Number, Language and Mathematics

Kim JooSoak*

Number is a major object in mathematics. Mathematics is a discipline which studies the properties of a number. The object is expressible by mathematical language, which has been devised more rigorously than natural language. However, the language is not thoroughly free from natural language. Countability of natural number is also originated from natural language. It is necessary to understand how language leads a number into mathematics, its' main playground.

Representation of number

A mathematical object[1], number, has various properties which are not fully understood yet. What has been a principal role in the invention of real number as an entity? It is a line, the geometric object, therefore it is probable that geometric properties are incorporated into the entity unwittingly. How this geometric object is interpreted with? It is a number.

First of all, it is decisive to understand how the points on line are coordinated. Suppose a straight line, which is homogeneous, infinitely and infinitesimally continuous. Pick an arbitrary point position around center and sign it. Define a unit interval and sign the point with the interval to the right direction. Apply the same procedure to the left direction. Label the points on the right direction successively with natural number. Apply the same procedure but prefixing an additional minus symbol to the label of points to the left direction. Important positions on the line are determined. The positions will be called integer later($\mathbb{Z}$).

The unit interval should be divided decimally for further representations. Sign the divided positions and label them successively with the number $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for a time. The same procedures with sub-intervals can be repeated countless as long as infinitesimally continuous line is assumed. The more there are sub-intervals, the more it becomes precise. The positions determined by this can be represented systematically with a decimal point notation.

It is expedient to recognize this procedure when they are represented schematically. Every interval for further representation is divided decimally like a

$$\underline{a_n⓪ \quad ① \quad ② \quad ③ \quad ④ \quad ⑤ \quad ⑥ \quad ⑦ \quad ⑧ \quad ⑨ \quad a_n'⓪,}$$

where each sign represents

$$a_n⓪\colon\ a_0.a_1a_2a_3\ldots a_n000\ldots$$

* soakjoo@hanmail.net

$$①: a_0.a_1a_2a_3...a_n100...$$
$$②: a_0.a_1a_2a_3...a_n200...$$
$$③: a_0.a_1a_2a_3...a_n300...$$
$$\vdots$$
$$⑦: a_0.a_1a_2a_3...a_n700...$$
$$⑧: a_0.a_1a_2a_3...a_n800...$$
$$⑨: a_0.a_1a_2a_3...a_n900...$$
$$a_n'⓪: a_0.a_1a_2a_3...a_n'000...$$

respectively. Each $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$[2-4], and a_n' is a successor of a_n in the set.
Now, the coordination of a line with decimal notation is completed. It's crucial to acknowledge the fact that only special points on the line are designated. There are only nine positions which are chosen, signed, and labelled, at each interval. Therefore it completes a decimal system with already designated one. The sub-intervals, however small, have an identical property due to their inductive structure. They are only insensible to our naked eye but the mechanisms are identical. Infinity in mathematics is defined by induction[4].

The line is classified into the coordinated positions and the infinitesimal intervals divided by the positions. The intervals actually occupy the line. The coordinated positions are fortunate ones as they are representable by decimal notation. The positions on the infinitesimal interval are not representable but they are more general ones from the perspective of points on line. The uncoordinated positions don't have any special property with regard to decimal system. Instead, the coordinated positions are fortunate and special ones. The special positions are reference to their neighbors when analyzing a line.

It is misleading to jump to the conclusion that a line is filled with coordinated positions as they are represented infinitely onto the infinitesimally continuous interval. It seems counter intuitive at first sight. A point is geometric object without magnitude. The point positions cannot fill the line, instead it is filled with infinitesimally continuous intervals. Filling is possible only when a magnitude is premised. A line is not filled with points but with intervals of certain magnitude. However, the line is not complete when a point is missing. It is legitimate to say that the line is complete when the coordinated positions and uncoordinated intervals are considered together.

There is a matter of nomenclature on this classification. It's related to the

organization of number. How the numbers originated from a line are organized?
We expect real number($\mathbb{R}$) to complete the continuous line without any missing element.[5] The coordinated positions and the positions on the intervals are called rational and irrational, respectively. Finally, they have formal and sophisticated names. There is a subtle difference between the definition and our intuition. There is no hole between any two rationals which are located close infinitesimally. Actually, it is an infinitesimal interval with certain magnitude. It is a misleading intuition that all real numbers are representable with decimal point notation.

The classification of number into rational and irrational is determined by the possibility of representation. There are well known irrationals such as $\sqrt{2}$, π, and e [6]. They don't have definite value when expressed decimally. Only their approximate values are expressed. Rigorously, they are unrepresentable such that expressed by their approximate values. Our expectation that the definite value of irrational has not yet known is wrong. It is indefinite by definition. That the definite value has not yet known is one thing, but that it is indefinite by definition is another. Some indefinite numbers may have definite value with the help of technology. They will be classified into rational when it happens.
In this explanation, the definiteness and indefiniteness of number confines to decimal representation. It means that irrational may have definite value with a different and specially devised representation.

A number on line was determined by coordination. However, it must be reminded that it is when you confront a difficulty in understanding the number more deeply. What means continuity of real number? Does it continuous and how? They need to be answered rigorously. Real number was originated from a line but it is. The line is classified into the coordinated positions and the uncoordinated intervals. It is the line itself continuous but also real number when it is defined to include the whole line. It's important to recognize a priori one. Real number is continuous by its' definition to complete a line.

It is countable when you identify an object from its' surrounding. You cannot count definitely when something is overlapped such that it is difficult to determine its' distinctiveness. It is countable when an object has distinctive features such that it is possible to represent. How does this principle is applied to a number on line? It is feasible by coordinating a line. Only the positions which fit to the rule to pick a point are chosen. The rule is decimal division.
There are countless points on the interval, however, only a special point is chosen. It is special and distinctive with regard to the rule of decimal choice. The choice itself means a speciality by rule. The chosen point is surrounded by unchosen points, which are unrepresentable as they do not fit to the rule. The

choice premises that there are points which cannot be chosen. The chosen point is countable due to its' distinctiveness by rule, but unchosen points are not so. It is a matter of choice from general ones.
The countable one and the uncountable ones are called rational and irrational, respectively. Now, it's back to mathematics. They correspond to a coordinated point and its' surroundings. Rational is representable with decimal notation such that it is definite and countable, but irrationals are not so. In a sense, decimal notation itself is a definition of rational.

There is no rule to represent all irrationals simultaneously with a unique notation, not to mention the decimal notation. It does not mean that irrational is absolutely not representable. The well known irrationals such as $\sqrt{2}$ [7], π [8], and *e* [6] have their unique representations to make them definite, respectively. The rule to make π definite does not guarantee that it also makes *e* definite. If you deploy them with power series, you will find a special rule of choice when a interval is divided and determined. You will find a clue on the relativity of a definite and representable point on line. As a point on line, irrational is not special one but the rule to make it definite is peculiar.

Rationals are representable and countable because they fit to the rule but irrationals are the opposite case. It is a matter of representation. Once a rule is chosen, then a interval is classified into the representable positions and their surroundings. When you represent a interval with a peculiar rule, you will confront the same problem with decimal representation. Generally speaking, the difference between rationals and irrationals is the possibility of representation of a point on line by the rule of notation given.
The power series representations below will provide you an insight into this matter.

There are many different representations on the same reference. Each natural number(n) also be represented differently depending on circumstances. Each has a usage and merit. The ordinary one is

$$n = \ldots a_3 a_2 a_1 a_0,$$

where $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ [2-4]. It's simple and most widely used in textbook.
It is possible to represent by power series

$$n = \sum_{i=0}^{\infty} a_i 10^i = a_0 + a_1 10^1 + a_2 10^2 + a_3 10^3 + \cdots,$$

where $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. It is not as simple as ordinary one but it provides an insight into its' history. It includes an algebraic symbol and makes each decimal place definite. The ordering of natural number by index i is identical to the position of number on line[3-4].

Natural number can also be represented schematically,

$$
\begin{array}{c}
\cdots 0000000000\mathbf{0}.00000000000 \cdots \\
\cdots 000000000\mathbf{0}1.00000000000 \cdots \\
\cdots 00000000\mathbf{0}02.00000000000 \cdots \\
\cdots 0000000\mathbf{0}003.00000000000 \cdots \\
\vdots \\
\cdots \mathbf{0}0000000010.00000000000 \cdots \\
\vdots
\end{array}
$$

where all decimal places are designated. It is intuitive when understanding a number on line.

Decimal point also be represented similarly because natural number is meta-language on this representation. The interval between zero and one is

$$r = a_0.a_1a_2a_3\cdots,$$

where $a_0 \in \{0\}$, and $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

The power series is

$$r = \sum_{i=0}^{\infty} a_i 10^{-i} = a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots,$$

where $a_0 \in \{0\}$, and $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. It is insightful into the decimal point notation. At each index i , a interval is divided decimally and it is divided again for further representation. It repeats infinitely into the infinitesimally continuous interval. Ordering of decimal point by index i is not identical to the position of number on line. This is different from the power series of natural number.

They are also schematically represented by

$$
\begin{array}{c}
\cdots 00000000000.\mathbf{0}0000000000 \cdots \\
\cdots 00000000000.1\mathbf{0}000000000 \cdots \\
\cdots 00000000000.20\mathbf{0}00000000 \cdots \\
\cdots 00000000000.300\mathbf{0}0000000 \cdots \\
\vdots
\end{array}
$$

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet\ \bullet\ \bullet\ 00000000000.01000000000\ \bullet\ \bullet\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{array}$$

where all places are designated. Each place includes only $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ by decimal notation.

Real number on line is represented by combing the notation of natural number and decimal point.

$$\begin{aligned} R = n + r &= \sum_{i=0}^{\infty} a_i 10^i + \sum_{j=0}^{\infty} b_j 10^{-j} \\ &= a_0 + a_1 10 + a_2 10^2 + a_3 10^3 + \bullet\bullet\bullet \\ &\quad + b_0 + b_1 10^{-1} + b_2 10^{-2} + b 10^{-3} \bullet\bullet\bullet \end{aligned}$$

, where $a_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $b_0 \in \{0\}$, and $b_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

When a unit interval is divided ternary, the sub-interval point representation is

$$r = a_0 \cdot a_1 a_2 a_3 \cdots,$$

or

$$r = \sum_{i=0}^{\infty} \frac{a_i}{3^i} = a_0 + \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \bullet\bullet\bullet$$

where $a_0 \in \{0\}$, $a_i \in \{0, 1, 2\}$, respectively. Number 3 is used instead of 10 of ternary system for notational convenience. Rational number 1/3 becomes more definite 0.1 with this representation instead of 0.333 • • • in decimal point representation.

Irrational indefinite by decimal system can be represented definitely by a unique method. The well known irrational *e* is indefinite with decimal system,

e= 2.718281...

, but it is represented definitely with power series

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \bullet\bullet\bullet \text{ [6].}$$

Here is a trick to make the number definite for sub-interval point representation. The following power series is a specially devised one for a definite expression of *e*. It is

$$r = \sum_{n=0}^{\infty} \frac{a_n}{n+1!} = \frac{a_0}{1!} + \frac{a_1}{2!} + \frac{a_2}{3!} + \frac{a_3}{4!} + \cdots$$

, where $a_0 \in \{0\}$, $a_1 \in \{0, 1\}$, $a_2 \in \{0, 1, 2\}$, $a_3 \in \{0, 1, 2, 3\}$, ..., respectively.

The division of unit interval to sub-interval is determined by factorial with an inductive property.

With this representation, the irrational *e* becomes definite

$$e = 2 + r,$$

where $r = 0.111\cdots\cdots$.

Irrational indefinite by the definition of decimal system is not always indefinite when it is represented by other unique method. It is possible to represent definitely with a unique one if it does not matter on the unification of representations. There is only a matter of conversion between different representations.

The explanation above must be confined to a number on line. A number on different references[9-11] may have unexpected properties. A substantial properties of the number lie in what it refers. The expressions 'fortunate', 'special', and 'general' are mentioned from a perspective of a point on line. They should not be confused with the intuition you already have.

Fallacy in naive set theory

Russell's paradox[4] is a legitimate example against the way Georg Cantor defines a terminology. In his use of word, there is a inherent contradiction or else a misleading interpretation due to the ambiguity of definition. These necessarily result in paradox which again nullifies his arguments. The diagonal argument[12] and Cantor's theorem[4] are good examples. Consistent use of word is important to make an argument logical. It should be faithful to the definition and axioms on which it is based.

Infinity in mathematics becomes definite by induction. Rigorous definition of natural number is also based on this. Peano's axioms[13] are set-theoretic version of natural number($\mathbb{N}$).

1) $\varnothing \in \mathbb{N}$.
2) For any $n \in \mathbb{N}$, there is one and only one successor $n' \in \mathbb{N}$.
3) If $m, n \in \mathbb{N}$ and if $m' = n'$, then $m = n$.
4) If $n \in \mathbb{N}$, then $n' \neq \varnothing$
5) Let $S \subset \mathbb{N}$. Suppose $n \in S$. If $n \in S$ implies $n' \in S$, then $S = \mathbb{N}$.

According to Cantor's diagonal argument, there is a number which cannot be corresponded to natural number. The number differs in diagonal places of enumerable numbers which are mapped bijectively to natural number. Therefore, Cantor concluded that the cardinality of real number is greater than natural number. It is an illogical jump.

The representations of natural number and decimal point are inversely related. It is clear from the schematic representations of each number, which are presented in previous sections. They are identical in that there are only decimal choices in each places of representation.

When diagonal argument is applied to natural number, it confronts a paradox.

"There is a number which is different from all natural number." Is this number a natural number or not? The number is definitely a natural number by appearance but it should be different from all natural numbers. How this is possible? If a number is different from all elements of the set, it cannot be included in the set.

The paradox originates from the misleading interpretations of definition. Infinite structure of natural number is based on mathematical induction. Inductive set is defined by the property 2) above. Here is a fallacy in our intuition. The word "all" only quantifies $n \in \mathbb{N}$ but not its successor $n' \in \mathbb{N}$. Rigorously speaking, the logic is "For all $n \in \mathbb{N}$, arbitrary one $n \in \mathbb{N}$ is chosen and then its' successor $n' \in \mathbb{N}$ is defined". Cantor's argument originates from the misunderstanding of definition, especially what the definition assumes implicitly.

There are two interpretations on the above question, inconsistent one and consistent one. Cantor's argument is inconsistent one such that it confronts a paradox. He implicitly assumed the word 'all' quantifies all n, $n' \in \mathbb{N}$, simultaneously, or he interpreted it arbitrarily depending on circumstance.

The consistent one is "The natural numbers which differ from all $n \in \mathbb{N}$ are $m>n$, where $m \in \mathbb{N}$. Here the word 'all' quantifies all numbers between 0 and n.

By the same principle, Cantor's argument on the cardinality[12] of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ is inconsistent. It should be faithful to the definition. For all $n \in \mathbb{N}$, how many integers possibly be represented with those? How these integers are mapped with all $n \in \mathbb{N}$ by one-to-one correspondence? The answer on cardinality can be replied from these questions. It is also clear from our intuition on the inclusion relationship $\mathbb{N} \in \mathbb{Z} \in \mathbb{Q}$. Integer includes all natural numbers and additional numbers which is different from the numbers. The same is applied to rational. Cantor's argument is possible as long as they are enumerable and infinite, however, it is inconsistent with the definition.

Cantor's theorem which states a set and its' power set should also be revised. He defined a special set to maintain his argument. He attributed the contradiction to the assumption of surjection, but it actually originated from misinterpretation of the set he himself defined. The definition is a major premise and an interpretation on the assumption of surjection must be a corollary. He maintained his argument without sufficient considerations on the matter of quantification.

Cantor developed a logic arbitrarily. According to his argument on the cardinality of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, a bijection is possible between $\mathbb{N}$ and its' power set $P(\mathbb{N})$ as long as they are enumerable and infinite. However, rigorous use of the word 'all' makes it definite between $n \in \mathbb{N}$ and $P(n)$. The cardinality between them also becomes clear.

The continuum hypothesis[4],[12] is based on the expectation that there is a absolute truth on the relationship between $\mathbb{N}$ and $\mathbb{R}$. The answer to this question can be found from a different and unexpected perspective. A distorted intuition needs to be corrected in advance. Decimal point representation does not complete the interval [0,1). It represents only a rational. A representable number of cases are identical to natural number as it is clear in the schematic representation of previous section.
The answer depends on how uncoordinated intervals are represented. There are many kinds of representation as in previous section, but the interpretation of cardinality must be faithful to the definition from which a logic develops. The hypothesis itself is not critical in the foundation of mathematics. A correct use of language is much more important.

The role of an language on number
Real number on line represents a point on it. It is classified into rational and irrational. Rational corresponds to a coordinated point. The point is recognizable by rule, therefore representable. It is definite by rule, therefore countable. If not so, it is the opposite case. Irrationals correspond to uncoordinated points on interval. Irrationals surround a rational. Some irrationals but not all once become recognizable by a unique rule such that they are representable, where they have the properties of rational. Our intuition on number has been mislead. It should be revised with rigorous ones above.

Understanding a language thoroughly means to grasp what it assumes, implicitly or unwittingly. Some languages materializes only when a premise is self-evident.
The number on line is a point without magnitude. However, the point is defined by the distance from origin. This enables the number two-faced. It has a magnitude when manipulated algebraically but not so when it represents a point on line. It can have any kinds of magnitude when it expresses a natural language. Sometimes, the elusive characteristics distorts our intuition to fill the line with numbers. Does it have magnitude? It depends on the reference what natural language is to express. It is fateful to struggle against the pitfall of language for rigorous reasoning.

I have no idea what is absolute truth on number. However, it is essential to use a language consistently to avoid paradox. When a language is evolved or applied to other fields, its' history must be recognized definitely. Fallacy is not in an entity itself but in the way of recognition. Coherent use of language makes a logic consistent as well.

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Reference

[1] Lou van den Dries. Mathematical Logic. Lecture Notes, 2016.
Retrived from https://faculty.math.illinois.edu/~vddries/main.pdf.

[2] Axiom of choice. Wikipedia(6 April 2020).
Retrived from https://en.wikipedia.org/wiki/Axiom_of_choice .

[3] Ivan Khatchatourian. Topology. Lecture Notes, 2018.
Retrived from http://www.math.toronto.edu/ivan/mat327/docs/notes/11-choice.pdf.

[4] Peter J. Cameron. Sets, Logic and Categories. Springer Science & Business Media, 1999. ISBN 1852330562, 9781852330569.

[5] Mark Dean. Real Numbers. Lecture Notes, 2014.
http://www.columbia.edu/~md3405/Real_Numbers_14.pdf.

[6] James Stewart. Calculus. Brooks/Cole, 2011. ISBN 0538497815, 9780538497817.

[7] Square root of 2. Wikipedia(18 February 2020).
Retrived from https://en.wikipedia.org/wiki/Square_root_of_2.

[8] Leibniz formula for π. Wikipedia(19 March 2020).
Retrived from https://en.wikipedia.org/wiki/Leibniz_formula_for_%CF%80.

[9] Benson Mates. Elementary Logic. Oxford University Press, 1972.
ISBN 019501491X, 9780195014914.

[10] Gottlob Frege. On Sense and Reference ["Über Sinn und Bedeutung"], Zeitschrift für Philosophie und philosophische Kritik, vol. 100, 1892.

[11] Willard V. Quine. On What There Is. The Review of Metaphysics, Vol. 2, No. 5, pp. 21-38, 1948.

[12] Richard H. Hammack. Book of Proof. ?, 2018. ISBN 0989472116, 9780989472111.

[13] Quo-Shin Chi. Foundation for Higher Mathematics. Lecture notes, 2019. Retrived from https://www.math.wustl.edu/~chi/.