Modal Ω-Logic: Automata, Neo-Logicism, and Set-theoretic Realism*

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Abstract

This essay examines the philosophical significance of Ω-logic in Zermelo-Fraenkel set theory with choice (ZFC). The dual isomorphism between algebra and coalgebra permits Boolean-valued algebraic models of ZFC to be interpreted as coalgebras. The modal profile of Ω-logical validity can then be countenanced within a coalgebraic logic, and Ω-logical validity can be defined via deterministic automata. I argue that the philosophical significance of the foregoing is two-fold. First, because the epistemic and modal profiles of Ω-logical validity correspond to those of second-order logical consequence, Ω-logical validity is genuinely logical, and thus vindicates a neo-logicist conception of mathematical truth in the set-theoretic multiverse. Second, the foregoing provides a modal-computational account of the interpretation of mathematical vocabulary, adducing in favor of a realist conception of the cumulative hierarchy of sets.

1 Introduction

This essay examines the philosophical significance of the consequence relation defined in the Ω-logic for set-theoretic languages. I argue that, as with second-order logic, the modal profile of validity in Ω-Logic enables the property to be epistemically tractable. Because of the dual isomorphism between algebras and coalgebras, Boolean-valued models of set theory can be interpreted as coalgebras. In Section **2**, I demonstrate how the modal profile of Ω-logical validity can be countenanced within a coalgebraic logic, and how Ω-logical validity can

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further be defined via automata. In Section **3**, I examine how models of epistemic modal algebras to which modal coalgebraic automata are dually isomorphic are availed of in the computational theory of mind. Finally, in Section **4**, the philosophical significance of the characterization of the modal profile of Ω-logical validity for the philosophy of mathematics is examined. I argue (i) that it vindicates a type of neo-logicism with regard to mathematical truth in the set-theoretic multiverse, and (ii) that it provides a modal and computational account of formal grasp of the concept of 'set', adducing in favor of a realist conception of the cumulative hierarchy of sets. Section **5** provides concluding remarks.

2 Definitions

In this section, I define the axioms of Zermelo-Fraenkel set theory with choice. I define the mathematical properties of the large cardinal axioms to which ZFC can be adjoined, and I provide a detailed characterization of the properties of Ω-logic for ZFC. Because Boolean-valued algebraic models of Ω-logic are dually isomorphic to coalgebras, a category of coalgebraic logic is then characterized which models both modal logic and deterministic automata. Modal coalgebraic models of automata are then argued to provide a precise characterization of the modal and computational profiles of Ω-logical validity.

2.1 Axioms[1]

- Empty set:

  $\exists x \forall u (u \notin x)$

[1] For a standard presentation, see Jech (2003). For detailed, historical discussion, see Maddy (1988,a).

- Extensionality:

  $x = y \iff \forall u(u \in x \iff u \in y)$

- Pairing:

  $\exists x \forall u(u \in x \iff u = a \vee u = b)$

- Union:

  $\exists x \forall u[u \in x \iff \exists v(u \in v \wedge v \in a)]$

- Separation:

  $\exists x \forall u[u \in x \iff u \in a \wedge \phi(u)]$

- Power Set:

  $\exists x \forall u(u \in x \iff u \subseteq a)$

- Infinity:

  $\exists x \varnothing \in x \wedge \forall u(u \in x \rightarrow \{u\} \in x)$

- Replacement:

  $\forall u \exists ! v \psi(u,v) \rightarrow \forall x \exists y (\forall u \in x)(\exists v \in y)\psi(u,v)$

- Choice:

  $\forall u[u \in a \rightarrow \exists v(v \in u)] \wedge \forall u,x[u \in a \wedge x \in a \rightarrow \exists v(v \in u \iff v \in x) \vee \neg v(v \in u \wedge v \in x)] \rightarrow \exists x \forall u[u \in a \rightarrow \exists ! v(v \in u \wedge u \in x)]$

2.2 Large Cardinals

Borel sets of reals are subsets of ω^ω or $\mathbb{R}$, closed under countable intersections and unions.[2] For all ordinals, a, such that $0 < a < \omega_1$, and $b < a$, ${\Sigma^0}_a$ denotes

[2] See Koellner (2013), for the presentation, and for further discussion, of the definitions in this and the subsequent paragraph.

the open subsets of ω^ω formed under countable unions of sets in $\Pi^0{}_b$, and $\Pi^0{}_a$ denotes the closed subsets of ω^ω formed under countable intersections of $\Sigma^0{}_b$.

Projective sets of reals are subsets of ω^ω, formed by complementations (ω^ω – u, for u$\subseteq\omega^\omega$) and projections [p(u) = $\{\langle x_1, \ldots, x_n\rangle \in \omega^\omega \mid \exists y \langle x_1, \ldots, x_n, y\rangle \in u\}$]. For all ordinals a, such that $0 < a < \omega$, $\Pi^1{}_0$ denotes closed subsets of ω^ω; $\Pi^1{}_a$ is formed by taking complements of the open subsets of ω^ω, $\Sigma^1{}_a$; and $\Sigma^1{}_{a+1}$ is formed by taking projections of sets in $\Pi^1{}_a$.

The full power set operation defines the cumulative hierarchy of sets, V, such that $V_0 = \varnothing$; $V_{a+1} = P(V_0)$; and $V_\lambda = \bigcup_{a<\lambda} V_a$.

In the inner model program (cf. Woodin, 2001, 2010, 2011; Kanamori, 2012,a,b), the definable power set operation defines the constructible universe, L($\mathbb{R}$), in the universe of sets V, where the sets are transitive such that a$\in$C $\iff$ a$\subseteq$C; $L(\mathbb{R}) = V_{\omega+1}$; $L_{a+1}(\mathbb{R}) = \mathrm{Def}(L_a(\mathbb{R}))$; and $L_\lambda(\mathbb{R}) = \bigcup_{a<\lambda}(L_a(\mathbb{R}))$.

Via inner models, Gödel (1940) proves the consistency of the generalized continuum hypothesis, ${\aleph_a}^{\aleph_a} = \aleph_{a+1}$, as well as the axiom of choice, relative to the axioms of ZFC. However, for a countable transitive set of ordinals, M, in a model of ZF without choice, one can define a generic set, G, such that, for all formulas, ϕ, either ϕ or $\neg\phi$ is forced by a condition, f, in G. Let M[G] = $\bigcup_{a<\kappa} M_a[G]$, such that $M_0[G] = \{G\}$; with $\lambda < \kappa$, $M_\lambda[G] = \bigcup_{a<\lambda} M_a[G]$; and $M_{a+1}[G] = V_a \cap M_a[G]$.[3] G is a Cohen real over M, and comprises a set-forcing extension of M. The relation of set-forcing, $\Vdash$, can then be defined in the ground model, M, such that the forcing condition, f, is a function from a finite subset of ω into {0,1}, and $f \Vdash$ u$\in$G if f(u) = 1 and $f \Vdash$ u$\notin$G if f(u) = 0. The cardinalities of an open dense ground model, M, and a generic extension, G, are identical, only if the countable chain condition (c.c.c.) is satisfied, such that, given a chain

[3]See Kanamori (2012,a: 2.1; 2012,b: 4.1), for further discussion.

– i.e., a linearly ordered subset of a partially ordered (reflexive, antisymmetric, transitive) set – there is a countable, maximal antichain consisting of pairwise incompatible forcing conditions. Via set-forcing extensions, Cohen (1963, 1964) constructs a model of ZF which negates the generalized continuum hypothesis, and thus proves the independence thereof relative to the axioms of ZF.[4]

Gödel (1946/1990: 1-2) proposes that the value of Orey sentences such as the GCH might yet be decidable, if one avails of stronger theories to which new axioms of infinity – i.e., large cardinal axioms – are adjoined.[5] He writes that: 'In set theory, e.g., the successive extensions can be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinatorial and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e. the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory . . . is replaceable by a proof from such an axiom of infinity. It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from present axioms plus some true assertion about the largeness of the universe of sets'.

For cardinals, x,a,C, $C \subseteq a$ is closed unbounded in a, if it is closed [if $x < C$ and $\bigcup(C \cap a) = a$, then $a \in C$] and unbounded ($\bigcup C = a$) (Kanamori, op. cit.: 360). A cardinal, S, is stationary in a, if, for any closed unbounded $C \subseteq a$, $C \cap S \neq \emptyset$ (op. cit.). An ideal is a subset of a set closed under countable unions, whereas

[4] See Kanamori (2008), for further discussion.

[5] See Kanamori (2007), for further discussion. Kanamori (op. cit.: 154) notes that Gödel (1931/1986: fn48a) makes a similar appeal to higher-order languages, in his proofs of the incompleteness theorems. The incompleteness theorems are examined in further detail, in Section **4.2**, below.

filters are subsets closed under countable intersections (361). A cardinal κ is regular if the cofinality of κ – comprised of the unions of sets with cardinality less than κ – is identical to κ. Uncountable regular limit cardinals are weakly inaccessible (op. cit.). A strongly inaccessible cardinal is regular and has a strong limit, such that if $\lambda < \kappa$, then $2^{\lambda} < \kappa$ (op. cit.).

Large cardinal axioms are defined by elementary embeddings.[6] Elementary embeddings can be defined thus. For models A,B, and conditions ϕ, j: A $\rightarrow$ B, $\phi\langle a_1, \ldots, a_n\rangle$ in A if and only if $\phi\langle j(a_1), \ldots, j(a_n)\rangle$ in B (363). A measurable cardinal is defined as the ordinal denoted by the critical point of j, crit(j) (Koellner and Woodin, 2010: 7). Measurable cardinals are inaccessible (Kanamori, op. cit.).

Let κ be a cardinal, and $\eta > \kappa$ an ordinal. κ is then η-strong, if there is a transitive class M and an elementary embedding, j: V $\rightarrow$ M, such that crit(j) = κ, j(κ) $>\eta$, and $V_{\eta} \subseteq M$ (Koellner and Woodin, op. cit.).

κ is strong if and only if, for all η, it is η-strong (op. cit.).

If A is a class, κ is η-A-strong, if there is a j: V $\rightarrow$ M, such that κ is η-strong and $j(A \cap V_{\kappa}) \cap V_{\eta} = A \cap V_{\eta}$ (op. cit.).

κ is a Woodin cardinal, if κ is strongly inaccessible, and for all $A \subseteq V_{\kappa}$, there is a cardinal $\kappa_A < \kappa$, such that κ_A is η-A-strong, for all η such that κ_{η}, $\eta < \kappa$ (Koellner and Woodin, op. cit.: 8).

κ is superstrong, if j: V $\rightarrow$ M, such that crit(j) = κ and $V_{j(\kappa)} \subseteq M$, which entails that there are arbitrarily large Woodin cardinals below κ (op. cit.).

Large cardinal axioms can then be defined as follows.

$\exists x \Phi$ is a large cardinal axiom, because:

(i) Φx is a Σ_2-formula;

[6] The definitions in the remainder of this subsection follow the presentations in Koellner and Woodin (2010) and Woodin (2010, 2011).

(ii) if κ is a cardinal, such that V $\models \Phi(\kappa)$, then κ is strongly inaccessible; and

(iii) for all generic partial orders $\mathbb{P} \in V_\kappa$, $V^{\mathbb{P}} \models \Phi(\kappa)$; I_{NS} is a non-stationary ideal; A^G is the canonical representation of reals in $L(\mathbb{R})$, i.e. the interpretation of A in M[G]; $H(\kappa)$ is comprised of all of the sets whose transitive closure is $< \kappa$ (cf. Rittberg, 2015); and $L(\mathbb{R})^{\mathbb{P}max} \models \langle H(\omega_2), \in, I_{NS}, A^G \rangle \models$ 'ϕ'. $\mathbb{P}$ is a homogeneous partial order in $L(\mathbb{R})$, such that the generic extension of $L(\mathbb{R})^{\mathbb{P}}$ inherits the generic invariance, i.e., the absoluteness, of $L(\mathbb{R})$. Thus, $L(\mathbb{R})^{\mathbb{P}max}$ is (i) effectively complete, i.e. invariant under set-forcing extensions; and (ii) maximal, i.e. satisfies all Π_2-sentences and is thus consistent by set-forcing over ground models (Woodin, ms: 28).

Assume ZFC and that there is a proper class of Woodin cardinals; $A \in \mathbb{P}(\mathbb{R}) \cap L(\mathbb{R})$; ϕ is a Π_2-sentence; and V(G), s.t. $\langle H(\omega_2), \in, I_{NS}, A^G \rangle \models$ 'ϕ': Then, it can be proven that $L(\mathbb{R})^{\mathbb{P}max} \models \langle H(\omega_2), \in, I_{NS}, A^G \rangle \models$ 'ϕ', where 'ϕ' := $\exists A \in \Gamma^\infty \langle H(\omega_1), \in, A \rangle \models \psi$.

The axiom of determinacy (AD) states that every set of reals, $a \subseteq \omega^\omega$ is determined, where κ is determined if it is decidable.

Woodin's (1999) Axiom (*) can be thus countenanced:

$AD^{L(\mathbb{R})}$ and $L[(\mathbb{P}\omega_1)]$ is a $\mathbb{P}$max-generic extension of $L(\mathbb{R})$,

from which it can be derived that $2^{\aleph_0} = \aleph_2$. Thus, $\neg$CH; and so CH is absolutely decidable.

2.3 Ω-Logic

For partial orders, $\mathbb{P}$, let $V^{\mathbb{P}} = V^{\mathbb{B}}$, where $\mathbb{B}$ is the regular open completion of ($\mathbb{P}$).[7] $M_a = (V_a)^M$ and ${M^{\mathbb{B}}}_a = ({V^{\mathbb{B}}}_a)^M = ({V_a}^{M^{\mathbb{B}}})$. *Sent* denotes a set of

[7] The definitions in this section follow the presentation in Bagaria et al. (2006).

sentences in a first-order language of set theory. T$\cup\{\phi\}$ is a set of sentences extending ZFC. *c.t.m* abbreviates the notion of a countable transitive $\in$-model. *c.B.a.* abbreviates the notion of a complete Boolean algebra.

Define a *c.B.a.* in V, such that $V^{\mathbb{B}}$. Let $V^{\mathbb{B}}{}_0 = \emptyset$; $V^{\mathbb{B}}{}_\lambda = \bigcup_{b<\lambda} V^{\mathbb{B}}{}_b$, with λ a limit ordinal; $V^{\mathbb{B}}{}_{a+1} = \{f\colon X \to \mathbb{B} \mid X \subseteq V^{\mathbb{B}}{}_a\}$; and $V^{\mathbb{B}} = \bigcup_{a \in On} V^{\mathbb{B}}{}_a$.

ϕ is true in $V^{\mathbb{B}}$, if its Boolean-value is $1^{\mathbb{B}}$, if and only if

$V^{\mathbb{B}} \models \phi$ iff $[\![\phi]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$.

Thus, for all ordinals, a, and every *c.B.a.* $\mathbb{B}$, $V^{\mathbb{B}}{}_a \equiv (V_a)^{V^{\mathbb{B}}}$ iff for all $x \in V^{\mathbb{B}}$, $\exists y \in V^{\mathbb{B}} [\![x = y]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$ iff $[\![x \in V^{\mathbb{B}}]\!]^{\mathbb{B}} = 1^{\mathbb{B}}$.

Then, $V^{\mathbb{B}}{}_a \models \phi$ iff $V^{\mathbb{B}} \models$ '$V_a \models \phi$'.

Ω-logical validity can then be defined as follows:

For T$\cup\{\phi\}\subseteq$*Sent*,

T $\models_\Omega \phi$, if for all ordinals, a, and *c.B.a.* $\mathbb{B}$, if $V^{\mathbb{B}}{}_a \models$ T, then $V^{\mathbb{B}}{}_a \models \phi$.

Supposing that there exists a proper class of Woodin cardinals and if T$\cup\{\phi\}\subseteq$*Sent*, then for all set-forcing conditions, $\mathbb{P}$:

T $\models_\Omega \phi$ iff $V^T \models$ 'T $\models_\Omega \phi$',

where T $\models_\Omega \phi \equiv \emptyset \models$ 'T $\models_\Omega \phi$'.

The Ω-Conjecture states that V $\models_\Omega \phi$ iff $V^{\mathbb{B}} \models_\Omega \phi$ (Woodin, ms). Thus, Ω-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic multiverse.

The soundness of Ω-Logic is defined by universally Baire sets of reals. For a cardinal, e, let a set A be e-universally Baire, if for all partial orders $\mathbb{P}$ of cardinality e, there exist trees, S and T on ω X λ, such that A = p[T] and if G$\subseteq\mathbb{P}$ is generic, then $p[T]^G = \mathbb{R}^G - p[S]^G$ (Koellner, 2013). A is universally Baire, if it is e-universally Baire for all e (op. cit.).

Ω-Logic is sound, such that V $\vdash_\Omega \phi \rightarrow$ V $\models_\Omega \phi$. However, the completeness of Ω-Logic has yet to be resolved.

Finally, in category theory, a category C is comprised of a class Ob(C) of objects a family of arrows for each pair of objects C(A,B) (Venema, 2007: 421). A functor from a category C to a category D, **E**: C $\rightarrow$ D, is an operation mapping objects and arrows of C to objects and arrows of D (422). An endofunctor on C is a functor, **E**: C $\rightarrow$ C (op. cit.).

A **E**-coalgebra is a pair $\mathbb{A} = (A, \mu)$, with A an object of C referred to as the carrier of $\mathbb{A}$, and μ: A $\rightarrow$ **E**(A) is an arrow in C, referred to as the transition map of $\mathbb{A}$ (390).

$\mathbb{A} = \langle A, \mu\text{: } A \rightarrow \mathbf{E}(A)\rangle$ is dually isomorphic to the category of algebras over the functor μ (417-418). If μ is a functor on categories of sets, then Boolean-algebraic models of Ω-logical validity are isomorphic to coalgebraic models.

The significance of the foregoing is that coalgebraic models may themselves be availed of in order to define modal logic and automata theory. Coalgebras provide therefore a setting in which the Boolean-valued models of set theory, the modal profile of Ω-logical validity, and automata can be interdefined. In what follows, $\mathbb{A}$ will comprise the coalgebraic model – dually isomorphic to the complete Boolean-valued algebras defined in the Ω-Logic of ZFC – in which modal similarity types and automata are definable. As a coalgebraic model of modal logic, $\mathbb{A}$ can be defined as follows (407):

For a set of formulas, Φ, let $\nabla\Phi := \Box \bigvee \Phi \wedge \bigwedge \Diamond\Phi$, where $\Diamond\Phi$ denotes the set $\{\Diamond\phi \mid \phi\in\Phi$ (op. cit.). Then,

$\Diamond\phi \equiv \nabla\{\phi, \mathrm{T}\}$,

$\Box\phi \equiv \nabla\varnothing \vee \nabla\phi$ (op. cit.).

Let an **E**-coalgebraic modal model, $\mathbb{A} = \langle S,\lambda,R[.]\rangle$, such that $\mathbb{S},s \Vdash \nabla\Phi$ if and only if, for all (some) successors σ of s∈S, $[\Phi,\sigma(s)\in\mathbf{E}(\Vdash_{\mathbb{A}})]$ (op. cit.).

A coalgebraic model of deterministic automata can be thus defined (391). An automaton is a tuple, $\mathbb{A} = \langle$A, a_I, C, δ, F$\rangle$, such that A is the state space of the automaton $\mathbb{A}$; $a_I \in$A is the automaton's initial state; C is the coding for the automaton's alphabet, mapping numerals to properties of the natural numbers; δ: A X C $\rightarrow$ A is a transition function, and F$\subseteq$ A is the collection of admissible states, where F maps A to {1,0}, such that F: A $\rightarrow$ 1 if a$\in$F and A $\rightarrow$ 0 if a$\notin$F (op. cit.). The determinacy of coalgebraic automata, the category of which is dually isomorphic to the Set category satisfying Ω-logical consequence, is secured by the existence of Woodin cardinals: Assuming ZFC, that λ is a limit of Woodin cardinals, that there is a generic, set-forcing extension G $\subseteq$ the collapse of $\omega < \lambda$, and that $\mathbb{R}^* = \bigcup\{\mathbb{R}^G[a] \mid$ a $< \lambda\}$, then $\mathbb{R}^* \models$ the axiom of determinacy (AD) (Koellner and Woodin, op. cit.: 10).

Finally, $\mathbb{A} = \langle$A, α:A $\rightarrow$ $\mathbf{E}$(A)$\rangle$ is dually isomorphic to the category of algebras over the functor α (417-418). For a category C, object A, and endofunctor $\mathbf{E}$, define a new arrow, α, s.t. α:$\mathbf{E}$A $\rightarrow$ A. A homomorphism, f, can further be defined between algebras $\langle$A, $\alpha\rangle$, and $\langle$B, $\beta\rangle$. Then, for the category of algebras, the following commutative square can be defined: (i) $\mathbf{E}$A $\rightarrow$ $\mathbf{E}$B ($\mathbf{E}f$); (ii) $\mathbf{E}$A $\rightarrow$ A (α); (iii) $\mathbf{E}$B $\rightarrow$ B (β); and (iv) A $\rightarrow$ B (f) (cf. Hughes, 2001: 7-8). The same commutative square holds for the category of coalgebras, such that the latter are defined by inverting the direction of the morphisms in both (ii) [A $\rightarrow$ $\mathbf{E}$A (α)], and (iii) [B $\rightarrow$ $\mathbf{E}$B (β)] (op. cit.).

Thus, $\mathbb{A}$ is the coalgebraic category for modal, deterministic automata, dually isomorphic to the complete Boolean-valued algebraic models of Ω-logical validity, as defined in the category of sets.

3 Epistemic Modal Algebras and the Computational Theory of Mind

Beyond the remit of Boolean-valued models of set-theoretic languages, models of epistemic modal algebras are availed of by a number of paradigms in contemporary empirical theorizing, including the computational theory of mind and the theory of quantum computability. In Epistemic Modal Algebra, the topological boolean algebra, A, can be formed by taking the powerset of the topological space, X, defined above; i.e., A = P(X). The domain of A is comprised of formula-terms – eliding propositions with names – assigned to elements of P(X), where the proposition-letters are interpreted as encoding states of information. The top element of the algebra is denoted '1' and the bottom element is denoted '0'. We interpret modal operators, f(x), – i.e., intensional functions in the algebra – as both concerning topological interiority, as well as reflecting *epistemic possibilities*. An Epistemic Modal-valued Algebraic structure has the form, F = $\langle A$, $D_{P(X)}$, $\rho\rangle$, where ρ is a mapping from points in the topological space to elements or regions of the algebraic structure; i.e., ρ : $D_{P(X)}$ x $D_{P(X)}$ $\rightarrow A$. A model over the Epistemic-Modal Topological Boolean Algebraic structure has the form M = ⟨F, V⟩, where V(a) $\leq \rho$(a) and V(a,b) $\wedge\ \rho$(a, b) $\leq$ V(b).[8]
For all $x_{x/a,\phi}$,y$\in A$:

f(1) = 1;

f(x) $\leq$ x;

f(x $\wedge$ y) = f(x) $\wedge$ f(y);

f[f(x)] = f(x);

V(a, a) > 0;

[8]See Lando (2015); McKinsey and Tarski (1944); and Rasiowa (1963), for further details.

$V(a, a) = 1;$

$V(a, b) = V(b, a);$

$V(a, b) \wedge V(b, c) \leq V(a, c);$

$V(a = a) = \rho(a, a);$

$V(a, b) \leq f[V(a, b)];$

$V(\neg\phi) = \rho(\neg\phi) - f(\phi);$

$V(\diamond\phi) = \rho\phi - f[- V(\phi)];$

$V(\Box\phi) = f[V(\phi)]$ (cf. Lando, op. cit.).[9]

Marcus (2001) argues that mental representations can be treated as algebraic rules characterizing the computation of operations on variables, where the values of a target domain for the variables are universally quantified over and the function is one-one, mapping a number of inputs to an equivalent number of outputs (35-36). Models of the above algebraic rules can be defined in both classical and weighted, connectionist systems: Both a single and multiple nodes can serve to represent the variables for a target domain (42-45). Temporal synchrony or dynamic variable-bindings are stored in short-term working memory (56-57), while information relevant to long-term variable-bindings are stored in registers (54-56). Examples of the foregoing algebraic rules on variable-binding include both the syntactic concatenation of morphemes and noun phrase reduplication in linguistics (37-39, 70-72), as well as learning algorithms (45-48). Conditions on variable-binding are further examined, including treating the binding relation between variables and values as tensor products – i.e., an application of a multiplicative axiom for variables and their values treated as vectors (53-54, 105-106). In order to account for recursively formed, complex representations, which he refers to as structured propositions, Marcus argues instead that the

[9]Note that, in cases of Boolean-valued epistemic topological algebras, models of corresponding coalgebras will be topological (cf. Takeuchi, 1985 for further discussion).

syntax and semantics of such representations can be modeled via an ordered set of registers, which he refers to as 'treelets' (108).

A strengthened version of the algebraic rules on variable-binding can be accommodated in models of epistemic modal algebras, when the latter are augmented by cylindrifications, i.e., operators on the algebra simulating the treatment of quantification, and diagonal elements.[10] By contrast to Boolean Algebras with Operators, which are propositional, cylindric algebras define first-order logics. Intuitively, valuation assignments for first-order variables are, in cylindric modal logics, treated as possible worlds of the model, while existential and universal quantifiers are replaced by, respectively, possibility and necessity operators ($\diamond$ and $\Box$) (Venema, 2013: 249). For first-order variables, $\{\mathrm{v}_i \mid i < \alpha\}$ with α an arbitrary, fixed ordinal, $\mathrm{v}_i = \mathrm{v}_j$ is replaced by a modal constant $\mathbf{d}_{i,j}$ (op. cit: 250). The following clauses are valid, then, for a model, M, of cylindric modal logic, with $\mathrm{E}_{i,j}$ a monadic predicate and T_i for i,j $< \alpha$ a dyadic predicate:

$\mathrm{M,w} \Vdash \mathrm{p} \iff \mathrm{w} \in \mathrm{V(p)}$;

$\mathrm{M,w} \Vdash \mathbf{d}_{i,j} \iff \mathrm{w} \in \mathrm{E}_{i,j}$;

$\mathrm{M,w} \Vdash \diamond_i \psi \iff$ there is a v with $\mathrm{wT}_i\mathrm{v}$ and $\mathrm{M,v} \vdash \psi$ (252).[11]

Finally, a cylindric modal algebra of dimension α is an algebra, $\mathbb{A} = \langle \mathrm{A}, +, \bullet, -, 0, 1, \diamond_i, \mathbf{d}_{ij} \rangle_{i,j<\alpha}$, where $\diamond_i$ is a unary operator which is normal ($\diamond_i 0 = 0$)

[10]See Henkin et al (op. cit.: 162-163) for the introduction of cylindric algebras, and for the axioms governing the cylindrification operators.

[11]Cylindric frames need further to satisfy the following axioms (op. cit.: 254):

1. $\mathrm{p} \to \diamond_i \mathrm{p}$
2. $\mathrm{p} \to \Box_i \diamond_i \mathrm{p}$
3. $\diamond_i \diamond_i \mathrm{p} \to \diamond_i \mathrm{p}$
4. $\diamond_i \diamond_j \mathrm{p} \to \diamond_j \diamond_i \mathrm{p}$
5. $\mathbf{d}_{i,i}$
6. $\diamond_i (\mathbf{d}_{i,j} \wedge \mathrm{p}) \to \Box_i (\mathbf{d}_{i,j} \to \mathrm{p})$

[Translating the diagonal element and cylindric (modal) operator into, respectively, monadic and dyadic predicates and universal quantification: $\forall \mathrm{xyz}[(\mathrm{T}_i\mathrm{xy} \wedge \mathrm{E}_{i,j}\mathrm{y} \wedge \mathrm{T}_i\mathrm{xz} \wedge \mathrm{E}_{i,j}\mathrm{z}) \to \mathrm{y} = \mathrm{z}]$ (op. cit.)]

7. $\mathbf{d}_{i,j} \iff \diamond_k (\mathbf{d}_{i,k} \wedge \mathbf{d}_{k,j})$.

and additive [$\diamond_i(\mathrm{x} + \mathrm{y}) = \diamond_i\mathrm{x} + \diamond_i\mathrm{y})$] (257).

The philosophical interest of cylindric modal algebras to Marcus' cognitive models of algebraic variable-binding is that variable substitution is treated in the modal algebras as a modal relation, while universal quantification is interpreted as necessitation. The interest of translating universal generalization into operations of epistemic necessitation is, finally, that – by identifying epistemic necessity with apriority – both the algebraic rules for variable-binding and the recursive formation of structured propositions can be seen as operations, the implicit knowledge of which is apriori.

In quantum information theory, let a constructor be a computation defined over physical systems. Constructors entrain nomologically possible transformations from admissible input states to output states (cf. Deutsch, 2013). On this approach, information is defined in terms of constructors, i.e., intensional computational properties. The foregoing transformations, as induced by constructors, are referred to as tasks. Because constructors encode the counterfactual to the effect that, were an initial state to be computed over, then the output state would result, modal notions are thus constitutive of the definition of the tasks at issue. There are, further, both topological and algebraic aspects of the foregoing modal approach to quantum computation.[12] The composition of tasks is formed by taking their union, where the union of tasks can be satisfiable while its component tasks might not be. Suppose, e.g., that the information states at issue concern the spin of a particle. A spin-state vector will be the sum of the probabilities that the particle is spinning either upward or downward. Suppose that there are two particles which can be spinning either upward or downward. Both particles can be spinning upward; spinning downward; particle-1 can be

[12]For an examination of the interaction between topos theory and an S4 modal axiomatization of computable functions, see Awodey et al. (2000).

spinning upward while particle-2 spins downward; and vice versa. The state vector, V which records the foregoing possibilities – i.e., the superposition of the states – will be equal to the product of the spin-state of particle-1 and the spin-state of particle-2. If the particles are both spinning upward or both spinning downward, then V will be .5. However – relative to the value of each particle vector, referred to as its eigenvalue – the probability that particle-1 will be spinning upward is .5 and the probability that particle-2 will be spinning downward is .5, such that the probability that both will be spinning upward or downward = .5 x .5 = .25. Considered as the superposition of the two states, V will thus be unequal to the product of their eigenvalues, and is said to be entangled. If the indeterminacy evinced by entangled states is interpreted as inconsistency, then the computational properties at issue might further have to be defined on a distribution of epistemic possibilities which permit of hyperintensional distinctions.[13]

4 Modal Coalgebraic Automata and the Philosophy of Mathematics

This section examines the philosophical significance of the Boolean-valued models of set-theoretic languages and the modal coalgebraic automata to which they are dually isomorphic. I argue that, similarly to second-order logical consequence, (i) the 'mathematical entanglement' of Ω-logical validity does not undermine its status as a relation of pure logic; and (ii) both the modal profile and model-theoretic characterization of Ω-logical consequence provide a guide

[13] The nature of the indeterminacy in question is examined in Saunders and Wallace (2008), Deutsch (2010), Hawthorne (2010), Wilson (2011), Wallace (2012: 287-289), Lewis (2016: 277-278), and Khudairi (ms). For a thorough examination of approaches to the ontology of quantum mechanics, see Arntzenius (2012: ch. 3).

to its epistemic tractability.[14] I argue, then, that there are several considerations adducing in favor of the claim that the interpretation of the concept of set constitutively involves modal notions. The role of the category of modal coalegebraic deterministic automata in (i) characterizing the modal profile of Ω-logical consequence, and (ii) being constitutive of the formal understanding-conditions for the concept of set, provides, then, support for a realist conception of the cumulative hierarchy.

4.1 Neo-Logicism

Frege's (1884/1980; 1893/2013) proposal – that cardinal numbers can be explained by specifying an equivalence relation, expressible in the signature of second-order logic and identity, on lower-order representatives for higher-order entities – is the first attempt to provide a foundation for mathematics on the basis of logical axioms rather than rational or empirical intuition. In Frege (1884/1980. cit.: 68) and Wright (1983: 104-105), the number of the concept, **A**, is argued to be identical to the number of the concept, **B**, if and only if there is a one-to-one correspondence between **A** and **B**, i.e., there is a bijective mapping, R, from **A** to **B**. With Nx: a numerical term-forming operator,

- $\forall \mathbf{A} \forall \mathbf{B} \exists \mathrm{R}[[\mathrm{Nx}{:}\ \mathbf{A} = \mathrm{Nx}{:}\ \mathbf{B} \equiv \exists \mathrm{R}[\forall \mathrm{x}[\mathbf{A}\mathrm{x} \rightarrow \exists \mathrm{y}(\mathbf{B}\mathrm{y} \wedge \mathrm{Rxy} \wedge \forall \mathrm{z}(\mathbf{B}\mathrm{z} \wedge \mathrm{Rxz} \rightarrow \mathrm{y} = \mathrm{z}))] \wedge \forall \mathrm{y}[\mathbf{B}\mathrm{y} \rightarrow \exists \mathrm{x}(\mathbf{A}\mathrm{x} \wedge \mathrm{Rxy} \wedge \forall \mathrm{z}(\mathbf{A}\mathrm{z} \wedge \mathrm{Rzy} \rightarrow \mathrm{x} = \mathrm{z}))]]]$.

Frege's Theorem states that the Dedekind-Peano axioms for the language of arithmetic can be derived from the foregoing abstraction principle, as augmented to the signature of second-order logic and identity.[15] Thus, if second-order logic

[14] The phrase, 'mathematical entanglement', is owing to Koellner (2010: 2).

[15] Cf. Dedekind (1888/1963) and Peano (1889/1967). See Wright (1983: 154-169) for a proof sketch of Frege's theorem; Boolos (1987) for the formal proof thereof; and Parsons (1964) for an incipient conjecture of the theorem's validity.

may be counted as pure logic, despite that domains of second-order models are definable via power set operations, then one aspect of the philosophical significance of the abstractionist program consists in its provision of a foundation for classical mathematics on the basis of pure logic as augmented with non-logical implicit definitions expressed by abstraction principles.

There are at least three reasons for which a logic defined in ZFC might not undermine the status of its consequence relation as being logical. The first reason for which the mathematical entanglement of Ω-logical validity might be innocuous is that, as Shapiro (1991: 5.1.4) notes, many mathematical properties cannot be defined within first-order logic, and instead require the expressive resources of second-order logic. For example, the notion of well-foundedness cannot be expressed in a first-order framework, as evinced by considerations of compactness. Let E be a binary relation. Let m be a well-founded model, if there is no infinite sequence, $a_0, \ldots, a_i$, such that $Ea_0, \ldots, Ea_{i+1}$ are all true. If m is well-founded, then there are no infinite-descending E-chains. Suppose that T is a first-order theory containing m, and that, for all natural numbers, n, there is a T with $n + 1$ elements, $a_0, \ldots, a_n$, such that $\langle a_0, a_1 \rangle, \ldots, \langle a_n, a_{n-1} \rangle$ are in the extension of E. By compactness, there is an infinite sequence such that that $a_0 \ldots a_i$, s.t. $Ea_0, \ldots, Ea_{i+1}$ are all true. So, m is not well-founded.

By contrast, however, well-foundedness can be expressed in a second-order framework:

$\forall X[\exists x Xx \rightarrow \exists x[Xx \wedge \forall y(Xy \rightarrow \neg Eyx)]]$, such that m is well-founded iff every non-empty subset X has an element x, s.t. nothing in X bears E to x.

One aspect of the philosophical significance of well-foundedness is that it provides a distinctively second-order constraint on when the membership relation in a given model is intended. This contrasts with Putnam's (1980) claim, that first-order models *mod* can be intended, if every set s of reals in *mod* is such that an ω-model in *mod* contains s and is constructible, such that – given the Downward Lowenheim-Skolem theorem[16] – if *mod* is non-constructible but has a submodel satisfying 's is constructible', then the model is non-well-founded and yet must be intended. The claim depends on the assumption that general understanding-conditions and conditions on intendedness must be co-extensive, to which I will return in Section **4.2**

A second reason for which Ω-logic's mathematical entanglement might not be pernicious, such that the consequence relation specified in the Ω-logic might be genuinely logical, may again be appreciated by its comparison with second-order logic. Shapiro (1998) defines the model-theoretic characterization of logical consequence as follows:

'(10) Φ is a logical consequence of [a model] Γ if Φ holds in all possibilities under every interpretation of the nonlogical terminology which holds in Γ' (148).

A condition on the foregoing is referred to as the 'isomorphism property', according to which 'if two models M, M' are isomorphic vis-a-vis the nonlogical items in a formula Φ, then M satisfies Φ if and only if M' satisfies Φ' (151).

Shapiro argues, then, that the consequence relation specified using second-order resources is logical, because of its modal and epistemic profiles. The epistemic tractability of second-order validity consists in 'typical soundness theorems, where one shows that a given deductive system is 'truth-preserving' (154). He writes that: '[I]f we know that a model is a good mathematical model of logical consequence (10), then we know that we won't go wrong using a sound deductive system. Also, we can know that an argument is a logical consequence . . . via a set-theoretic proof in the metatheory' (154-155).

[16]For any first-order model M, M has a submodel M' whose domain is at most denumerably infinite, s.t. for all assignments s on, and formulas $\phi(x)$ in, M', $M,s \Vdash \phi(x) \iff M',s \Vdash \phi(x)$.

The modal profile of second-order validity provides a second means of accounting for the property's epistemic tractability. Shapiro argues, e.g., that: 'If the isomorphism property holds, then in evaluating sentences and arguments, the only 'possibility' we need to 'vary' is the size of the universe. If enough sizes are represented in the universe of models, then the modal nature of logical consequence will be registered ... [T]he only 'modality' we keep is 'possible size', which is relegated to the set-theoretic metatheory' (152).

Shapiro's remarks about the considerations adducing in favor of the logicality of non-effective, second-order validity generalize to Ω-logical validity. In the previous section, the modal profile of Ω-logical validity was codified by the dual isomorphism between complete Boolean-valued algebraic models of Ω-logic and the category, $\mathbb{A}$, of coalgebraic modal logics. As with Shapiro's definition of logical consequence, where Φ holds in all possibilities in the universe of models and the possibilities concern the 'possible size' in the set-theoretic metatheory, the Ω-Conjecture states that $V \models_{\Omega} \phi$ iff $V^{\mathbb{B}} \models_{\Omega} \phi$, such that Ω-logical validity is invariant in all set-forcing extensions of ground models in the set-theoretic multiverse.

Finally, the epistemic tractability of Ω-logical validity is secured, both – as on Shapiro's account of second-order logical consequence – by its soundness, but also by its isomorphism to the coalgebraic category of deterministic automata, where the determinacy thereof is again secured by the existence of Woodin cardinals.

4.2 Set-theoretic Realism

In this section, I argue, finally, that the modal profile of Ω-logic can be availed of in order to account for the understanding-conditions of the concept of set, and thus crucially serve as part of the argument for set-theoretic realism.

Putnam (op. cit.: 473-474) argues that defining models of first-order theories is sufficient for both understanding and specifying an intended interpretation of the latter. Wright (1985: 124-125) argues, by contrast, that understanding-conditions for mathematical concepts cannot be exhausted by the axioms for the theories thereof, even on the intended interpretations of the theories. He suggests, e.g., that:

'[I]f there really were uncountable sets, their existence would surely have to flow from the concept of set, as intuitively satisfactorily explained. Here, there is, as it seems to me, no assumption that the content of the ZF-axioms cannot exceed what is invariant under all their classical models. [Benacerraf] writes, e.g., that: 'It is granted that they are to have their 'intended interpretation': 'e' is to mean set-membership. Even so, and conceived as encoding the intuitive concept of set, they fail to entail the existence of uncountable sets. So how can it be true that there are such sets? Benacerraf's reply is that the ZF-axioms are indeed faithful to the relevant informal notions only if, in addition to ensuring that 'E' means set-membership, we interpret them so as to observe the constraint that 'the universal quantifier has to mean all or at least all sets' (p. 103). It follows, of course, that if the concept of set does determine a background against which Cantor's theorem, under its intended interpretation, is sound, there is more to the concept of set that can be explained by communication of the intended sense of 'e' and the stipulation that the ZF-axioms are to hold. And the residue is contained, presumably, in the informal explanations to which, Benacerraf reminds us, Zermelo intended his formalization to answer. At least, this must be so if the 'intuitive concept of set' is capable of being explained at all. Yet it is notable that Benacerraf nowhere ventures to supply the missing informal explanation – the story which will pack enough into the extension of 'all sets' to yield Cantor's theorem, under its intended interpretation, as a highly non-trivial corollary'(op. cit).

In order to provide the foregoing explanation in virtue of which the concept of set can be shown to be associated with a realistic notion of the cumulative hierarchy, I will argue that there are several points in the model theory and epistemology of set-theoretic languages at which the interpretation of the concept of set constitutively involves modal notions. The aim of the section will thus be to provide a modal foundation for mathematical platonism.

One point is in the coding of the signature of the theory, T, in which Gödel's incompleteness theorems are proved (cf. Halbach and Visser, 2014). Relative to,

(i) a choice of coding for an ω-complete, recursively axiomatizable language, L, of T – i.e. a mapping between properties of numbers and properties of terms and formulas in L;

(ii) a predicate, phi; and

(iii) a fixed-point construction:

Let phi express the property of 'being provable', and define (iii) such that, for all consistent theories T of L, there are sentences, p_{phi}, corresponding to each formula, phi(x), in T, s.t. for 'm' := p_{phi},

$\mid\!-_T$ p_{phi} iff phi(m).

One can then construct a sentence, 'm' := ¬phi(m), such that L is incomplete (the first incompleteness theorem).

Moreover, L cannot prove its own consistency:

If:

$\mid\!-_T$ 'm' iff ¬phi(m),

Then:

$\vdash_T$ C $\rightarrow$ m.

Thus, L is consistent only if L is inconsistent (the second incompleteness theorem).

In the foregoing, the choice of coding bridges the numerals in the language with the properties of the target numbers. The choice of coding is therefore intensional, and has been marshalled in order to argue that the very notion of syntactic computability – via the equivalence class of partial recursive functions, λ-definable terms, and the transition functions of discrete-state automata such as Turing machines – is constitutively semantic (cf. Rescorla, 2015). Further points at which intensionality can be witnessed in the phenomenon of self-reference in arithmetic are introduced by Reinhardt (1986). Reinhardt (op. cit.: 470-472) argues that the provability predicate can be defined relative to the minds of particular agents – similarly to Quine's (1968) and Lewis' (1979) suggestion that possible worlds can be centered by defining them relative to parameters ranging over tuples of spacetime coordinates or agents and locations – and that a theoretical identity statement can be established for the concept of the foregoing minds and the concept of a computable system.

In the previous section, intensional computational properties were defined via modal coalgebraic deterministic automata, where the coalgebraic categories are dually isomorphic to the category of sets in which Ω-logical validity was defined. Coalgebraic modal logic was shown to elucidate the modal profile of Ω-logical consequence in the Boolean-valued algebraic models of set theory. The intensionality witnessed by the choice of coding may therefore be further witnessed by the modal automata specified in the foregoing coalgebraic logic.

A second point at which understanding-conditions may be shown to be constitutively modal can be witnessed by the conditions on the epistemic entitlement to assume that the language in which Gödel's second incompleteness theorem is proved is consistent (cf. Dummett, 1963/1978; Wright, 1985). Wright (op. cit.: 91, fn.9) suggests that '[T]o treat [a] proof as establishing consistency is implicitly to exclude any doubt ... about the consistency of first-order number theory'. Wright's elaboration of the notion of epistemic entitlement, appeals to a notion of rational 'trust', which he argues is recorded by the calculation of 'expected epistemic utility' in the setting of decision theory (2004; 2014: 226, 241). Wright notes that the rational trust subserving epistemic entitlement will be pragmatic, and makes the intriguing point that 'pragmatic reasons are not a special genre of reason, to be contrasted with e.g. epistemic, prudential, and moral reasons' (2012: 484). Crucially, however, the very idea of expected epistemic utility in the setting of decision theory makes implicit appeal to the notion of possible worlds, where the latter can again be determined by the coalgebraic logic for modal automata.

A third consideration adducing in favor of the thought that grasp of the concept of set might constitutively possess a modal profile is that the concept can be defined as an intension – i.e., a function from possible worlds to extensions. The modal similarity types in the coalgebraic modal logic may then be interpreted as dynamic-interpretational modalities, where the dynamic-interpretational modal operator has been argued to entrain the possible reinterpretations both of the domains of the theory's quantifiers (cf. Fine, 2005, 2006), as well as of the intensions of non-logical concepts, such as the membership relation (cf. Uzquiano, 2015).[17]

[17]For an examination of the philosophical significance of modal coalgebraic automata beyond the philosophy of mathematics, see Baltag (2003). Baltag (op. cit.) proffers a colagebraic semantics for dynamic-epistemic logic, where coalgebraic functors are intended to record the informational dynamics of single- and multi-agent systems. For an algebraic characterization of dynamic-epistemic logic, see Kurz and Palmigiano (2013). For further discussion, see

The fourth consideration avails directly of the modal profile of Ω-logical consequence. While the above dynamic-interpretational modality will suffice for possible reinterpretations of mathematical terms, the absoluteness and generic invariance of the consequence relation is such that, if the Ω-conjecture is true, then Ω-logical validity is invariant in all possible set-forcing extensions of ground models in the set-theoretic multiverse. The truth of the Ω-conjecture would thereby place an indefeasible necessary condition on a formal understanding of the intension for the concept of set.

5 Concluding Remarks

In this essay I have examined the philosophical significance of the isomorphism between Boolean-valued algebraic models of modal Ω-logic and modal coalgebraic models of automata. I argued that – as with the property of validity in second-order logic – Ω-logical validity is genuinely logical, and thus entails a type of neo-logicism in the foundations of mathematics. I argued, then, that modal coalegebraic deterministic automata, which characterize the modal profile of Ω-logical consequence, are constitutive of the interpretation of mathematical concepts such as the membership relation. The philosophical significance of modal Ω-logic is thus that it can be availed of to vindicate both a neo-logicist foundation for set theory and a realist interpretation of the cumulative hierarchy of sets.

Khudairi (ms). The latter proceeds by examining undecidable sentences via the epistemic interpretation of multi-dimensional intensional semantics. See Reinhardt (1974), for a similar epistemic interpretation of set-theoretic languages, in order to examine the reduction of the incompleteness of undecidable sentences on the counterfactual supposition that the language is augmented by stronger axioms of infinity; and Maddy (1988,b), for critical discussion. Chihara (2004) argues, as well, that conceptual possibilities can be treated as imaginary situations with regard to the construction of open-sentence tokens, where the latter can then be availed of in order to define nominalistically adequate arithmetic properties.

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