Cognitivism about Epistemic Modality

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Abstract

This paper aims to vindicate the thesis that cognitive computational properties are abstract objects implemented in physical systems. I avail of the equivalence relations countenanced in Homotopy Type Theory, in order to specify an abstraction principle for intensional, computational properties. The homotopic abstraction principle for intensional mental functions provides an epistemic conduit into our knowledge of cognitive algorithms as abstract objects. I examine, then, how intensional functions in Epistemic Modal Algebra are deployed as core models in the philosophy of mind, Bayesian perceptual psychology, the program of natural language semantics in linguistics, and in quantum information theory, and I argue that this provides abductive support for the truth of homotopic abstraction. Epistemic modality can thus be shown to be both a compelling and a materially adequate candidate for the fundamental structure of mental representational states, comprising a fragment of the Language of Thought.

1 Introduction

This essay aims to vindicate the thesis that cognitive computational properties are abstract objects implemented in physical systems.[1] A recent approach to the foundations of mathematics is Homotopy Type Theory.[2] In Homotopy Type Theory, homotopies can be defined as equivalence relations on intensional functions. In this essay, I argue that homotopies can thereby figure in abstraction principles for intensional, cognitive computational properties.[3] Homotopies for intensional functions thus comprise identity criteria for some cognitive mech-

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[1] Cf. Turing (1950); Putnam (1967); Newell (1973); Fodor (1975); and Pylyshyn (1978).

[2] Cf. The Univalent Foundations Program (2013).

[3] For the first proposal to the effect that abstraction principles can be used to define abstracta such as cardinal number, see Frege (1884/1980: 68; 1893/2013: 20). For the locus classicus of the contemporary abstractionist program, see Hale and Wright (2001).

anisms. The philosophical significance of the foregoing is twofold. First, the proposal demonstrates how epistemic modality is a viable candidate for a fragment of the Language of Thought.[4] The identity of intensional functions and epistemic modal operators can be witnessed via a unique, epistemic interpretation of the algebraic semantics of modal logic. Second, the proposal serves to delineate one conduit for our epistemic access to cognitive algorithms as abstract objects.[5]

In Section **2**, I provide an abstraction principle for cognitive algorithms, by availing of the equivalence relations countenanced in Homotopy Type The-

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[4]Given a metalanguage, a precedent to the current approach – which models thoughts and internal representations via possible worlds model theory – can be found in Wittgenstein (1921/1974: 2.15-2.151, 3-3.02).

[5]Bealer (1982) proffers a *non-modal* algebraic logic for intensional entities – i.e., properties, relations, and propositions – which avails of a λ-definable variable-binding abstraction operator (op. cit.: 46-48, 209-210). Bealer reduces modal notions to logically necessary conditions-cum-properties, as defined in his non-modal algebraic logic (207-209). The present approach differs from the foregoing by: (i) countenancing a *modal* algebra, on an epistemic interpretation thereof; (ii) treating the abstraction operator as a Fregean function from concepts to objects, rather than as a λ-operator; (iii) availing of the univalence axiom in Homotopy Type Theory – which collapses identity and isomorphism – in order to provide an equivalence relation for the abstraction principle pertinent to (ii); and (iv) demonstrating how the model is availed of in various branches of the cognitive sciences, such that Epistemic Modal Algebra may be considered a viable candidate for the Language of Thought.

Katz (1998) proffers a view of the epistemology of abstracta, according to which the propositions of the Language of Thought are abstract, and on which the syntax and the semantics for the propositions are innate (35). Katz suggests that the proposal is consistent with both a Fregean approach to propositions, according to which they are thoughts formed by the concatenation of concepts, and a Russellian approach, according to which they are structured tuples of non-conceptual entities (36). He endorses, ultimately, an approach to propositions which eschews Frege's (1892/1997) distinction between sense and reference, yet according to which they are 'abstract senses', or thoughts, correlated to natural language sentence types (114-115). That propositions are abstract is argued, then, to suffice for knowledge of abstract entities (op. cit.). One difference between Katz's proposal and the one here presented is that Katz rejects modal approaches to propositions, because the latter cannot countenance reductio proofs based on counterfactuals with impossible antecedents (38; cf. Lewis, 1973: I.6). Following, Lewis (op. cit.), the present approach does not avail of impossible worlds, i.e., worlds at which true contradictions obtain; and thus counterfactuals with impossible antecedents are vacuously true. If so, then Katz's argument against modal approaches to propositions can be circumvented. A second difference is that, on Katz's approach, the necessity of mathematical truths is argued to consist in reductio proofs, such that the relevant formulas will be true on all interpretations, and thus true of logical necessity (39). However, the endeavor to reduce the necessity of mathematical truths to the necessity of logical consequence would result in the preclusion, both of cases of informal proofs in mathematics, which can, e.g., involve diagrams (cf. Azzouni, 2004; Giaquinto, 2008: 1.2), and of mathematical truths which obtain in axiomatizable, yet non-logical mathematical languages such as Euclidean geometry. Finally, Katz rejects abstraction principles, and thus implicit definitions for abstract objects (105-106).

ory. In Section **3**, I define a topological boolean algebraic semantics for modal logic, where the modalities defined on the algebra range over states of information.[6] In Section **4**, I describe how models of Epistemic Modal Algebra are availed of when perceptual representational states are modeled in Bayesian perceptual psychology; when speech acts are modeled in natural language semantics; when transformations are defined in quantum information theory; and when knowledge, belief, intentional action, and rational intuition are modeled in philosophical approaches to the nature of propositional attitudes. This provides abductive support for the claim that Epistemic Modal Algebra is both a compelling and materially adequate candidate for a fragment of the Language of Thought. In Section **5**, I argue that the proposal (i) resolves objections to the relevant abstraction principles advanced by both Dean (2016) and Linnebo and Pettigrew (2014), and (ii) reveals a commitment to *in re* platonism in the appeal to cognitive mechanisms in Azzouni's (2013: 9.1-9.2; forthcoming) version of nominalism. Section **6** provides concluding remarks.

2   An Abstraction Principle for Cognitive Algorithms

In this section, I specify a homotopic abstraction principle for intensional functions. Intensional isomorphism, as a jointly necessary and sufficient condition for the identity of intensions, is first proposed in Carnap (1947: §14). The isomorphism of two intensional structures is argued to consist in their logical, or L-, equivalence, where logical equivalence is co-extensive with the notions of

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[6]The epistemic modality can be interpreted so as to target notions of conceivability ($\diamond$) and apriority ($\square \equiv \neg\diamond\neg$); belief and knowledge; and psychophysical possibilities. See Section **4**, for further discussion.

both analyticity (§2) and synonymy (§15). Carnap writes that: '[A]n expression in S is L-equivalent to an expression in S' if and only if the semantical rules of S and S' together, without the use of any knowledge about (extra-linguistic) facts, suffice to show that the two have the same extension' (p. 56), where semantical rules specify the intended interpretation of the constants and predicates of the languages (4).[7] The current approach differs from Carnap's by defining the equivalence relation necessary for an abstraction principle for epistemic intensions on Voevodsky's (2006) Univalence Axiom, which collapses identity with isomorphism in the setting of intensional type theory.[8] In the following section, I define, then, a class of models for Epistemic Modal Algebra.

Topological Semantics

In the topological semantics for modal logic, a frame is comprised of a set of points in topological space, X, and an accessibility relation, R:

F = $\langle$X,R$\rangle$;

X = $(X_x)_{x \in X}$; and

R = $(Rxy)_{x,y \in X}$ iff $R_x \subseteq X_x$ x X_x, s.t. if Rxy, then $\exists o \subseteq X$, with x$\in$o s.t. $\forall y \in o(Rxy)$,

where the set of points accessible from a privileged node in the space is said to be open.[9] A model defined over the frame is a tuple, M = $\langle$F,V$\rangle$, with V a

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[7]For criticism of Carnap's account of intensional isomorphism, based on Carnap's (1937: 17) 'Principle of Tolerance' to the effect that pragmatic desiderata are a permissible constraint on one's choice of logic, see Church (1954: 66-67).

[8]Note further that, by contrast to Carnap's approach, epistemic intensions are here distinguished from linguistic intensions (cf. Author (ms_1,), for further discussion), and the current work examines the philosophical significance of the convergence between epistemic intensions and formal, rather than natural, languages. For a translation from type theory to set theory – which is of interest to, inter alia, the definability of epistemic intensions in the setting of set theory (cf. Author (ms_2, ms_3, ms_4,) – see Linnebo and Rayo (2012). For topological Boolean-valued models of epistemic set theory – i.e., a variant of ZF with the axioms augmented by epistemic modal operators interpreted as informal provability and having a background logic satisfying S4 – see Scedrov (1985), Flagg (1985), and Goodman (1990).

[9]In order to ensure that the Kripke semantics matches the topological semantics, X must

valuation function from subsets of points in F to propositonal variables taking the values 0 or 1. Necessity is interpreted as an interiority operator on the space:

M,x $\Vdash$ $\Box\phi$ iff $\exists o \subseteq X$, with x$\in$o, such that $\forall y \in o$ M,y $\Vdash$ ϕ.

Homotopy Theory

Homotopy Theory countenances the following identity, inversion, and concatenation morphisms, which are identified as continuous paths in the topology. The formal clauses, in the remainder of this section, evince how homotopic morphisms satisfy the properties of an equivalence relation.[10]

p : [0,1] $\rightarrow$ X, with p(0) = x and p(1) = y;

f : $X_1 \rightarrow X_2$;

g : $X_1 \rightarrow X_2$;

H : X_1 x {0,1} $\rightarrow X_2$, $H_{x,0}$ = f(x) and $H_{x,1}$ = g(x).

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Reflexivity

$\forall$x,y:A$\forall$p(p : x $=_A$ y) : τ(x,y,p), with A and τ designating types, 'x:A' interpreted as 'x is a token of type A', **refl** a dependent function of reflexivity, and U designating a universe of elements, e:

$\forall\alpha$:A$\exists$e(α) : τ(α, α, refl_α);

p,q : (x $=_A$ y)

$\exists$r$\in$e : p $=_{(x=_A y)}$ q

$\exists\mu$: r $=_{(p=_{(x=_A y)}q)}$ s.

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further be Alexandrov; i.e., closed under arbitrary unions and intersections. Thanks here to Peter Milne.

[10]The definitions and proofs at issue can be found in the Univalent Foundations Program (op. cit.: ch. 2.0-2.1).

The Induction Principle

If:

$\forall$x,y:A$\forall$p(p : x $=_A$ y)$\exists\tau[\tau$(x,y,p)] $\land$ $\forall\alpha$:A$\exists$e(α) : $\tau(\alpha$, α, $\texttt{refl}_\alpha$)

Then:

$\forall$x,y:A$\exists$p(p : x $=_A$ y)$\exists$e[$\texttt{ind}_{=_A}$(τ,e,x,y,p) : τ(x,y,p), such that

$\texttt{ind}_{=_A}$(τ,e,α,α,$\texttt{refl}_\alpha$) $\equiv$ e(α)].

Symmetry

$\forall$A$\forall$x,y:A$\exists$H$_\Sigma$(x=y $\rightarrow$ y=x)

H$_\Sigma$:= p $\mapsto$ p^{-1}, such that

$\forall$x:A($\texttt{refl}_x$ $\equiv$ $\texttt{refl}_x{}^{-1}$).

Transitivity

$\forall$A$\forall$x,y:A$\exists$H$_T$(x=y $\rightarrow$ y=z $\rightarrow$ x=z)

H$_T$:= p $\mapsto$ q $\mapsto$ p $\bullet$ q, such that

$\forall$x:A[$\texttt{refl}_x$ $\bullet$ $\texttt{refl}_x$ $\equiv$ $\texttt{refl}_x$].

Homotopic Abstraction

For all type families A,B, there is a homotopy:

H := [(f $\sim$ g) :$\equiv$ $\prod_{x:A}$(f(x) = g(x)], where

$\prod_{f:A\rightarrow B}$[(f $\sim$ f) $\land$ (f $\sim$ g $\rightarrow$ g $\sim$ f) $\land$ (f $\sim$ g $\rightarrow$ g $\sim$ h $\rightarrow$ f $\sim$ h)],

such that, via Voevodsky's (op. cit.) Univalence Axiom, for all type families

A,B:U, there is a function:

$\texttt{idtoeqv}$: (A $=_U$ B) $\rightarrow$ (A $\simeq$ B),

which is itself an equivalence relation:

(A $=_U$ B) $\simeq$ (A $\simeq$ B).

Abstraction principles for intensional computational properties take, then, the form:

- $\exists$f,g[f(x) = g(x)] $\simeq$ [f(x) $\simeq$ g(x)].

3   Epistemic Modal Algebra

In Epistemic Modal Algebra, the topological boolean algebra, *A*, can be formed by taking the powerset of the topological space, X, defined above; i.e., A = P(X). The domain of *A* is comprised of formula-terms – eliding propositions with names – assigned to elements of P(X), where the proposition-letters are interpreted as encoding states of information. The top element of the algebra is denoted '1' and the bottom element is denoted '0'. We interpret modal operators, f(x), – i.e., intensional functions in the algebra – as both concerning topological interiority, as well as reflecting *epistemic possibilities*. An Epistemic Modal-valued Algebraic structure has the form, F = $\langle A, D_{P(X)}, \rho \rangle$, where ρ is a mapping from points in the topological space to elements or regions of the algebraic structure; i.e., $\rho : D_{P(X)}$ x $D_{P(X)} \to A$. A model over the Epistemic-Modal Topological Boolean Algebraic structure has the form M = $\langle$F, V$\rangle$, where V(a) $\leq \rho$(a) and V(a,b) $\wedge \rho$(a, b) $\leq$ V(b).[11] For all x$_{x/a,\phi}$,y$\in A$:

f(1) = 1;

f(x) $\leq$ x;

f(x $\wedge$ y) = f(x) $\wedge$ f(y);

f[f(x)] = f(x);

V(a, a) > 0;

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[11]See Lando (2015); McKinsey and Tarski (1944); and Rasiowa (1963), for further details.

V(a, a) = 1;

V(a, b) = V(b, a);

V(a, b) $\wedge$ V(b, c) $\leq$ V(a, c);

V(a = a) = ρ(a, a);

V(a, b) $\leq$ f[V(a, b)];

V($\neg\phi$) = ρ($\neg\phi$) $-$ f(ϕ);

V($\diamond\phi$) = $\rho\phi$ $-$ f[$-$ V(ϕ)];

V($\square\phi$) = f[V(ϕ)] (cf. Lando, op. cit.).

4 Examples in Philosophy, Cognitive Science, and Quantum Information Theory

The foregoing possible worlds model theory is availed of by a number of paradigms in contemporary empirical theorizing: the computational theory of mind, Bayesian perceptual psychology, natural language semantics, and the theory of quantum information.

In Bayesian perceptual psychology, the problem of underdetermination is resolved by availing of a gradational possible worlds model. The visual system is presented with a set of possibilities with regard, e.g., to the direction of a light source. So, for example, the direction of light might be originating from above, or it might be originating from below. The visual system computes the constancy, i.e. the likelihood that one of the possibilities is actual.[12] The computation of the perceptual constancy is an unconscious statistical inference, as anticipated by Helmholtz's (1878) conjecture.[13] The constancy places, then, a

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[12] Cf. Mamassian et al. (2002).

[13] For the history of the integration of algorithms and computational modeling into contemporary visual psychology, see Johnson-Laird (2004).

condition on the accuracy of the attribution of properties – such as boundedness and volume – to distal particulars.[14]

In the program of natural language semantics in empirical and philosophical linguistics, the common ground or 'context set' is the set of possibilities presupposed by a community of speakers.[15] Kratzer (1979: 121) refers to cases in which the above possibilities are epistemic as an 'epistemic conversational background', where the epistemic possibilities are a subset of objective or circumstantial possibilities (op. cit.). Modal operators are then defined on the space, encoding the effects of various speech acts in entraining updates on the context set.[16] So, e.g., assertion is argued to provide a truth-conditional update on the context set, whereas there are operator updates, the effects of which are not straightforwardly truth-conditional and whose semantic values must then be defined relative to an array of intensional parameters (including a context – agent, time, location, et al. – and a tuple of indices).

In quantum information theory, let a constructor be a computation defined over physical systems. Constructors entrain nomologically possible transformations from admissible input states to output states (cf. Deutsch, 2013). On this approach, information is defined in terms of constructors, i.e., intensional computational properties. The foregoing transformations, as induced by constructors, are referred to as tasks. Because constructors encode the counterfactual to the effect that, were an initial state to be computed over, then the output state would result, modal notions are thus constitutive of the definition of the tasks

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[14]Cf. Burge (2010), and Rescorla (2013), for further discussion. A distinction ought to be drawn between unconscious perceptual representational states – as targeted in Burge (op. cit.) – and the inquiry into whether the properties of phenomenal consciousness have accuracy-conditions – where phenomenal properties are broadly construed, so as to include, e.g., color-phenomenal properties, as well as the property of being aware of one's perceptual states.

[15]Cf. Stalnaker (1978).

[16]Cf. Kratzer (op. cit.); Stalnaker (op. cit.); Lewis (1980); Heim (1992); Veltman (1996); von Fintel and Heim (2011); and Yalcin (2012).

at issue. There are, further, both topological and algebraic aspects of the foregoing modal approach to quantum computation.[17] The composition of tasks is formed by taking their union, where the union of tasks can be satisfiable while its component tasks might not be. Suppose, e.g., that the information states at issue concern the spin of a particle. A spin-state vector will be the sum of the probabilities that the particle is spinning either upward or downward. Suppose that there are two particles which can be spinning either upward or downward. Both particles can be spinning upward; spinning downward; particle-1 can be spinning upward while particle-2 spins downward; and vice versa. The state vector, V which records the foregoing possibilities – i.e., the superposition of the states – will be equal to the product of the spin-state of particle-1 and the spin-state of particle-2. If the particles are both spinning upward or both spinning downward, then V will be .5. However – relative to the value of each particle vector, referred to as its eigenvalue – the probability that particle-1 will be spinning upward is .5 and the probability that particle-2 will be spinning downward is .5, such that the probability that both will be spinning upward or downward = .5 x .5 = .25. Considered as the superposition of the two states, V will thus be unequal to the product of their eigenvalues, and is said to be entangled. If the indeterminacy evinced by entangled states is interpreted as inconsistency, then the computational properties at issue might further have to be hyperintensional.[18]

Finally, Epistemic Modal Algebra, as a fragment of the Language of Thought, is able to delineate the fundamental structure of the propositional attitudes targeted in 20th century philosophy; notably knowledge, belief, intentional action,

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[17]For an examination of the interaction between topos theory and an S4 modal axiomatization of computable functions, see Awodey et al. (2000).

[18]The nature of the indeterminacy in question is examined in Hawthorne (2010). For a thorough examination of approaches to the ontology of quantum mechanics, see Arntzenius (2012: ch. 3).

and rational propositional intuition. Author (ms₁) argues, e.g., that the types of intention – acting intentionally; referring to an intention as an explanation for one's course of action; and intending to pursue a course of action in the future – can be modeled as modal operators, whose semantic values are defined relative to an array of intensional parameters. Field (2001: 85-86) argues that – because possible worlds models of intentional states entrain logical equivalence for contents that ought to be distinct – such models underdetermine the identification of the intentions subserving agents' behavior . The objection can be answered, in virtue of there being unique arrays of intensional parameters relative to which each of the types of intention (modeled as modal operators) receive their value. E.g., an agent can be said to act intentionally iff her 'intention-in-action' receives a positive semantic value, where a necessary condition on the latter is that there is at least one world in her epistemic modal space at which – relative to a context of a particular time and location, which constrains the admissibility of her possible actions as defined at a first index, and which subsequently constrains the outcome thereof as defined at a second index – the intention is realized:

$[\![\text{Intenton-in-Action}(\phi)]\!]_w = 1$ only if $\exists w' [\![\phi]\!]^{w', c(=t,l), a, o} = 1$.

The agent's intention to pursue a course of action at a future time – i.e., her 'intention-for-the-future' – can receive a positive value only if there is a possible world and a future time, relative to which the possibility that a state, ϕ, is realized can be defined. Thus:

$[\![\text{Intention-for-the-future}(\phi)]\!]_w = 1$ only if $\exists w' \forall t \exists t' [t < t' \wedge [\![\phi]\!]^{w', t'} = 1]$.

In the setting of epistemic logic, epistemic necessity can further be modeled in a relational semantics encoding the property of knowledge, whereas epistemic possibility might encode the property of belief (cf. Hintikka, 1962; Fagin et al.,

1995; Meyer and van der Hoek, 1995; Williamson, 2009; Author, ms$_2$). Finally, Author (ms$_3$) treats Gödel's (1953) conception of rational propositional intuition as a modal operator in the setting of a bimodal, dynamic provability logic, and demonstrates how – via correspondence theory – the notion of 'intuition-of', i.e. a property of awareness of one's cognitive states, can be shown to be formally equivalent to the notion of 'intuition-that', i.e. a modal operator concerning the value of the propositional state at issue.[19]

5   Objections and Replies

Dean (2016) raises two issues for a proposal similar to the foregoing, namely that algorithms – broadly construed – can be defined via abstraction principles which specify equivalence relations between implementations of computational properties in isomorphic machines.[20] Dean's candidate abstraction principle for algorithms as abstracts is: that the algorithm implemented by M_1 = the algorithm implemented by M_2 iff $M_1 \simeq M_2$.[21] Both issues target the uniqueness of the algorithm purported to be identified by the abstraction principle.

The first issue generalizes Benacerraf's (1965) contention that, in the reduction of number theory to set theory, there must be, and is not, a princi-

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[19]The correspondence results between modal propositional and first-order logic are advanced in van Benthem (1983; 1984/2003) and Janin and Walukiewicz (1996). Availing of correspondence theory in order to account for the relationship between the notions of 'intuition-of' and 'intuition-that' resolves an inquiry posed by Parsons (1993: 233). As a dynamic interpretational modality, rational intuition can further serve as a guide to possible reinterpretations both of quantifier domains (cf. Fine, 2005) and of the extensions of mathematical vocabulary such as the membership-relation (cf. Uzquiano, 2015). This provides an account of Gödel's (op. cit.; 1961) suggestion that rational intuition can serve as a guide to conceptual elucidation.

[20]Fodor (2000: 105, n.4) and Piccinini (2004) note that the identification of mental states with their functional roles ought to be distinguished from identifying those functional roles with abstract computations. Conversely, a computational theory of mind need not be committed to the identification of abstract, computational operations with the functional organization of a machine. Identifying abstract computational properties with the functional organization of a creature's mental states is thus a choice point, in theories of the nature of mental representation.

[21]Cf. Moschovakis (1998).

pled reason for which to prefer the identification of natural numbers with von Neumann ordinals (e.g., $2 = \{\emptyset,\{\emptyset\}\}$), rather than with Zermelo ordinals (i.e., order-types of well-orderings).[22] The issue is evinced by the choice of whether to define algorithms as isomorphic *iterations* of state transition functions (cf. Gurevich, 1999), or to define them as isomorphic *recursions* of functions which assign values to a partially ordered set of elements (cf. Moschovakis, op. cit.). Linnebo and Pettigrew (2014: 10) argue similarly that, for two 'non-rigid' structures which admit of non-trivial automorphisms, one can define a graph which belies their isomorphism. E.g., let an abstraction principle be defined for the isomorphism between S and S*, such that

$\forall$S,S*[$\mathbf{A}$S = $\mathbf{A}$S* iff $\langle$S, R$_1$ ... R$_n\rangle \simeq \langle$S*, R*$_1$ ... R*$_n\rangle$].

However, if there is a graph, G, such that:

S = \{v$_1$, v$_2$\}, and R = \{$\langle$v$_1$, v$_2\rangle$, $\langle$v$_2$, v$_1\rangle$\},

then one can define an automorphism, f : G $\simeq$ G, such that f(v$_1$) = v$_2$ and f(v$_2$) = v$_1$, such that S* = \{v$_1$\} while R* = \{$\langle$v*$_1$, v*$_1\rangle$\}. Then S* has one element via the automorphism, while S has two. So, S and S* are not, after all, isomorphic.

The second issue is that complexity is crucial to the identity criteria of algorithms. Two algorithms might be isomorphic, while the decidability of one algorithm is proportional to a deterministic *polynomial* function of the size of its input – with k a member of the natural numbers, N, and TIME referring to the relevant complexity class: $\bigcup_{k \in N}$TIME(n^k) – and the decidability of the second algorithm will be proportional to a deterministic *exponential* function of the size of its input – $\bigcup_{k \in N}$TIME(2^{n^k}). The deterministic polynomial time complexity class is a subclass of the deterministic exponential time complexity

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[22]Cf. Zermelo (1908/1967) and von Neumann (1923/1967). Well-orderings are irreflexive, transitive, binary relations on all non-empty sets, which define a least or distinguished element in the sets.

class. However, there are problems decidable by algorithms only in polynomial time (e.g., the problem of primality testing, such that, for any two natural numbers, the numbers possess a greatest common divisor equal to 1), and only in exponential time (familiarly from logic, e.g., the problem of satisfiability – i.e., whether, for a given formula, there exists a model which can validate it – and the problem of validity – i.e. whether a satisfiable formula is valid).[23]

Both issues can be treated by noting that Dean's discussion targets abstraction principles for the very notion of a computable function, rather than for abstraction principles for cognitive computational properties. It is a virtue of homotopic abstraction principles for cognitive intensional functions that both the temporal complexity class to which the functions belong, and the applications of the model, are subject to variation. Variance in the cognitive roles, for which Epistemic Modal Algebra provides a model, will crucially bear on the nature of the representational properties unique to the interpretation of the intensional functions at issue. Thus, e.g., when the internal representations in the Language of Thought – as modeled by Epistemic Modal Algebra – subserve perceptual representational states, then their contents will be individuated by both the computational constancies at issue and the external, environmental properties – e.g., the properties of lightness and distance – of the perceiver.[24] A further virtue of the foregoing is that variance in the coding of Epistemic Modal Algebras – i.e. in the types of information over which the intensional functions will be defined – by constrast to a restriction of the Language of Thought to mathematical languages such as Peano arithmetic, permits homotopic abstraction principles to circumvent the Burali-Forti paradox for implicit definitions

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[23]For further discussion, see Dean (2015).

[24]The computational properties at issue can also be defined over non-propositional information states, such as cognitive maps possessed of geometric rather than logical structure. See, e.g., O'Keefe and Nadel (1978); Camp (2007); and Rescorla (2009).

based on isomorphism.[25]

The examples of instances of Epistemic Modal Algebra – witnessed by the possible worlds models in Bayesian perceptual psychology, linguistics, quantum information theory, and philosophy of mind – provide abductive support for the existence of the intensional functions specified in homotopic abstraction principles. The philosophical significance of independent, abductive support for the existence of epistemic modalities in the philosophy of mind and cognitive science is that the latter permits a circumvention of the objections to the abstractionist foundations of number theory that have accrued since its contemporary founding (cf. Wright, 1983). Eklund (2006) suggests, e.g., that the existence of the abstract objects which are the referents of numerical term-forming operators might need to be secured, prior to assuming that the abstraction principle for cardinal number is true. While Hale and Wright (2009) maintain, in response, that the truth of the relevant principles will be prior to the inquiry into whether the terms defined therein refer, they provide a preliminary endorsement of an 'abundant' conception of properties, according to which identifying the sense of a predicate will be sufficient for predicate reference.[26] One aspect of the significance of empirical and philosophical instances of models of Epistemic Modal Algebra is thus that, by providing independent, abductive support for the truth of the homotopic abstraction principles for cognitive algorithms, the proposal remains neutral on the status of 'sparse' versus 'abundant' conceptions of properties.[27] Another aspect of the philosophical significance of possible worlds semantics

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[25]Cf. Burali-Forti (1897/1967). Hodes (1984) and Hazen (1985) note that abstraction principles based on isomorphism with unrestricted comprehension entrain the paradox.

[26]For identity conditions on abundant properties – where the domain of properties, in the semantics of second-order logic, is a subset of the domain of objects, and the properties are definable in a metalanguage by predicates whose satisfaction-conditions have been fixed – see Hale (2013). For a generalization of the abundant conception, such that the domain of properties is isomorphic to the powerset of the domain of objects, see Cook (2014).

[27]Cf. Rayo (2003: 318).

being availed of in Bayesian vision science, empirical linguistics, and quantum information theory, is that it belies the purportedly naturalistic grounds for Quine's (1963/1976) scepticism of *de re* modality.

A final objection derives from the work of Azzouni (2013: 9.1-9.2; forthcoming). Azzouni (op. cit.) argues for a position which he refers to as 'quantifier neutralism', according to which the truth-conditions for quantifiers are sufficient for identifying the latter, without requiring reference to domains. Thus, reference to the purported domains over which quantifiers range is a superfluous and misleading metaphor. On this approach, the truth-conditions for quantifiers will be theory-relative. In the absence of objects comprising domains over which the quantifiers are purported to range, there is thus no reason to believe that the truth-conditions in mathematical and scientific theories are ontologically-committing. He suggests, then, that the explanation of the belief that there are objects derives from the 'involuntary' perception thereof (forthcoming: 282). The experience as of objects is then attributed to 'internal cognitive mechanisms' (286; see also pp. 8-9, 264, and the remainder of section 7.5).[28]

Azzouni's suggestion that there are 'internal cognitive mechanisms' subserving apparent object perception might betray an ontological commitment to there being abstracta; namely, the intensional, cognitive computational properties countenanced in Bayesian perceptual psychology. Azzouni (op. cit.: 7.8 - 7.9) anticipates and endeavors to counter the foregoing reply, by noting that the optimality of theories proffered in logic, metaphysics, and the natural sciences requires no commitment to the existence of the entities mentioned therein.

Azzouni's argument against the existence of objects – concreta and abstracta alike – hinges on the individuation-conditions for quantifiers. He argues, as

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[28]For an examination of the psychological mechanisms necessary for object-perception, see, inter alia, Treisman and Gelade (1980); Spelke and Kinzler (2007); and Scholl and Flombaum (2010).

noted, that objectual, first-order quantifiers can be defined via metalinguistic truth-conditions (op. cit.: 10). The metalinguistic truth-conditions take, e.g., the form: $\exists x.(Px)$ iff $\exists X.\delta X$ (op. cit.). However, the foregoing theory of quantification is unduly restrictive. For example, the truth-condition for probability quantifiers is given by the formula:

$(Px \geq r)[\phi x]$,

where x is a bound second-order variable; P(x) encodes the probability that the set $\{x{:}\phi x\}$ is less than or equal to r; *r is an element of the intersection between a subset of hereditarily countable sets and the real interval, [0,1]*; and the quantifier, $(Px \geq r)$, is closed under finite conjunctions and disjunctions [cf. Keisler (1985: 509-510); Barwise and Feferman (1985: 507)]. Crucially, the metalinguistic truth-condition for probability quantifiers makes explicit reference to the real numbers.[29] The claim, then, that quantifiers can be defined without reference to domains of objects does not generalize. So, if ontological commitment is generally to be eschewed of, the argument must proceed via a different means.

6   Concluding Remarks

In this essay, the equivalence relations countenanced in Homotopy Type Theory were availed of, in order to specify an abstraction principle for intensional, computational properties. The homotopic abstraction principle for intensional mental functions provides an epistemic conduit into our knowledge of cognitive algorithms as abstract objects. Because intensional functions in Epistemic Modal Algebra are deployed as core models in the philosophy of mind, Bayesian

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[29]Note that, by contrast to Field's (1980/2016: 55-56) approach, Azzouni's eschewal of ontological commitment to concrete objects would preclude redefining the reals as relations of betweeness and congruence on points of spacetime.

visual psychology, natural language semantics, and quantum information theory, there is independent abductive support for the truth of homotopic abstraction. Epistemic modality may thereby be recognized as both a compelling and a materially adequate candidate for the fundamental structure of mental representational states, and as thus comprising a fragment of the Language of Thought.

References

Arntzenius, F. 2012. *Space, Time, and Stuff*. Oxford University Press.

Awodey, S., L. Birkedal, and D. Scott. 2000. Local Realizability Toposes and a Modal Logic for Computability. Technical Report No. CMU-PHIL-99.

Azzouni, J. 2004. Proof and Ontology in Euclidean Mathematics. In T. Kjeldsen, S. Pedersen, and L. Sonne-Hansen (eds.), *New Trends in the History and Philosophy of Mathematics*. University Press of Southern Denmark.

Azzouni, J. 2013. *Semantic Perception: How the Illusion of a Common Language Arises and Persists*. Oxford University Press.

Azzouni, J. Forthcoming. *Talking about Something: Metametaphysics, Quantifiers, and What There Is*.

Barwise, J., and S. Feferman (eds.). 1985. *Model-Theoretic Logics*. Springer-Verlag.

Bealer, G. 1982. *Quality and Concept*. Oxford University Press.

enacerraf, P. 1965. What Numbers Could Not Be. *Philosophical Review*, 74.

van Benthem, J. 1983. *Modal Logic and Classical Logic*. Bibliopolis.

van Benthem, J. 1984/2003. Correspondence Theory. In D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic, 2nd Edition, Volume 3*. Kluwer.

Burali-Forti, C. 1897/1967. A Question on Transfinite Numbers (trans. J. van Heijenoort). In van Heijenoort (ed.), *From Frege to Gödel*. Harvard University Press.

Burge, T. 2010. *Origins of Objectivity*. Oxford University Press.

Camp, E. 2007. Thinking with Maps. *Philosophical Perspectives*, 21.

Carnap, R. 1937. *The Logical Syntax of Language*, tr. A. Smeaton. Routledge.

Carnap, R. 1947. *Meaning and Necessity*. University of Chicago Press.

Church, A. 1954. Intensional Isomorphism and the Identity of Belief. *Philosophical Studies*, 5:5.

Cook, R. 2014. Possible Predicates and Actual Properties. *Synthese*, DOI 10.1007/s11229-014-0592-1.

Dean, W. 2015. Computational Complexity Theory. *Stanford Encyclopedia of Philosophy*.

Dean, W. 2016. Algorithms and the Mathematical Foundations of Computer Science. In L. Horsten and P. Welch (eds.), *Gödel's Disjunction*. Oxford University Press.

Deutsch, D. 2013. Constructor Theory. *Synthese*, 190.

Eklund, M. 2006. Neo-Fregean Ontology. *Philosophical Perspectives*, 20.

Fagin, R., Halpern, J., Moses, Y., and M. Vardi. 1995. *Reasoning about Knowledge*. MIT Press.

Field, H. 1980/2016. *Science without Numbers*, 2nd ed. Oxford University Press.

Field, H. 2001. *Truth and the Absence of Fact*. Oxford University Press.

Fine, K. 2005. Our Knowledge of Mathematical Objects. In T. Gendler and J. Hawthorne (eds.), *Oxford Studies in Epistemology, Vol. 1*. Oxford University Press.

von Fintel, K. and I. Heim. 2011. *Intensional Semantics*. MIT.

Flagg, R. 1985. Epistemic Set Theory is a Conservative Extension of Intuitionistic Set Theory. *Journal of Symbolic Logic*, 50:4.

Fodor, J. 1975. *The Language of Thought*. Harvard University Press.

Fodor, J. 2000. *The Mind Doesn't Work that Way*. MIT Press.

Frege, G. 1884/1980. *The Foundations of Arithmetic*, 2nd ed., trans. J.L. Austin. Northwestern University Press.

Frege, G. 1892/1997. 'On *Sinn* and *Bedeutung*', tr. M. Black. In M. Beaney (ed.), *The Frege Reader*. Blackwell.

Frege, G. 1893/2013. *Basic Laws of Arithmetic, Vol. I-II*, trans. and ed. P. Ebert, M. Rossberg, C. Wright, and R. Cook. Oxford University Press.

Giaquinto, M. 2008. Visualizing in Mathematics. In P. Mancosu (ed.), *The Philosophy of Mathematical Practice*. Oxford University Press.

Gödel, K. 1953,III-V. Is Mathematics Syntax of Language? In Gödel (1995), *Collected Works, Volume III*, eds. S. Feferman, J. Dawson, W. Goldfarb, C. Parsons, and R. Solovay. Oxford University Press.

Gödel, K. 1961. The Modern Development of the Foundations of Mathematics in the Light of Philosophy. In Gödel (1995).

Goodman, N.D. 1990. Topological Models of Epistemic Set Theory. *Annals of Pure and Applied Mathematics*, 46.

Gurevich, Y. 1999. The Sequential ASM Thesis. *Bulletin of the European Association for Theoretical Computer Science*, 67.

Hale, B. 2013. Properties and the Interpretation of Second-order Logic. *Philosophia Mathematica*, 3:21.

Hale, B., and C. Wright. 2009. The Metaontology of Abstraction. In D. Chalmers, D. Manley, and R. Wasserman (eds.), *Metametaphysics*. Oxford University Press.

Hale, B., and C. Wright. 2001. *The Reason's Proper Study*. Oxford University Press.

Hawthorne, J. 2010. A Metaphysician Looks at the Everett Interpretation. In S. Saunders, J. Barrett, A. Kent, and D. Wallace (eds.), *Many Worlds? Everett, Quantum Theory, and Reality*. Oxford University Press.

Hazen, A. 1985. Review of Crispin Wright's *Frege's Conception of Numbers as Objects. Australasian Journal of Philosophy*, 63:2.

Heim, I. 1992. Presupposition Projection and the Semantics of Attitude Verbs. *Journal of Semantics*, 9.

Helmholtz, H. von. 1878/1977. The Facts in Perception. In Helmholtz, *Epistemological Writings*, trans. M. Lowe, eds. R. Cohen and Y. Elkana. D. Reidel Publishing.

Hintikka, J. 1962. *Knowledge and Belief.* Cornell University Press.

Hodes, H. 1984. Logicism and the Ontological Commitments of Arithmetic. *Journal of Philosophy*, 81:3.

Janin, D., and I. Walukiewicz. 1996. On the Expressive Completeness of the Propositional μ-Calculus with respect to Monadic Second-order Logic. In U. Montanari and V. Sasone (eds.), *Proceedings CONCUR, '96.* Springer.

Johnson-Laird, P.N. 2004. The History of Mental Models. In K. Manktelow and M. Chung (eds.), *Psychology of Reasoning.* Psychology Press.

Katz, J. 1998. *Realistic Rationalism.* MIT Press.

Keisler, H.J. 1985. Probability Quantifiers. In J. Barwise and S. Feferman (op. cit.).

Kratzer, A. 1979. Conditional Necessity and Possibility. In R. Brle, U. Egli, and A.v. Stechow (eds.), *Semantics from Different Points of View.* Springer.

Lando, T. 2015. First Order S4 and Its Measure-theoretic Semantics. *Annals of Pure and Applied Logic*, 166.

Lewis, D. 1973. *Counterfactuals.* Blackwell.

Lewis, D. 1980. Index, Context, and Content. In S. Kanger and S. Öhman (eds.), *Philosophy and Grammar.* D. Reidel Publishing. Reprinted in Lewis (1998), *Papers in Philosophical Logic.* Cambridge University Press.

Linnebo, Ø, and A. Rayo. 2012. Hierarchies, Ontological and Ideological. *Mind*, 121:482.

Linnebo, Ø., and R. Pettigrew. 2014. Two Types of Abstraction for Structuralism. *Philosophical Quarterly*, doi: 10.1093/pq/pqt044.

Mamassian, P., M. Landy, and L. Maloney. 2002. Bayesian Modelling of Visual Perception. In R. Rao and M. Lewicki (eds.), *Probabilistic Models of the Brain.* MIT Press.

McKinsey, J., and A. Tarski. 1944. The Algebra of Topology. *The Annals of Mathematics, Second Series*, 45:1.

eyer, J.J., and W. van der Hoek. 1995. *Epistemic Logic for AI and Computer Science.* Cambridge University Press.

Moschovakis, Y. 1998. On the Founding of a Theory of Algorithms. In H. Dales and G. Olivieri (eds.), *Truth in Mathematics.* Clarendon Press.

Newell, A. 1980. Physical Symbol Systems. *Cognitive Science*, 4.

O'Keefe, J., and L. Nadel. 1978. *The Hippocampus as a Cognitive Map*. Oxford University Press.

Parsons, C. 1993. On Some Difficulties concerning Intuition and Intuitive Knowledge. *Mind*, 102:406.

Piccinini, G. 2004. Functionalism, Computationalism, and Mental States. *Studies in History and Philosophy of Science*, 35.

Putnam, H. 1967. Psychological Predicates. In W. Capitan and D. Merrill (eds.) *Art, Mind, and Religion*. University of Pittsburgh Press.

Pylyshyn, Z. 1978. Computational Models and Empirical Constraints. *Brain and Behavioral Sciences*, 1.

Quine, W.V. 1963. Necessary Truth. In Quine (1976), *The Ways of Paradox*, 2nd ed. Harvard University Press.

Rayo, A. 2003. Success by Default? *Philosophia Mathematica*, 3:11.

Rescorla, M. 2009. Cognitive Maps and the Language of Thought. *British Journal for the Philosophy of Science*, 60.

Rescorla, M. 2013. Bayesian Perceptual Psychology. In M. Mohan (ed.), *The Oxford Handbook of the Philosophy of Perception*. Oxford University Press.

Scedrov, A. 1985. Extending Gödel's Modal Interpretation to Type Theory and Set Theory. In Shapiro (ed.), *Intensional Mathematics*. Elsevier Science Publishers.

Scholl, B., and J.I. Flombaum. 2010. Object Persistence. In B. Goldstein (ed.), *Encyclopedia of Perception, Vol. 2*. SAGE Publications.

Spelke, E., and K. Kinzler. 2007. Core Knowledge. *Developmental Science*, 10:1.

Stalnaker, R. 1978. Assertion. In P. Cole (ed.), *Syntax and Semantics, Vol. 9*. Academic Press.

Treisman, A., and G. Gelade. 1980. A Feature-integration Theory of Attention. *Cognitive Psychology*, 12:16.

Turing, A. 1950. Computing Machinery and Intelligence. *Mind*, 59: 236.

The Univalent Foundations Program. 2013. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study.

Uzquiano, G. 2015. Varieties of Indefinite Extensibility. *Notre Dame Journal of Formal Logic*, 58:1.

Veltman, F. 1996. Defaults in Update Semantics. *Journal of Philosophical Logic*, 25:3.

Voevodsky, V. 2006. A Very Short Note on the Homotopy λ-Calculus. Unpublished.

von Neumann, J. 1923/1967. On the Introduction of Transfinite Numbers (trans. van Heijenoort). In van Heijenoort (op. cit.).

Williamson, T. 2009. Some Computational Constraints in Epistemic Logic. In S. Rahman, J. Symons, D. Gabbay, and J. van Bendegem (eds.), *Logic, Epistemology, and the Unity of Science*. Springer.

Wittgenstein, L. 1921/1974. *Tractatus Logico-Philosophicus*, trans. D.F. Pears and B.F. McGuinness. Routledge.

Wright, C. 1983. *Frege's Conception of Numbers as Objects*. Aberdeen University Press.

Yalcin, S. 2012. Bayesian Expressivism. *Proceedings of the Aristotelian Society*, doi: 10.1111/j.1467-9264.2012.00329.x

Zermelo, E. 1908/1967. A New Proof of the Possibility of a Well-ordering (trans. S. Bauer-Mengelberg). In van Heijenoort (op. cit.).