Non-Transitive Self-Knowledge: Luminosity via Modal μ-Automata

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Abstract

This essay provides a novel account of iterated epistemic states. The essay argues that states of epistemic determinacy might be secured by countenancing self-knowledge on the model of fixed points in monadic second-order modal logic, i.e. the modal μ-calculus. Despite the epistemic indeterminacy witnessed by the invalidation of modal axiom 4 in the sorites paradox – i.e. the KK principle: $\Box\phi \rightarrow \Box\Box\phi$ – an epistemic interpretatation of the Kripke functors of a μ-automaton permits the iterations of the transition functions to entrain a principled means by which to account for necessary conditions on self-knowledge.

This essay provides a novel account of self-knowledge, which avoids the epistemic indeterminacy witnessed by the invalidation of modal axiom 4 in epistemic logic; i.e. the KK principle: $\Box\phi \rightarrow \Box\Box\phi$. The essay argues, by contrast, that – despite the invalidation of modal axiom 4 on its epistemic interpretation – states of epistemic determinacy might yet be secured by countenancing self-knowledge on the model of fixed points in monadic second-order modal logic, i.e. the modal μ-calculus.

Counterinstances to modal axiom 4 – which records the property of transitivity in labeled transition systems, i.e., the relational semantics for modal logic[1] – have been argued to occur within various interpretations of the sorites paradox. Suppose, e.g., that a subject is presented with a bounded continuum, the incipient point of which bears a red color hue and the terminal point of which bears an orange color hue. Suppose, then, that the cut-off points between the points

[1] Cf. Kripke (1963).

ranging from red to orange are indiscriminable, such that the initial point, a, is determinately red, and matches the next apparent point, b; b matches the next apparent point, c; and thus – by transitivity – a matches c. Similarly, if b matches c, and c matches d, then b matches d. The sorites paradox consists in that iterations of transitivity would entail that the initial and terminal points in the bounded continuum are phenomenally indistinguishable. However, if one takes transitivity to be the culprit in the sorites, then eschewing the principle would entail a rejection of the corresponding modal axiom (4), which records the iterative nature of the relation. Given the epistemic interpretation of the axiom – namely, that knowledge that a point has a color hue entails knowing that one knows that the point has that color hue – a resolution of the paradox which proceeds by invalidating axiom 4 subsequently entrains the result that one can know that one of the points has a color hue, and yet not know that they know that the point has that color hue (Williamson, 1990: 107-108; 1994: 223-244; 2001: chs. 4-5). The non-transitivity of phenomenal indistinguishability can then provide a structural barrier to higher-order knowledge of one's first-order states. The foregoing result holds, furthermore, in the probabilistic setting, such that the evidential probability that a proposition has a particular value may be certain – i.e., be equal to 1 – while the iteration of the evidential probability operator – recording the evidence with regard to that evidence – is yet equal to 0. Thus, one may be certain on the basis of one's evidence that a proposition has a particular value, while the higher-order evidence with regard to one's evidence adduces entirely against that valuation (Williamson, 2014).

The argument eschews 'safety' as a necessary condition on knowledge, for which Williamson's (2001) approach explicitly argues and as codified by margin-for-error principles of the form: $\forall x \forall \phi [K^{m+1} \phi(x) \rightarrow K^{m} \phi(x+1)]$' (Williamson, 2001: 128; Gómez-Torrente, 2002: 114); i.e., that if one knows – relative to a margin which ranges over a world accessible from the actual world, m – that an object satisfies a property, then a distinct similar object satisfies that property in the actual world. Intuitively, the safety condition ensures that if one knows that a predicate is satisfied, then one knows that the predicate is satisfied in relevantly similar worlds. Williamson targets the inconsistency of margin-for-error principles, the luminosity principle ['$\forall$x$\forall\phi$[ϕ(x) $\rightarrow$ Kϕ(x)'], and the characterization of the sorites as occurring when an object satisfies a property, such that similar objects would further do so. The triad evinces, arguably, that the safety condition is not satisfied in the sorites, s.t. knowledge does not obtain, and the luminosity principle is false. In cases, further, in which conditions on knowledge are satisfied, epistemic indeterminacy is supposed to issue from the non-transitivity of the accessibility relation on worlds (1994: 242).

One of the primary virtues of the present proposal is thus that it targets the property of transitivity directly, because transitivity both engenders the sorites paradox on the assumption that the states are known and the property is codified by the epistemic modal axiom for transitivity, i.e., 4 or the KK principle. By so doing, it permits a uniform interpretation of transitivity in the sorites – as codified by the KK principle – such that it applies not only to epistemic accessibility relations whose obtaining is relevant to the safety condition, but further to the logical property and its explanatory role in engendering the paradox.

A second virtue adducing in favor of the foregoing, 'epistemicist' approach to vagueness – which takes the latter to be a phenomenon of epistemic indeterminacy – is that vagueness can be explained without having to revise the underlying logic. The epistemicist approach is consistent with classical logical laws, such as e.g. the law of excluded middle; and thus it can determinately be the case that a point has a color hue; determinately be the case that the next subsequent point has a distinguishable color hue; and one can in principle know where in the continuum the cut-off between the two points lies – yet vagueness will consist in the logical limits – i.e. the non-transitivity – of one's state of knowledge. Thus, one will not in principle be able to know that they know the point at which the color hues are dissimilar.

In this essay, I endeavor to provide a novel account which permits the retention of both classical logic as well as a modal approach to the phenomenon of vagueness, while salvaging the ability of subjects to satisfy necessary conditions on self-knowledge. I will argue that – despite the invalidity of modal axiom 4, given the non-transitivity of the similarity relation – a distinct means of securing an iterated state of knowledge concerning one's first-order knowledge that a particular state obtains is by availing of fixed point, non-deterministic automata in the setting of coalgebraic modal logic. Propositional modal logic is equivalent to the bisimulation-invariant fragment of fixed point monadic second-order logic.[2] The fixed point higher-order modal logic is referred to as the modal μ-calculus, where μ(x) is an operator recording a least fixed point. Despite the non-transitivity of sorites phenomena – such that, on its epistemic interpretation, the subsequent invalidation of modal axiom 4 entails structural, higher-order epistemic indeterminacy – the modal μ-calculus provides a natural setting in which a least fixed point can be defined with regard to the states instantiated by non-deterministic modal automata. In virtue of recording iterations of particular states, the least fixed points witnessed by non-deterministic modal automata provide, then, an escape route from the conclusion that the invalidation of the KK principle provides an exhaustive and insuperable obstruction

[2] Cf. Janin and Walukiewicz (1996).

to self-knowledge. Rather, the least fixed points countenanced in the modal μ-calculus provide another conduit into subjects' knowledge to the effect that they know that a state has a determinate value. Thus, because of the fixed points definable in the modal μ-calculus, the non-transitivity of the similarity relation is yet consistent with necessary conditions on epistemic determinacy and self-knowledge, and the states at issue can be luminous to the subjects who instantiate them.

In the remainder of the essay, we introduce labeled transition systems, the modal μ-calculus, and non-deterministic Kripke (i.e., μ-) automata. We recount then the sorites paradox in the setting of the modal μ-calculus, and demonstrate how the existence of fixed points enables there to be iterative phenomena which ensure that – despite the invalidation of modal axiom 4 – iterations of mental states can be secured, and can thereby be luminous.

A labeled transition system is a tuple comprised of a set of worlds, M; a valuation, V, from M to its powerset, P(M); and a family of accessibility relations, R. So LTS = ⟨M,V,R⟩ (cf. Venema, 2012: 7). A Kripke coalgebra combines V and R into a Kripke functor, σ_R; i.e. the set of binary morphisms from M to P(M) (op. cit.: 7-8). Thus for an s∈M, σ(s) := [σ_V(s), σ_R(s)] (op. cit.). Satisfaction for the system is defined inductively as follows: For a formula ϕ defined at a state, s, in M,

$[\![\phi]\!]^M$ = V(s) [3]

$[\![\neg\phi]\!]^M$ = S – V(s)

$[\![\bot]\!]^M = \emptyset$

$[\![\mathrm{T}]\!]^M$ = M

$[\![\phi \vee \psi]\!]^M = [\![\phi]\!]^M \cup [\![\psi]\!]^M$

[3] Alternatively, M,s $\Vdash \phi$ if s∈V(ϕ) (9).

$[\![\phi \wedge \psi]\!]^M = [\![\phi]\!]^M \cap [\![\psi]\!]^M$

$[\![\diamond_s \phi]\!]^M = \langle \mathrm{R}_s \rangle [\![\phi]\!]^M$

$[\![\Box_s \phi]\!]^M = [\mathrm{R}_s][\![\phi]\!]^M$, with

$\langle \mathrm{R}_s \rangle(\phi) :=$ {s'∈S | R_s[s'] $\cap\, \phi \neq \emptyset$} and

$[\mathrm{R}_s](\phi) :=$ {s'∈S | R_s[s'] $\subseteq \phi$} (9).

In propositional dynamic logic (PDL), $\langle \pi \rangle \phi$ abbreviates that some execution of a non-deterministic computable program entrains the information state contained in ϕ, where computability is here defined in accord with the Church-Turing thesis that a function is effectively computable if and only if it is partial and recursive, as co-extensive with the class of λ-definable terms and the class of finite, discrete-state automata such as Turing machines (cf. Church, 1936; Turing, 1937). $[\pi]\phi$ abbreviates that all executions of a non-deterministic computable program entrains the information state contained in ϕ. Complex operations in propositional dynamic logics may then obtain (cf. Blackburn and van Benthem, 2007: 59-61). A Choice principle states that the union of π_1 and π_2 may be formed, such that the logic executes either π_1 or π_2. A Composition principle states that there is an operation π_1;π_1, such that the logic first executes π_1 and then executes π_2. An Iteration principle defines a program, π*, where π* entrains the execution of π a finite number of times. Finally, a Test principle defines a program, π?, where π? can comprise the following algorithms: '(π?;a) $\cup$ ($\neg\pi$?;b)', which states that if a program π obtains, then a obtains, else b obtains; 'a;($\neg\pi$?;a)*;π?', which states that the logic will repeat the execution, a, a finite number of times until the program π is tested; and '(π?;a)*;$\neg\pi$?', which states that while a program π is being executed a finite number of times, do a (op. cit.: 59-60).

The modal μ-calculus is then defined as follows. Recall again the foregoing

Iteration principle from PDL, $\langle\pi^*\rangle\phi$ (Venema, op. cit.: 25). In our Kripke colagebra, we thus have M,s $\Vdash \langle\pi^*\rangle\phi \iff (\phi \vee \diamond_s\langle\pi^*\rangle\phi)$ (op. cit.). $\langle\pi^*\rangle\phi$ is thus said to be the *fixed point* for the equation, x $\iff \phi \vee \diamond$x, where the value of the formula is a function of the value of x conditional on the constancy in value of ϕ (op. cit., 38). The smallest solution of the formula, x $\iff \phi \vee \diamond$x, is written μ.x$\phi \vee \diamond$x (25). The value of the least fixed point is, finally, defined more specifically thus:

$[\![\mu.\mathrm{x}\phi \vee \diamond\mathrm{x}]\!] = \mathrm{V}(\phi) \cup \langle\mathrm{R}\rangle([\![\mu.\mathrm{x}\phi \vee \diamond\mathrm{x}]\!])$ (38).

A non-deterministic automaton is a tuple $\mathbb{A} = \langle$A, δ, Acc, $\mathrm{a}_I\rangle$, with A a finite set of states, a_I being the initial state of A; δ is a transition function s.t. δ: A $\rightarrow$ P(A); and Acc $\subseteq$ A is an acceptance condition which specifies admissible conditions on δ (60, 66). A Muller acceptance condition is defined as a subset of A, $\alpha \subseteq$ P(A), such that $\mathrm{Acc}_\alpha := \{\mathrm{s}{\in}\mathrm{A}^\omega \mid \mathrm{Inf(s)} \in \alpha)\}$ (intuitively: the admissible states are the infinite states in a deployment of the transition function) (60). A Büchi condition is a subset $\beta \subseteq$ A, such that $\mathrm{Acc}_\beta := \{\mathrm{s}{\in}\mathrm{A}^\omega \mid \mathrm{Inf(s)} \cap \beta \neq \emptyset\}$ (intuitively: an operation of the transition function passes through the state s infinitely often) (op. cit.). Finally, a parity condition is defined via a mapping, Ω: A $\rightarrow \omega$, such that $\mathrm{Acc}_\Omega := \{\mathrm{s}{\in}\mathrm{A}^\omega \mid \max\{\Omega(\mathrm{s}) \mid \mathrm{s} \in \mathrm{Inf(s)}\}$ is an even number$\}$ (intuitively: Ω is the largest natural number occurring infinitely often in the sequence of states figuring as input to δ, and such that the automaton accepts a particular infinite state iff Ω is even (op. cit.).

Let two Kripke models $\mathbb{A} = \langle$A, a$\rangle$ and $\mathbb{S} = \langle$S, s$\rangle$, be bisimilar if and only if there is is a non-empty binary relation, Z $\subseteq \mathbb{A}$ x $\mathbb{S}$, which is satisfied, if:

(i) For all a$\in\mathbb{A}$and s$\in\mathbb{S}$, if aZs, then a and s satisfy the same proposition letters;

(ii) *The forth condition.* If aZs and $\mathrm{R}_\triangle$a,$\mathrm{v}_1 \ldots \mathrm{v}_n$, then there are $\mathrm{v}_1 \ldots \mathrm{v'}_n$ in

$\mathbb{S}$, s.t.

• for all i $(1 \leq i \leq n)$ v$_i$Zv'$_i$, and

• R'$_\triangle$s,v'$_1$ … v'$_n$;

(iii) *The back condition.* If aZs and R'$_\triangle$s,v'$_1$ … v'$_n$, then there are v$_1$ … v$_n$ in $\mathbb{A}$, s.t.

• for all i $(1 \leq i \leq n)$ v$_i$Zv'$_i$ and

• R$_\triangle$a,v$_1$ … v$_n$ (cf. Blackburn et al, 2001: 64-65).

Bisimulations may be redefined as *relation liftings*. We let, e.g., a Kripke functor, **K**, be such that there is a relation **K!** $\subseteq$ **K**(A) x **K**(A') (17). Let Z be a binary relation s.t. Z $\subseteq$ A x A' and P!Z $\subseteq$ P(A) x P(A'), with

P!Z := {(X,X') | ∀x∈X∃x'∈X' with (x,x')∈Z ∧ ∀x'∈X'∃x∈X with (x,x')∈Z}

(op. cit.). Then, we can define the relation lifting, **K!**, as follows:

K! := {[(π,X), (π',X')] | π = π' and (X,X')∈P!Z} (op. cit.).

Finally, given the Kripke functor, **K**, **K** can be defined as the μ-automaton, i.e., the tuple $\mathbb{A} = \langle$A, δ, Ω, a$_I\rangle$, with a$_I$∈A defined again as the initial state in the set of states A; Ω defined once more as the foregoing parity acceptance condition; and δ defined as a mapping such that δ : A → P$_\exists$(**K**A), where the ∃ subscript indicates that (a,s)∈A x S → {(a',s) ∈ **K**(A) x S | a' ∈ δ(a)} = Ω(a), and (a',s) ∈ **K**(A) x S if and only if {Z ⊆ A x S | [a', σ(s)] ∈ **K!**(Z)} = 0 (93).

The philosophical significance of the foregoing can now be witnessed by defining the μ-automata on an alphabet; in particular, a non-transitive set comprising a bounded real-valued, ordered sequence of chromatic properties. Whereas Ω, in the above parity mapping, is identified with the largest even number occurring infinitely often in the alphabet over which the automaton is defined, the Muller acceptance condition would appear to be more suitable for a background language of real-valued, chromatic properties. The Kripke functor whose acceptance conditions are Muller permits us, subsequently, to define fixed points relative to arbitrary points comprising the non-transitive sequence. Thus, although the non-transitivity of the ordered sequence of color hues belies modal axiom 4, such that one can know that a particular point in the sequence has a particular value although not know that one knows that the point satisfies that value, the perceived constancy of the chromatic values, ϕ, in the non-transitive set of colors nevertheless permits every sequential input state in the μ-automaton to define a fixed point. With $\delta^{M}{}_{x}$(x') := $\langle\delta\rangle^{S[x \mapsto x']}$, the transition function can then satisfy the Muller condition relative to each point in the continuum, such that $\delta^{M}{}_{x}$(x') iff V(ϕ $\cup$ $\langle$R$\rangle$(x') iff V(ϕ) $\cup$ $\langle$R$\rangle$($[\![\mu$.xϕ $\vee$ $\diamond$x$]\!]$) (38).

The epistemicist approach to vagueness relies, as noted, on the epistemic interpretation of the modal operator, such that the invalidation of transitivity and modal axiom 4 ($\Box\phi \rightarrow \Box\Box\phi$) can be interpreted as providing a barrier to a necessary condition on self-knowledge. Crucially, μ-automata can receive a similar epistemic interpretation. Thus, interpreting the μ-automaton's Kripke functors epistemically permits the *iterations* of the set of functions – as defined by the fixed points relative to the arbitrary points in the ordered continuum – to provide a principled means – distinct from the satisfaction of the KK principle – by which to account for the pertinent iterations of epistemic states unique to an agent's self-knowledge.

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