantics, the value of a formula or term relative to a first parameter ranging over epistemic scenarios determines the value of the formula or term relative to a second parameter ranging over metaphysically possible worlds. The dependence is recorded by 2D-intensions. Chalmers (2006: 102) provides a conditional analysis of 2D-intensions to characterize the dependence: "Here, in effect, a term's subjunctive intension depends on which epistemic possibility turns out to be actual. / This can be seen as a mapping from scenarios to subjunctive intensions, or equivalently as a mapping from (scenario, world) pairs to extensions. We can say: the two-dimensional intension of a statement S is true at (V, W) if V verifies the claim that W satisfies S. If $[A]_1$ and $[A]_2$ are canonical descriptions of V and W, we say that the two-dimensional intension is true at (V, W) if $[A]_1$ epistemically necessitates that $[A]_2$ subjunctively necessitates S. A good heuristic here is to ask "If $[A]_1$ is the case, then if $[A]_2$ had been the case, would S have been the case?". Formally, we can say that the two-dimensional intension is true at (V, W) iff '$\Box_1([A]_1 \rightarrow \Box_2([A]_2 \rightarrow S))$' is true, where '$\Box_1$' and '$\Box_2$' express epistemic and subjunctive necessity respectively".

- 2D-Hyperintension:

  $\texttt{2D}(x) = \lambda s \lambda w [\![x]\!]^{s,w} = 1.$

We can then combine topics with truthmakers rather than worlds, thus countenancing multi-hyperintensional semantics, i.e. topic-sensitive epistemic two-dimensional truthmaker semantics:

[14] See Author (ms_1) through (ms_n) for further discussion.

- Topic-sensitive Epistemic Hyperintension:

  $\texttt{pri}_t(x) = \lambda s \lambda \mathrm{t}.[\![x]\!]^{s \cap t, s \cap t}$, with s a truthmaker from an epistemic state space.

- Topic-sensitive Subjunctive Hyperintension:

  $\texttt{sec}_{v_@ \cap t}(\mathrm{x}) = \lambda w \lambda \mathrm{t}.[\![x]\!]^{v_@ \cap t, w \cap t}$, with w a truthmaker from a metaphysical state space.

- Topic-sensitive 2D-Hyperintension:

  $\texttt{2D}(x) = \lambda s \lambda w \lambda \mathrm{t}[\![\mathrm{x}]\!]^{s \cap t, w \cap t} = 1.$

We here propose a topic-sensitive truthmaker semantics for dynamic epistemic logic and dynamic interpretational modalities.

The language of public announcement logic has the following syntax (see Baltag and Renne, 2016):

$$\phi := \mathrm{p} \mid \phi \wedge \phi \mid \neg\phi \mid [\mathrm{a}]\phi \mid [\phi!]\psi$$

[a]ϕ is interpreted as the 'the agent knows ϕ'. $[\phi!]\psi$ is an announcement formula, and is intuitively interpreted as "whenever ϕ is true, ψ is true after we eliminate all not-ϕ possibilities (and all arrows to and from these possibilities)".

Semantics for public announcement logic is as follows:

M, w $\Vdash \phi$ if and only if w$\in$V(ϕ)

M, w $\Vdash \phi \wedge \psi$ if and only if M, w $\Vdash \phi$ and M, w $\Vdash \psi$

M, w $\Vdash \neg\phi$ if and only if M, w $\nVdash \phi$

M, w $\Vdash$ [a]ϕ if and only if M, v $\Vdash \phi$ for each v satisfying wR_av

M, w $\Vdash [\phi!]\psi$ if and only if M, w $\nVdash \phi$ or M$[\phi!]$, w $\Vdash \psi$,

where M$[\phi!]$ = (W$[\phi!]$, R$[\phi!]$, V$[\phi!]$) is defined by

W$[\phi!]$:= (v$\in$W | M, v $\Vdash \phi$) (intuitively, "retain only the worlds where ϕ is true") (op. cit.),

x$R[\phi!]_a$y if and only if xR_ay (intuitively, "leave arrows between remaining words unchanged"), and

v$\in$V$[\phi!]$(p) if and only if v$\in$V(p) (intuitively, "leave the valuation the same at remaining worlds").

My proposal is that both announcement formulas, $[\phi!]\psi$, and Fine and Uzquiano's dynamic modalities ought to be rendered hyperintensional, such that the box operators are interpreted as topic-sensitive necessary truthmakers. The dynamic interpretational modalities can just take the clause for A(ϕ) as above. For announcement formulas, $[\phi!]\psi$ if and only if either (i) for all s$\in$P there is no s'$\in$P such that s' $\sqcup$ s $\in$P and s' $\vdash \phi$ or (ii) M$[\phi!]$, s $\vdash \psi$,

where M$[\phi!]$ = $\langle$S$[\phi!]$, $\leq[\phi!]$, v$[\phi!]\rangle$ is defined by

S$[\phi!]$:= s'$\in$S | M, s' $\vdash \phi$ (intuitively, retain only states which verify ϕ),

$\leq[\phi!]$ if and only if s $\leq$ s' (intuitively, leave relations between remaining states unchanged), and

v$[\phi!]$ if and only if v: Prop $\rightarrow$ (2^S x 2^S) which assigns a bilateral proposition $\langle\phi^+, \phi^-\rangle$ to $\phi\in$Prop (intuitively, leave the valuation the same at remaining states). States are topic-sensitive such that s in the foregoing abbreviates s ∩ t. Thus topic-sensitive truthmakers, conceived as states of intuition, can receive an interpretation on which they induce reinterpretations of (hyper-)intensions and quantifier domains, and thus effect conceptual elucidation.

4 Concluding Remarks

In this note, I have endeavored to outline the modal logic of Gödel's conception of intuition, in order both (i) to provide a formally tractable foundation thereof, and thus (ii) to answer the primary objection to the notion as a viable approach to the epistemology of mathematics. I have been less concerned with providing a defense of the general approach from the array of objections that have been proffered in the literature. Rather, I have sought to demonstrate how the mechanisms of rational intuition can be formally codified and thereby placed on a secure basis.

Among, e.g., the most notable remaining objections, Koellner (2009) has argued that the best candidates for satisfying Gödel's conditions on being intrinsically justified are reflection principles, which state that the height of the hierarchy of sets, V, cannot be constructed 'from below', because, for all true formulas in V, the formulas will be true in a proper initial segment of the hierarchy. Koellner's limitative results are, then, to the effect that reflection principles cannot exceed the use of second-order parameters without entraining inconsistency or triviality (op. cit.). Another crucial objection is that the properties of rational intuition, as a species of cognitive phenomenology, lack clear and principled criteria of individuation. Burgess (2014) notes, e.g., that the role of rational intuition in alighting upon mathematical truths might be distinct from the functions belonging to what he terms a 'heuristic' type of intuition. The constitutive role of the latter might be to guide a mathematician's non-algorithmic insight as she pursues an informal proof. A similar objection is advanced in Cappelen (2012: 3.2-3.3), who argues that – by contrast to the properties picked out by theoretical terms such as 'utility function' – terms purporting to designate cognitive phenomenal properties both lack paradigmatic criteria of individuation and must thereby be a topic of disagreement, in virtue of the breadth of variation in the roles that the notion has been intended to satisfy. Williamson (2020) advances an argument against whether intuitions – as understood by Boghossian (2020) and described above – are fit for purpose in internalist epistemologies, as well as an argument against intuition's theoretical significance in general. As explicated by Boghossian (2020: 222), Williamson's argument against whether intuitions are fit for purpose for the internalist runs as follows:

'1. To have reason to believe in a certain type of mental state, it must either be consciously available or reasonably postulated.

2. Intuitions are not consciously available (mere introspection does not reveal them).

3. Hence, to have reason to believe in intuitions, they must be reasonably

postulated.

4. If a type of mental state is postulated, then it is not consciously available.

5. If a type of mental state is to serve an internalist purpose, it must be consciously available.

6. Hence, even if we had reason to believe in intuitions, they would not be able to serve an internalist purpose' (op. cit.).

Our proposal above would reject premise 2 of Williamson's argument, because intuitions are defined as real phenomenal properties, and thus as being consciously available.

Williamson's argument against the theoretical significance of intuitions in general is that they are 'in danger of justifying bigoted beliefs' (Williamson, 2020: 237). He mentions the possibility of there being e.g. Nazis with consistent belief sets (238), and asks of intuitions: 'Why should they be impervious to all the usual distortions from ignorance and error, bigotry and bias' (213)? Arguably, there are not sufficient constraints on intuition to rule out that intuitions provide prima facie justification even for the most unethical belief sets.

The foregoing issues notwithstanding, I have endeavored to demonstrate that – as with the property of knowledge – an approach to the notion of intuition-that which construes the notion as a modal operator, and the provision thereof with a philosophically defensible logic, might be sufficient to counter the objection that the very idea of rational intuition is mysterious and constitutively unconstrained.

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