Modal Cognitivism and Modal Expressivism

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Abstract

This paper aims to provide a mathematically tractable background against which to model both modal cognitivism and modal expressivism. I argue that epistemic modal algebras, endowed with a hyperintensional truthmaker semantics, comprise a materially adequate fragment of the language of thought. I demonstrate, then, how modal expressivism can be regimented by modal coalgebraic automata, to which the above epistemic modal algebras are categorically dual. I examine, in particular, the virtues unique to the modal expressivist approach here proffered in the setting of the foundations of mathematics, by contrast to competing approaches based upon both the inferentialist approach to concept-individuation and the codification of speech acts via intensional semantics.

1 Introduction

This essay endeavors to reconcile two approaches to the modal foundations of thought: modal cognitivism and modal expressivism. The novel contribution of the paper is its argument for a reconciliation between the two positions, by providing a hybrid account in which both internal cognitive architecture, on the model of epistemic possibilities, as well as modal automata, are accommodated, while retaining what is supposed to be their unique and inconsistent roles.

The notions of cognitivism and expressivism here targeted concern the role of internal – rather than external – factors in countenancing the nature of thought and information (cf. Fodor, 1975; Haugeland, 1978). Possible worlds or hyperintensional semantics is taken then to provide the most descriptively adequate means of countenancing the structure of the foregoing.[1] Whereas the type

[1] Delineating cognitivism and expressivism by whether the positions avail of internal repre-

of modal cognitivism examined here assumes that thoughts and information take exclusively the form of internal representations, the target modal expressivist proposals assume that information states are exhaustively individuated by both linguistic behavior and conditions external to the cognitive architecture of agents.

Modal cognitivism is thus the proposal that the internal representations comprising the language of thought can be modeled via either a possible world or hyperintensional semantics. Modal expressivism has, in turn, been delineated in two ways. On the first approach, the presuppositions shared by a community of speakers have been modeled as possibilities (cf. Kratzer, 1979; Stalnaker, 1978, 1984). Speech acts have in turn been modeled as modal operators which update the common ground of possibilities, the semantic values of which are then defined relative to an array of intensional parameters (Stalnaker, op. cit.; Veltman, 1996; Yalcin, 2007). On the second approach, the content of concepts is supposed to be individuated via the ability to draw inferences. Modally expressive normative inferences are taken then to have the same subjunctive form as that belonging to the alethic modal profile of descriptive theoretical concepts (Brandom, 2014: 211-212).[2] Both the modal approach to shared information and the speech acts which serve to update the latter, and the inferential approach to concept-individuation, are consistent with mental states having semantic values

sentations is thus orthogonal to the eponymous dispute between realists and antirealists with regard to whether mental states are truth-apt, i.e., have a representational function, rather than being non-representational and non-factive even if real (cf. Dummett, 1959; Blackburn, 1984; Price, 2013).

[2]Brandom writes, e.g.: 'For modal *expressivism* tells us that modal vocabulary makes explicit normatively significant relations of subjunctively robust material consequence and incompatibility among claimable (hence propositional) contents in virtue of which ordinary empirical descriptive vocabulary *describes* and does not merely *label*, *discriminate*, or *classify*. And modal *realism* tells us that there are modal facts, concerning the subjunctively robust relations of material consequence and incompatibility in virtue of which ordinary empirical descriptive properties and facts are determinate. Together, these two claims give a definite sense to the possibility of the correspondence of modal claimings with modal facts' (op. cit.: 2012).

or truth-conditional characterizations.

So defined, the modal cognitivist and modal expressivist approaches have been assumed to be in constitutive opposition. While the cognitivist proposal avails of modal resources in order to model the internal representations comprising an abstract language of thought, the expressivist proposal targets informational properties which extend beyond the remit of internal cognitive architecture: both the form and the parameters relevant to determining the semantic values of linguistic utterances, where the informational common ground is taken to be reducible to possibilities; and the individuation of the contents of concepts on the basis of inferential behavior.

In this paper, I provide a background mathematical theory, in order to account for the reconciliation of the cognitivist and expressivist proposals. I avail, in particular, of the duality between Boolean-valued models of epistemic modal algebras and coalgebras; i.e., labeled transition systems defined in the setting of category theory.[3] The mappings of coalgebras permit of flexible interpretations, such that they are able to characterize both modal logics as well as discrete-state automata. I argue that the correspondence between epistemic modal algebras and modal coalgebraic automata is sufficient then for the provision of a mathematically tractable, modal foundation for thought and action, which wholly captures both the modal cognitivist and modal expressivist proposals.

In Section **2**, I provide the background mathematical theory, in order to account for the reconciliation of the cognitivist and expressivist proposals.

In Section **3**, I provide reasons adducing in favor of modal cognitivism, and argue for the material adequacy of epistemic modal algebras as a fragment of

[3]For an algebraic characterization of dynamic-epistemic logic, see Kurz and Palmigiano (2013). Baltag (2003) develops a coalgebraic semantics for dynamic-epistemic logic, where coalgebraic mappings are intended to record the informational dynamics of single- and multi-agent systems. The current approach differs from the foregoing by examining the duality between static epistemic modal algebras and coalgebraic automata in a single-agent system.

the language of thought.

In Section **4**, modal coalgebraic automata are argued, finally, to be preferred as models of modal expressivism, by contrast to the speech-act and inferentialist approaches, in virtue of the advantages accruing to the model in the philosophy of mathematics. The interest in modal coalgebraic automata consists, in particular, in the range of mathematical properties that can be recovered on the basis thereof.[4] By contrast to the above competing approaches to modal expressivism, the mappings of modal coalgebraic automata are able both to model and explain elementary embeddings in the category of sets; the intensions of mathematical terms; as well as the modal profile of Ω-logical consequence.

Section **5** provides concluding remarks.

2 The Hybrid Proposal

2.1 Epistemic Modal Algebra

An epistemic modal algebra is defined as $U = \langle \mathrm{A}, 0, 1, \neg, \cap, \cup, \mathbf{l}, \mathbf{m} \rangle$, with A a set containing 0 and 1 (Bull and Segerberg, 2001: 28).[5]

$\mathbf{l}1 = 1$,

$\mathbf{l}(\mathrm{a} \cap \mathrm{b}) = \mathbf{l}\mathrm{a} \cap \mathbf{l}\mathrm{b}$

$\mathbf{m}\mathrm{a} = \neg\mathbf{l}\neg\mathrm{a}$,

$\mathbf{m}0 = 0$,

$\mathbf{m}(\mathrm{a} \cup \mathrm{b}) = \mathbf{m}\mathrm{a} \cup \mathbf{m}\mathrm{b}$, and

$\mathbf{l}\mathrm{a} = \neg\mathbf{m}\neg\mathrm{a}$ (op. cit.).

A valuation v on U is a function from propositional formulas to elements of

[4]See Wittgenstein (2001: IV, 4-6, 11, 30-31), for a prescient expressivist approach to the modal profile of mathematical formulas.

[5]Boolean algebras with operators were introduced by Jonsson and Tarski (1951, 1952).

the algebra, which satisfies the following conditions:

$v(\neg A) = \neg v(A)$,

$v(A \wedge B) = v(A) \cap v(B)$,

$v(A \vee B) = v(A) \cup v(B)$,

$v(\blacksquare A) = \mathbf{l}v(A)$, and

$v(\diamond A) = \mathbf{m}v(A)$ (op. cit.).

A frame F = ⟨W,R⟩ consists of a set W and a binary relation R on W (op. cit.). R[w] denotes the set {v∈W | (w,v)∈R}. A valuation V on F is a function such that V(A,x) ∈ {1,0} for each propositional formula A and x∈W, satisfying the following conditions:

V(¬A,x) = 1 iff V(A,x) = 0,

V(A ∧ B,x) = 1 iff V(A,x) = 1 and V(B,x) = 1,

V(A ∨ B,x) = 1 iff V(A,x) = 1 or V(B,x) = 1 (op. cit.)

Chalmers defines epistemic possibility as not being apriori ruled out (2011: 63, 66),[6] i.e. as the dual of epistemic necessity or apriority (65),[7] $\diamond\phi$ iff $\neg\blacksquare\neg\phi$, and as being true at an epistemic scenario i.e. epistemically possible world (62, 64)[8]. I concur that epistemic possibility is the dual of epistemic necessity i.e. apriority, but argue in this paper for an epistemic two-dimensional truthmaker semantics which avails of hyperintensional epistemic states, i.e. epistemic truthmakers or verifiers for a proposition, which comprise a state space (Fine 2017a,b; Hawke and Özgün, forthcoming). Epistemic states are parts of epis-

[6] 'One might also adopt a conception on which every proposition that is not logically contradictory is deeply epistemically possible, or on which ever proposition that is not ruled out a priori is deeply epistemically possible. In this paper, I will mainly work with the latter understanding' (63). 'For example, a sentence s is deeply epistemically possible when the thought that s expresses cannot be ruled out a priori' (66).

[7] 'We can say that s is deeply epistemically necessary when s is a priori: that is when s expresses actual or potential a priori knowledge' (65).

[8] 'For all sentences s, s is epistemically possible iff there exists a scenario [i.e. epistemically possible world - HK] such that w verifies s' (64), where '[w]hen w verifies s, we can say that s is true at w' (63)

temically possible worlds, rather than whole worlds themselves. Apriority is thus redefined in the hyperintensional semantics.

According to truthmaker semantics for epistemic logic, a modalized state space model is a tuple $\langle S, P, \leq, v \rangle$, where S is a non-empty set of states, i.e. parts of the elements in A in the foregoing epistemic modal algebra U, P is the subspace of possible states where states s and t comprise a fusion when $s \sqcap t \in P$, $\leq$ is a partial order, and v: Prop $\rightarrow$ (2^S x 2^S) assigns a bilateral proposition $\langle p^+, p^- \rangle$ to each atom p$\in$Prop with p^+ and p^- incompatible (Hawke and Özgün, forthcoming: 10-11). Exact verification ($\vdash$) and exact falsification ($\dashv$) are recursively defined as follows (Fine, 2017a: 19; Hawke and Özgün, forthcoming: 11):

$s \vdash p$ if $s \in [\![p]\!]^+$

(s verifies p, if s is a truthmaker for p i.e. if s is in p's extension);

$s \dashv p$ if $s \in [\![p]\!]^-$

(s falsifies p, if s is a falsifier for p i.e. if s is in p's anti-extension);

$s \vdash \neg p$ if $s \dashv p$

(s verifies not p, if s falsifies p);

$s \dashv \neg p$ if $s \vdash p$

(s falsifies not p, if s verifies p);

$s \vdash p \wedge q$ if $\exists t,u$, $t \vdash p$, $u \vdash q$, and $s = t \sqcap u$

(s verifies p and q, if s is the fusion of states, t and u, t verifies p, and u verifies q);

$s \dashv p \wedge q$ if $s \dashv p$ or $s \dashv q$

(s falsifies p and q, if s falsifies p or s falsifies q);

$s \vdash p \vee q$ if $s \vdash p$ or $s \vdash q$

(s verifies p or q, if s verifies p or s verifies q);

$s \dashv p \vee q$ if $\exists t,u$, $t \dashv p$, $u \dashv q$, and $s = t \sqcap u$

(s falsifies p or q, if s is the fusion of the states t and u, t falsifies p, and u falsifies q);

$s \vdash \forall x\phi(x)$ if $\exists s_1, \ldots, s_n$, with $s_1 \vdash \phi(a_1), \ldots, s_n \vdash \phi(a_n)$, and $s = s_1 \sqcap \ldots \sqcap s_n$

[s verifies $\forall x\phi(x)$ "if it is the fusion of verifiers of its instances $\phi(a_1), \ldots, \phi(a_n)$" (Fine, 2017c)];

$s \dashv \forall x\phi(x)$ if $s \dashv \phi(a)$ for some individual a in a domain of individuals (op. cit.)

[s falsifies $\forall x\phi(x)$ "if it falsifies one of its instances" (op. cit.)];

$s \vdash \exists x\phi(x)$ if $s \vdash \phi(a)$ for some individual a in a domain of individuals (op. cit.)

[s verifies $\exists x\phi(x)$ "if it verifies one of its instances $\phi(a_1), \ldots, \phi(a_n$" (op. cit.)];

$s \dashv \exists x\phi(x)$ if $\exists s_1, \ldots, s_n$, with $s_1 \dashv \phi(a_1), \ldots, s_n \dashv \phi(a_n)$, and $s = s_1 \sqcap \ldots \sqcap s_n$ (op. cit.)

[s falsifies $\exists x\phi(x)$ "if it is the fusion of falsifiers of its instances" (op. cit.)];

s exactly verifies p if and only if $s \vdash p$ if $s \in [\![p]\!]$;

s inexactly verifies p if and only if $s \triangleright p$ if $\exists s' \leq S$, $s' \vdash p$; and

s loosely verifies p if and only if, $\forall t$, s.t. $s \sqcup t$, $s \sqcup t \vdash p$, where $\sqcup$ is the relation of compatibility (35-36);

$s \vdash A\phi$ if and only if for all $t \in P$ there is a $t' \in P$ such that $t' \sqcap t \in P$ and $t' \vdash \phi$[9];

[9] In epistemic two-dimensional semantics, epistemic possibility is defined as the dual of apriority or epistemic necessity, i.e. as not being ruled-out apriori ($\neg\blacksquare\neg$), and follows Chalmers (2011: 66). Apriority receives, however, different operators depending on whether it is defined in truthmaker semantics or possible worlds semantics. Both operators are admissible, and the definition in terms of truthmakers is here taken to be more fundamental. The definition of apriority here differs from that of DeRose (1991: 593-594) – who defines the epistemic possibility of P as being true iff "(1) no member of the relevant community knows that P is false and (2) there is no relevant way by which members of the relevant community can come to know that P is false" – by defining epistemic possibility in terms of apriority rather than

s $\dashv$ Aϕ if and only if there is a t∈P such that for all u∈P either t $\sqcap$ u∉P or u $\dashv$ ϕ, where Aϕ denotes the apriority of ϕ.[10]

In order to account for two-dimensional indexing, we augment the model, M, with a second state space, S*, on which we define both a new parthood relation, $\leq$*, and partial function, V*, which serves to map propositions in a domain, D, to pairs of subsets of S*, {1,0}, i.e. the verifier and falsifier of p, such that $[\![p]\!]^+ = 1$ and $[\![p]\!]^- = 0$. Thus, M = ⟨S, S*, D, $\leq$, $\leq$*, V, V*⟩. The two-dimensional hyperintensional profile of propositions may then be recorded by defining the value of p relative to two parameters, c,i: c ranges over subsets of S, and i ranges over subsets of S*.

(*) M,s∈S,s*∈S* $\vdash$ p iff:

(i) $\exists c_s [\![p]\!]^{c,c} = 1$ if $s \in [\![p]\!]^+$; and

(ii) $\exists i_{s*} [\![p]\!]^{c,i} = 1$ if $s^* \in [\![p]\!]^+$

(Distinct states, s,s*, from distinct state spaces, S,S*, provide a multi-dimensional verification for a proposition, p, if the value of p is provided a truthmaker by s. The value of p as verified by s determines the value of p as verified by s*).

We say that p is hyper-rigid iff:

knowledge. It differs from that of Huemer (2007: 129) – who defines the epistemic possibility of P as it not being the case that P is epistemically impossible, where P is epistemically impossible iff P is false, the subject has justification for ¬P "adequate for dismissing P", and the justification is "Gettier-proof" – by not availing of impossibilities, and rather availing of the duality between apriority as epistemic necessity and epistemic possibility.

[10] A more natural clause for apriority in truthmaker semantics might perhaps be thought to be 's $\vdash$ A(ϕ) iff there is a t∈P such that for all t'∈P t'∈P and t' $\vdash$ ϕ', because the latter echoes the clause for the necessity operator according to which necessity is truth at all accessible worlds, 'M,w $\Vdash$ $\Box$(ϕ) iff ∀w'[If R(w,w'), then M,w' $\Vdash$ ϕ]'. However, appealing to a single state that comprises a fusion with all possible states and is a necessary verifier is arguably preferable to the claim that necessity be recorded by there being all states comprising a fusion with a first state serving to verify a proposition p, because the latter claim is silent about whether the corresponding verifier of p in the fusion of all of those states is necessary. Thanks here to Peter Hawke. Note as well that the clauses for apriority here tie the notion to states of information, by contrast to the proposal in Edgington (2004: 6) according to which "a priori knowledge is independent of the state of information".

(**) M,s∈S,s*∈S* ⊢ p iff:

(i) $\forall c'_s [\![p]\!]^{c,c'} = 1$ if s∈$[\![p]\!]^+$; and

(ii) $\forall i_{s*} [\![p]\!]^{c,i} = 1$ if s*∈$[\![p]\!]^+$

Epistemic (primary), subjunctive (secondary), and 2D hyperintensions can be defined as follows, where hyperintensions are functions from states to extensions, and intensions are functions from worlds to extensions. Epistemic two-dimensional truthmaker semantics receives substantial motivation by its capacity (i) to model conceivability arguments involving hyperintensional metaphysics, and (ii) to avoid the problem of mathematical omniscience entrained by intensionalism about propositions[11]:

- Epistemic Hyperintension:

  $\texttt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$, with s a state in the state space defined over the foregoing epistemic modal algebra, U

- Subjunctive Hyperintension:

  $\texttt{sec}_{v_@}(x) = \lambda w.[\![x]\!]^{v_@,w}$, with w a state in metaphysical state space W

In epistemic two-dimensional semantics, the value of a formula or term relative to a first parameter ranging over epistemic scenarios determines the value of the formula or term relative to a second parameter ranging over metaphysically possible worlds. The dependence is recorded by 2D-intensions. Chalmers (2006: 102) provides a conditional analysis of 2D-intensions to characterize the dependence: 'Here, in effect, a term's subjunctive intension depends on which epistemic possibility turns out to be actual. / This can be seen as a mapping from scenarios to subjunctive intensions, or equivalently as a mapping from (scenario, world) pairs to extensions. We can say: the two-dimensional intension of

[11]See Khudairi (ms_1) through (ms_n) for further discussion.

a statement S is true at (V, W) if V verifies the claim that W satisfies S. If D_1 and D_2 are canonical descriptions of V and W, we say that the two-dimensional intension is true at (V, W) if D_1 epistemically necessitates that D_2 subjunctively necessitates S. A good heuristic here is to ask "If D_1 is the case, then if D_2 had been the case, would S have been the case?". Formally, we can say that the two-dimensional intension is true at (V, W) iff '$\Box_1(D_1 \rightarrow \Box_2(D_2 \rightarrow S))$' is true, where '$\Box_1$' and '$\Box_2$' express epistemic and subjunctive necessity respectively'.

- 2D-Hyperintension:

  $\mathtt{2D}(x) = \lambda s \lambda w [\![\mathrm{x}]\!]^{s,w} = 1.$

An abstraction principle for epistemic hyperintensions can be defined as follows:

For all types, A,B, and where 'x:X' interpreted as 'x is a token of type X', there is a homotopy[12]:

H := $[(\mathrm{f} \sim \mathrm{g}) :\equiv \prod_{x:A}(\mathrm{f(x)} = \mathrm{g(x)})]$, where
$\prod_{f:A \rightarrow B}[(\mathrm{f} \sim \mathrm{f}) \wedge (\mathrm{f} \sim \mathrm{g} \rightarrow \mathrm{g} \sim \mathrm{f}) \wedge (\mathrm{f} \sim \mathrm{g} \rightarrow \mathrm{g} \sim \mathrm{h} \rightarrow \mathrm{f} \sim \mathrm{h})]$,
such that, via Voevodsky's (2006) Univalence Axiom, for all type families A,B:U, there is a function:
$\mathtt{idtoeqv} : (\mathrm{A} =_U \mathrm{B}) \rightarrow (\mathrm{A} \simeq \mathrm{B})$,
which is itself an equivalence relation:
$(\mathrm{A} =_U \mathrm{B}) \simeq (\mathrm{A} \simeq \mathrm{B})$.

Abstraction principles for epistemic hyperintensions take, then, the form:

- $\exists$f,g$[\mathrm{f(x)} = \mathrm{g(x)}] \simeq [\mathrm{f(x)} \simeq \mathrm{g(x)}]$.

[12] A homotopy is a continuous mapping or path between a pair of functions.

2.2 Modal Coalgebraic Automata

Modal coalgebraic automata can be thus characterized. Let a category C be comprised of a class Ob(C) of objects and a family of arrows for each pair of objects C(A,B) (Venema, 2007: 421). A functor from a category C to a category D, **E**: C $\rightarrow$ D, is an operation mapping objects and arrows of C to objects and arrows of D (422). An endofunctor on C is a functor, **E**: C $\rightarrow$ C (op. cit.).

A **E**-coalgebra is a pair $\mathbb{A}$ = (A, μ), with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\rightarrow$ **E**(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390).

As, further, a coalgebraic model of modal logic, $\mathbb{A}$ can be defined as follows (407):

For a set of formulas, Φ, let $\nabla\Phi := \Box \bigvee \Phi \wedge \bigwedge \diamond\Phi$, where $\diamond\Phi$ denotes the set $\{\diamond\phi \mid \phi{\in}\Phi\}$ (op. cit.). Then,

$\diamond\phi \equiv \nabla\{\phi, \mathrm{T}\}$,

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.).

$[\![\nabla\Phi]\!] = \{\mathrm{w}{\in}\mathrm{W} \mid \mathrm{R[w]} \subseteq \bigcup \{[\![\phi]\!] \mid \phi{\in}\Phi\}$ and $\forall\phi{\in}\Phi$, $[\![\phi]\!] \cap \mathrm{R[w]} \neq \emptyset\}$ (Fontaine, 2010: 17).

Let an **E**-coalgebraic modal model, $\mathbb{A} = \langle \mathrm{S},\lambda,\mathrm{R}[.]\rangle$, such that $\mathbb{S}$,s $\Vdash \nabla\Phi$ if and only if, for all (some) successors σ of s$\in$S, $[\Phi,\sigma(\mathrm{s}){\in}\mathbf{E}(\Vdash_{\mathbb{A}})]$ (Venema, 2007: 407), with $\mathbf{E}(\Vdash_{\mathbb{A}})$ a relation lifting of the satisfaction relation $\Vdash_{\mathbb{A}} \subseteq$ S x Φ. Let a functor, **K**, be such that there is a relation $\overline{\mathbf{K}} \subseteq \mathbf{K}$(A) x $\mathbf{K}$(A') (Venema, 2012: 17)). Let Z be a binary relation s.t. Z $\subseteq$ A x A' and $\wp\overline{Z} \subseteq \wp$(A) x $\wp$(A'), with

$\wp\overline{Z}$:= {(X,X') | $\forall$x$\in$X$\exists$x'$\in$X' with (x,x')$\in$Z $\wedge$ $\forall$x'$\in$X'$\exists$x$\in$X with (x,x')$\in$Z} (op. cit.). Then, we can define the relation lifting, $\overline{\mathbf{K}}$, as follows:

$\overline{\mathbf{K}}$:= {[(π,X), (π',X')] | π = π' and (X,X')$\in\wp\overline{Z}$} (op. cit.).

The relation lifting, $\overline{\mathbf{K}}$, associated with the functor, **K**, satisfies the following properties (Enqvist et al, 2019: 586):

- $\overline{\mathbf{K}}$ extends $\mathbf{K}$. Thus $\overline{\mathbf{K}}f = \mathbf{K}f$ for all functions $f\colon X_1 \to X_2$;

- $\overline{\mathbf{K}}$ preserves the diagonal. Thus $\overline{\mathbf{K}}\mathrm{Id}_X = \mathrm{Id}_{KX}$ for any set X and functor, Id, where Id_C maps a set S to the product S x C (583, 586);

- $\overline{\mathbf{K}}$ is monotone. R $\subseteq$ Q implies $\overline{\mathbf{K}}$R $\subseteq$ $\overline{\mathbf{K}}$Q for all relations R,Q $\subseteq$ X_1 x X_2;

- $\overline{\mathbf{K}}$ commutes with taking converse. $\overline{\mathbf{K}}R^\circ = (\overline{\mathbf{K}}R)^\circ$ for all relations R $\subseteq$ X_1 x X_2;

- $\overline{\mathbf{K}}$ distributes over relation composition. $\overline{\mathbf{K}}$(R ; Q) = $\overline{\mathbf{K}}$R ; $\overline{\mathbf{K}}$Q, for all relations R $\subseteq$ X_1 x X_2 and Q $\subseteq$ X_2 x X_3, provided that the functor $\mathbf{K}$ preserves weak pullbacks (op. cit.). Venema and Vosmaer (2014: §4.2.2) define a weak pullback as follows: 'A weak pullback of two morphisms f : X $\to$ Z and g : Y $\to$ Z with a shared codomain Z is a pair of morphisms p_X : P $\to$ X and p_Y : P $\to$ Y with a shared domain P, such that (1) $f \circ p_X = g \circ p_Y$, and (2) for any other pair of morphisms q_X : Q $\to$ X and q_Y : Q $\to$ Y with $f \circ q_X = g \circ q_Y$, there is a morphism q : Q $\to$ P such that $p_X \circ q = q_X$ and $p_Y \circ q = q_Y$. This pullback is "weak" because we are not requiring q to be unique. Saying that [a set functor] T : **Set** $\to$ **Set** preserves weak pullbacks means that if p_X : P $\to$ X and p_Y : P $\to$ Y form a weak pullback of f : X $\to$ Z and g : Y $\to$ Z, then $\mathrm{T}p_X$: TP $\to$ TX and $\mathrm{T}p_Y$: TP $\to$ TY form a weak pullback of Tf : TX $\to$ TZ and Tg : TY $\to$ TZ'.

A coalgebraic model of deterministic automata can finally be thus defined (Venema, 2007: 391). An automaton is a tuple, $\mathbb{A} = \langle$A, a_I, C, δ, F$\rangle$, such that

A is the state space of the automaton $\mathbb{A}$; $a_I \in A$ is the automaton's initial state; C is the coding for the automaton's alphabet, mapping numerals to properties of the natural numbers; δ: A X C $\rightarrow$ A is a transition function, and F$\subseteq$ A is the collection of admissible states, where F maps A to {1,0}, such that F: A $\rightarrow$ 1 if a$\in$F and A $\rightarrow$ 0 if a$\notin$F (op. cit.).

Modal automata are defined over a modal one-step language (Venema, 2020: 7.2). With A being a set of propositional variables the set, $\mathtt{Latt}$(X), of lattice terms over X has the following grammar:

$$\pi ::= \perp \mid \top \mid x \mid \pi \wedge \pi \mid \pi \vee \pi,$$

with x$\in$X and $\pi \in \mathtt{Latt}$(A) (op. cit.).

The set, $\mathtt{1ML}$(A), of modal one-step formulas over A has the following grammar:

$$\alpha \in A ::= \perp \mid \top \mid \diamond\pi \mid \Box\pi \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \text{ (op. cit.).}$$

A modal P-automaton $\mathbb{A}$ is a triple, (A, Θ, a_I), with A a non-empty finite set of states, $a_I \in A$ an initial state, and the transition map

Θ: A x $\wp$P $\rightarrow$ $\mathtt{1ML}$(A)

maps states to modal one-step formulas (op. cit.: 7.3).

The crux of the reconciliation between algebraic models of cognitivism and the formal foundations of modal expressivism is based on the duality between categories of algebras and coalgebras: $\mathbb{A} = \langle A, \alpha{:}A \rightarrow \mathbf{E}(A)\rangle$ is dual to the category of algebras over the functor α (417-418). For a category C, object A, and endofunctor $\mathbf{E}$, define a new arrow, α, s.t. $\alpha{:}\mathbf{E}A \rightarrow A$. A homomorphism, f, can further be defined between algebras $\langle A, \alpha\rangle$, and $\langle B, \beta\rangle$. Then, for the category of algebras, the following commutative square can be defined: (i) $\mathbf{E}$A $\rightarrow$ $\mathbf{E}$B ($\mathbf{E}f$); (ii) $\mathbf{E}$A $\rightarrow$ A (α); (iii) $\mathbf{E}$B $\rightarrow$ B (β); and (iv) A $\rightarrow$ B (f) (cf.

Hughes, 2001: 7-8). The same commutative square holds for the category of coalgebras, such that the latter are defined by inverting the direction of the morphisms in both (ii) [A $\rightarrow$ **E**A (α)], and (iii) [B $\rightarrow$ **E**B (β)] (op. cit.).

The significance of the foregoing is twofold. First and foremost, the above demonstrates how a formal correspondence can be effected between algebraic models of cognition and coalgebraic models which provide a natural setting for modal logics and automata. The second aspect of the philosophical significance of modal coalgebraic automata is that – as a model of modal expressivism – the proposal is able to countenance fundamental properties in the foundations of mathematics, and circumvent the issues accruing to the attempt so to do by the competing expressivist approaches.

3 Material Adequacy

The material adequacy of epistemic modal algebras as a fragment of the representational theory of mind is witnessed by the prevalence of possible worlds and hyperintensional semantics – the model theory for which is algebraic (cf. Blackburn et al., 2001: ch. 5) – in cognitive psychology and artificial intelligence.

Contemporary vision science endeavors to account for the issue of underdetermination, with regard to the transition from the receipt of retinal lightwave spectra to the perceptual representations of physical particulars. In order to account for the transition, the visual system is taken to be comprised of implicit computations that are governed by the Bayesian probability calculus, and the probability measure is interpreted as a function of likelihood (cf. Mamassian et al, 2002; Burge, 2010; Rescorla, 2013). The visual system is presented with a distribution of possibilities, concerning e.g. whether light is emanating from above or emanating from below. The set of possibilities is pointed, as the visual system calculates the likelihood that one of the possibilities is actual. The possibility assigned the highest likelihood of being actual is referred to as a perceptual constancy. The designated possibility places, then, a condition on the accuracy of the attribution of properties, such as boundedness and volume, to distal, physical objects.

In artificial intelligence, the subfield of knowledge representation draws on epistemic logic, where belief and knowledge are interpreted as necessity operators (Meyer and van der Hoeck, 1995; Fagin et al., 1995). Possibility and necessity may receive other interpretations in mental terms, such as that of conceivability and apriority (i.e. truth in all epistemic possibilities, or inconceivability that not ϕ). The language of thought hypothesis maintains that thinking occurs in a mental language with a computational syntax and a semantics. The philosophical significance of cognitivism about epistemic modality is that it construes epistemic intensions and hyperintensions as abstract, computational functions in the mind, and thus provides an explanation of the relation that human beings bear to epistemic possibilities. Intensions and hyperintensions are semantically imbued abstract functions comprising the computational syntax of the language of thought. The functions are semantically imbued because they are defined relative to a parameter ranging over either epistemically possible worlds or epistemic states in a state space, and extensions or semantic values are defined for the functions relative to that parameter. Cognitivism about epistemic modality argues that thoughts are composed of epistemic intensions or hyperintensions. Cognitivism about epistemic modality provides a metaphysical explanation or account of the ground of thoughts, arguing that they are grounded in epistemic possibilities and either intensions or hyperintensions which are themselves internal representations comprising the syntax and semantics for a mental language. This is consistent with belief and knowledge being countenanced in an epistemic logic for artificial intelligence, as well. Epistemic possibilities are constitutively related to thoughts, and figure furthermore in the analysis of notions such as apriority and conceivability, as well as belief and knowledge in epistemic logic for artificial intelligence.[13]

4 Precedent

The proposal that possible worlds semantics comprises the model for thoughts and propositions is anticipated by Wittgenstein (1921/1974); Chalmers (2011); and Jackson (2011). Their approaches depart, however, from the one here examined in the following respects.

Wittgenstein writes: 'Logical pictures can depict the world. / A picture has a logico-pictorial form in common with what it depicts. / A picture depicts reality by representing a possibility of existence and non-existence of states of affairs. / A picture represents a possible situation in logical space. / A picture contains the possibility of the situation that it represents ... A logical picture of facts is a thought. / 'A state of affairs is thinkable': what this means is that we can picture it to ourselves. / The totality of true thoughts is a picture of the world. / A thought contains the possibility of the situation of which it is the thought. What is thinkable is possible too' (op. cit.: 2.19-2.203, 3-3.02).

Wittgenstein (op. cit.: 1-1.1) has been interpreted as endorsing an identity theory of propositions, which does not distinguish between internal thoughts and external propositions (cf. McDowell, 1994: 27; and Hornsby, 1997: 1-3).

[13] I claim only that epistemic intensions and hyperintensions – i.e. functions from epistemically possible worlds or epistemic states to extensions – are computable functions comprising a fragment of the language of thought, leaving it open whether the mind is more generally a Turing machine. I thus hope to avoid taking a position here on whether human cognition is generally computational in light of Gödel's (1931) incompleteness theorems. For further discussion, see Gödel (1951), the essays in Horsten and Welch (2016), and Koellner (2018a,b).

How the identity theory of propositions is able to accommodate Wittgenstein's suggestion that a typed hierarchy of propositions can be generated – only if the class of propositions has a general form and the sense of propositions over which operations range is invariant by being individuated by the possibilities figuring as their truth and falsity conditions (cf. Wittgenstein, 1979: 21/11/16, 23/11/16, 7/11/17; and Potter, 2009: 283-285 for detailed discussion) – is an open question. Wittgenstein (1921/1974: 5.5561) writes that 'Hierarchies are and must be independent of reality', although provides no account of how the independence can be effected.

Jackson (2008: 48-50) distinguishes between personal and subpersonal theories by the role of neural science in individuating representational states (cf. Shea, 2013, for further discussion), and argues in favor of a 'personal-level implicit theory' for the possible worlds semantics of mental representations.

Chalmers' approach comes closest to the one here proffered, because he argues for a hybrid cognitivist-expressivist approach as well, according to which epistemic intensions – i.e. functions from epistemically possible worlds to extensions – are individuated by their inferential roles (2012a: 462-463; ms). Chalmers endorses what he refers to as "anchored inferentialism", and in particular "acquaintance inferentialism" for intensions, according to which "there is a limited set of primitive concepts, and all other concepts are grounded in their inferential role with respect to these concepts", where "the primitive concepts are acquaintance concepts" (463, 466) and "[a]cquaintance concepts may include phenomenal concepts and observational concepts: primitive concepts of phenomenal properties, spatiotemporal properties, and secondary qualities" (2010b: 11). According to Chalmers, "anchored inferential role determines a primary intension. The relevant role can be seen as an internal (narrow or short-armed) role, so that the content is a narrow content" (5). The inferences in question are taken to be "suppositional" inferences, from a base class of truths, PQTI – i.e. truths about physics, consciousness, and indexicality, and a that's all truth – determining canonical specifications of epistemically possible worlds, to other truths (3). With regard to how suppositional inference, i.e. "scrutability", plays a role in the definitions of intensions, Chalmers writes that "[t]he primary intension of [a sentence] S is true at a scenario [i.e. epistemically possible world] w iff D epistemically necessitates S, where D is a canonical specification of w", where "D epistemically necessitates S iff a conditional of the form 'D $\rightarrow$ S' is apriori" and the apriori entailment is the relation of scrutability (2006). Chalmers (2012a: 245) is explicit about this: "The intension of a sentence S (in a context) is true at a scenario w iff S is a priori scrutable from D (in that context), where D is a canonical specification of w (that is, one of the epistemically complete sentences in the equivalence class of w) ... A Priori Scrutability entails that this sentence S is a priori scrutable (for me) from a canonical specification D of my actual scenario, where D is something along the lines of $PQTI$". "The secondary intension of S is true at a world w iff D metaphysically necessitates S", where "D metaphysically necessitates S when a subjunctive conditional of the form 'if D had been the case, S would have been the case' is true" (op. cit.). Thus, suppositional inference, i.e. scrutability, determines the intensions of two-dimensional semantics.

In this paper, intensions and hyperintensions are countenanced as semantically imbued functions. Intensions and hyperintensions as functions comprise the computational syntax for the language of thought, but they are semantically imbued because they are functions from epistemic possibilities to extensions.[14]

[14]An anticipation of this proposal is Tichy (1969), who defines intensions as Turing machines.

This is consistent with the inferences of scrutability playing a role in the individuation of intensions and hyperintensions, but whereas Chalmers grounds inferences in dispositions (2010: 10; ms), I claim that the inferences drawn from the canonical specifications of epistemic possibilities to arbitrary truths are apriori computations between mental representations.

I assume a dissociation between the natural language semantics for epistemic modals and an account of mental states as epistemic possibilities or hyperintensional epistemic states. However, my expressivism about epistemic modality might be thought to adduce in favor of expressivism about epistemic modals. Relativists about epistemic modals either relativize content or relativize truth to a context of assessment (Starr, 2012: 3; Egan and Weatherson, 2011: 11-14). According to content relativism, epistemic modals express different propositions in different contexts of assessment (Starr, op. cit.). According to truth relativism, epistemic modals express the same proposition, which is true relative to some assessors and false relative to others, such that truth is a three-place relation between a world, a judge, and a proposition, i.e. a centered world and a proposition (Starr, op. cit.: 3, 5). Thus, X believes that stealing is wrong is an ascription of belief in a centered proposition, i.e. a de se belief (Beddor, 2019). As Egan and Weatherson (2011: 14-15, 17) and Yalcin (2011: 307) point out, utterances with epistemic modals on the truth relativist proposal thus express second-order states (cf. Beddor, op. cit.).

That epistemic modal beliefs are second-order on both the contextualist and the truth relativist proposal adduces against the merits of the view. Yalcin (op. cit.: 308) argues that non-human animals can entertain states expressed by epistemic modals, and we here follow him in thinking that, by taking epistemic modal beliefs to be second-order de se ascriptions, the contextualist and truth relativist proposal would preclude young children and non-human animals from entertaining epistemic possibilities. However, young children and non-human animals, while lacking the capacity to entertain second-order states, nevertheless entertain epistemic possibilities. The foregoing thus adduces in favor of the expressivist proposal that epistemic modals express first-order states of mind.

In the the remainder of the paper, I endeavor to demonstrate the advantages accruing to the present approach to countenancing modal expressivism via modal coalgebraic automata, via a comparison of the theoretical strength of the proposal when applied to characterizing the fundamental properties of the foundations of mathematics, by contrast to the competing approaches to modal expressivism and the limits of their applications thereto.

5 Modal Expressivism and the Philosophy of Mathematics

When modal expressivism is modeled via speech acts on a common ground of presuppositions, the application thereof to the foundations of mathematics is limited by the manner in which necessary propositions are characterized.[15]

Because for example a proposition is taken, according to the proposal, to be identical to a set of possible worlds, all necessarily true mathematical formulas can only express a single proposition; namely, the set of all possible worlds (cf. Stalnaker, 1978; 2003: 51). Thus, although distinct set-forming operations will

[15]See Khudairi (ms) for an application of the epistemic interpretation of two-dimensional semantics to account for the modal profile of large cardinal axioms and Orey sentences; i.e. mathematical propositions that are undecidable relative to the axioms of a given language. (For the origins of two-dimensional intensional semantics, see Kamp, 1967; Vlach, 1973; and Segerberg, 1973.) The distinction between epistemic and metaphysical possibilities, as they pertain to the values of mathematical formulas, is anticipated by Gödel's (1951: 11-12) distinction between mathematics in its subjective and objective senses, where the former targets all "demonstrable mathematical propositions", and the latter includes "all true mathematical propositions".

be codified by distinct axioms of a language of set theory, the axioms will be assumed to express the same proposition: The axiom of Pairing in set theory – which states that a unique set can be formed by combining an element from each of two extant sets: $\forall x,y.\exists z.\forall w.w \in z \iff w = x \vee w = y$ – will be supposed to express the same proposition as the Power Set axiom – which states that a set can be formed by taking the set of all subsets of an extant set: $\forall x.\exists y.\forall z.z \in y \iff z \subseteq x$. However, that distinct operations – i.e., the formation of a set by selecting elements from two extant sets, by contrast to forming a set by collecting all of the subsets of a single extant set – are characterized by the different axioms is readily apparent. As Williamson (2016: 244) writes: "...if one follows Robert Stalnaker in treating a proposition as the set of (metaphysically) possible worlds at which it is true, then all true mathematical formulas literally express the same proposition, the set of all possible worlds, since all true mathematical formulas literally express necessary truths. It is therefore trivial that if one true mathematical proposition is absolutely provable, they all are. Indeed, if you already know one true mathematical proposition (that 2 + 2 = 4, for example), you thereby already know them all. Stalnaker suggests that what mathematicians really learn are in effect new contingent truths about which mathematical formulas we use to express the one necessary truth, but his view faces grave internal problems, and the conception of the content of mathematical knowledge as contingent and metalinguistic is in any case grossly implausible."

Thomasson (2007) argues for a version of modal expressivism which she refers to as 'modal normativism', according to which alethic modalities are to be replaced by deontic modalities taking the form of object-language, modal indicative conditionals (op. cit.: 136, 138, 141). The modal indicative conditionals serve to express constitutive rules pertaining, e.g., to ontological dependencies which state that: 'Necessarily, if an entity satisfying a property exists then a distinct entity satisfying a property exists' (143-144), and generalizes to other expressions, such as analytic conditionals which state, e.g., that: 'Necessarily, if an entity satisfies a property, such as being a bachelor, then the entity satisfies a distinct yet co-extensive property, such as being unmarried' (148).

A virtue of Thomasson's interpretation of modal indicative conditionals as expressing both analytic and ontological dependencies is that it would appear to converge with the 'If-thenist' proposal in the philosophy of mathematics. 'If-thenism' is an approach according to which, if an axiomatized mathematical language is consistent, then (i) one can either bear epistemic attitudes, such as fictive acceptance, toward the target system (cf. Leng, 2010: 180) or (ii) the system (possibly) exists [cf. Russell (op. cit.: §1); Hilbert (1899/1980: 39); Menger (1930/1979: 57); Putnam (1967); Shapiro (2000: 95); Chihara (2004: Ch. 10); and Awodey (2004: 60-61)].[16] However, there are at least two issues for the modal normativist approach in the setting of the philosophy of mathematics.

One general issue for the proposal is that the treatment of quantification remains unaddressed, given that there are translations from modal operators, such as figure in modal indicatives, into existential and universal quantifiers.[17]

[16]See Leng (2009), for further discussion. Field (1980/2016: 11-21; 1989: 54-65, 240-241) argues in favor of the stronger notion of conservativeness, according to which consistent mathematical theories must be satisfiable by internally consistent theories of physics. More generally, for a class of assertions, A, comprising a theory of fundamental physics, and a class of sentences comprising a mathematical language, M, any sentences derivable from A+M ought to be derivable from A alone. Another variation on the 'If-thenist' proposal is witnessed in Field (2001: 333-338), who argues that the existence of consistent forcing extensions of set-theoretic ground models adduces in favor of there being a set-theoretic pluriverse, and thus entrains indeterminacy in the truth-values of undecidable sentences. For a similar proposal, which emphasizes the epistemic role of examining how instances of undecidable sentences obtain and fail so to do relative to forcing extensions in the set-theoretic pluriverse, see Hamkins (2012: §7).

[17]The formal correspondence between modalities and quantifiers is anticipated by Aristotle (*De Interpretatione*, 9; *De Caelo*, I.12), who defines the metaphysical necessity of a proposition as its being true at all times. For detailed discussion of Aristotle's theory, see Waterlow (1982). For a contemporary account of the multi-modal logic for metaphysical and temporal

A second issue for the normative indicative conditional approach is that Thomasson's normative modalities are unimodal. They are thus not sufficiently fine-grained to capture distinctions such as Gödel's (op. cit.) between mathematics in its subjective and objective senses.[18] Further distinctions between the types of mathematical modality can be delineated which permit epistemic types of mathematical possibility to serve as a guide as to whether a formula is metaphysically mathematically possible.[19] The convergence between epistemic and metaphysical mathematical modalities can be countenanced via a two-dimensional intensional semantics. Thus, by eschewing alethic modalities for unimodal, normative indicatives, the normative modalities are unable to account for the relation between the alethic interpretation of modality and, e.g., logical mathematical modalities treated as consistency operators on languages (cf. Field, 1989: 249-250, 257-260; Leng: 2007; 2010: 258), or for the convergence between epistemic possibilities concerning decidability and their bearing on the metaphysical modal status of undecidable sentences.

According, finally, to Brandom's (op. cit.) modal expressivist approach, terms are individuated by their rules of inference, where the rules are taken to have a modal profile translatable into the counterfactual forms taken by the transition functions of automata (cf. Brandom, 2008: 142). In order to countenance the metasemantic truth-conditions for the object-level, pragmatic abilities captured by the automata's counterfactual transition states, Brandom

modalities, see Dorr and Goodman (2019). For contemporary accounts of the correspondence between modal operators and quantifiers see von Wright (1952/1957); Montague (1960/1974: 75); Lewis (1975/1998; 1981/1998); Kratzer (op. cit.; 1981/2012); and Kuhn (1980). For the history of modal logic, see Goldblatt (2006).

[18]See footnote 14 for the relevant definitions.

[19]See Khudairi (ms) for further discussion. A precedent is Reinhardt (1974: 199-200), who proposes the use of imaginary sets, classes, and projections, as 'imaginary experiments' (204), in order to ascertain the consequences of accepting new axioms for ZF which might account for the reduction of the incompleteness of Orey sentences. See Maddy (1988,b), for critical discussion.

augments a first-order language comprised of a stock of atomic formulas with an incompatibility function (141). An incompatibility function, I, is defined as the incoherence of the union of two sentences, where incoherence is a generalization of the notion of inconsistency to nonlogical vocabulary.

$\mathrm{x} \cup \mathrm{y} \in Inc \iff \mathrm{x} \in I(\mathrm{y})$ (141-142).

Incompatibility is supposed to be a modal notion, such that the union of the two sentences is incompossible (126). A sentence, β is an incompatibility-consequence, $\Vdash_I$, of a sentence, α, iff there is no sequence of sentences, $<\gamma_1, \ldots, \gamma_n>$, such that it can be the case that $\alpha \Vdash_I <\gamma_1, \ldots, \gamma_n>$, yet not be the case that $\beta \Vdash_I <\gamma_1, \ldots, \gamma_n>$ (125). To be incompatible with a necessary formula is to be compatible with everything that does not entail the formula (129-130). Dually, to be incompatible with a possible formula is to be incompatible with everything compatible with something compatible with the formula (op. cit.).

There are at least two, general issues for the application of Brandom's modal expressivism to the foundations of mathematics.

The first issue is that the mathematical vocabulary – e.g., the set-membership relation, $\in$ – is axiomatically defined. I.e., the membership relation is defined by, inter alia, the Pairing and Power Set axioms of set-theoretic languages. Thus, mathematical terms have their extensions individuated by the axioms of the language, rather than via a set of inference rules that can be specified in the absence of the mention of truth values. Even, furthermore, if one were to avail of modal notions in order to countenance the intensions of the mathematical vocabulary at issue – i.e., functions from terms in intensional contexts to their extensions – the modal profile of the intensions is orthogonal to the properties encoded by the incompatibility function. Fine (2006) avails, e.g., of a dynamic logic in order to countenance the possibility of reinterpreting the intensions at issue, and of thus accounting for variance in the range of the domains of quantifier expressions. The dynamic possibilities are specified as operational conditions on tracking increases in the size of the cardinality of the universe (Fine, 2005). Uzquiano (2015b) argues that it is always possible to reinterpret the intensions of non-logical vocabulary, as one augments one's language with stronger axioms of infinity and climbs thereby farther up the cumulative hierarchy of sets. The reinterpretations of, e.g., the concept of set are effected by the addition of new large cardinal axioms, which stipulate the existence of larger inaccessible cardinals. However, it is unclear how the incompatibility function – i.e., a modal operator defined via Boolean negation and a generalized condition on inconsistency – might similarly be able to model the intensions pertaining to the ontological expansion of the cumulative hierarchy.

The second issue is that Brandom's inferential expressivist semantics is not compositional (Brandom, 2008: 135-136). While the formulas of the semantics are recursively formed – because the decomposition of complex formulas into atomic formulas is decidable[20] – formulas in the language are not compositional, because they fail to satisfy the subformula property to the effect that the value of a logically complex formula is calculated as a function of the values of the component logical connectives applied to subformulas therein (op. cit.).[21]

By contrast to the limits of Brandom's approach to modal expressivism, modal coalgebraic automata can circumvent both of the issues mentioned in the foregoing. In response to the first issue, concerning the axiomatic individuation

[20] Let a decision problem be a propositional function which is feasibly decidable, if it is a member of the polynomial time complexity class; i.e., if it can be calculated as a polynomial function of the size of the formula's input [see Dean (2015) for further discussion].

[21] Note that Incurvati and Schlöder (2020) advance a multilateral inferential expressivist semantics for epistemic modality which satisfies the subformula property. (Thanks here to Luca Incurvati.) Incurvati and Schlöder (2021) extend the semantics to normative vocabulary, but it is an open question whether their semantics is adequate for mathematical vocabulary as well.

and intensional profiles of mathematical terms, mappings of modal coalgebraic automata can be interpreted in order to provide a precise delineation of the intensions of the target vocabulary [cf. Khudairi (ms)]. In response, finally, to the second of the above issues, the values taken by modal coalgebraic automata are both decidable and computationally feasible, while the duality of coalgebras to Boolean-valued models of modal algebras ensures that the formulas therein retain their compositionality. The decidability of coalgebraic automata can further be witnessed by the role of modal coalgebras in countenancing the modal profile of Ω-logical consequence, where – given a proper class of Woodin cardinals – the values of mathematical formulas can remain invariant throughout extensions of the ground models comprising the set-theoretic universe (cf. Woodin, 2010; Khudairi, ms). The individuation of large cardinals can further be characterized by the mappings of modal coalgebraic automata, when the latter are interpreted so as to countenance the elementary embeddings constitutive of large cardinal axioms in the category of sets.

6 Concluding Remarks

In this essay, I have endeavored to account for a mathematically tractable background against which to model both modal cognitivism and modal expressivism. I availed, to that end, of the duality between epistemic modal algebras and modal coalgebraic automata. Epistemic modal algebras were shown to comprise a materially adequate fragment of the language of thought, given that models thereof figure in both cognitive psychology and artificial intelligence. It was then shown how the approach to modal expressivism here proffered, as regimented by the modal coalgebraic automata to which the epistemic modal algebras are dual, avoids the pitfalls attending to the competing modal expressivist approaches based upon both the inferentialist approach to concept-individuation and the approach to codifying the speech acts in natural language via intensional semantics. The present modal expressivist approach was shown, e.g., to avoid the limits of the foregoing in the philosophy of language, as they concerned the status of necessary propositions; the inapplicability of inferentialist-individuation to mathematical vocabulary; and failures of compositionality. Countenancing modal expressivism via modal coalgebraic automata was shown, then, to be able to account for both the intensions of mathematical terms and possible reinterpretations thereof; for the modal profile of Ω-logical consequence in the category of sets; and for the elementary embeddings constitutive of large cardinal axioms in set-theoretic languages.

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