Modal and Hyperintensional Cognitivism and Modal and Hyperintensional Expressivism

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Abstract

This paper aims to provide a mathematically tractable background against which to model both modal and hyperintensional cognitivism and modal and hyperintensional expressivism. I argue that epistemic modal algebras, endowed with a hyperintensional, topic-sensitive epistemic two-dimensional truthmaker semantics, comprise a materially adequate fragment of the language of thought. I demonstrate, then, how modal expressivism can be regimented by modal coalgebraic automata, to which the above epistemic modal algebras are categorically dual. I examine five methods for modeling the dynamics of conceptual engineering for intensions and hyperintensions. I develop a novel topic-sensitive truthmaker semantics for dynamic epistemic logic, and develop a novel dynamic epistemic two-dimensional hyperintensional semantics. I examine then the virtues unique to the modal and hyperintensional expressivist approaches here proffered in the setting of the foundations of mathematics, by contrast to competing approaches based upon both the inferentialist approach to concept-individuation and the codification of speech acts via intensional semantics.

1 Introduction

This essay endeavors to reconcile two approaches to the modal foundations of thought: modal and hyperintensional cognitivism and modal and hyperintensional expressivism. The novel contribution of the essay is its argument for a reconciliation between the two positions, by providing a hybrid account in which both internal cognitive architecture, on the model of epistemic possibilities, as well as modal automata, are accommodated, while retaining what is supposed to be their unique and inconsistent roles.

The notions of cognitivism and expressivism here targeted concern the role of internal – rather than external – factors in countenancing the nature of thought

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and information (see Fodor, 1975; Haugeland, 1978). Possible worlds or hyperintensional semantics is taken then to provide the most descriptively adequate means of countenancing the structure of the foregoing.[1] Whereas the type of modal and hyperintensional cognitivism examined here assumes that thoughts and information take exclusively the form of internal representations, the target modal and hyperintensional expressivist proposals assume that information states are exhaustively individuated by both linguistic behavior and conditions external to the cognitive architecture of agents.

Modal and hyperintensional cognitivism is thus the proposal that the internal representations comprising the language of thought can be modeled via either a possible world or hyperintensional semantics.[2] Modal expressivism has,

[1]Delineating cognitivism and expressivism by whether the positions avail of internal representations is thus orthogonal to the eponymous dispute between realists and antirealists with regard to whether mental states are truth-apt, i.e., have a representational function, rather than being non-representational and non-factive, even if real (see Dummett, 1959; Blackburn, 1984; Price, 2013).

[2]See Fodor (1975). I endorse (i) 'the representational theory of thought'. (Rescorla, 2024: 1.1) writes: 'Fodor (1981: 177–203; 1987: 16–26) proposes a theory of propositional attitudes that assigns a central role to mental representations. A mental representation is a mental item with semantic properties (such as a denotation, or a meaning, or a truth-condition, etc.). To believe that p, or hope that p, or intend that p, is to bear an appropriate relation to a mental representation whose meaning is that p. For example, there is a relation belief* between thinkers and mental representations, where the following biconditional is true no matter what English sentence one substitutes for "p":

'X believes that p iff there is a mental representation S such that X believes* S and S means that p.

'More generally:

'(1) Each propositional attitude A corresponds to a unique psychological relation A*, where the following biconditional is true no matter what sentence one substitutes for "p": X As that p iff there is a mental representation S such that X bears A* to S and S means that p.

'On this analysis, mental representations are the most direct objects of propositional attitudes. A propositional attitude inherits its semantic properties, including its truth-condition, from the mental representation that is its object.

'Proponents of (1) typically invoke functionalism to analyze A*. Each psychological relation A* is associated with a distinctive functional role: a role that S plays within your mental activity just in case you bear A* to S. When specifying what it is to believe* S, for example, we might mention how S serves as a basis for inferential reasoning, how it interacts with desires to produce actions, and so on. Precise functional roles are to be discovered by scientific psychology. Following Schiffer (1981), it is common to use the term "belief-box" as a placeholder for the functional role corresponding to belief*: to believe* S is to place S in your belief box. Similarly for "desire-box", etc.

'According to Fodor (1987: 17), thinking consists in chains of mental events that instantiate mental representations:

'(2) Thought processes are causal sequences of tokenings of mental representations. A paradigm example is deductive inference: I transition from believing* the premises to believing* the conclusion. The first mental event (my belief* in the premises) causes the second (my belief* in the conclusion).

'(1) and (2) fit together naturally as a package that one might call the representational theory of thought (RTT). RTT postulates mental representations that serve as the objects of propositional attitudes and that constitute the domain of thought processes'; (ii) 'the compositionality of mental representations: Compositionality of mental representations (COMP): Mental representations have a compositional semantics: complex representations are composed of simple constituents, and the meaning of a complex representation depends upon the meanings of its constituents together with the constituency structure into which those constituents in turn, been delineated in two ways. On the first approach, the presuppositions shared by a community of speakers have been modeled as possibilities (see Kratzer, 1979; Stalnaker, 1978, 1984). Speech acts have in turn been modeled as modal operators which update the common ground of possibilities, the semantic values of which are then defined relative to an array of intensional parameters (Stalnaker, op. cit.; Veltman, 1996; Yalcin, 2007). On the second approach, the content of concepts is supposed to be individuated via the ability to draw inferences. Modally expressive normative inferences are taken then to have the same subjunctive form as that belonging to the alethic modal profile of descriptive theoretical concepts (Brandom, 2014: 211-212).[3] Both the modal approach to shared information and the speech acts which serve to update the latter, and the inferential approach to concept-individuation, are consistent with mental states having semantic values or truth-conditional characterizations. Hyperintensional expressivism is countenanced by Hawke (2024: 1120, 1127-1129) and is defined by way of combining a topic-sensitive epistemic truthmaker semantics and a two-component assertability semantics: 'A formal assertibility semantics models the *assertibility relation* $\Vdash$, holding between a unified body of information $\mathbf{s}$ (an *information state*) and a meaningful declarative ϕ, exactly when: were an agent's knowledge state to contain exactly information $\mathbf{s}$, she would be correct to assert ϕ, from a purely semantic and epistemic perspective [...] We assume that an information state can be identified with a proposition and use $\mathbf{I}$ to denote the set of all information states. Call a subset of $\mathbf{I}$ a *cognitive feature* [...]

'**Definition 1 (Expressed Feature)** Relative to a model and an account of $\Vdash$, the cognitive feature expressed by ϕ is: $[\![\phi]\!] := \{\boldsymbol{s} \in \mathbf{I} : \boldsymbol{s} \Vdash \phi\}$.

'So, ϕ expresses the type of information state that renders ϕ assertible.

'A *TF frame* has five components: W, T, @, knowledge function $\mathbf{K}$, and

are arranged' (Rescorla, 2024: 1.2); (iii) that mental representations are logically structured: 'Logically structured mental representations (LOGIC): Some mental representations have logical structure. The compositional semantics for these mental representations resembles the compositional semantics for logically structured natural language expressions' (Rescorla, 2024: 1.3); (iv) 'the classical computational theory of mind (CCTM). According to CCTM, the mind is a computational system similar in important respects to a Turing machine, and certain core mental processes are computations similar in important respects to computations executed by a Turing machine' (Rescorla, 2024: 3); and reject (v) 'the formal-syntactic conception of computation (FSC). According to FSC, computation manipulates symbols in virtue of their formal syntactic properties but not their semantic properties' [op. cit.; see Elohim (2024), and Rescorla (2015)]. Chalmers (2023) endorses the representational language of thought hypothesis without the classical computational language of thought hypothesis. Chalmers endorses the representational language of thought hypothesis with 'subsymbolic versions of nonclassical computational LOT'. See Chalmers (1990); Kleyko et al. (2022); Piantadosi (2021).

[3] Brandom writes, e.g.: 'For modal *expressivism* tells us that modal vocabulary makes explicit normatively significant relations of subjunctively robust material consequence and incompatibility among claimable (hence propositional) contents in virtue of which ordinary empirical descriptive vocabulary *describes* and does not merely *label*, *discriminate*, or *classify*. And modal *realism* tells us that there are modal facts, concerning the subjunctively robust relations of material consequence and incompatibility in virtue of which ordinary empirical descriptive properties and facts are determinate. Together, these two claims give a definite sense to the possibility of the correspondence of modal claimings with modal facts' (op. cit.: 2012).

belief function **B**. W and @ are as before. T is a set of possible *topics*; call a subset of T a *subject matter* (denoted **m**). We now model a proposition, or information state, as a pair $\langle \mathbf{i}, \mathbf{m} \rangle$: an intension **i** plus a subject matter **m**. The first component gives the verification/truth conditions of a proposition; the second fixes what it is about. A proposition is *veridical at w* iff its intension includes w, and veridical iff it is *veridical* at @.

'Per fragmentation, an acceptance state is now modeled as a set of propositions, called fragments. Thus, **K** and **B** map a world to a set of propositions: $\mathbf{K}(w)$ is Smith's *total knowledge state* at w and $\mathbf{B}(w)$ is Smith's *total belief state* at w. We stipulate that every proposition in $\mathbf{K}(w)$ is veridical at w and that every knowledge fragment is a type of belief fragment: $\mathbf{K}(w) \subseteq \mathbf{B}(w)$, for all w.

'**Definition 7 (FaTE)** ['Fragmented and Topic-sensitive Expressivism' (27)]

'For arbitrary $\mathbf{s}$, p, ϕ, and ψ, relative to TF model T:

$\mathbf{s} \Vdash p$ iff $\mathtt{t}(p) \subseteq \mathbf{s}$ and $\mathbf{s} \subseteq \mathtt{v}(p)$

$\mathbf{s}\ \ p$ iff $\mathtt{t}(p) \subseteq \mathbf{s}$ and $\mathbf{s} \cap \mathtt{v}(p) = \emptyset$

$\mathbf{s} \Vdash \neg\phi$ iff $\mathbf{s}\ \ \phi$

$\mathbf{s}\ \ \neg\phi$ iff $\mathbf{s} \Vdash \phi$

$\mathbf{s} \Vdash \phi \wedge \psi$ iff $\mathbf{s} \Vdash \phi$ and $\mathbf{s} \Vdash \psi$

$\mathbf{s}\ \ \phi \wedge \psi$ iff there are $\mathbf{u}$ and $\mathbf{v}$ s.t. $\mathbf{s} = \mathbf{u} \cup \mathbf{v}$ and $\mathbf{u}\ \ \phi$ and $\mathbf{v}\ \ \psi$

$\mathbf{s} \Vdash \Diamond\phi$ iff $\mathtt{t}(\phi) \subseteq \mathbf{s}$ and $\mathbf{s}\ \ \phi$

$\mathbf{s}\ \ \Diamond\phi$ iff $\mathbf{s}\ \ \phi$

$\mathbf{s} \Vdash K\phi$ iff $\mathtt{t}(K\phi) \subseteq \mathbf{s}$ and $\forall w{\in}\mathbf{s}$: $\exists \mathbf{k}{\in}\mathbf{K}(w)$: $\mathbf{k} \Vdash \phi$

$\mathbf{s}\ \ K\phi$ iff $\mathtt{t}(K\phi) \subseteq \mathbf{s}$ and $\forall w{\in}\mathbf{s}$: $\forall \mathbf{k}{\in}\mathbf{K}(w)$: $\mathbf{k} \nVdash \phi$

$\mathbf{s} \Vdash B\phi$ iff $\mathtt{t}(B\phi) \subseteq \mathbf{s}$ and $\forall w{\in}\mathbf{s}$: $\exists \mathbf{b}{\in}\mathbf{B}(w)$: $\mathbf{b} \Vdash \phi$

$\mathbf{s}\ \ B\phi$ iff $\mathtt{t}(B\phi) \subseteq \mathbf{s}$ and $\forall w{\in}\mathbf{s}$: $\forall \mathbf{b}{\in}\mathbf{B}(w)$: $\mathbf{b} \nVdash \phi$'.

So defined, the modal and hyperintensional cognitivist and modal and hyperintensional expressivist approaches have been assumed to be in constitutive opposition. While the cognitivist proposal avails of modal resources in order to model the internal representations comprising an abstract language of thought, the expressivist proposal targets informational properties which extend beyond the remit of internal cognitive architecture: both the form and the parameters relevant to determining the semantic values of linguistic utterances, where the informational common ground is taken to be reducible to possibilities; and the individuation of the contents of concepts on the basis of inferential behavior.

In this essay, I provide a background mathematical theory, in order to account for the reconciliation of the cognitivist and expressivist proposals. I avail, in particular, of the duality between Boolean-valued models of epistemic modal algebras and coalgebras; i.e., labeled transition systems defined in the setting of category theory.[4] The mappings of coalgebras permit of flexible interpretations, such that they are able to characterize both modal logics as well as discrete-state automata. I argue that the correspondence between epistemic modal algebras

[4]For an algebraic characterization of dynamic-epistemic logic, see Kurz and Palmigiano (2013). Baltag (2003) develops a coalgebraic semantics for dynamic-epistemic logic, where coalgebraic mappings are intended to record the informational dynamics of single- and multi-agent systems.

and modal coalgebraic automata is sufficient then for the provision of a mathematically tractable, modal foundation for thought and action.

In Section **2**, I provide the background mathematical theory, in order to account for the reconciliation of the cognitivist and expressivist proposals.

In Section **3**, I provide reasons adducing in favor of modal and hyperintensional cognitivism, and argue for the material adequacy of epistemic modal algebras as a fragment of the language of thought.

In Section **4**, I compare my approach with those advanced in the historical and contemporary literature.

In Section **5**, I provide new models for the dynamics of conceptual engineering of intensions and hyperintensions. The first method is via announcements in dynamic epistemic logic. The second method is via dynamic interpretational modalities which redefine intensions and hyperintensions which reassign topics to atomic formulas. The third method is via dynamic hyperintensional belief revision. The fourth method is via rendering epistemic two-dimensional semantics dynamic, such that updates to the epistemic space for the first parameter of a formula will determine an update to the metaphysical space for the second parameter of the formula. The fifth method models updates to two-dimensional intensions via the Logic of Epistemic Dependency in the parameter for epistemic space which then constrains interventions to structural equation models in the parameter for metaphysical space.[5]

In Section **6**, I countenance a hyperintensional construal of the Epistemic Church-Turing Thesis, to ground my dynamic two-dimensional semantics.

In Section **7**, I examine reasons adducing in favor of an expressivist natural language semantics for epistemic modals, to complement the metaphysical expressivism for epistemic modality examined in the chapter.

In Section **8**, modal coalgebraic automata are argued, finally, to be preferred as models of modal expressivism, by contrast to the speech-act and inferentialist approaches, in virtue of the advantages accruing to the model in the philosophy of mathematics. The interest in modal coalgebraic automata consists, in particular, in the range of mathematical properties that can be recovered on the basis thereof.[6] By contrast to the above competing approaches to modal expressivism, the mappings of modal coalgebraic automata are able both to model and explain elementary embeddings; the intensions of mathematical terms; as well as the modal profile of Ω-logical consequence.

[5] For the origins of two-dimensional intensional semantics, see Kamp, 1967; Vlach, 1973; and Segerberg, 1973. Kant (1787/1998) anticipates two-dimensional semantics by inquiring into the objective validity of the categories in the Transcendental Deduction in the *Critique of Pure Reason*. See Book I of the 'Transcendental Analytic', the 'Analytic of Concepts', which includes the Metaphysical Deduction (A66–83, B92–116) and the Transcendental Deduction (A84–130, B116–169.) The distinction between epistemic and metaphysical possibilities, as they pertain to the values of mathematical formulas, is anticipated by Gödel's (1951: 11-12) distinction between mathematics in its subjective and objective senses, where the former targets all 'demonstrable mathematical propositions', and the latter includes 'all true mathematical propositions'.

[6] See Wittgenstein (2001: IV, 4-6, 11, 30-31), for a prescient expressivist approach to the modal profile of mathematical formulas.

Section **9** provides concluding remarks.

2 The Hybrid Proposal

2.1 Epistemic Modal Algebra

An epistemic modal algebra is defined as $U = \langle$A, 0, 1, ¬, ∩, ∪, **l**, **m**$\rangle$, with A a set containing 0 and 1 (Bull and Segerberg, 2001: 28).[7]

l1 = 1,
l(a ∩ b) = **l**a ∩ **l**b
ma = ¬**l**¬a,
m0 = 0,
m(a ∪ b) = **m**a ∪ **m**b, and
la = ¬**m**¬a (op. cit.).

A valuation v on U is a function from propositional formulas to elements of the algebra, which satisfies the following conditions:
v(¬A) = ¬v(A),
v(A ∧ B) = v(A) ∩ v(B),
v(A ∨ B) = v(A) ∪ v(B),
v(□A) = **l**v(A), and
v(◇A) = **m**v(A) (op. cit.).

A frame F = ⟨W, R⟩ consists of a set W and a binary relation R on W (op. cit.). R[w] denotes the set {v∈W | (w,v)∈R}. A valuation V on F is a function such that V(A, x)∈{1,0} for each propositional formula A and x∈W, satisfying the following conditions:
V(¬A, x) = 1 iff V(A, x) = 0,
V(A ∧ B, x) = 1 iff V(A, x) = 1 and V(B, x) = 1,
V(A ∨ B, x) = 1 iff V(A, x) = 1 or V(B, x) = 1 (op. cit.)

2.1.1 Epistemic Two-dimensional Truthmaker Semantics

Chalmers endorses a principle of plenitude according to which 'For all sentences s, s is epistemically possible iff there exists a scenario [i.e. epistemically possible world - D.E.] such that w verifies s' (2011: 64), where '[w]hen w verifies s, we can say that s is true at w' (63). In this essay, I accept, instead, a hyperintensional truthmaker approach to epistemic possibility, defined by the notion of exact verification in a state space, where states are parts of whole worlds (Fine 2017a,b; Hawke and Özgün, 2023). According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple ⟨S, P, ≤, v⟩, where S is a non-empty set of states, i.e. parts of the elements in A in the foregoing epistemic modal algebra U, P is the subspace of possible states where states s and t comprise a fusion when s ⊔ t∈P, ≤ is a partial order, and v: Prop → (2^S x 2^S) assigns a bilateral proposition $\langle p^+, p^-\rangle$ to each atom p∈Prop with p^+ and p^- incompatible (Hawke and Özgün, 2023). Exact verification (⊢) and exact

[7] Boolean algebras with operators were introduced by Jónsson and Tarski (1951, 1952).

falsification ($\dashv$) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, 2023):

s $\vdash$ p if s$\in[\![p]\!]^+$

(s verifies p, if s is a truthmaker for p i.e. if s is in p's extension);

s $\dashv$ p if s$\in[\![p]\!]^-$

(s falsifies p, if s is a falsifier for p i.e. if s is in p's anti-extension);

s $\vdash$ ¬p if s $\dashv$ p

(s verifies not p, if s falsifies p);

s $\dashv$ ¬p if s $\vdash$ p

(s falsifies not p, if s verifies p);

s $\vdash$ p $\wedge$ q if $\exists$v,u, v $\vdash$ p, u $\vdash$ q, and s = v $\sqcup$ u

(s verifies p and q, if s is the fusion of states, v and u, v verifies p, and u verifies q);

s $\dashv$ p $\wedge$ q if s $\dashv$ p or s $\dashv$ q

(s falsifies p and q, if s falsifies p or s falsifies q);

s $\vdash$ p $\vee$ q if s $\vdash$ p or s $\vdash$ q

(s verifies p or q, if s verifies p or s verifies q);

s $\dashv$ p $\vee$ q if $\exists$v,u, v $\dashv$ p, u $\dashv$ q, and s = v $\sqcup$ u

(s falsifies p or q, if s is the fusion of the states v and u, v falsifies p, and u falsifies q);

s $\vdash$ $\forall$xϕ(x) if $\exists$s$_1$, ..., s$_n$, with s$_1$ $\vdash$ ϕ(a$_1$), ..., s$_n$ $\vdash$ ϕ(a$_n$), and s = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$

[s verifies $\forall$xϕ(x) "if it is the fusion of verifiers of its instances ϕ(a$_1$), ..., ϕ(a$_n$)" (Fine, 2017c)];

s $\dashv$ $\forall$xϕ(x) if s $\dashv$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s falsifies $\forall$xϕ(x) "if it falsifies one of its instances" (op. cit.)];

s $\vdash$ $\exists$xϕ(x) if s $\vdash$ ϕ(a) for some individual a in a domain of individuals (op. cit.)

[s verifies $\exists$xϕ(x) "if it verifies one of its instances ϕ(a$_1$), ..., ϕ(a$_n$)" (op. cit.)];

s $\dashv$ $\exists$xϕ(x) if $\exists$s$_1$, ..., s$_n$, with s$_1$ $\dashv$ ϕ(a$_1$), ..., s$_n$ $\dashv$ ϕ(a$_n$), and s = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$ (op. cit.)

[s falsifies $\exists$xϕ(x) "if it is the fusion of falsifiers of its instances" (op. cit.)];

s exactly verifies p if and only if s $\vdash$ p if s$\in[\![p]\!]$;

s inexactly verifies p if and only if s $\triangleright$ p if $\exists$s' $\leq$ S, s' $\vdash$ p; and

s loosely verifies p if and only if, $\forall$v, s.t. s $\sqcup$ v $\vdash$ p, where $\sqcup$ is the relation of compatibility (35-36);

s $\vdash$ Aϕ if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u$\in$P and u' $\vdash$ ϕ, where Aϕ denotes the apriority of ϕ[8]; and

[8] In epistemic two-dimensional semantics, epistemic possibility is defined as the dual of apriority or epistemic necessity, i.e. as not being ruled-out apriori (¬□¬), and follows Chalmers (2011: 66). Apriority receives, however, different operators depending on whether it is defined in truthmaker semantics or possible worlds semantics. Both operators are admissible, and the definition in terms of truthmakers is here taken to be more fundamental. The definition of apriority here differs from that of DeRose (1991: 593-594) – who defines the epistemic

s $\dashv$ Aϕ if and only if there is a v$\in$P such that for all u$\in$P either v $\sqcup$ u$\notin$P or u $\dashv$ ϕ;

s $\vdash$ A(Aϕ) if and only if for all u$\in$P there is a u'$\in$P such that u' $\sqcup$ u $\in$P and u' $\vdash$ ϕ and there is a u"$\in$P such that u' $\sqcup$ u"$\in$P and u" $\vdash$ ϕ;

s $\vdash$ A($\forall$xϕ(x)) if and only if for all u$\in$P there is a u'$\in$P such that u $\vdash$ [u' $\vdash$ $\exists$s$_1$, ..., s$_n$, with s$_1$ $\vdash$ ϕ(a$_1$), ..., s$_n$ $\vdash$ ϕ(a$_n$), and u' = s$_1$ $\sqcup$ ... $\sqcup$ s$_n$];

s $\vdash$ A($\exists$xϕ(x)) if and only if or all u$\in$P there is a u'$\in$P such that u $\vdash$ [u' $\vdash$ ϕ(a)] for some individual a in a domain of individuals (op. cit.).

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions:[9]

- Epistemic Hyperintension:

  $\texttt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$, with s a state in the state space defined over the foregoing epistemic modal algebra, U;

- Subjunctive Hyperintension:

  $\texttt{sec}_{v@}(x) = \lambda w.[\![x]\!]^{v@,w}$, with w a state in metaphysical state space W;

In epistemic two-dimensional semantics, the value of a formula or term relative to a first parameter ranging over epistemic scenarios determines the value of the formula or term relative to a second parameter ranging over metaphysically possible worlds. The dependence is recorded by 2D-intensions. Chalmers (2006: 102) provides a conditional analysis of 2D-intensions to characterize the dependence: 'Here, in effect, a term's subjunctive intension depends on which epistemic possibility turns out to be actual. / This can be seen as a mapping from scenarios to subjunctive intensions, or equivalently as a mapping from (scenario, world) pairs to extensions. We can say: the two-dimensional intension of a statement S is true at (V, W) if V verifies the claim that W satisfies S. If [A]$_1$ and [A]$_2$ are canonical descriptions of V and W, we say that the two-dimensional intension is true at (V, W) if [A]$_1$ epistemically necessitates that [A]$_2$ subjunctively necessitates S. A good heuristic here is to ask "If [A]$_1$ is the case, then if [A]$_2$ had been the case, would S have been the case?". Formally, we can say that the two-dimensional intension is true at(V, W) iff '$\Box_1$([A]$_1$ $\rightarrow$ $\Box_2$([A]$_2$ $\rightarrow$ S))' is true, where '$\Box_1$' and '$\Box_2$' express epistemic and subjunctive necessity respectively'.

possibility of P as being true iff "(1) no member of the relevant community knows that P is false and (2) there is no relevant way by which members of the relevant community can come to know that P is false" – by defining epistemic possibility in terms of apriority rather than knowledge. It differs from that of Huemer (2007: 129) – who defines the epistemic possibility of P as it not being the case that P is epistemically impossible, where P is epistemically impossible iff P is false, the subject has justification for $\neg$P "adequate for dismissing P", and the justification is "Gettier-proof" – by not availing of impossibilities, and rather availing of the duality between apriority as epistemic necessity and epistemic possibility.

[9]The notation for intensions follows the presentation in Chalmers and Rabern (2014: 211-212) and von Fintel and Heim (2011).

- 2D-Hyperintension:

  $\mathtt{2D}(x) = \lambda s \lambda w [\![\mathrm{x}]\!]^{s,w} = 1.$

An abstraction principle for two-dimensional hyperintensions can be defined as follows:

For all types, A,B, there is a homotopy[10]:

$\mathrm{H} := [(\mathrm{f} \sim \mathrm{g}) :\equiv \prod_{x:A}(\mathrm{f(x)} = \mathrm{g(x)})]$, where
$\prod_{f:A \to B}[(\mathrm{f} \sim \mathrm{f}) \wedge (\mathrm{f} \sim \mathrm{g} \to \mathrm{g} \sim \mathrm{f}) \wedge (\mathrm{f} \sim \mathrm{g} \to \mathrm{g} \sim \mathrm{h} \to \mathrm{f} \sim \mathrm{h})]$,
such that, via Voevodsky's (2006) Univalence Axiom, for all type families A,B:U, there is a function:
$\mathtt{idtoeqv} : (\mathrm{A} =_U \mathrm{B}) \to (\mathrm{A} \simeq \mathrm{B})$,
which is itself an equivalence relation:
$(\mathrm{A} =_U \mathrm{B}) \simeq (\mathrm{A} \simeq \mathrm{B})$.

Abstraction principles for two-dimensional hyperintensions take, then, the form of a function type equivalence[11]:

- $\forall \mathrm{x}[\#\mathrm{f(x)} = \#\mathrm{g(x)}] \simeq [\mathrm{f(x)} \simeq \mathrm{g(x)}]$.[12]

2.2 Modal Coalgebraic Automata

Modal coalgebraic automata can be thus characterized. Let a category C be comprised of a class Ob(C) of objects and a family of arrows for each pair of objects C(A,B) (Venema, 2007: 421). A functor from a category C to a category D, **E**: C $\to$ D, is an operation mapping objects and arrows of C to objects and arrows of D (422). An endofunctor on C is a functor, **E**: C $\to$ C (op. cit.).

A **E**-coalgebra is a pair $\mathbb{A} = (\mathrm{A}, \mu)$, with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\to$ **E**(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390).

As, further, a coalgebraic model of modal logic, $\mathbb{A}$ can be defined as follows (407):

For a set of formulas, Φ, let $\nabla\Phi := \Box \bigvee \Phi \wedge \bigwedge \Diamond\Phi$, where $\Diamond\Phi$ denotes the set $\{\Diamond\phi \mid \phi \in \Phi\}$ (op. cit.). Then,

$\Diamond\phi \equiv \nabla\{\phi, \mathrm{T}\}$,

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.).

[10] '[A] *homotopy* between a pair of continuous maps f: $X \to Y$ and g: $X \to Y$ is a continuous map H: X x [0, 1] $\to Y$ satisfying $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$' (Awodey et al., 2013: 1164). '[T]he logical notion of identity $a = b$ of two objects a,b: A of the same type A can be understood as the existence of a path p: $a \rightsquigarrow b$ from point a to point b in the space A. This also means that two functions f,g: $A \to B$ are identical just in case they are homotopic, since a homotopy is just a family of paths p_x: $f(x) \rightsquigarrow g(x)$ in B, one for each x:A. In type theory, for every type A there is a (formerly somewhat mysterious) type Id_A of identities between objects of A; in homotopy type theory, this is just the *path space* A^I of all continuous maps $I \to A$ from the unit interval' (op. cit.: 1165).

[11] See Awodey (2019), for a discussion of the relation between senses and equivalence types.

[12] See chapter **3**, for further discussion.

$[\![\nabla\Phi]\!]$ = {w∈W | R[w] ⊆ $\bigcup$ {$[\![\phi]\!]$ | ϕ∈Φ} and ∀ϕ∈Φ, $[\![\phi]\!]$ ∩ R[w] ≠ ∅} (Fontaine, 2010: 17).

Let an **E**-coalgebraic modal model, $\mathbb{A}$ = ⟨S, λ, R[.]⟩, where λ(s) is 'the collection of proposition letters true at s in S, and R[s] is the successor set of s in S', such that $\mathbb{S}$,s ⊩ ∇Φ if and only if, for all (some) successors σ of s∈S, [Φ, σ(s)∈**E**($\Vdash_{\mathbb{A}}$)] (Venema, 2007: 407), with **E**($\Vdash_{\mathbb{A}}$) a relation lifting of the satisfaction relation $\Vdash_{\mathbb{A}}$ ⊆ S x Φ. Let a functor, **K**, be such that there is a relation $\overline{\mathbf{K}}$ ⊆ **K**(A) x **K**(A') (Venema, 2012: 17)). Let Z be a binary relation s.t. Z ⊆ A x A' and $\wp\overline{Z}$ ⊆ ℘(A) x ℘(A'), with

$\wp\overline{Z}$:= {(X, X') | ∀x∈X∃x'∈X' with (x, x')∈Z ∧ ∀x'∈X'∃x∈X with (x, x')∈Z} (op. cit.). Then, we can define the relation lifting, $\overline{\mathbf{K}}$, as follows:

$\overline{\mathbf{K}}$:= {[(π, X), (π', X')] | π = π' and (X, X')∈$\wp\overline{Z}$} (op. cit.), with π a projection mapping of $\overline{\mathbf{K}}$.[13]

The relation lifting, $\overline{\mathbf{K}}$, associated with the functor, **K**, satisfies the following properties (Enqvist et al, 2019: 586):

- $\overline{\mathbf{K}}$ extends **K**. Thus $\overline{\mathbf{K}}f = \mathbf{K}f$ for all functions f: $X_1 \to X_2$;
- $\overline{\mathbf{K}}$ preserves the diagonal. Thus $\overline{\mathbf{K}}\mathrm{d}_X = \mathrm{Id}_{KX}$ for any set X and functor, Id, where Id_C maps a set S to the product S x C (583, 586);
- $\overline{\mathbf{K}}$ is monotone. R ⊆ Q implies $\overline{\mathbf{K}}$R ⊆ $\overline{\mathbf{K}}$Q for all relations R,Q ⊆ X_1 x X_2;
- $\overline{\mathbf{K}}$ commutes with taking converse. $\overline{\mathbf{K}}R^\circ = (\overline{\mathbf{K}}R)^\circ$ for all relations R ⊆ X_1 x X_2;
- $\overline{\mathbf{K}}$ distributes over relation composition. $\overline{\mathbf{K}}$(R ; Q) = $\overline{\mathbf{K}}$R ; $\overline{\mathbf{K}}$Q, for all relations R ⊆ X_1 x X_2 and Q ⊆ X_2 x X_3, provided that the functor **K** preserves weak pullbacks (op. cit.). Venema and Vosmaer (2014: §4.2.2) define a weak pullback as follows: 'A weak pullback of two morphisms f : X → Z and g : Y → Z with a shared codomain Z is a pair of morphisms p_X : P → X and p_Y : P → Y with a shared domain P, such that (1) $f \circ p_X = g \circ p_Y$, and (2) for any other pair of morphisms q_X : Q → X and q_Y : Q → Y with $f \circ q_X = g \circ q_Y$, there is a morphism q : Q → P such that $p_X \circ q = q_X$ and $p_Y \circ q = q_Y$. This pullback is "weak" because we are not requiring q to be unique. Saying that [a set functor] T : **Set** → **Set** preserves weak pullbacks means that if p_X : P → X and p_Y : P → Y form a weak pullback of f : X → Z and g : Y → Z, then Tp_X : TP → TX and Tp_Y : TP → TY form a weak pullback of Tf : TX → TZ and Tg : TY → TZ'.

A coalgebraic model of deterministic automata can finally be thus defined (Venema, 2007: 391). An automaton is a tuple, $\mathbb{A}$ = ⟨A, a_I, C, Ξ, F⟩, such that

[13] The projections of a relation R, with R a relation between two sets X and Y such that R ⊆ X x Y, are

X ⟵(π_1) R (π_2)⟶ Y such that π_1((x, y)) = x, and π_2((x, y)) = y. See Rutten (2019: 240).

A is the state space of the automaton $\mathbb{A}$; $a_I \in A$ is the automaton's initial state; C is the coding for the automaton's alphabet, mapping numerals to the natural numbers; Ξ: A X C $\rightarrow$ A is a transition function, and F $\subseteq$ A is the collection of admissible states, where F maps A to $\{1,0\}$, such that F: A $\rightarrow$ 1 if a$\in$F and A $\rightarrow$ 0 if a$\notin$F (op. cit.).

Modal automata are defined over a modal one-step language (Venema, 2020: 7.2). With A being a set of propositional variables the set, $\mathtt{Latt}$(X), of lattice terms over X has the following grammar:

$$\phi ::= \bot \mid \top \mid \mathrm{x} \mid \phi \wedge \phi \mid \phi \vee \phi,$$

with x$\in$X and $\phi \in \mathtt{Latt}$(A) (op. cit.).

The set, $\mathtt{1ML}$(A), of modal one-step formulas over A has the following grammar:

$$\alpha \in \mathrm{A} ::= \bot \mid \top \mid \Diamond\phi \mid \Box\phi \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \text{ (op. cit.).}$$

A modal P-automaton $\mathbb{A}$ is a triple, (A, Θ, a_I), with A a non-empty finite set of states, $a_I \in$A an initial state, and the transition map

Θ: A x $\wp$P $\rightarrow$ $\mathtt{1ML}$(A)

maps states to modal one-step formulas (op. cit.: 7.3).

The crux of the reconciliation between algebraic models of cognitivism and the formal foundations of modal expressivism is based on the duality between categories of algebras and coalgebras: $\mathbb{A} = \langle \mathrm{A}, \alpha{:}\mathrm{A} \rightarrow \mathbf{E}(\mathrm{A})\rangle$ is dual to the category of algebras over the functor α (417-418). For a category C, object A, and endofunctor $\mathbf{E}$, define a new arrow, α, s.t. $\alpha{:}\mathbf{E}\mathrm{A} \rightarrow \mathrm{A}$. A homomorphism, f, can further be defined between algebras $\langle \mathrm{A}, \alpha\rangle$, and $\langle \mathrm{B}, \beta\rangle$. Then, for the category of algebras, the following commutative square can be defined: (i) $\mathbf{E}$A $\rightarrow$ $\mathbf{E}$B ($\mathbf{E}f$); (ii) $\mathbf{E}$A $\rightarrow$ A (α); (iii) $\mathbf{E}$B $\rightarrow$ B (β); and (iv) A $\rightarrow$ B (f) (see Hughes, 2001: 7-8). The same commutative square holds for the category of coalgebras, such that the latter are defined by inverting the direction of the morphisms in both (ii) [A $\rightarrow$ $\mathbf{E}$A (α)], and (iii) [B $\rightarrow$ $\mathbf{E}$B (β)] (op. cit.)

The significance of the foregoing is twofold. First and foremost, the above demonstrates how a formal correspondence can be effected between algebraic models of cognition and coalgebraic models which provide a natural setting for modal logics and automata. The second aspect of the philosophical significance of modal coalgebraic automata is that – as a model of modal expressivism – the proposal is able to countenance fundamental properties in the foundations of mathematics, and circumvent the issues accruing to the attempt so to do by the competing expressivist approaches.

3 Material Adequacy

The material adequacy of epistemic modal algebras as a fragment of the representational theory of mind is witnessed by the prevalence of possible worlds and hyperintensional semantics – the model theory for which is algebraic (see Blackburn et al., 2001: ch. 5) – in cognitive psychology and artificial intelligence.

In artificial intelligence, the subfield of knowledge representation draws on epistemic logic, where belief and knowledge are interpreted as necessity operators (Meyer and van der Hoek, 1995; Fagin et al., 1995). Possibility and necessity may receive other interpretations in mental terms, such as that of conceivability and apriority (i.e. truth in all epistemic possibilities, or inconceivability that not ϕ). The language of thought hypothesis maintains that thinking occurs in a mental language with a computational syntax and a semantics. The philosophical significance of cognitivism about epistemic modality and hyperintensionality is that it construes epistemic intensions and hyperintensions as abstract, computational functions in the mind, and thus provides an explanation of the relation that human beings bear to epistemic possibilities. Intensions and hyperintensions are semantically imbued abstract functions comprising the computational syntax of the language of thought. The functions are semantically imbued because they are defined relative to a parameter ranging over either epistemically possible worlds or epistemic states in a state space, and extensions or semantic values are defined for the functions relative to that parameter. Cognitivism about epistemic modality or hyperintensionality argues that thoughts are composed of epistemic intensions or hyperintensions. Cognitivism about epistemic modality provides a metaphysical explanation or account of the ground of thoughts, arguing that they are grounded in epistemic possibilities and either intensions or hyperintensions which are themselves internal representations comprising the syntax and semantics for a mental language. This is consistent with belief and knowledge being countenanced in an epistemic logic for artificial intelligence, as well. Epistemic possibilities are constitutively related to thoughts, and figure furthermore in the analysis of notions such as apriority and conceivability, as well as belief and knowledge in epistemic logic for artificial intelligence.

My claim is only that epistemic intensions and hyperintensions – i.e. functions from epistemically possible worlds or epistemic states to extensions – are computable functions comprising a fragment of the language of thought, leaving it open whether the mind is more generally a Turing machine. I thus hope to avoid taking a position here on whether human cognition is generally computational in light of Gödel's (1931/1986) incompleteness theorems. See Elohim (2024), for proofs of the incompleteness theorems. A theory is recursively enumerable if the valid strings in the theory can be enumerated by a Turing machine. A theory is recursive if the Turing machine halts on every input. Gödel's disjunction claims that either (I) the mind is a Turing machine and thus there are sentences which are undecidable, i.e. not provable, because (i) formal theories are recursively enumerable, i.e. formalizable by Turing machines, and (ii) the first incompleteness theorem entails that, in *consistent* formal systems, the provability via the recursive enumerability of sentences is distinct from the truth of Gödel sentences (1931/1986: 195), or (II) the mind surpasses the computability via the recursive enumerability of sentences in a Turing machine, and currently undecidable sentences are provable i.e. decidable owing to (i) mathematical intuition instead of computable mechanism, and (ii) Gödel's acceptance of rational optimism. For further discussion, see Gödel (1951); Lucas (1961); Penrose (1989; 1994); the essays in Horsten and Welch (2016); and Koellner (2018a,b). See Elohim (2024), for further discussion. I account for the convergence between modal and hyperintensional computational automata and rational intuition in Elohim (2024).

4 Precedent

The proposal that possible worlds semantics comprises the model for thoughts and propositions is anticipated by Wittgenstein (1921/1974: 2.15-2.151, 3-3.02); Chalmers (2011); and Jackson (2011). Their approaches depart, however, from the one here examined in the following respects.

Wittgenstein (op. cit.: 1-1.1) has been interpreted as endorsing an identity theory of propositions, which does not distinguish between internal thoughts and external propositions (see McDowell, 1994: 27; and Hornsby, 1997: 1-3). How the identity theory of propositions is able to accommodate Wittgenstein's suggestion that a typed hierarchy of propositions can be generated – only if the class of propositions has a general form and the sense of propositions over which operations range is invariant by being individuated by the possibilities figuring as their truth and falsity conditions (see Wittgenstein, 1979: 21/11/16, 23/11/16, 7/11/17; and Potter, 2009: 283-285 for detailed discussion) – is an open question. Wittgenstein (1921/1974: 5.5561) writes that 'Hierarchies are and must be independent of reality', although provides no account of how the independence can be effected.

Jackson (2008: 48-50) distinguishes between personal and subpersonal theories by the role of neural science in individuating representational states (see Shea, 2013, for further discussion), and argues in favor of a 'personal-level implicit theory' for the possible worlds semantics of mental representations.

Chalmers' approach comes closest to the one here proffered, because he argues for a hybrid cognitivist-expressivist approach as well, according to which epistemic intensions – i.e. functions from epistemically possible worlds to extensions – are individuated by their inferential roles (2012: 462-463). Chalmers endorses what he refers to as 'anchored inferentialism', and in particular 'acquaintance inferentialism' for intensions, according to which 'there is a limited set of primitive concepts, and all other concepts are grounded in their inferential role with respect to these concepts', where 'the primitive concepts are acquaintance concepts' (463, 466) and '[a]cquaintance concepts may include phenomenal concepts and observational concepts: primitive concepts of phenomenal properties, spatiotemporal properties, and secondary qualities' (2010b: 11). According to Chalmers, 'anchored inferential role determines a primary intension. The relevant role can be seen as an internal (narrow or short-armed) role, so that the content is a narrow content' (5). The inferences in question are taken to be 'suppositional' inferences, from a base class of truths, $PQTI$ – i.e. truths about physics, consciousness, and indexicality, and a that's all truth – determining canonical specifications of epistemically possible worlds, to other truths (3). With regard to how suppositional inference, i.e. 'scrutability', plays a role in the definitions of intensions, Chalmers writes that '[t]he primary intension of [a sentence] S is true at a scenario [i.e. epistemically possible world] w iff [A] epistemically necessitates S, where [A] is a canonical specification of w', where '[A] epistemically necessitates S iff a [material] conditional of the form '[A] → S' is apriori' and the apriori material entailment is the relation of scrutability (2006).[14] Chalmers (2012: 245) is explicit about this: 'The intension of a sentence S (in a context) is true at a scenario w iff S is a priori scrutable from [A] (in that context), where [A] is a canonical specification of w (that is, one of the epistemically complete sentences in the equivalence class of w) ... A Priori Scrutability entails that this sentence S is a priori scrutable (for me) from a canonical specification [A] of my actual scenario, where [A] is something along the lines of $PQTI$'. 'The secondary intension of S is true at a world w iff [A] metaphysically necessitates S', where '[A] metaphysically necessitates S when a subjunctive conditional of the form 'if [A] had been the case, S would have been the case' is true' (op. cit.). Thus, suppositional inference, i.e. scrutability, determines the intensions of two-dimensional semantics.

On the approach advanced here, intensions and hyperintensions are countenanced as semantically imbued functions. Intensions and hyperintensions as functions comprise the computational syntax for the language of thought, but they are semantically imbued because they are functions from epistemic possibilities to extensions.

An anticipation of this proposal is Tichy (1969), who defines intensions as Turing machines. Adriaans (2020) provides an example of intensions modeled using a Turing machine, as well.[15] The expression

$U_j(\overline{T_i}x) = y$

has the following components. 'The universal Turing machine U_j is a **context** in which the computation takes place. It can be interpreted as a **possible computational world** in a modal interpretation of computational semantics.

[14]'We can define a priori scrutability in parallel to definitional entailment: a sentence S is a priori scrutable from (or a priori entailed by) a class of sentences C if S can be logically derived from some members of C along with some a priori truths. Given weak assumptions, the right-hand side is equivalent to the claim that there is a conjunction D of sentences in C such that the material conditional "If D, then S" (which is equivalent to "¬(D ∧ ¬S)" is a priori' (Chalmers, 2012: 7). Chalmers (private correspondence) writes: '[I] use strict implication (a priori material implication) [i.e. $\Box_1(p \rightarrow q)$; see Chalmers, 2006], not material implication, so avoid the paradoxes of the latter, and accept the paradoxes of the former'. Mares (2024) writes: '[T]he strict implication (p $\Box\!\!\rightarrow$ q) is true whenever it is not possible that p is true and q is false — i.e., $\neg\Diamond(p \wedge \neg q)$. Among the paradoxes of strict implication are the following:

'$(p \wedge \neg p) \rightarrow q$,

'$p \rightarrow (q \rightarrow q)$,

'$p \rightarrow (q \vee \neg q)$.

'The first asserts that a contradiction strictly implies every proposition; the second and third imply that every proposition strictly implies a tautology'.

[15]Approaches to conceiving of intensions as computable functions have been pursued, as well, by Muskens (2005), Moschovakis (2006), and Lappin (2014). The computational complexity of algorithms for intensions has been investigated by Mostowski and Wojtyniak (2004), Mostowski and Szymanik (2012), and Kalocinski and Godziszewski (2018).

/ The sequences of symbols $\overline{T_i}x$ and y are **well-formed data**. / The sequence $\overline{T_i}$ is a self-delimiting description of a program and it can be interpreted as a piece of well-formed **instructional data**. / The sequence $\overline{T_i}x$ is an **intension**. The sequence y is the corresponding **extension**. / The expression $U_j(\overline{T_i}x) = y$ states the result of the program $\overline{T_i}x$ in world U_j is y. It is a **true sentence**'.

I will avail, in this book, of Adriaans (2020)'s definition of intensions as Turing machines. The variable, x, in the (hyper-)intension, $\overline{T_i}x$, ranges over epistemically possible worlds or states and metaphysically possible worlds or states, and $\overline{T_i}x$ is a function from epistemic states verifying sentences, where the epistemic states are taken as actual, to the value of the sentences verified by metaphysical states, to the sentences' extensions.

This is consistent with the inferences of scrutability playing a role in the individuation of intensions and hyperintensions, but whereas Chalmers grounds inferences in dispositions (2010: 10; 2021), I claim that the inferences drawn from the canonical specifications of epistemic possibilities to arbitrary truths are apriori computations between mental representations.

Schroeder (2008) provides a protracted examination of variations on the expression relation. Schroeder argues that expressivists ought to opt for an assertability account of the expression relation, such that the propositions expressed by sentences are governed by assertability conditions for the sentences rather than their truth conditions, and the expression thus doesn't concern the conveyance of information but rather norms on correct assertion of the sentence. He writes: 'Every sentence in the language is associated with conditions in which it is semantically correct to use that sentence assertorically ... Assertability conditions, so conceived, are a device of the semantic theorist. They are not a kind of information that speakers intend to convey. So there is no sense in which a community of speakers could get by, managing to communicate information to each other about the world, by means of assertability conditions alone. It is only because some assertability conditions mention beliefs, and beliefs have contents about the world, that speakers can manage to convey information about the world' (op. cit.: 108, 110). The present account is not committed to Schroeder's proposed assertability expressivism. However, I note in Section **2.6** that Hawke and Steinert-Threlkeld (2021)'s assertability semantics for epistemic modals is consistent with the model-theoretic account of expressivism here advanced. The present account might also converge with a view which Schroeder attributes to Gibbard (1990, 2003), which he refers to as indicator expressivism, according to which mental states do not express propositional contents, but rather express ur-contents owing to an agent's intentions (§4.1). Ur-contents differ from propositional contents, by the differences in their roles in expressing normative and non-normative contents. Schroeder objects to the appeal to ur-contents, arguing that they play a role too similar to that of propositional contents because they convey descriptive information, while Gibbard simultaneously rejects the similarity (107). I think that because ur-contents express normative contents rather than non-normative ones, they are sufficiently distinct from propositional contents, and that it is innocuous for them to be descriptive in part. The present model-theoretic account of expressivism might thus be thought to be consistent with indicator expressivism.

5 Conceptual Engineering of Intensions and Hyperintensions

How can intensions and hyperintensions be revised, given that they are here countenanced as computable functions comprising the syntax of the language of thought? Note that the epistemically possible worlds or hyperintensional truthmakers, and the topics to which they are sensitive, which figure as input to intensions and hyperintensions, can be externally individuated. If so, then they are susceptible to updates by external sources. One might want further to engage in the project of the conceptual engineering one's intensions and hyperintensions, perhaps in order to engage in an ameliorative project relevant to using more socially just concepts (see Haslanger, 2012, 2020 for further discussion). Conceptual engineering of intensions and hyperintensions can then be effected by five methods. The first is via announcements in dynamic epistemic logic. The second method is via dynamic interpretational modalities which concern the possible reassignment of topics to atomic formulas. The third method is via dynamic hyperintensional belief revision. We here propose a novel truthmaker semantics for the first and second methods.

The language of public announcement logic has the following grammar (see Baltag and Renne, 2016):

$$\phi := \mathrm{p} \mid \phi \wedge \phi \mid \neg\phi \mid [\mathrm{a}]\phi \mid [\phi!]\psi$$

$[\mathrm{a}]\phi$ is interpreted as the 'the agent knows ϕ'. $[\phi!]\psi$ is an announcement formula, and is intuitively interpreted as 'whenever ϕ is true, ψ is true after we eliminate all not-ϕ possibilities (and all arrows to and from these possibilities)'.

Semantics for public announcement logic is as follows:

$\langle \mathrm{M, w}\rangle \Vdash \phi$ if and only if $\mathrm{w}\in\mathrm{V}(\phi)$

$\langle \mathrm{M, w}\rangle \Vdash \phi \wedge \psi$ if and only if $\mathrm{M, w} \Vdash \phi$ and $\mathrm{M, w} \Vdash \psi$

$\langle \mathrm{M, w}\rangle \Vdash \neg\phi$ if and only if $\mathrm{M, w} \nVdash \phi$

$\langle \mathrm{M, w}\rangle \Vdash [\mathrm{a}]\phi$ if and only if $\mathrm{M, w} \Vdash \phi$ for each v satisfying $\mathrm{wR}_a\mathrm{v}$

$\langle \mathrm{M, w}\rangle \Vdash [\phi!]\psi$ if and only if $\mathrm{M, w} \nVdash \phi$ or $\mathrm{M}[\phi!]$, $\mathrm{w} \Vdash \psi$,

where $\mathrm{M}[\phi!] = (\mathrm{W}[\phi!], \mathrm{R}[\phi!], \mathrm{V}[\phi!])$ is defined by

$\mathrm{W}[\phi!] := (\mathrm{v}\in\mathrm{W} \mid \mathrm{M, v} \Vdash \phi)$ (intuitively, 'retain only the worlds where ϕ is true' (op. cit.),

$\mathrm{xR}[\phi!]_a\mathrm{y}$ if and only if $\mathrm{xR}_a\mathrm{y}$ (intuitively, 'leave arrows between remaining words unchanged'), and

$\mathrm{v}\in\mathrm{V}[\phi!](\mathrm{p})$ if and only if $\mathrm{v}\in\mathrm{V}(\mathrm{p})$ (intuitively, 'leave the valuation the same at remaining worlds').

Fine (2006) and Uzquiano (2015) countenance interpretational modalities. Fine (2005b)'s modality is simultaneously postulational, dynamic, and prescriptive. The dynamic modality is interpreted so as to concern the execution of computer programs which entrain e.g. the introduction of objects into a domain which conform to a certain property. Fine (2006) advances a postulational interpretational modality which concerns the possible reinterpretation of quantifier domains in accounting for indefinite extensibility. Uzquiano's modality is interpretational and also relevant to capturing the property of indefinite extensibility. The modality is mathematical, and concerns the possible reinterpretations of the intensions of non-logical vocabulary such as the membership relation, $\in$.

In this chapter, I propose to render Fine's and Uzquiano's interpretational modalities dynamic. The dynamic interpretational modalities are interpreted as program executions which entrain reinterpretations of intensions as well as reinterpretations of hyperintensions which reassign topics to atomic formulas.

My proposal is that both announcement formulas, $[\phi!]\psi$, and Fine and Uzquiano's modalities ought to be rendered hyperintensional, such that the box operators are further interpreted as necessary truthmakers as specified in the clause for A(ϕ) above. The dynamic interpretational modalities can just take the clause for A(ϕ). For announcement formulas, $[\phi!]\psi$ if and only if either (i) for all t$\in$P there is no t'$\in$P such that t' $\sqcup$ t $\in$P and t' $\vdash$ ϕ or (ii) M[ϕ!], s $\vdash$ ψ,

where M[ϕ!] $= \langle$S[ϕ!], $\leq$[ϕ!], v[ϕ!]$\rangle$ is defined by

S[ϕ!] := s'$\in$S | M, s' $\vdash$ ϕ (intuitively, retain only states which verify ϕ),

$\leq$[ϕ!] if and only if s $\leq$ s' (intuitively, leave relations between remaining states unchanged), and

v[ϕ!] if and only if v: Prop $\rightarrow$ (2^S x 2^S) which assigns a bilateral proposition $\langle\phi^+, \phi^-\rangle$ to $\phi\in$Prop (intuitively, leave the valuation the same at remaining states).

This would suffice for what Chalmers (2020) refers to as conceptual re-engineering, rather than 'de novo' conceptual engineering, of intensions and hyperintensions. Conceptual re-engineering concerns the refinement or replacement of extant concepts, while de novo engineering concerns the introduction of new concepts. The third method for conceptual re-engineering contents would be via Berto and Özgün (2021)'s logic for dynamic hyperintensional belief revision, which includes a topic-sensitive upgrade operator. On this method, the worlds and topics for formulas are both updated in cases of belief revision.

A fourth novel method can be countenanced, namely making epistemic two-dimensional semantics dynamic. On this approach, an epistemic action such as an announcement which updates the first, epistemic parameter for a formula would entrain an update to a second parameter ranging over metaphysically possible worlds or states in a state space. Using two-dimensional (hyper-)intensions, such that the value of a formula relative to a first parameter ranging over epistemic states determines the value of the formula relative to a second parameter ranging over metaphysical states, an update (announcement, epistemic action) to the epistemic space over which the first parameter of a formula ranges induces an update to the metaphysical space over which a second parameter for a formula ranges. With M* a model including a class of epistemic states, S, and a class of metaphysical states, W, two-dimensional updates have the form:

M*, w $\Vdash$ [ϕ!]ψ if and only if M*, w $\nVdash$ ϕ or M*[ϕ!], w $\Vdash$ ψ,

where M*[ϕ!] = (S[ϕ!], W[ϕ!]$^{S[\phi!]}$, R[ϕ!], V[ϕ!]). W[ϕ!]$^{S[\phi!]}$ records the dynamic two-dimensional update of metaphysical states, W, conditional on the update of epistemic states, S, and the rest is defined as above.

A fifth method for modeling updates might be via the interventions of structural equation models which reassign values to exogenous variables which then determines the values of endogenous variables (see e.g. Pearl, 2009).[16] Using two-dimensional (hyper-)intensions, the updates to the epistemic parameter of a formula might be modeled using Baltag (2016)'s Logic of Epistemic Dependency. As Baltag writes: 'An *epistemic dependency formula* $K_a^{x_1,\ldots,x_n}y$ says that an agent knows the value of some variable y *conditional* on being given the values of the variables $x_1, \ldots, x_n \ldots$ if we use the abbreviation $(w(\overrightarrow{x}))$ = $(v(\overrightarrow{x}))$ for the conjunction $(w(x_1)) = (v(x_1)) \wedge (w(x_n)) = (v(x_n))$, then we put

$w \Vdash K_a^{x_1,\ldots,x_n}y$ iff $\forall v \sim_a w$ $(w(\overrightarrow{x})) = (\mathrm{v}(\overrightarrow{x})) \Rightarrow v(y) = w(y)$.

In words: an agent knows y given $x_1, \ldots, x_n$ if the value of y is the same in all the epistemic alternatives that agree with the actual world on the values of $x_1, \ldots, x_n$. This operator has connections with Dependence Logic and allows us to "pre-encode" the dynamics of the value-announcement operator [!x]ϕ' (136).

Epistemic updates via announcements would then, via two-dimensional intensions and hyperintensions, induce an intervention in the metaphysical space in the parameter defining the second dimension of a formula, by reassigning values of exogenous variables so as to constrain the values of endogenous variables in structural equations.

6 Two-dimensional Hyperintensionality and the Epistemic Church-Turing Thesis

The Epistemic Church-Turing Thesis can receive a similar two-dimensional hyperintensional formalization. Carlson (2016: 132) presents the schema for the Epistemic Church-Turing Thesis as follows:

With $\Box$ interpreted as a knowledge operator, '$\Box\forall x\exists y\Box\phi \rightarrow \exists e\Box\forall x\exists y[E(e, x, y) \wedge \phi]$,

'where e does not occur free in ϕ and E is a fixed formula of L_{PA} [i.e. the language of Peano Arithmetic] with free variables v_0, v_1, v_2 such that, letting N be the standard model of arithmetic,

'N $\Vdash$ E(e, x, y)[e, x, y | a, m, n]

'iff on input m, the ath Turing machine halts and outputs n. For convenience, we will write $\{t_1\}\{t_2\} \simeq t_3$ for $E(t_1, t_2, t_3)$ when t_1, t_2, t_3 are terms'. Carlson defines $(x_1, \ldots, x_n) \mid (y_1, \ldots, y_1)$ as denoting the 'function which maps x_i to y_i for each i = 1, ..., n' (op. cit.: 130). Hyperintensionally reformalized, the Epistemic Church-Turing Thesis is then:

[16]Thanks here to Hannes Leitgeb for mentioning interventions in structural equation models with regard to a possible example of updates in metaphysical space.

$A\forall x\exists y A\phi \rightarrow \exists e A\forall x\exists y[E(e, x, y) \wedge \phi]$.

The two-dimensional hyperintensional profile of the Epistemic Church-Turing Thesis can be countenanced by adding a topic-sensitive truthmaker from a metaphysical state space and making its value dependent on the value of the epistemically necessary truthmaker A(ϕ). Thus:

$A^{(w\cap t)}\forall x\exists y A^{(w\cap t)}\phi \rightarrow \exists e A^{(w\cap t)}\forall x\exists y[E(e, x, y) \wedge \phi]$.

An application of the two-dimensional Epistemic Church-Turing Thesis is to the above dynamic epistemic two-dimensional semantics. Two-dimensional Turing machines can be availed of in order to provide mechanistic, constructive definitions of the epistemic actions and metaphysical interventions and their dependence in the two-dimensional semantics. Aside from defining epistemic hyperintensions as computable functions, where the functions comprise a fragment of the computable syntax of the language of thought, I record here my preference for non-mechanistic approaches to epistemic modality, such as the interpretation thereof as informal provability or as an inference package.

In the remainder of the essay, I outline an expressivist semantics for epistemic modality. I endeavor, then, to demonstrate the advantages accruing to the present approach to countenancing modal expressivism via modal coalgebraic automata, via a comparison of the theoretical strength of the proposal when applied to characterizing the fundamental properties of the foundations of mathematics, by contrast to the competing approaches to modal expressivism and the limits of their applications thereto.

7 Expressivist Semantics for Epistemic Possibility

I assume a dissociation between the natural language semantics for epistemic modals and an account of mental states as epistemic possibilities or hyperintensional epistemic states. However, my expressivism about epistemic modality might be thought to adduce in favor of expressivism about epistemic modals.

Let expressivism about a domain of discourse be the claim that an utterance from that domain expresses a mental state, rather than states a fact (Hawke and Steinert-Threlkeld, 2021). Hawke and Steinert-Threlkeld (op. cit., 480) distinguish between semantic expressivism and pragmatic expressivism. Expressivism about epistemic possibility takes the property expressed by $\Diamond\phi$ to be $\{\mathbf{s} \subseteq W\text{: } \mathbf{s} \nVdash \neg p\}$, where **s** is a state of information, W is a set of possible worlds, and $\mathbf{s} \Vdash \phi$ if and only if ϕ is assertible relative to **s**, if and only if the state of information is compatible with ϕ (op. cit.). Semantic expressivism incorporates a 'psychologistic semantics' according to which the value of ϕ is a partial function from information states to truth-values, such that 'the mental type expressed by ϕ is characterized in terms of the assertibility relation $\Vdash$' and 'the definition of $\Vdash$ is an essential part of that of $[\![\ ]\!]$' (481). Pragmatic expressivism rejects the psychologistic semantics condition, and 'allows for a *gap* between the compositional semantic theory and $\Vdash$' (op. cit.).

Hawke and Steinert-Threlkeld (op. cit.) argue that satisfying the following conditions is a desideratum of any expressivist account about epistemic possibility (§3.5):

(Weak) Wide-scope Free Choice (**WFC** (§3.1)):
$\Diamond p \vee \Diamond\neg p \Vdash \Diamond p \wedge \Diamond\neg p$
Disjunctive Inheritence (**DIN** (§3.2)):
$(\Diamond p \wedge q) \vee r \Vdash [\Diamond(p \wedge q) \wedge q] \vee r$
Disjunctive Syllogism and Schroeder's Constraints (§3.4):
DSF $\{\Diamond\neg q,\ p \vee \Box q \nVdash p\}$
SCH $\{\Diamond\neg p,\ p \vee \Box q \nVdash \Box q\}$

DSF and **SCH** record the failure of disjunctive syllogism in the presence of epistemic contradictions.

WFC is vindicated by the contention that when someone asserts p $\vee$ $\neg$p, they neither believe p nor believe $\neg$p, and so are in a position to assert both $\Diamond$p and $\Diamond\neg$p.

DIN is vindicated by the equivalence of the content of the utterances, e.g.,
(1) David is at home and might be watching a film.
(2) David is at home and might be watching a film at home (§3.2).

Hawke and Steinert-Threlkeld's modal propositional assertibility semantics is then as follows (§5.1).

Reading $t \subseteq s$: $[\![\phi]\!]^t \neq 1$ as 's refutes ϕ':

- if p is an atom: $[\![p]\!]^s = 1$ iff $s \subseteq V(p)$
  if p is an atom $[\![p]\!]^s = 0$ iff s refutes p
- $[\![\neg\phi]\!]^s = 1$ iff $[\![\phi]\!]^s = 0$
  $[\![\neg\phi]\!]^s = 0$ iff $[\![\phi]\!]^s = 1$
- $[\![\phi \wedge \psi]\!]^s = 1$ iff $[\![\phi]\!]^s = 1$ and $[\![\psi]\!]^s = 1$
  $[\![\phi \wedge \psi]\!]^s = 0$ iff s refutes $\phi \wedge \psi$
- $[\![\phi \vee \psi]\!]^s = 1$ iff there exists s_1, s_2 such that $s = s_1 \cup s_2$, $[\![\phi]\!]^{s_1} = 1$ and $[\![\psi]\!]^{s_2} = 1$
  $[\![\phi \vee \psi]\!]^s = 0$ iff s refutes $\phi \vee \psi$
- $[\![\Diamond\phi]\!]^s = 1$ iff $[\![\phi]\!]^s \neq 0$
  $[\![\Diamond\phi]\!]^s = 0$ iff s refutes $\Diamond\phi$
- $\Box\phi := \neg\Diamond\neg\phi$
- $\Diamond\phi := \neg\Box\neg\phi$.[17]

[17] I have revised the previous clause, and further added this clause to Hawke and Steinert-Threlkeld's model. The clause states that epistemic possibility is defined as the dual of apriority or epistemic necessity, i.e. as not being ruled-out apriori ($\neg\Box\neg$), and follows Chalmers (2011: 66).

Unlike Yalcin's (2007) domain semantics (4.1), Veltman's (1996) update semantics (4.2), and Moss' (2015; 2018) probabilistic semantic expressivism (6.2), Hawke and Steinert-Threlkeld's assertibility semantics satisfies **WFC**, **DIN**, **DSF**, and **SCH** (Hawke and Steinert-Threlkeld, 2020: 507). As a preliminary, suppose

Proposition 1 If ϕ is $\Diamond$-free, then s $\Vdash \Diamond\phi$ holds iff there exists w$\in$s such that: $\{w\} \Vdash \phi$ (op. cit.).

Proof: s $\Vdash \Diamond\phi$ holds iff $[\![\phi]\!]^s \neq 0$. $[\![\phi]\!]^s = 0$ iff $[\![\phi]\!]^{\{w\}} = 0$ for every w$\in$s. So, $[\![\phi]\!]^s \neq 0$ iff $[\![\phi]\!]^w \neq 0$ for some w$\in$s iff $\{w\} \Vdash \phi$ for some w$\in$s (op. cit.).

For **WFC**, suppose that s $\Vdash \Diamond$p $\vee$ $\Diamond\neg$p. So, there exists s_1, s_2 that cover s and $s_1 \Vdash \Diamond$p and $s_2 \Vdash \Diamond\neg$p. By Proposition 1, there exist u,v$\in$s such that $\{u\}$ $\Vdash$ p and $\{v\} \Vdash \neg$p. Thus, s $\Vdash \Diamond$p and s $\Vdash \Diamond\neg$p (op. cit.).

For **DIN**, suppose that s $\Vdash$ ($\Diamond$p $\wedge$ q) $\vee$ r. So, there exists s_1, s_2, such that s = $s_1 \cup s_2$ with $s_1 \Vdash \Diamond$p, $s_1 \Vdash$ q, and $s_2 \Vdash$ r. For every w$\in s_1$, $\{w\} \Vdash$ q. There also exists u$\in s_1$ such that $\{u\} \Vdash$ p. Hence, $\{u\} \Vdash$ p $\wedge$ q and – by Proposition 1 – $s_1 \Vdash \Diamond$(p $\wedge$ q). Thus s $\Vdash$ [$\Diamond$(p $\wedge$ q) $\wedge$ q] $\vee$ r (op. cit.).

For **DSF** and **SCH**, suppose that there is an s such that every world in s is either a p $\wedge$ $\neg$q world or a $\neg$p $\wedge$ q world. Suppose that there exists at least one p $\wedge$ $\neg$q world in s and at least one $\neg$p $\wedge$ q world in s (op. cit.).

8 Modal Expressivism and the Philosophy of Mathematics

When modal expressivism is modeled via speech acts on a common ground of presuppositions, the application thereof to the foundations of mathematics is limited by the manner in which necessary propositions are characterized.

Because for example a proposition is taken, according to the proposal, to be identical to a set of possible worlds, all necessarily true mathematical formulas can only express a single proposition; namely, the set of all possible worlds (see Stalnaker, 1978; 2003: 51). Thus, although distinct set-forming operations will be codified by distinct axioms of a language of set theory, the axioms will be assumed to express the same proposition: The axiom of Pairing in set theory – which states that a unique set can be formed by combining an element from each of two extant sets: $\exists x \forall u(u \in x \iff u = a \vee u = b)$ – will be supposed to express the same proposition as the Power Set axiom – which states that a set can be formed by taking the set of all subsets of an extant set: $\exists x \forall u(u \in x \iff u \subseteq a)$. However, that distinct operations – i.e., the formation of a set by selecting elements from two extant sets, by contrast to forming a set by collecting all of the subsets of a single extant set – are characterized by the different axioms is readily apparent. As Williamson (2016a: 244) writes: '...if one follows Robert Stalnaker in treating a proposition as the set of (metaphysically) possible worlds at which it is true, then all true mathematical formulas literally express the same proposition, the set of all possible worlds, since all true mathematical formulas literally express necessary truths. It is therefore trivial that if one true mathematical proposition is absolutely provable, they all are. Indeed, if you already know one true mathematical proposition (that 2 + 2 = 4, for example), you thereby already know them all. Stalnaker suggests that what mathematicians really learn are in effect new contingent truths about which mathematical formulas we use to express the one necessary truth, but his view faces grave internal problems, and the conception of the content of mathematical knowledge as contingent and metalinguistic is in any case grossly implausible.'

Thomasson (2007) argues for a version of modal expressivism which she refers to as 'modal normativism', according to which alethic modalities are to be replaced by deontic modalities taking the form of object-language, modal indicative conditionals (op. cit.: 136, 138, 141). The modal indicative conditionals serve to express constitutive rules pertaining, e.g., to ontological dependencies which state that: 'Necessarily, if an entity satisfying a property exists then a distinct entity satisfying a property exists' (143-144), and generalizes to other expressions, such as analytic conditionals which state, e.g., that: 'Necessarily, if an entity satisfies a property, such as being a bachelor, then the entity satisfies a distinct yet co-extensive property, such as being unmarried' (148). A virtue of Thomasson's interpretation of modal indicative conditionals as expressing both analytic and ontological dependencies is that it would appear to converge with the 'If-thenist' proposal in the philosophy of mathematics. 'If-thenism' is an approach according to which, if an axiomatized mathematical language is consistent, then (i) one can either bear epistemic attitudes, such as fictive acceptance, toward the target system (see Leng, 2010: 180) or (ii) the system (possibly) exists [see Russell (op. cit.: §1)]; Hilbert (1899/1980: 39); Menger (1930/1979: 57); Putnam (1967); Shapiro (2000: 95); Chihara (2004: Ch. 10); and Awodey (2004: 60-61)].[18]

According, finally, to Brandom's (op. cit.) modal expressivist approach, terms are individuated by their rules of inference, where the rules are taken to have a modal profile translatable into the counterfactual forms taken by the transition functions of automata (see Brandom, 2008: 142). In order to countenance the metasemantic truth-conditions for the object-level, pragmatic abilities captured by the automata's counterfactual transition states, Brandom augments a first-order language comprised of a stock of atomic formulas with an

[18]See Leng (2009), for further discussion. Field (1980/2016: 11-21; 1989: 54-65, 240-241) argues in favor of the stronger notion of conservativeness, according to which consistent mathematical theories must be satisfiable by internally consistent theories of physics. More generally, for a class of assertions, A, comprising a theory of fundamental physics, and a class of sentences comprising a mathematical language, M, any sentences derivable from A + M ought to be derivable from A alone. Another variation on the 'If-thenist' proposal is witnessed in Field (2001: 333-338), who argues that the existence of consistent forcing extensions of set-theoretic ground models adduces in favor of there being a set-theoretic pluriverse, and thus entrains indeterminacy in the truth-values of undecidable sentences. For a similar proposal, which emphasizes the epistemic role of examining how instances of undecidable sentences obtain and fail so to do relative to forcing extensions in the set-theoretic pluriverse, see Hamkins (2012: §7).

incompatibility function (141). An incompatibility function, I, is defined as the incoherence of the union of two sentences, where incoherence is a generalization of the notion of inconsistency to nonlogical vocabulary.

x $\cup$ y $\in$ $Inc \iff$ x $\in$ I(y) (141-142).

Incompatibility is supposed to be a modal notion, such that the union of the two sentences is incompossible (126). A sentence, β is an incompatibility-consequence, $\Vdash_I$, of a sentence, α, iff there is no sequence of sentences, $<\gamma_1, \ldots, \gamma_n>$, such that it can be the case that $\alpha \Vdash_I <\gamma_1, \ldots, \gamma_n>$, yet not be the case that $\beta \Vdash_I <\gamma_1, \ldots, \gamma_n>$ (125). To be incompatible with a necessary formula is to be compatible with everything that does not entail the formula (129-130). Dually, to be incompatible with a possible formula is to be incompatible with everything compatible with something compatible with the formula (op. cit.).

There are at least two, general issues for the application of Brandom's modal expressivism to the foundations of mathematics.

The first issue is that the mathematical vocabulary – e.g., the set-membership relation, $\in$ – is axiomatically defined. I.e., the membership relation is defined by, inter alia, the Pairing and Power Set axioms of set-theoretic languages. Thus, mathematical terms have their extensions individuated by the axioms of the language, rather than via a set of inference rules that can be specified in the absence of the mention of truth values. Even, furthermore, if one were to avail of modal notions in order to countenance the intensions of the mathematical vocabulary at issue – i.e., functions from terms or sentences in worlds to their extensions – the modal profile of the intensions is orthogonal to the properties encoded by the incompatibility function. Fine (2006) avails, e.g., of postulational interpretational modalities in order to countenance the possibility of reinterpreting quantifier domains, and of thus accounting for variance in the range of the domains of quantifier expressions. The interpretational possibilities are specified as operational conditions on tracking increases in the size of the cardinality of the universe. Uzquiano (2015b) argues that it is always possible to reinterpret the intensions of non-logical vocabulary, as one augments one's language with stronger axioms of infinity and climbs thereby farther up the cumulative hierarchy of sets. The reinterpretations of, e.g., the concept of set are effected by the addition of new large cardinal axioms, which stipulate the existence of larger inaccessible cardinals. However, it is unclear how the incompatibility function – i.e., a modal operator defined via Boolean negation and a generalized condition on inconsistency – might similarly be able to model the intensions pertaining to the ontological expansion of the cumulative hierarchy.

The second issue is that Brandom's inferential expressivist semantics is not compositional (Brandom, 2008: 135-136). While the formulas of the semantics are recursively formed – because the decomposition of complex formulas into atomic formulas is decidable[19] – formulas in the language are not compositional, because they fail to satisfy the subformula property to the effect that the value

[19] Let a decision problem be a propositional function which is feasibly decidable, if it is a member of the polynomial time complexity class; i.e., if it can be calculated as a polynomial function of the size of the formula's input [see Dean (2021) for further discussion].

of a logically complex formula is calculated as a function of the values of the component logical connectives applied to subformulas therein (op. cit.).[20]

By contrast to the limits of Brandom's approach to modal expressivism, modal coalgebraic automata can circumvent both of the issues mentioned in the foregoing. In response to the first issue, concerning the axiomatic individuation and intensional profiles of mathematical terms, mappings of modal coalgebraic automata can be interpreted in order to provide a precise delineation of the (hyper-)intensions of the target vocabulary. In response, finally, to the second of the above issues, the values taken by modal coalgebraic automata are both decidable and computationally feasible, while the duality of coalgebras to Boolean-valued models of modal algebras ensures that the formulas therein retain their compositionality. The decidability of coalgebraic automata can further be witnessed by the role of modal coalgebras in countenancing the modal profile of Ω-logical consequence, where – given a proper class of Woodin cardinals – the values of mathematical formulas can remain invariant throughout extensions of the ground models comprising the set-theoretic universe (see Woodin, 2010; and Elohim (2019). The individuation of large cardinals can further be characterized by the functors of modal coalgebras, when the latter are interpreted so as to countenance the elementary embeddings constitutive of large cardinal axioms in category theory.

9 Concluding Remarks

In this essay, I have endeavored to account for a mathematically tractable background against which to model both modal and hyperintensional cognitivism and modal and hyperintensional expressivism. I availed, to that end, of the duality between epistemic modal and hyperintensional algebras and modal and hyperintensional coalgebraic automata. Epistemic modal and hyperintensional algebras were shown to comprise a materially adequate fragment of the language of thought, given that models thereof figure in both cognitive psychology and artificial intelligence. With regard to conceptual engineering of intensions and hyperintensions, I introduced a novel topic-sensitive truthmaker semantics for dynamic epistemic logic as well as a novel dynamic epistemic two-dimensional hyperintensional semantics. It was then shown how the approach to modal and hyperintensional expressivism here proffered, as regimented by the modal and hyperintensional coalgebraic automata to which the epistemic modal and hyperintensional algebras are dual, avoids the pitfalls attending to the competing modal and hyperintensional expressivist approaches based upon both the inferentialist approach to concept-individuation and the approach to codifying the speech acts in natural language via intensional semantics. The present modal

[20]Note that Incurvati and Schlöder (2020) advance a multilateral inferential expressivist semantics for epistemic modality which satisfies the subformula property. (Thanks here to Luca Incurvati.) Incurvati and Schlöder (2021) extend the semantics to normative vocabulary, but it is an open question whether the semantics is adequate for mathematical vocabulary as well.

and hyperintensional expressivist approach was shown, e.g., to avoid the limits of the foregoing in the philosophy of mathematics, as they concerned the status of necessary propositions; the inapplicability of inferentialist-individuation to mathematical vocabulary; and failures of compositionality.

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