

# Fixed Points in the Epistemic Hyperintensional $\mu$ -Calculus and the KK Principle

David Elohim\*

Written: January 24, 2016. Revised: June 4, 2024

## Abstract

This essay provides a novel account of iterated epistemic states. The essay argues that states of epistemic determinacy might be secured by countenancing iterated epistemic states on the model of fixed points in the modal  $\mu$ -calculus. Despite the epistemic indeterminacy witnessed by the invalidation of modal axiom 4 in the sorites paradox – i.e. the KK principle:  $\Box\phi \rightarrow \Box\Box\phi$  – a epistemic hyperintensional  $\mu$ -automaton permits fixed points to entrain a principled means by which to iterate epistemic states and account thereby for necessary conditions on self-knowledge. The epistemic hyperintensional  $\mu$ -calculus is applied to the iteration of the epistemic states of a single agent instead of the common knowledge of a group of agents, and is thus a novel contribution to the literature.

This essay provides a novel account of self-knowledge, which avoids the epistemic indeterminacy witnessed by the invalidation of modal axiom 4 in epistemic logic; i.e. the KK principle:  $\Box\phi \rightarrow \Box\Box\phi$ . The essay argues, by contrast, that – despite the invalidation of modal axiom 4 on its epistemic interpretation – states of epistemic determinacy might yet be secured by countenancing self-knowledge on the model of fixed points in the modal  $\mu$ -calculus.

Counterinstances to modal axiom 4 – which records the property of transitivity in labeled transition systems<sup>1</sup> – have been argued to occur within various interpretations of the sorites paradox. Suppose, e.g., that a subject is presented with a bounded continuum, the incipient point of which bears a red color hue and the terminal point of which bears an orange color hue. Suppose, then, that the cut-off points between the points ranging from red to orange are indiscriminable, such that the initial point, a, is determinately red, and matches the next apparent point, b; b matches the next apparent point, c; and thus – by transitivity – a matches c. Similarly, if b matches c, and c matches d, then b matches d. The sorites paradox consists in that iterations of transitivity would

---

\*I changed my name, from Hasen Joseph Khudairi and Timothy Alison Bowen, to David Elohim, in April, 2024. Please cite this paper and my published book and articles under ‘Elohim, David’.

<sup>1</sup>See Kripke (1963).

entail that the initial and terminal points in the bounded continuum are phenomenally indistinguishable. However, if one takes transitivity to be the culprit in the sorites, then eschewing the principle would entail a rejection of the corresponding modal axiom (4), which records the iterative nature of the relation.<sup>2</sup> Given the epistemic interpretation of the axiom – namely, that knowledge that a point has a color hue entails knowing that one knows that the point has that color hue – a resolution of the paradox which proceeds by invalidating axiom 4 subsequently entrains the result that one can know that one of the points has a color hue, and yet not know that they know that the point has that color hue (Williamson, 1990: 107-108; 1994: 223-244; 2001: chs. 4-5).

The non-transitivity of phenomenal indistinguishability corresponds to the non-transitivity of epistemic accessibility. As Williamson (1994: 242) writes: "The example began with the non-transitive indiscriminability of days in the height of the tree, and moved on to a similar phenomenon for worlds. It seems that this can always be done. Whatever  $x$ ,  $y$  and  $z$  are, if  $x$  is indiscriminable from  $y$ , and  $y$  from  $z$ , but  $x$  is discriminable from  $z$ , then one can construct miniature worlds  $w_x$ ,  $w_y$  and  $w_z$  in which the subject is presented with  $x$ ,  $y$  and  $z$  respectively, everything else being relevantly similar. The indiscriminability of the objects is equivalent to the indiscriminability of the corresponding worlds, and therefore to their accessibility. The latter is therefore a non-transitive relation too." The foregoing result holds, furthermore, in the probabilistic setting, such that the evidential probability that a proposition has a particular value may be certain – i.e., be equal to 1 – while the iteration of the evidential probability operator – recording the evidence with regard to that evidence – is yet equal to 0. Thus, one may be certain on the basis of one's evidence that a proposition has a particular value, while the higher-order evidence with regard to one's evidence adduces entirely against that valuation (Williamson, 2014).

In the foregoing argument, 'safety' figures as a necessary condition on knowledge, and is codified by margin-for-error principles of the form:  $\forall x \forall \phi [K^{m+1} \phi(x) \rightarrow K^m \phi(x+1)]$ , with  $m$  a natural number (Williamson, 2001: 128; Gómez-Torrente, 2002: 114). Intuitively, the safety condition ensures that if one knows that a predicate is satisfied, then one knows that the predicate is satisfied in relevantly similar worlds. Williamson targets the inconsistency of margin-for-error principles, the luminosity principle [ $\forall x \forall \phi [\phi(x) \rightarrow K\phi(x)]$ ], and the characterization of the sorites as occurring when an initial state satisfies a condition, e.g. being red, and a terminal state satisfies a distinct condition, e.g. being orange. As Srinivisan (2013: 4) writes: 'By [the luminosity principle], if C obtains in  $\alpha_0$ , then S knows that C obtains in  $\alpha_0$ . By [margin-for-error principles], if S knows that C obtains in  $\alpha_0$ , then C obtains in  $\alpha_1$ . By [the characterization of the sorites], C does obtain in  $\alpha_0$ ; therefore, C obtains in  $\alpha_1$ . Similarly, we can establish that C also obtains in  $\alpha_2, \alpha_3, \alpha_4, \dots, \alpha_n$ . But according to [the characterization of the sorites] C doesn't obtain in  $\alpha_n$ . Thus we arrive at a contradiction'. The triad evinces that the luminosity principle is false, given the plausibility of margin-for-error principles and the characterization of the sorites.

<sup>2</sup>For more on non-transitivist approaches to the sorites, see Zardini (2019).

In cases, further, in which conditions on knowledge are satisfied, epistemic indeterminacy is supposed to issue from the non-transitivity of the accessibility relation on worlds (1994: 242).

The anti-luminosity argument can be availed of to argue against the KK principle. If states are not luminous, then knowing that  $\phi$  will not entail that one knows that one knows that  $\phi$ . A different argument is presented, as well, in Williamson (2001: ch. 5, p. 115-116). Suppose the following:

(1<sub>i</sub>) If K that x is i+1 inches tall, then  $\neg K\neg x$  is i inches tall

(If an agent knows that some object is i+1 inches tall, then for all the agent knows the object is i inches tall); and

(C) ‘If p and all members of the set X are pertinent propositions, p is a logical consequence of X, and [an agent] knows each member of X, then he knows p’ (op. cit.: 116).

Suppose that:

(2<sub>i</sub>) An agent knows that the object is not i inches tall.

By the KK principle, (3<sub>i</sub>) follows from (2<sub>i</sub>).

(3<sub>i</sub>) An agent knows that she knows that an object is not i inches tall.

Suppose a proposition (q) which states that the object is i+1 inches tall. By (1), then the agent knows that  $\neg(2_i)$ . However, if (3<sub>i</sub>), then the agent knows (2<sub>i</sub>). Thus, (q)  $\rightarrow$  (2<sub>i</sub>)  $\wedge$   $\neg(2_i)$ . Thus – by (C) – (1<sub>i</sub>) and (3<sub>i</sub>) imply that the agent knows  $\neg(q)$ :

(2<sub>i+1</sub>) the agent knows that the object is not i+1 inches tall.

Thus, from (KK), (C), and (2<sub>i</sub>), we can infer (2<sub>i+1</sub>).

Repeating the argument for values of i ranging from 0 to 664, we have

(2<sub>0</sub>) An agent knows that the object is not 0 inches tall.

(2<sub>664</sub>) An agent knows that the object is not 664 inches tall.

However, suppose that the object is in fact 664 inches tall and grant the factivity of knowledge (modal axiom T:  $\Box\phi \rightarrow \phi$ ). Then (2<sub>664</sub>) is false. So, from (1), (2<sub>0</sub>), (C), and (KK), we can derive a false conclusion, (2<sub>664</sub>).

(C) is a principle of deductive closure, and thus arguably ought to be preserved. Williamson takes (2<sub>i</sub>) to be a truism, and (1) to be defensible. He thus argues that we ought to reject the KK principle.

In this essay, I endeavor to provide a novel account which permits the retention of both classical logic as well as a modal approach to the phenomenon of vagueness, while salvaging the ability of subjects to satisfy necessary conditions on there being iterated epistemic states. I will argue that – despite the invalidity of modal axiom 4 – a distinct means of securing an iterated state of knowledge concerning one’s first-order knowledge that a particular state obtains is by availing of fixed point, non-deterministic automata in the setting of coalgebraic modal logic.

The modal  $\mu$ -calculus is equivalent to the bisimulation-invariant fragment of monadic second-order logic.<sup>3</sup>  $\mu(x)$ . is an operator recording a least fixed point. Despite the non-transitivity of sorites phenomena – such that, on its epistemic interpretation, the subsequent invalidation of modal axiom 4 entails structural,

<sup>3</sup>Cf. Janin and Walukiewicz (1996).

higher-order epistemic indeterminacy – the modal  $\mu$ -calculus provides a natural setting in which a least fixed point can be defined with regard to the states instantiated by non-deterministic modal automata. In virtue of recording iterations of particular states, the least fixed points witnessed by non-deterministic modal automata provide, then, an escape route from the conclusion that the invalidation of the KK principle provides an exhaustive and insuperable obstruction to self-knowledge. Rather, the least fixed points countenanced in the modal  $\mu$ -calculus provide another conduit into subjects' knowledge to the effect that they know that a state has a determinate value. Thus, because of the fixed points definable in the modal  $\mu$ -calculus, the non-transitivity of the similarity relation is yet consistent with necessary conditions on epistemic determinacy and self-knowledge, and the states at issue can be luminous to the subjects who instantiate them.

In the remainder of the essay, we introduce labeled transition systems, the modal  $\mu$ -calculus, and non-deterministic Kripke (i.e.,  $\mu$ -) automata. We recount then the sorites paradox in the setting of the modal  $\mu$ -calculus, and demonstrate how the existence of fixed points enables there to be iterative phenomena which ensure that – despite the invalidation of modal axiom 4 – iterations of mental states can be secured, and can thereby be luminous.

A labeled transition system is a tuple comprised of a set of worlds,  $M$ ; a valuation,  $V$ , from  $M$  to its powerset,  $\wp(M)$ ; and a family of accessibility relations,  $R$ . So  $LTS = \langle M, V, R \rangle$  (cf. Venema, 2012: 7). A Kripke coalgebra combines  $V$  and  $R$  into a Kripke functor,  $\sigma$ ; i.e. the set of binary morphisms from  $M$  to  $\wp(M)$  (op. cit.: 7-8). Thus for an  $s \in M$ ,  $\sigma(s) := [\sigma_V(s), \sigma_R(s)]$  (op. cit.). Satisfaction for the system is defined inductively as follows: For a formula  $\phi$  defined at a state,  $s$ , in  $M$ ,

$$\begin{aligned}
\llbracket \phi \rrbracket^M &= V(s) \text{ }^4 \\
\llbracket \neg \phi \rrbracket^M &= S - V(s) \\
\llbracket \perp \rrbracket^M &= \emptyset \\
\llbracket \top \rrbracket^M &= M \\
\llbracket \phi \vee \psi \rrbracket^M &= \llbracket \phi \rrbracket^M \cup \llbracket \psi \rrbracket^M \\
\llbracket \phi \wedge \psi \rrbracket^M &= \llbracket \phi \rrbracket^M \cap \llbracket \psi \rrbracket^M \\
\llbracket \diamond_d \phi \rrbracket^M &= \langle R_d \rangle \llbracket \phi \rrbracket^M \\
\llbracket \square_d \phi \rrbracket^M &= [R_d] \llbracket \phi \rrbracket^M, \text{ with} \\
\langle R_d \rangle(\phi) &:= \{s \in S \mid R_d[s] \cap \phi \neq \emptyset\} \text{ and} \\
[R_d](\phi) &:= \{s \in S \mid R_d[s] \subseteq \phi\} \text{ (9)} \\
\llbracket \mu x. \phi \rrbracket &= \bigcap \{U \subseteq M \mid \llbracket \phi \rrbracket \subseteq U\} \text{ (Fontaine, 2010: 18)} \\
\llbracket \nu x. \phi \rrbracket &= \bigcup \{U \subseteq M \mid U \subseteq \llbracket \phi \rrbracket\} \text{ (op. cit.; Fontaine and Place, 2010),}
\end{aligned}$$

A Kripke coalgebra can be represented as the pair  $(M, \sigma: S \rightarrow \mathbf{KA})$  (Venema, 2020: 8.1)

In our Kripke colagebra, we have  $M, s \Vdash \langle \pi^* \rangle \phi \iff (\phi \vee \diamond_s \langle \pi^* \rangle \phi)$  (Venema, 2012: 25).  $\langle \pi^* \rangle \phi$  is thus said to be the *fixed point* for the equation,  $x \iff \phi \vee \diamond x$ , where the value of the formula is a function of the value of  $x$  conditional on the constancy in value of  $\phi$  (38). The smallest solution of the formula,  $x \iff$

<sup>4</sup>Alternatively,  $M, s \Vdash \phi$  if  $s \in V(\phi)$  (9).

$\phi \vee \diamond x$ , is written  $\mu x. \phi \vee \diamond x$  (25). The value of the least fixed point is, finally, defined more specifically thus:

$$\llbracket \mu x. \phi \vee \diamond x \rrbracket = V(\phi) \cup \langle R \rangle (\llbracket \mu x. \phi \vee \diamond x \rrbracket) \quad (38).$$

A non-deterministic automaton is a tuple  $\mathbb{A} = \langle A, \Xi, \text{Acc}, a_I \rangle$ , with  $A$  a finite set of states,  $a_I$  being the initial state of  $A$ ;  $\Xi$  is a transition function s.t.  $\Xi: A \rightarrow \wp(A)$ ; and  $\text{Acc} \subseteq A$  is an acceptance condition which specifies admissible conditions on  $\Xi$  (60, 66).

Let two Kripke models  $\mathbb{A} = \langle A, a \rangle$  and  $\mathbb{S} = \langle S, s \rangle$ , be bisimilar if and only if there is a non-empty binary relation,  $Z \subseteq A \times S$ , which is satisfied, if:

- (i) For all  $a \in A$  and  $s \in S$ , if  $aZs$ , then  $a$  and  $s$  satisfy the same proposition letters;
- (ii) *The forth condition.* If  $aZs$  and  $R_{\Delta a, v_1} \dots v_n$ , then there are  $v'_1 \dots v'_n$  in  $S$ , s.t.
  - for all  $i$  ( $1 \leq i \leq n$ )  $v_i Z v'_i$ , and
  - $R'_{\Delta s, v'_1} \dots v'_n$ ;
- (iii) *The back condition.* If  $aZs$  and  $R'_{\Delta s, v'_1} \dots v'_n$ , then there are  $v_1 \dots v_n$  in  $A$ , s.t.
  - for all  $i$  ( $1 \leq i \leq n$ )  $v_i Z v'_i$  and
  - $R_{\Delta a, v_1} \dots v_n$  (cf. Blackburn et al, 2001: 64-65).

Bisimulations may be redefined as *relation liftings*. We let, e.g., a Kripke functor,  $\mathbf{K}$ , be such that there is a relation  $\overline{\mathbf{K}} \subseteq \mathbf{K}(A) \times \mathbf{K}(A')$  (Venema, 2020: 81). Let  $Z$  be a binary relation s.t.  $Z \subseteq A \times A'$  and  $\wp Z \subseteq \wp(A) \times \wp(A')$ , with  $\wp Z := \{(X, X') \mid \forall x \in X \exists x' \in X' \text{ with } (x, x') \in Z \wedge \forall x' \in X' \exists x \in X \text{ with } (x, x') \in Z\}$  (op. cit.). Then, we can define the relation lifting,  $\overline{\mathbf{K}}$ , as follows:

$\overline{\mathbf{K}} := \{[(\pi, X), (\pi', X')] \mid \pi = \pi' \text{ and } (X, X') \in \wp Z\}$  (op. cit.), with  $\pi$  a projection mapping of  $\overline{\mathbf{K}}$ .<sup>5</sup>

The relation lifting,  $\overline{\mathbf{K}}$ , associated with the functor,  $\mathbf{K}$ , satisfies the following properties (Enqvist et al, 2019: 586):

- $\overline{\mathbf{K}}$  extends  $\mathbf{K}$ . Thus  $\overline{\mathbf{K}}f = \mathbf{K}f$  for all functions  $f: X_1 \rightarrow X_2$ ;
- $\overline{\mathbf{K}}$  preserves the diagonal. Thus  $\overline{\mathbf{K}}\text{Id}_X = \text{Id}_{\mathbf{K}X}$  for any set  $X$  and functor,  $\text{Id}$ , where  $\text{Id}_C$  maps a set  $S$  to the product  $S \times C$  (583, 586);
- $\overline{\mathbf{K}}$  is monotone.  $R \subseteq Q$  implies  $\overline{\mathbf{K}}R \subseteq \overline{\mathbf{K}}Q$  for all relations  $R, Q \subseteq X_1 \times X_2$ ;
- $\overline{\mathbf{K}}$  commutes with taking converse.  $\overline{\mathbf{K}}R^\circ = (\overline{\mathbf{K}}R)^\circ$  for all relations  $R \subseteq X_1 \times X_2$ ;
- $\overline{\mathbf{K}}$  distributes over relation composition.  $\overline{\mathbf{K}}(R ; Q) = \overline{\mathbf{K}}R ; \overline{\mathbf{K}}Q$ , for all relations  $R \subseteq X_1 \times X_2$  and  $Q \subseteq X_2 \times X_3$ , provided that the functor  $\mathbf{K}$  preserves weak pullbacks (op. cit.). Venema and Vosmaer (2014: §4.2.2)

<sup>5</sup>The projections of a relation  $R$ , with  $R$  a relation between two sets  $X$  and  $Y$  such that  $R \subseteq X \times Y$ , are  $X \longleftarrow (\pi_1) R (\pi_2) \longrightarrow Y$  such that  $\pi_1((x, y)) = x$ , and  $\pi_2((x, y)) = y$ . See Rutten (2019: 240).

define a weak pullback as follows: 'A weak pullback of two morphisms  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  with a shared codomain  $Z$  is a pair of morphisms  $p_X : P \rightarrow X$  and  $p_Y : P \rightarrow Y$  with a shared domain  $P$ , such that (1)  $f \circ p_X = g \circ p_Y$ , and (2) for any other pair of morphisms  $q_X : Q \rightarrow X$  and  $q_Y : Q \rightarrow Y$  with  $f \circ q_X = g \circ q_Y$ , there is a morphism  $q : Q \rightarrow P$  such that  $p_X \circ q = q_X$  and  $p_Y \circ q = q_Y$ . This pullback is "weak" because we are not requiring  $q$  to be unique. Saying that [a set functor]  $T : \mathbf{Set} \rightarrow \mathbf{Set}$  preserves weak pullbacks means that if  $p_X : P \rightarrow X$  and  $p_Y : P \rightarrow Y$  form a weak pullback of  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$ , then  $Tp_X : TP \rightarrow TX$  and  $Tp_Y : TP \rightarrow TY$  form a weak pullback of  $Tf : TX \rightarrow TZ$  and  $Tg : TY \rightarrow TZ$ .

Finally, given the Kripke functor,  $\mathbf{K}$ ,  $\mathbf{K}$  can be defined as the  $\mu$ -automaton, i.e., the tuple  $\mathbb{A} = \langle A, \Xi, a_I \rangle$ , with  $a_I \in A$  defined again as the initial state in the set of states  $A$ ; and  $\Xi$  defined as a mapping such that  $\Xi : A \rightarrow \wp_{\exists}(\mathbf{K}A)$ , where the  $\exists$  subscript indicates that  $(a,s) \in A \times S \rightarrow \{(a',s) \in \mathbf{K}(A) \times S \mid a' \in \Xi(a)\}$  (93). The duality between the categories of coalgebras,  $\mathbf{A}$ , and algebras,  $\mathbf{S}$ , and the definition of the functor,  $\mathbf{K}$ , as an expression relation expressed by  $\mu$ -automata, provide an account of expressivism for self-knowledge.<sup>6</sup>

The philosophical significance of the foregoing can now be witnessed by defining the  $\mu$ -automata on an alphabet; in particular, a non-transitive set comprising a bounded real-valued, ordered sequence of terms for chromatic properties. Although the non-transitivity of the ordered sequence of terms for color hues belies modal axiom 4, such that one can know that a particular point in the sequence has a particular value although not know that one knows that the point satisfies that value, terms for chromatic values,  $\phi$ , in the non-transitive set of color terms nevertheless permits every sequential input state in the  $\mu$ -automaton to define a fixed point. In order for there to be least and greatest fixed points, there must be monotone operators defined on complete lattices. As Venema (2020: A-2) writes: 'A *partial order* is a structure  $\mathbb{P} = \langle P, \leq \rangle$  such that  $\leq$  is a reflexive, transitive and antisymmetric relation on  $P$ . Given a partial order  $\mathbb{P}$ , an element  $p \in P$  is an *upper bound* (*lower bound, respectively*) of a set  $X \subseteq P$  if  $p \geq x$  for all  $x \in X$  ( $p \leq x$  for all  $x \in X$ ). If the set of upper bounds of  $X$  has a minimum, this element is called the *least upper bound, supremum, or join* of  $X$ , notation:  $\bigvee X$ . Dually, the *greatest lower bound, infimum, or meet* of  $X$ , if existing, is denoted as  $\bigwedge X$  ... A partial order  $\mathbb{P}$  is called a *lattice* if every two-element subset of  $P$  has both an infimum and a supremum; in this case, the notation is as follows:  $p \wedge q := \bigwedge \{p, q\}$ ,  $p \vee q := \bigvee \{p, q\}$  ... A partial order  $\mathbb{P}$  is called a *complete lattice* if every subset of  $P$  has both an infimum and a supremum ... A complete lattice will usually be denoted as a structure  $\mathbb{C} = \langle C, \bigvee, \bigwedge \rangle$ . 'Let  $\mathbb{P}$  and  $\mathbb{P}'$  be two partial orders and let  $f : P \rightarrow P'$  be some map. Then  $f$  is called *monotone* or

<sup>6</sup>See Elohim (2024), for further discussion of modal and hyperintensional expressivism. For further discussion of the convergence between expressivism and self-knowledge, see Wittgenstein (1953/2009); Bar-On (2004); and Bar-On and Wright (2023). For discussion of the convergence between self-knowledge and Wright (2001)'s intention-based view of rule-following, see Coliva (2012) and Elohim (2024).

order preserving if  $f(x) \leq f(y)$  whenever  $x \leq y \dots$ ' (3.1). 'Let  $\mathbb{P} = \langle P, \leq \rangle$  be a partial order, and let  $f: P \rightarrow P$  be some map. Then an element  $p \in P$  is called a *prefixpoint* of  $f$  if  $f(p) \leq p$ , a *postfixpoint* of  $f$  if  $p \leq f(p)$ , and a *fixpoint* if  $f(p) = p$ . The sets of prefixpoints, postfixpoints, and fixpoints of  $f$  are denoted respectively as  $\text{PRE}(f)$ ,  $\text{POS}(f)$  and  $\text{FIX}(f)$ . / In case the set of fixpoints of  $f$  has a least (respectively greatest) member, this element is denoted  $\text{LFP}.f$  ( $\text{GFP}.f$ , respectively)' (3-2). The Knaster-Tarski Theorem says, then, that, for a complete lattice,  $\mathbb{C} = \langle C, \vee, \wedge \rangle$ , with  $f: C \rightarrow C$  being monotone,  $f$  has both a least and greatest fixpoint,  $\text{LFP}.f = \bigwedge \text{PRE}(f)$ , and  $\text{GFP}.f = \bigvee \text{POS}(f)$  (op. cit.).<sup>7</sup>

The epistemicist approach to vagueness relies, as noted, on the epistemic interpretation of the modal operator, such that the invalidation of transitivity and modal axiom 4 ( $\Box\phi \rightarrow \Box\Box\phi$ ) can be interpreted as providing a barrier to a necessary condition on self-knowledge.<sup>8</sup> Crucially,  $\mu$ -automata can receive a similar epistemic interpretation.<sup>9</sup> An epistemic interpretation of a  $\mu$ -automaton is just such that the automaton operates over epistemically possible worlds. The automaton can thus be considered a model for an epistemic agent. The transition function accounts for the transition from one epistemic state to another, e.g. as one proceeds along the stages of a continuum. A fixed point operator on a given epistemic state, e.g.  $\Box(\phi)$  where  $\Box$  is interpreted so as to mean knowledge-that, amounts to one way to iterate the state. If one knows a proposition  $\phi$ , the least fixed point operation,  $\mu x.(\Box(\phi))$ , records an iteration of the epistemic state, knowledge of knowledge, and similarly for belief. Thus, interpreting the  $\mu$ -automaton epistemically permits the fixed points relative to the arbitrary points in the ordered continuum to provide a principled means – distinct from the satisfaction of the KK principle – by which to account for the pertinent iterations of epistemic states unique to an agent's self-knowledge.

Fixed points for the iteration of epistemic states can be obtained, as with the Knaster-Tarski Theorem, by Lambek (1968)'s Theorem according to which 'If [an endofunctor,  $F: C \rightarrow C$ , for all categories,  $C$ ,] has an initial algebra  $\alpha: F(X) \rightarrow X$ , then  $X$  is isomorphic to  $F(X)$  via  $\alpha$ '.  $X$  is the smallest fixed point of  $F$  and the terminal coalgebra,  $F(X)$ , is the largest fixed point of  $F$  (<https://ncatlab.org/nlab/show/initial+algebra+of+an+endofunctor>).<sup>10</sup>

Fixed points for the iteration of epistemic states can, too, be obtained from

(i) the Kleene fixed point theorem which states: 'Let  $f: P \rightarrow P$  be a monotone function on a poset [i.e. partially ordered set: reflexive,  $a \leq a$ ; transitive, if  $a \leq b \wedge b \leq c \rightarrow a \leq c$ ; anti-symmetric, if  $a \leq b \wedge b \leq a$ , then  $a = b$  –

<sup>7</sup>Knaster (1928); Tarski (1955).

<sup>8</sup>Williamson (p.c.) writes: 'My argument against KK is compatible with there being lots of cases of knowledge that one knows. Indeed, the picture of knowledge it suggests (with margins for error) suggests that there is plenty of knowledge, as I think there is. Right now, I know that I know that I'm typing on my laptop. The failure of KK is a limitation on self-knowledge but not a very severe one.'

<sup>9</sup>For more on the epistemic  $\mu$ -calculus, see Bulling and Jamroga (2011); Bozianu et al (2013); and Dima et al (2014). For an examination of the modal  $\mu$ -calculus and common knowledge, see Alberucci (2002).

<sup>10</sup>Lambek (1968: §2); Adámek et al. (forthcoming).

D.E. ]  $P$ . If  $P$  has a least element  $\perp$  and joins of increasing sequences, and if  $f$  preserves joins of increasing sequences, then a least fixed point of  $f$  can be constructed as the join of the increasing sequence:

$\perp \leq f(\perp) \leq f^2(\perp) \leq \dots$  (<https://ncatlab.org/nlab/show/Kleene%5B%27s+fixed+point+theorem%5D>),<sup>11</sup> and

(ii) the Pohlová-Adámek fixed point theorem which states: ‘Let  $C$  be a category with an initial object  $0$  and transfinite composition of length  $\omega$ , hence colimits [i.e. sums] of sequences  $\omega \rightarrow C$  (where  $\omega$  is the first infinite ordinal), and suppose  $F: C \rightarrow C$  preserves colimits of  $C^\omega$ -chains. Then the colimit  $\gamma$  of the chain

$$0 \rightarrow_i F(0) \rightarrow_{F(i)} \dots F^{(n)}(0) \rightarrow_{F^{(n)}(0)} F^{(n+1)}(0) \rightarrow \dots$$

carries a structure of the initial  $F$ -algebra’ (<https://ncatlab.org/nlab/show/Adámek%5B%27s+fixed+point+theorem%5D>).<sup>12</sup>

The fixed point operators in the modal  $\mu$ -calculus can be rendered hyperintensional, by defining the elements in the sets in the semantics for the operators above, such that they are hyperintensional parts of epistemically possible worlds, rather than whole epistemically possible worlds. [See Fine, 2017a,b,c, for a presentation of truthmaker semantics, and Elohim (2024) for detailed discussion of multi-hyperintensional semantics, which incorporates atomic topics, truthmakers, subject matter similarity relations between truthmakers, and two-dimensional indexing.] The semantics for each operator can then remain as presented in the foregoing, while changing the sets and their subsets to hyperintensional epistemic states or verifiers instead of worlds.

The fixed point approach to iterated epistemic states will provide a compelling alternative to the KK principle, if Williamson’s argument against the KK principle does not hold for all ancestral relations of knowledge but rather only for specific applications of luminosity and modal axiom 4.<sup>13</sup> If Williamson’s argument does not generalize to all ancestral relations of knowledge, then one can avoid the objection that the fact that  $\mu x.(\Box(\phi))$  entails that one knows that one knows that  $\phi$  is such that the state collapses just to KK such that the state would rarely be satisfied in light of the argument against the KK principle. An iteration procedure via a fixed point operation on a knowledge state is distinct from an application of the KK principle, i.e. an application of modal axiom 4, and provides a novel formal method for accounting for the iteration of epistemic states.

<sup>11</sup>Kleene (1971); Cousot and Cousot (1979).

<sup>12</sup>Pohlová (1973); Adámek (1974).

<sup>13</sup>Thanks here to Jon Litland for the objection.



## References

- Adámek, J. 1974. Free Algebras and Automata Realizations in the Language of Categories. *Commentationes Mathematicae Universitatis Carolinae*, Vol. 15, No. 4.
- Adámek, J., S. Milius, and L. Moss. Forthcoming. *Initial Algebras and Terminal Coalgebras: The Theory of Fixed Points of Functors*. Cambridge University Press.
- Alberucci, L. 2002. The Modal  $\mu$ -Calculus and Logics of Common Knowledge. PhD Thesis, Universität Bern.
- Bar-On, D. 2004. *Speaking My Mind: Expression and Self-Knowledge*. Oxford University Press.
- Bar-On, D., and C. Wright. 2023. *Expression and Self-Knowledge*. John Wiley and Sons.
- Blackburn, P., and J. van Benthem. 2007. Modal Logic: A Semantic Perspective. In Blackburn, van Benthem, and F. Wolter (eds.), *Handbook of Modal Logic*. Elsevier.
- Blackburn, P., M. de Rijke, and Y. Venema. 2001. *Modal Logic*. Cambridge University Press.
- Bozianu, R. C. Dima, and C. Enea. 2013. Model checking an Epistemic  $\mu$ -calculus with Synchronous and Perfect Recall. In B. Schipper (ed.), *Proceedings of the 14th Conference on Theoretical Aspects of Rationality and Knowledge – TARK 2013*.
- Bulling, N., and W. Jamroga. 2011. Alternating Epistemic  $\mu$ -Calculus. *IJCAI'11: Proceedings of the Twenty-Second international joint conference on Artificial Intelligence - Volume One*.
- Cousot, P., and R. Cousot. 1979. Constructive Versions of Tarski's Fixed Point Theorem. *Pacific Journal of Mathematics*, Vol. 82, No. 1.
- Church, A. 1936. An Unsolvable Problem of Elementary Number Theory. *American Journal of Mathematics*, 58:2.
- Coliva, A, (ed.). 2012. *Mind, Meaning, and Knowledge: Themes from the Philosophy of Crispin Wright*. Oxford University Press.
- Dima, C., B. Maubert, and S. Pinchinat. 2014. The Expressive Power of Epistemic  $\mu$ -Calculus. arXiv.
- Elohim, D. 2024. *Forms of Luminosity: Epistemic Modality and Hyperintensionality in Mathematics*. Ph.D. Dissertation, Arché, University of St Andrews. Amazon.
- Janin, D., and I. Walukiewicz. 1996. On the Expressive Completeness of the Propositional  $\mu$ -Calculus with respect to Monadic Second-order Logic. In U. Montanari and V. Sasone (eds.), *Proceedings CONCUR, '96*. Springer.

- Fine, K. 2017a. A Theory of Truthmaker Content I: Conjunction, Disjunction, and Negation. *Journal of Philosophical Logic*, 46:6.
- Fine, K. 2017b. A Theory of Truthmaker Content II: Subject-matter, Common Content, Remainder, and Ground. *Journal of Philosophical Logic*, 46:6.
- Fine, K. 2017c. Truthmaker Semantics. In B. Hale, C. Wright, and A. Miller (eds.), *A Companion to Philosophy of Language*. Blackwell.
- Fontaine, G. 2010. *Modal Fixpoint Logic*. ILLC Dissertation Series DS-2010-09.
- Fontaine, G., and T. Place. 2010. Frame Definability for Classes of Trees in the  $\mu$ -calculus. In P. Hlineny and A. Kucera (eds.), *Mathematical Foundations of Computer Science 2010*. Springer.
- Gómez-Torrente, M. 2002. Vagueness and Margin for Error Principles. *Philosophy and Phenomenological Research*, 64:1.
- Kleene, S. 1971. *Introduction to Metamathematics*. Woltors-Noordhoff Publishing and North-Holland Publishing.
- Knaster, B. 1928. Un Théorème sur les Fonctions d'Ensembles. *Annales Polonici Mathematici*, 6.
- Kripke, S. 1963. Semantical Considerations on Modal Logic. *Acta Philosophica Fennica*, 16.
- Lambek, J. 1968. A Fixpoint Theorem for Complete Categories. *Mathematische Zeitschrift*, Vol. 103.
- Pohlová, V. 1973. On Sums in Generalized Algebraic Categories. *Czechoslovak Mathematical Journal*, Vo. 23, No. 2.
- Rutten, J. 2019. *The Method of Coalgebra*. CWI.
- Srinivisan, A. 2013. Are We Luminous? *Philosophy and Phenomenological Research*, doi: 10.1111/phpr.12067.
- Tarski, A. 1955. A Lattice-Theoretical Fixpoint Theorem and Its Applications. *Pacific Journal of Mathematics*, Vol. 5, No. 2.
- Venema, Y. 2012. Lectures on the Modal  $\mu$ -Calculus.
- Venema, Y. 2020. Lectures on the Modal  $\mu$ -Calculus.
- Wittgenstein, L. 2009. *Philosophical Investigations*, 4th ed., tr. G.E.M. Anscombe, P.M.S. Hacker, and J. Schulte. Blackwell Publishing.
- Williamson, T. 1990. *Identity and Discrimination*. Basil Blackwell.
- Williamson, T. 1994. *Vagueness*. Routledge.
- Williamson, T. 2001. *Knowledge and Its Limits*. Oxford University Press.
- Williamson, T. 2014. Very Improbable Knowing. *Erkenntnis*, DOI 10.1007/s10670-013-9590-9.
- Wright, C. 2001. *Rails to Infinity: Essays on Themes from Wittgenstein's Philosophical Investigations*. Harvard University Press.
- Zardini, E. 2019. Non-transitivism and the Sorites Paradox. In S. Oms and Zardini (eds.), *The Sorites Paradox*. Cambridge University Press.