ASPECTS OF A LOGICAL THEORY OF ASSERTION AND INFERENCE

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Abstract. The aim here is to investigate assertion and inference as notions of logic. Assertion will be explained in terms of its purpose, which is to give interlocutors the right to request the assertor to do a certain task. The assertion is correct if, and only if, the assertor knows how to do this task. Inference will be explained as an assertion equipped with what I shall call a justification profile, a strategy for making good on the assertion. The inference is valid if, and only if, correctness is preserved from the premiss assertions to the conclusion assertion. Most of this is a spelling out of views on assertion and inference presented by Per Martin-Löf in lectures since 2015.

1. Introduction

The works of Per Martin-Löf and Dag Prawitz stand apart from much contemporary work in logic by the prominence they give to the notion of inference. Although inference has been much discussed in the recent philosophical literature, that discussion has taken place within the context of naturalistic epistemology rather than logic. One might agree—and I think Martin-Löf and Prawitz both would agree—that inference is an epistemological notion. One might still think that also logic ought to have a say in a theory of inference. This is not an inconsistent position if one holds, as both Martin-Löf and Prawitz do, that logic partly overlaps with epistemology.

According to a traditional definition of logic, it is the science of reasoning. Reasoning, of course, includes inference, so under this definition, the theory of inference falls wholly within the compass of logic. An alternative view is that of Harman (1986), for whom logic is the theory of consequence and, as such, not immediately relevant to the theory of reasoning. Others might grant that logic includes a theory of inference, but hold that, as far as logic is concerned, the theory of inference is subservient to the theory of consequence. In particular, they might explain the validity of inference in terms of the holding of consequence (see Sundholm, 1998, 2012). For both Martin-Löf and Prawitz, by contrast, logic includes a theory of inference that is not reducible to the theory of consequence.

The validity of inference has been a main theme in the works of Martin-Löf and Prawitz, reaching back, in Prawitz's case, to his definition of the validity of argument in (Prawitz, 1971, 1973), and in Martin-Löf's case, to the introduction of meaning explanations for his type theory, first published in (Martin-Löf, 1982). More recently, Martin-Löf was forced to rethink the definition of validity of inference after becoming aware, in 2009, of a circularity in a previously assumed definition. The outcome of the reassessment, as documented in various transcribed lectures since 2015, is the topic of (Klev, 2024). That article aims at succinctness and makes a point of not drawing connections to the works of other authors, even where such connections are obvious. The aim of the present article is to discuss in some more detail the theoretical foundations underlying Martin-Löf's definition of the validity of inference and to draw a few connections to the works of some other authors.

Whereas the article (Klev, 2024) is purely expository, the present article is partly expository and partly exploratory. An inference, however that notion is explained, involves premisses and conclusion. The explanation of what an inference is therefore requires an explanation of what the premisses and the conclusion of an inference are. Martin-Löf and Prawitz have both argued that the premisses and the conclusion of an inference are assertions, or judgements, rather than, say, propositions or sentences. It turns out that clarifying the notion of assertion takes one a long way towards clarifying the notions of inference and validity of inference. Sections 2 and 3 of the present article offer a detailed presentation of Martin-Löf's theory of assertion. The rest of the article deals with inference more specifically. This part of the article is not entirely expository in nature. In particular, the proposed account of inference in Section 4 is the author's own. The definition of the validity of inference presented in Section 5, by contrast, is Martin-Löf's.

Not only inference, but also assertion has been much discussed in recent philosophical literature. In spite of the wide scope of that discussion, as witnessed for instance by the almost thousand pages long *Oxford Handbook of Assertion* (Goldberg, 2020), it has not interacted much with logic. Logic conceived of as the theory of consequence would indeed not seem to have much place for a theory of assertion. Logic conceived of as including a theory of inference, however, must also include a theory of assertion, since the premisses and conclusion of an inference are assertions. Having accepted the logico-grammatical category of assertion into logic, a choice arises whether to explain the correctness of assertion either as the truth of its content or in some other way less obviously dependent on the notion of truth of its content (see Sundholm, 2004). As we shall see in Section 3, Martin-Löf has followed the latter path. Logic, for Martin-Löf as well as for Prawitz, thus includes an autonomous theory of assertion and inference—autonomous in the sense of not being subservient to a theory of propositions equipped with a relation of consequence.

By a logical theory of assertion and inference I understand a theory of assertion and inference *qua* notions of logic. The theory developed here is autonomous in the sense just described. Assertion is explained teleologically, in terms of its purpose. Correctness of assertion is explained in terms of knowledge how to do certain tasks. An inference is explained as an assertion presented in a certain way. Validity of inference is explained as the preservation of correctness from premisses to conclusion. My discussion will be especially geared towards Martin-Löf's constructive type theory (though I shall not assume that the reader is familiar with this language). Type theory includes a large number of rules of inference, hence a proper philosophical understanding of type theory requires a proper philosophical understanding of inference and related notions. The theory of assertion and inference developed here aims to provide at least some such understanding. Since predicate logic is included in constructive type theory, the theory serves equally well as an autonomous theory of assertion and inference for predicate logic.

2. Assertion

2.1. **The nature of premisses and conclusion.** An account of inference must begin with an account of what the premisses and conclusion of an inference are. A number of alternatives may be considered.

A formula in the sense of contemporary logic is an element of an inductively defined domain. It is thus an object on a par with a natural number or a list of natural numbers (Sundholm, 2002). Just as it makes little sense to speak of an inference from natural numbers as premisses to a natural number as conclusion, so it makes little sense to speak of an inference from certain formulae as premisses to a formula as conclusion.

The notion of proposition may seem a more promising candidate. However, although propositions are not inductively generated in the way formulae are, they are understood in contemporary logic as objects. In that sense, they are on a par with natural numbers, lists of natural numbers, and indeed formulae. One way of seeing this is by considering the notion of propositional function, a central notion in logic after Frege's introduction of function/argument structure. A function is always a function from one or more types, $\alpha_1, \dots, \alpha_n$, the domain of the function, to a type, or family of types, β, the co-domain of the function. A propositional function is, by definition, a function whose co-domain is the type of propositions. The notion of propositional function therefore presupposes that propositions form a type. The inhabitants of a type, however, are objects. (Here and in what follows, I use "object" of the inhabitants of any type, including function types. This use of the term is therefore wider than Frege's.) Since the premisses and conclusion of an inference are not objects, the notion of proposition is therefore unfit for serving that role.

Frege, in his writings, gave pride of place to the notion of judgement. Once the notion of judgement is recognized as properly logical, it is the obvious candidate to fill the office we are seeking to fill: the premisses and conclusion of an inference are judgements. In drawing an inference, we make a judgement—the conclusion—on the basis of certain other judgements—the premisses. Frege was simply following the tradition (e.g. Arnauld and Nicole, 1662; Kant, 1800) when he took the premisses and conclusion of an inference to be judgements.[1] The term "judgement" later fell out of favour with logicians and logically minded philosophers (e.g. Carnap, 1934, p. 1), before it was reinstated by Martin-Löf (1982) as a means to distinguish terminologically the logico-grammatical category of the theorems in his type theory from the logico-grammatical category of propositions (see Martin-Löf, 2011). It is now standardly employed in the literature on type theory and indeed in computer science logic more broadly. Upon introducing the term "judgement", Martin-Löf also started calling the premisses and conclusion of an inference by that name (Martin-Löf, 1982, pp. 161, 166). Before explaining Martin-Löf's most recent account of judgement in the following sections, let me mention two other candidates for the role of premisses and conclusion in an inference.

The notion of sentence is primarily a grammatical or linguistic one and seems for that reason unfit for serving a logical role. It does not belong to the same

[1] The tradition also uses term "proposition" for judgement, or for the verbal expression of a judgement. "Proposition" receives its modern sense only with (Russell, 1903); see Martin-Löf (1996, Lecture 1).

conceptual sphere as the premisses and conclusion of an inference. A sentence is the primary linguistic vehicle by means of which we can express the premisses and conclusion of an inference. That premisses and conclusion can be so expressed does, however, not mean that they are sentences. We can express hopes and requests by sentences, but that does not make sentences themselves into hopes and requests.

In the recent philosophical literature on inference, the premisses and conclusion of an inference are often called beliefs. As far as I know, no notion of belief has at present been delineated that constitutes a logico-grammatical category separate from that of proposition or judgement (or judgemental content, a notion that we shall discuss below). In theories of belief revision and in doxastic logic, for instance, beliefs are treated as propositions. On the basis of the foregoing discussion, we see that this is in tension with taking the premisses and conclusion of an inference to be beliefs. Treating beliefs as judgements, by contrast, is more promising in this respect.

2.2. **Speech acts and mental acts.** I shall use the terms "judgement" and "assertion" interchangeably, though for the most part I shall prefer "assertion". Whereas judgement is naturally thought of as an interior act, assertion is naturally thought of as a corresponding exterior act. Whatever the order of priority is between the two, judgement and assertion share the same formal, or logical, structure, and that justifies using the terms interchangeably in this article.

The logical structure in question is the force/content structure, common to speech acts and mental, or intentional, acts alike. It may be written schematically as

$$\Phi\, C$$

where Φ is the force and C is the content. This is the general logical structure, at the interior level, of mental, or intentional, acts, and at the exterior level, of speech acts. At the exterior level, it might be more natural speak of mood rather than force, but because of the parallelism between the two levels, we may use "force" in both places. Husserl (1901, Investigation V §20) emphasized that the two components—which he called act quality and act matter, rather than force and content—may vary independently of each other. One might assert that there is life on Mars, wish that there is life on Mars, or question whether there is life on Mars. Here the content stays the same, while the force varies. On the other hand, one might assert, not only that there is life on Mars, but also that it rained yesterday or that every map is colourable by four colours. Here the force stays the same, while the content varies.

The force of an act determines which kind of act it is. Judgements and assertions have the assertoric force, which Martin-Löf writes using the turnstile symbol, $\vdash$. In this he follows Whitehead and Russell (1910, p. 8) rather than Frege, for whom the turnstile was a composite symbol, consisting of a horizontal content stroke and a vertical judgement stroke. The structure of a judgement, or of an assertion, is, accordingly, written as follows:

$$\vdash C$$

We shall now explain each component in turn: first, the assertoric content, C, and then, the assertoric force, $\vdash$.

2.3. **Assertoric content.** It is common to regard both the content of a speech act and the content of a mental, or intentional, act as a proposition, where proposition is understood as in modern logic, hence as an object of a certain type. The objecthood of propositions, however, make them unfit for playing the role of content. An object cannot be asserted, or wished, or asked as a question. For instance, a set of possible worlds or a type cannot be asserted, or wished, or asked as a question. What can be asserted, or wished, or asked as a question is the content that the actual world is a member of a set, A, of possible worlds or the content that a type, A, is inhabited. One must distinguish the proposition A from the content that A is true, however the notions of proposition and truth of proposition are explained.

proposition	content
set A of possible worlds	the actual world is a member of A
type A of proof objects	the type A is inhabited

Considerations along these lines led Martin-Löf (2003) to distinguish the notion of proposition from the notion of content of a speech act or intentional act. Assertoric force applies to a content and not to a proposition.

Every proposition, A, gives rise to the content that A is true, as outlined in the table above for two different conceptions of proposition and a corresponding conception of truth of proposition. We thus recognize a content-forming operation, which we write as a postfixed "true", that takes a proposition A and yields the content

$$A \text{ true}$$

This is the form of content on which propositional logic is based, but it is far from being the only form of content that one might consider, as we shall see presently.

Martin-Löf's explanation of the notion of content is central to his theory of assertion: the content of a speech act, or intentional act, is a task—something to do (Martin-Löf, 2017a,b). This explanation lifts to the level of content the explanation of proposition as an expectation, or intention (Heyting, 1931), or as a task (Kolmogorov, 1932). In illustrating this explanation through examples of forms of content and the tasks associated with them, I will take for granted Martin-Löf's meaning explanations for his type theory (Martin-Löf, 1982, 1984, 1993). Other meaning explanations might identify other tasks with the forms of content considered here.

The task identified with a content of the form

$$A \text{ true}$$

is performed by exhibiting a proof of the proposition A. A proof of the proposition A is not to be confused with a demonstration of the judgment

$$\vdash A \text{ true}$$

A proof of a proposition, A, also called a proof object, may be regarded as a truthmaker of A (Sundholm, 1994). Under the propositions-as-types principle, a proof of A is an object of a certain type, namely the type of proofs of A.

In the language of type theory one exhibits a proof of A by making a judgement of the form

$$\vdash a : A$$

Here is another task: $a : A$. It is performed by exhibiting a canonical proof, c, of A and evaluating a to c. The form of the canonical proofs of A is determined by the introduction rule (or rules) for the outermost logical operator in A. (We are assuming here that A is a canonical proposition, meaning that its outermost operator is associated with a formation rule.) To evaluate a means systematically to unfold the defined objects employed in the build-up of a (cf. Martin-Löf, 2021, pp. 505-506). For instance, if a has the form $\mathbf{fst}(\langle b, c\rangle)$, then one step of evaluation, making use of the definitional equation $\mathbf{fst}(\langle b, c\rangle) \equiv b$, replaces the defined object a by the corresponding defining object b. If b is in canonical form, then the evaluation terminates with b as value. Otherwise, the evaluation continues.

If c is a canonical proof of A, then the task

$$c : A$$

is performed by exhibiting the immediate subpart (or -parts) of c. For instance, if c has the form $\langle d, e\rangle$, then the task is performed by exhibiting the objects d and e together with their types.

The type of functions from type A to type B is written $(A)B$. A task of the form

$$f : (A)B$$

is performed by, firstly, performing the task $f(a) : B$ whenever one is provided with an object a of type A, and secondly, performing the task $f(a) = f(a') : B$ whenever one is provided with identical objects a and a' of type A.[2] These are, as it were, conditional tasks: *if* provided with an object, a, of type A, then perform $f(a) : B$; and *if* provided with identical objects, a and a', of type A, then perform $f(a) = f(a') : B$. We shall meet with a closely related task in our discussion of hypothetical assertion in Section 2.6 below.

It should be noted that none of the tasks C above are formulated as: demonstrate $\vdash C\,!$ Indeed, although demonstrating $\vdash C$ is a well-defined task, it cannot serve as an explication of the content C, on pain of circularity. (We shall not know what it means to demonstrate $\vdash C$ unless C has already been explained.) Nor are any of the tasks above formulated as: demonstrate $\vdash C_1, \ldots, \vdash C_n\,!$, where these assertions could serve as the premisses in an inference with conclusion $\vdash C$. The notions of inference and demonstration play no role in the formulation of these tasks. The tasks rather involve notions such as evaluation to canonical form, as in "evaluate a to the canonical proof c of $A\,!$"; analysis into parts, as in "analyze $\langle b, c\rangle$ in $B \wedge C$ as b in B and c in $C\,!$"; and arbitrary instantiation, as in "given any a in A, do $f(a) : B\,!$".

2.4. **Assertoric force.** Assertion is characterized formally by the presence of assertoric force. If a speech act has the assertoric force, then it is an assertion, and it will be treated as such by interlocutors, unless the context requires otherwise (for instance, if the speaker is a serial liar). Martin-Löf refers to Bally (1932), who based his linguistic theory on the force/content distinction, rather than on the subject/predicate distinction. Within such a theory, being able to recognize the force of an utterance is considered part and parcel of linguistic understanding. Against this background, a purely formal characterization of assertion as the presence of assertoric force makes good sense.

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[2] On the notion of identity in type theory, see (Klev, 2022).

Besides this formal characterization, Martin-Löf also offers a contentual characterization of assertion. He proposes that acts quite generally are to be explained teleologically, in terms of their purpose. To state the purpose of assertion, one needs recourse to the notion of an interlocutor, someone who receives the assertion, in addition to the assertor, who makes the assertion. In the typical case, the interlocutor will be another person, but it may also be the assertor, in a "dialogue of the soul with itself" (Plato *Sophist* 263e). This observation justifies regarding assertion and judgement as having the same purpose.

The purpose of an assertion—thus runs Martin-Löf's suggestion—is to grant interlocutors the right to request the assertor to perform the task that constitutes the content of the assertion. The purpose of an assertion, $\vdash C$, is to grant interlocutors the right to request the assertor to perform C. Such a request creates an obligation on the part of the assertor to perform C. An assertion fulfilling its purpose thus creates a conditional obligation on the part of the assertor to perform the task C if requested to do so by an interlocutor. This formulation of the purpose of assertion presupposes, of course, that it makes sense to speak of the task constituting the content of an assertion—which it does, since the content of an assertion has been explained to be a task.

Martin-Löf is not the first to let the notion of purpose be central to an account of speech acts. Searle (1969, ch. 3) called the purpose of a speech act its essential feature, later adopting the technical term "illocutionary point" for this feature (Searle, 1975; Searle and Vanderveken, 1985). With each illocutionary force, Φ, there is associated an illocutionary point, namely, the purpose of performing a speech act with force Φ. Searle's account of the illocutionary point of assertion differs from Martin-Löf's: for Searle, the illocutionary point of an assertion is to represent an actual state of affairs (Searle, 1969, p. 66), or "how the world is" (Searle and Vanderveken, 1985, p. 94).

Martin-Löf's account of assertion bears close resemblance to that of Brandom (1983), though there is at least one important point of difference. Brandom distinguishes two purposes, or functions, of assertion. Firstly, by asserting J, the assertor gives others the right to assert J and to employ that assertion in their reasoning. Brandom calls this the dimension of endorsement of assertion. Secondly, by asserting J, the assertor takes on the conditional responsibility of justifying J if challenged. This responsibility is fulfilled by making one or more further assertions from which J may be inferred, which assertions may be challenged in turn. Brandom calls this the commitment dimension of assertion.

Martin-Löf agrees with Brandom that assertions made by one person may be employed as the premisses of an inference drawn by others (see Section 5.1 below). Martin-Löf differs from Brandom in how he conceives of the commitment dimension of assertion. For Brandom, an assertor of J takes on a commitment, if challenged, to provide premisses from which J may be inferred. Invoking inference here, however, in what is, in effect, part of an explanation of assertion, creates a threat of circularity. Since the explanation of inference itself must involve the notion of assertion, Brandom appears to want to explain assertion and inference in terms of each other. Martin-Löf avoids the threat of circularity, since he regards the responsibility undertaken in making an assertion J to be that of performing the task constituting the content of J, and as we have seen, that task is not defined in terms of inference.[3]

The idea that to assert is to take on responsibility of some kind was central to Peirce's account of assertion (Brock, 1981, fn. 30), as Martin-Löf has noted in his lectures (Martin-Löf, 2015, 2016, 2020). The nature of this responsibility appears not to have been clarified with the same level of detail in Peirce's work as it later has been by Brandom, Martin-Löf and others (e.g. Watson, 2004; Rescorla, 2009). Peirce seems primarily to concentrate on the adverse consequences that awaits an assertor whose assertion turns out to be incorrect. These may be legal, such as accusation, or even conviction, of perjury, or they may be reputational, such as damage to the credibility of the assertor. Responsibility to justify or otherwise make good on one's assertion seems to play less of a role in Peirce's account. This distinguishes Peirce's account from that of Brandom and Martin-Löf (cf. Shapiro, 2020).

2.5. **Assertion and request.** Martin-Löf's teleological explanation of assertion invokes the notion of request: an assertion that fulfils its purpose grants interlocutors the right to request the assertor to do the task that is the content of his assertion. Request, on the other hand, is explained partly in terms of assertion, since a request is always made in response to an assertion. Just as the acts of question and answer, or the acts of command and obeying, assertion and request are explained as a couple (Martin-Löf, 2020, p. 88).

When philosophers treat assertion as one part of a couple, the other part they have in mind is usually not request, but rather rejection. Already Aristotle gave prominence to the distinction between affirmation and denial (e.g. *On Interpretation* 6). More recently, Rumfitt (2000) has argued that taking rejection to be a primitive speech act besides assertion allows for the development of an inferentialist semantics validating classical logic. Rumfitt equips each formula of propositional logic with either a plus or a minus sign, indicating, respectively, assertion and rejection (ibid. p. 803). Rather than illocutionary forces, however, these signs are, to my mind, better regarded as indicating content-forming operators. The plus sign stands for the true-operator above, forming the content A true from the proposition A, whereas the minus sign stands for a corresponding false-operator. Under this reading, one may accept Rumfitt's rules as meaning determining for the logical connectives, yet resist recognizing rejection as a separate speech act. Where the constructive and the classical logician differ is, not in which illocutionary forces they recognize, but rather in how they explain the content-forming false-operator: whereas the constructive logician takes it to be defined in terms of propositional negation, as in (Martin-Löf, 1995), the classical logician takes it to be primitive. The argument that we must recognize a separate speech act of rejection in order to make sense of Rumfitt's plus and minus signs is therefore not conclusive. We are free to regard request as the counterpart of assertion.

[3]Brandom (1983, p. 641) uses the apt term "task responsibility", introduced by Baier (1966). For Baier, task responsibility, which includes the responsibility to perform specific tasks, differs, for instance, from responsibility in the sense of accountability and responsibility in the sense of liability.

That a request may always be made in response to an assertion is the content of the following rule:

$$\frac{\vdash C}{?\, C}$$

If the speech act above the line has been made, then the speech act below the line may be made. The speech act below the line is a request, as indicated by the question mark. With this diagram in mind, Martin-Löf's teleological explanation of assertion can be displayed diagrammatically as follows:

$$\text{(Ass)} \qquad \frac{\vdash C \qquad ?\, C}{C \text{ done}}$$

Provided an assertion, $\vdash C$, is made, then an interlocutor may respond by requesting the assertor to do C, thereby creating an obligation on the part of the assertor, who must respond by doing C. The diagram (Ass) is therefore to be read as: if the speech acts above the line have been made, then the act below the line *must* be made. A diagram giving a rule of inference is, by contrast, to be read as: if the speech acts above the line have been made, then the speech act below the line *may* be made.

A remark by Martin-Löf (2020, 2022) that I shall not be able to explore further here is that (Ass) has the form of an elimination rule for the assertion sign: the assertion sign occurs in the left-hand "premiss", but not in the conclusion.[4] The corresponding introduction rules are the usual inference rules, be they introduction or elimination rules in the standard sense—or rules falling outside the introduction/elimination pattern altogether, for that matter.

Whereas the schematic rule (Ass) may be regarded as meaning determining for the assertion sign, specific instances of it may be regarded as meaning determining for various forms of content. By instantiating the entirely schematic "C" by a less schematic indication of content, the resulting instance of (Ass) may be read as a description of tasks having that form. For example, let C be A true, where A is a proposition. We have explained this task as that of exhibiting a proof of A, where proof is to be understood in an objectual sense, as a proof object. The corresponding instance of (Ass) is the following rule:

$$\frac{\vdash A \text{ true} \qquad ?\, A \text{ true}}{\vdash a : A}$$

To perform the task $a : A$, one must exhibit a canonical proof, c, of A and evaluate a to c. Using an arrow, $\rightarrow$, to indicate evaluation, this task may be presented as the following instance of (Ass):

$$\begin{array}{c} \vdash a : A \qquad ?\, a : A \\ \hline \vdash c : A \\ a \rightarrow c : A \end{array}$$

The various rules that arise in this way from (Ass) by instantiating C with the forms of content employed in Martin-Löf's type theory are called the *dialogue rules* of type theory. These rules have been discussed by Martin-Löf and by the author in lectures, but a thorough written account remains to be made (though see Klev, 2024). For present purposes, the details do not matter. Only two observations are important. Firstly, the dialogue rules are not rules of inference. Secondly, the dialogue rules for a form of content, C, spell out the commitments undertaken

[4]If the task C involves making one or more further assertions, then the assertion sign will appear in the conclusion. Each task that makes up the content of one of these assertions must, however, be simpler—in some sense—than C, so that after finitely many applications of (Ass), one reaches a task that does not involve the making of further assertions.

by someone asserting $\vdash C$. The notion of inference is thus not involved in the explanation of assertion, not even in the more formal explanation in terms of rules. It is the dialogue rules that explain assertion.

2.6. **Assumptions.** Let us pause to consider an objection to the thesis that the premisses and conclusion of an inference are assertions. In an inference made under assumptions, it might be argued, the premisses are not all assertions. A premiss might depend on one or more assumptions, or might indeed be an assumption itself, and such a premiss is not an assertion. The objector might have in mind derivations in natural deduction, where assumptions occur at leaf positions and remain open until they are discharged. Any step in the derivation made while an assumption is still open may be understood contentually as an assertion made under an assumption, and that, the objector might say, is not an outright assertion, but some other speech act, perhaps better called "conditional assertion", following a suggestion of Quine (1952, § 3).[5]

This is not the place to discuss the merits of admitting a speech act of conditional assertion or a speech act of assumption. It suffices for us to see that we can do well without either. The clue is the sequent formulation of natural deduction, used by Gentzen (1936), where the nodes in a derivation are labelled by sequents, $\Gamma \Rightarrow A$, rather than single formulae. To the left of the arrow is a record, Γ, of all assumptions open at the node in question. A leaf node introducing an assumption is labelled by a tautologous sequent, $A \Rightarrow A$.

We can manage without postulating novel speech acts of conditional assertion and assumption because we may think of assertoric force as attaching directly to a sequent—or, more precisely, to a content of conditional form, the content represented by a sequent. Instead of introducing novel illocutionary forces, we thus introduce a novel form of content. In the language of propositional logic the conditional form of content is

$$A_1 \text{ true}, \ldots, A_n \text{ true} \Rightarrow B \text{ true}$$

where all the A's and B are propositions. This contrasts with categorical content, which has the simpler form A true. We call an assertion whose content has conditional form a hypothetical assertion.

It remains to explain content of conditional form. In this we follow the meaning explanations of hypothetical judgements first given in (Martin-Löf, 1982). Our explanation does, however, have a simpler form than Martin-Löf's, since it serves a simpler language, namely the language of propositional logic rather than the language of type theory.

Just as any other content, a content of conditional form is a task. The task

$$A \text{ true}, \Gamma \Rightarrow B \text{ true} \tag{Cond}$$

is performed by asserting

$$\vdash \Gamma \Rightarrow B \text{ true}$$

provided an interlocutor has asserted $\vdash A$ true. In the form of a dialogue rule, the task may be presented as follows:

$$\frac{\vdash A \text{ true}, \Gamma \Rightarrow B \text{ true} \qquad \vdash A \text{ true}}{\vdash \Gamma \Rightarrow B \text{ true}} \tag{Hyp}$$

[5]Quine (ibid.) appears to attribute this suggestion to Philip Rhinelander.

The right-hand assertion above the line is made by an opponent as part of the request

$$? \ A \text{ true}, \Gamma \Rightarrow B \text{ true}$$

Upon receiving this request from an opponent, the proponent must make the assertion below the line, $\vdash \Gamma \Rightarrow B$ true. The task (Cond) is thus a conditional one: to assert $\vdash \Gamma \Rightarrow B$ true provided an interlocutor asserts $\vdash A$ true. We have already met with conditional tasks when discussing content of the form $f : (A)B$.

The monological—in contrast to dialogical—understanding of the rule (Hyp) is just a modus-ponens-like rule for the sequent arrow. In (Klev, 2023) that rule is taken as meaning determining for the sequent arrow and shown to entail a more general rule—a cut rule—where the right-hand premiss itself is a hypothetical assertion. Two changes are necessary when passing from the monological to the dialogical understanding of the rule (Hyp): Firstly, the right-hand premiss is to be asserted by the opponent, rather than by the proponent, who asserts the left-hand premiss. Secondly, the proponent becomes obligated to assert the conclusion provided both of the premisses have been asserted. By contrast, even if the premisses of a rule of inference have all been asserted, one is not obligated to assert the conclusion—one is permitted, not obligated.

3. Correctness of assertion

3.1. **Knowledge account of correctness of assertion.** According to the knowledge account of assertion (Williamson, 2000, ch. 11), assertion is the unique speech act governed by the constitutive rule that one may assert J only if one knows J. That this rule is constitutive of assertion is taken to mean that it somehow captures the essence of assertion. For Martin-Löf, the essence of assertion is captured by its purpose, which is to commit the assertor to performing a certain task if requested to do so by an interlocutor. In the terminology of MacFarlane (2011), this is a commitment account of assertion. (We may add that it is a *teleological* commitment account. The significance of this qualification will become clearer in Section 3.3 below.) Martin-Löf does, however, also hold a version of the knowledge account—not as an account of assertion, but as an account of the correctness of assertion. He reformulates the knowledge account as an account of correctness: an assertion J is correct if, and only if, the assertor knows J.

This is not saying much unless some account of what it is to know an assertion is provided. Martin-Löf provides an account relying crucially on his characterization of assertoric content as a task: to know $\vdash C$ is to know how to do the task C. Knowing an assertion is thus an instance of knowledge-how, or practical knowledge.

Instead of stipulating outright that to know $\vdash C$ is to know how to do C, Martin-Löf (2019, 2022) reaches this equation in three steps.

In the first step, knowing $\vdash C$ is identified with knowing C to be true. The notion of truth involved here is truth of content, not truth of proposition (nor truth of assertion, for which we have adopted the term "correctness"). Various explanations can be given of this notion of truth (Sundholm, 2004). If the only form of content recognized is the form A true, where A is a proposition, then truth of content could be explained in terms of truth of the ingredient proposition, A. Another alternative is to say that C is true if, and only if, it can be judged with evidence. The latter formulation, close to ones that can be found in Brentano's late writings (Brentano, 1930, pp. 135, 139), could be spelled out by a constructivist as "C is true if, and only if, the judgment $\vdash C$ can be demonstrated". We shall return to this conception of truth of content below.

Martin-Löf's explanation is neither of these. The second step of his analysis equates the truth of C with the doability of C. With reference to Husserl's account of truth and evidence in terms of the fulfilment of an intention (Husserl, 1901, Investigation VI §39), Martin-Löf calls this step phenomenological.

The third and last step is the characteristically constructive one, in which knowledge of the doability of the task C is identified with knowledge how to do C. The resulting list of equations is presented as follows by Martin-Löf (2022):

$$\begin{aligned} & \text{to know } \vdash C \\ = \; & \text{to know that } C \text{ is true} \\ = \; & \text{to know that } C \text{ is doable} \\ = \; & \text{to know how to do } C \end{aligned}$$

3.2. **Assertoric versus apodeictic knowledge.** Knowing J in the sense of knowing how to perform the task that is the content of J is the unqualified mode of knowing an assertion. Let us call this *assertorically* knowing J. A more elevated mode of knowing J is to have demonstrated or otherwise scientifically justified J. Let us call this apodeictically knowing J. These are different modes of knowledge, since having demonstrated J is different from knowing how to perform the task that is the content of J.

Every piece of apodeictic knowledge is also a piece of assertoric knowledge. The argument for this inclusion given in (Martin-Löf, 2019) shows that the conclusion of every demonstration is known assertorically. (The argument involves the definition of the validity of an inference, which will be given in Section 5 below.) To illustrate that not every piece of assertoric knowledge is a piece of apodeictic knowledge, Martin-Löf (2016) cites knowledge by testimony, knowledge that we have on trust.

In order to see that knowledge by testimony is assertoric knowledge, it is useful to distinguish between direct and indirect ways of performing a task. When specifying a task C by giving its dialogue rules or otherwise, we describe the direct way of performing it. For instance, the direct way of performing a task of the form A true is to exhibit a proof of A. We cannot lay down, once and for all, all the indirect ways of performing the task that is the content of an assertion J, but we can point to a prominent way of doing so: reference to someone else who has made J. For instance, I can perform the task that is the content of the assertion

(V) The Vltava River is 430 kilometres long

by referring to a trusted source, such as an encyclopaedia or my geography teacher. That is how I fulfil the obligation created by an interlocutor's challenging my assertion of (V). Reference in this sense is therefore among the indirect ways of performing the task that is the content of (V). Since I thus know how to perform this task, albeit indirectly, my knowledge of (V) counts as assertoric knowledge.

In earlier lectures, Martin-Löf had not yet recognized the mode of assertoric knowledge, hence knowing J was explained in terms of apodeictic knowledge only (Martin-Löf, 1996, Second Lecture). The distinction between these two modes of knowledge was introduced in (Martin-Löf, 2015) and has since been fundamental to his account of assertion and inference. The terminology "apodeictic" versus "assertoric" knowledge, introduced in (Klev, 2024), was inspired by connections pointed out in (Martin-Löf, 2016) between these modes of knowledge and Kant's assertoric and apodeictic modalities of judgement. Martin-Löf had previously spoken about knowledge in the qualified versus knowledge in the unqualified sense (2015); knowledge in the strong versus knowledge in the weak sense (2016); and knowledge$_1$ and knowledge$_2$ (2019). The extension of "assertoric" from naming a modality of judgement to naming a mode of knowledge is not entirely novel: Brentano (1925, pp. 71, 165) speaks of both assertoric judgement and assertoric knowledge. (An internet search yields several similar results.)

Both assertoric and apodeictic knowledge may be understood as true justified belief, provided the terms involved in this traditional definition of knowledge are appropriately understood. By belief one must understand judgement. By true belief one must understand correct judgement. The difference between assertoric and apodeictic knowledge lies in the form of justification required by each. In order to justify a piece of assertoric knowledge, J, one must perform the task that is the content of J when prompted to do so by an interlocutor. In order to justify a piece of apodeictic knowledge, J, one must provide a demonstration of J.

3.3. **Ought implies can.** According to the ought-implies-can principle, one is obligated to do C only if one can do C. Applied to Martin-Löf's teleological explanation of assertion, it follows that an assertion need not necessarily fulfil its purpose. Indeed, although the creation of an obligation to do C is the purpose of asserting $\vdash C$, it is no part of Martin-Löf's account that every assertion fulfils this purpose. If I cannot do C, then, by the ought-implies-can principle, there is no obligation for me to do C, hence, in particular, no obligation for me to do C upon request. My assertion of $\vdash C$ therefore fails to fulfil its purpose, namely, it fails to create the conditional obligation that I do C if requested to by an interlocutor. It is an assertion, since its force is the assertoric force, but it is not a purpose-fulfilling assertion.

In other (perhaps more common) versions of the commitment account of assertion, the creation of a suitable obligation is a necessary condition for a speech act to count as an assertion. Shapiro (2020, p. 80), for instance, explains a "generic dialectical norm account" of assertion by saying that assertion is "constitutively characterized" by the assertor's taking on an obligation to respond in certain ways to appropriate challenges. According to such an account, a speech act that fails to create a suitable obligation is not an assertion. The ought-implies-can principle may still hold, namely, if the obligation created by an assertion can always be met. Shapiro includes retraction as one of the ways in which the assertor may meet the obligation created by an assertion. Since every assertion can be retracted, the obligation created by an assertion can always be met. Shapiro's generic dialectical norm account of assertion is therefore compatible with the ought-implies-can principle, even if it takes every assertion to create an obligation.

For Martin-Löf, by contrast, not every assertion fulfils the purpose of creating a suitable obligation. Martin-Löf (2022) notes that being purpose fulfilling is a natural notion of correctness for acts quite generally and for the speech act of assertion in particular. The question then arises of how the two notions of correctness of assertion relate to each other: correctness in the sense of the knowledge account and correctness in the sense of being purpose fulfilling. Martin-Löf (2020, p. 90) argues as follows that they are extensionally equivalent.

If an assertor knows how to do C, then his assertion $\vdash C$ fulfils its purpose, since a conditional obligation to do C upon request is indeed created. In the other direction, Martin-Löf relies on the ought-implies-can principle. For a conditional obligation to be created by an assertion, the assertor must, by the ought-implies-can principle, be able to perform the task that is its content. Hence, if a conditional obligation is indeed created by an assertion of $\vdash C$, the assertor must know how to perform C, whence the assertion is correct in the sense of the knowledge account.

For Martin-Löf, therefore, the assertion $\vdash C$ creates an obligation on the part of the assertor to do C if requested to by an interlocutor if, and only if, it is a correct assertion.

3.4. **Incorrectness.** An assertion is incorrect if, and only if, it fails to fulfil its purpose as an assertion: it fails to give interlocutors the right to request the assertor to perform the task that is the content of the assertion. It fails to do so because the assertor is not able to perform the task that constitutes the content of the assertion, whence, by the ought-implies-can principle, no obligation is in fact created. An incorrect assertion is thus an assertion in form only. It is an assertion much in the sense that a blunt knife is a knife: it has the form of a knife, but it fails to fulfil the purpose of a knife.

The notion of an incorrect assertion is useful in some reflective contexts. An assertion, $\vdash C$, was sincerely made: the assertor believed he would be able to perform the task C upon request. It turns out, however, that he is not so able. His assertion was incorrect. It does not follow that he knows, or that we know, C to be undoable. Perhaps, unbeknownst to the assertor, someone else is able to do C. Knowing C to be undoable means knowing positively, as it were, that C is undoable. The assertor's failure to do C is not enough. It may be difficult to say in general what the required positive knowledge consists in, but for a task of the form A true, the criterion is clear: to know A true to be undoable requires knowing that the proposition A has no proof, that it is uninhabited as a type. Martin-Löf (2013) shows that this positive form of knowledge satisfies a version of excluded middle: from the undoability of the task A true, the doability of the task $\neg A$ true may be inferred. Lifting this to the level of assertion, we obtain: from the incorrectness of $\vdash A$ true, the correctness of $\vdash \neg A$ true may be inferred.

4. Inference

4.1. **What is an inference?** Thomas Aquinas, in the opening of his commentary on Aristotle's *On Interpretation*, distinguished three operations of the intellect: the formation of concepts, the making of judgements, and reasoning. According to Thomas, the order of priority between these operations is as listed here, since reasoning is made up of judgements, and a judgement is made up of concepts. Just as previous commentators on Aristotle had done, Thomas notes that the same order is implicit in the traditional ordering of the books of Aristotle's Organon: its first book, the *Categories*, deals with concepts, its second book, *On Interpretation*, with judgement, and its third, fourth and fifth books, the *Prior* and *Posterior Analytics* and the *Topics*, with reasoning. As late as the 1930s, these three acts of the mind, ordered in the same way, forms the basis of a handbook in Scholastic formal logic (Maritain, 1933).

Kant and Frege are famous for having inverted the order of the first two operations, taking judgement to be more fundamental than concept (Heis, 2014). Frege may well be regarded as advocating also an inversion of the order of the last two operations, since he took inferring to be a special case of judging. In Frege's published writings, the view is clearly expressed in (Frege, 1906, p. 387):

> An inference [...] is a judgement that is made according to logical laws on the basis of judgements already made.

The same view, without the reference to logical laws, is found in a *Nachlass* piece that Frege's editors date to some time between 1879 and 1891 (Frege, 1983, p. 3):

> To judge while being conscious of other truths as grounds of justification is *to infer*.

In these quotations, which are representative of Frege's views, Frege in effect identifies the act of inference with the act of judging the conclusion of the inference. Inferring is a peculiar mode of judging, namely it is to make a judgement, the conclusion, with reference to certain other judgements, the premisses, as one's basis or ground.

The same view seems to be widely accepted in recent literature beginning with Boghossian (2014), and I will adopt it here. Prawitz (2015) prefers a different view. He takes inference to be a composite act, composed, namely, of the premiss assertions, the conclusion assertion, and "the claim that the latter is supported by the former" (p. 67). Inference is an "act of a more complex kind than that of judgement or assertion" (p. 68). Prawitz does, as far as I know, not discuss whether this composite act is equipped with assertoric force, with some other illocutionary force, or with no force whatsoever. I have not seen this last—no force—view worked out anywhere, but the two first views—assertoric force or some other force—are documented in the literature.

Neta (2013) holds that to infer $\vdash C$ from $\vdash C_1, \ldots, \vdash C_n$ is to make an assertion whose content has the form

$$C_1, \ldots, C_n \text{ and therefore } C$$

(That is Neta's view adapted to the grammar of assertion assumed here. A rendering more faithful to Neta's presentation would employ propositions, A, rather than contents, C.) This is a complex assertion, in the sense that its content is complex. Asserting it involves, in effect, asserting the premisses, asserting the conclusion, and asserting that a certain relation, called the basing relation, holds between them. This so-called basing relation would seem to correspond to what Prawitz speaks of as support: the premisses support the conclusion.

Wedgwood (2012) and Hlobil (2019) postulate a separate illocutionary force, which Hlobil calls inferential force, that attaches to a structured collection of contents, namely the contents, $C_1, \ldots, C_n$, of the premisses and the content, C, of the conclusion. Neither author says much about the structure in question, but what they do say suggests that the contents are structured in conditional form, $C_1, \ldots, C_n \Rightarrow C$. If this is how we are to understand the operand of inferential force, however, it is difficult to see what the difference is between an inferential act in this sense and a hypothetical assertion. Perhaps a further development of the view, as Hlobil admits is necessary, will reveal such a difference.

These various views contrast with the view, which I wish to defend here, that there is no separate act of inference. There is no separate act of passing from the premisses to the conclusion. There are acts of asserting the premisses and the conclusion, but there is no separate act of inferring the conclusion from the premisses. If we are to speak of an act of inference at all, it seems most appropriate to follow Frege and identify it with the act of asserting the conclusion. Inference so understood is, however, not a species of assertion. Rather, it is an assertion presented in a certain way, namely with reference to certain previously made assertions as its justification.

I will say that an inference is an assertion equipped with a justification profile, where a justification profile is an indication as to how the assertor intends to make good on his assertion. The profile may be assumed to have the form

$$\frac{\vdash C_1 \quad \dots \quad \vdash C_n}{\vdash C}$$

where $\vdash C$ is the assertion to which the profile is attached, and $\vdash C_1, \dots, \vdash C_n$ are assertions already made. By making the inference, the assertor not only asserts $\vdash C$, but also indicates that, if challenged to do the task C, he will take for granted that the tasks $C_1, \dots, C_n$ can all be done.

The purpose of inference is to ease the burden of making good on our assertions. We must always be prepared to make good on any assertion that we make. We make good on the assertion $\vdash C$ by performing the task C if requested to do so by an interlocutor. Performing the task C might, however, be quite an involved undertaking. The purpose of inference is to render the undertaking less involved. That $\vdash C$ has been inferred from $\vdash C_1, \dots, \vdash C_n$ means that we intend to complete the performance of C by relying on performances of the premiss tasks $C_1, \dots, C_n$. Performances of these tasks are taken for granted, hence we do not need to carry them out as part of our performing C.

As an example, consider the inference of $\vdash A$ true from $\vdash A \wedge B$ true. Performing the task A true, viz., exhibiting a proof of the proposition A, might be a complex undertaking. If, however, we take for granted a proof d of $A \wedge B$, the task A true is readily performed: exhibit the proof $\mathbf{fst}(d)$. Further examples will be given in Section 5.2 below.

4.2. **The taking condition.** Our account of inference satisfies Boghossian's *taking condition* (Boghossian, 2014), according to which any account of inference must portray the assertor as taking the premisses to support the conclusion. That the assertor takes the premisses $\vdash C_1, \dots, \vdash C_n$ to support his assertion $\vdash C$ means, for us, that the assertor takes for granted that he shall be able to perform C provided the tasks $C_1, \dots, C_n$ can be done (by himself or by others).

Of the various options proposed by Boghossian of construing the taking condition, our account fits best with the counterfactual construal.[6] Under this construal, inference is characterized by the truth of the following counterfactual conditional (Boghossian, 2014, p. 10): were an assertor asked (by others or by himself) why he believes the conclusion, he would cite the premisses as his reasons. The question why the assertor believes $\vdash C$ is rendered in our account as the request that he do

[6]Boghossian does not himself favour this construal, but Warren (2022) does.

C. The assertor's citing $\vdash C_1, \ldots, \vdash C_n$ as his reasons is rendered as his taking for granted performances of the tasks $C_1, \ldots, C_n$.

Lacking from the current account, but involved, apparently, in any account of inference according to Boghossian, is a reference to causality. Boghossian (p. 5) calls inference a causal process, namely where taking certain judgements $J_1, \ldots, J_n$ already made to support a further judgement J is the cause of the assertor's making J. (It is thus not the making of the premiss judgements $J_1, \ldots, J_n$, but the assertor's taking $J_1, \ldots, J_n$ to support J, that causes his conclusion judgement J.) This appeal to causality, which is widespread in the literature,[7] might do good work in a naturalistic epistemology, but it is out of place in a logical account of inference, which is our aim here. In logic we are not interested in questions of causality, in what causes us to make a certain judgement. Our interest is rather in questions of form, in questions of meaning, and in questions of truth, correctness, and validity—what is the logical form of judgement; how are judgements to be explained semantically; and what is it for a judgement to be correct? The difference between the aims of logic and those of naturalistic epistemology is well known and widely accepted since the works of Frege (1893, Preface) and Husserl (1900). I am mentioning it here merely to draw attention to it, not to suggest that the contemporary literature on inference is confused about it.

5. The validity of inference

5.1. **Definition of validity of inference.** An inference is valid if, and only if, correctness is preserved from the premisses to the conclusion: the conclusion, $\vdash C$, is correct provided all of the premisses, $\vdash C_1, \ldots, \vdash C_n$, are correct. That an assertion is correct means that it is assertorically known. Preservation of correctness may therefore be glossed as the preservation of assertoric knowledge: the conclusion $\vdash C$ is assertorically known provided all of the premisses, $\vdash C_1, \ldots, \vdash C_n$, are assertorically known. Spelling out assertoric knowledge in terms of ability, or knowledge how, we obtain the following characterization: the assertor knows how to perform the task C provided he knows how to do each of the tasks $\vdash C_1, \ldots, \vdash C_n$.

The assertions $\vdash C_1, \ldots, \vdash C_n$ might be your own, or they might be that of your interlocutors. Assertoric knowledge of these assertions is, therefore, assumed to be either your own or that of your interlocutors. In the latter case, preservation of assertoric knowledge means that you assertorically know the conclusion under the assumption that they assertorically know the premisses. Since the interlocutors know how to perform the tasks $C_1, \ldots, C_n$, the concluder may, in performing the task C, turn to them for help. Building on a remark by Sundholm, Martin-Löf (2022) notes that the validity of an inference "is tantamount to the concluder's ability to perform C when given this help". The explanation of the validity of inference in terms of preservation of correctness is thus flexible enough to encompass a dialogical, or interactive, conception of inference—where the premisses are assumed to be made by interlocutors—as well as the more traditional monological conception.

An inference is an assertion, $\vdash C$, equipped with a justification profile. If challenged to do the task C, the assertor will take for granted that the premiss tasks $C_1, \ldots, C_n$ can all be done. Validity means that the assertor knows how to do C

[7]E.g. Wedgwood (2006); Broome (2013); Warren (2022).

provided he knows how to do the tasks $C_1, \ldots, C_n$. Hence, if the inference is valid, the assertor's defence strategy will succeed: he will be able to perform C granted performances of each of the tasks $C_1, \ldots, C_n$. On the other hand, if the strategy is successful, then the assertor will know how to perform C provided he knows how to perform each of the tasks $C_1, \ldots, C_n$. The validity of an inference is therefore equivalent to the success of the justification profile with which it is equipped.

It is crucial to the definition of validity that correctness of assertion has been explained in terms of assertoric knowledge. Suppose that correctness was instead explained in terms of apodeictic knowledge: the assertion J is correct if, and only if, the assertor has demonstrated J. A demonstration is a chain of inferences, all of which, of course, have to be valid. Martin-Löf (2019) gives the following rule as meaning determining for the metalinguistic predicate "dem" of being demonstrated:

$$\frac{\text{dem}\, J_1 \ \ldots\ \text{dem}\, J_n \qquad \text{val}\left(\dfrac{J_1 \ \ldots\ J_n}{J}\right)}{\text{dem}\, J}$$

If correctness were explained in terms of apodeictic knowledge, the given explanation of validity as preservation of correctness would be circular: the validity of inference would be explained in terms of demonstration, and demonstration would be explained in terms of validity of inference, as in the displayed diagram. Martin-Löf's view is that such a circularity must be avoided.[8] The notion of assertoric knowledge and the consequent explanation of correctness allow us indeed to avoid it.

5.2. **Examples.** In order to illustrate Martin-Löf's definition of the validity of inference, let us consider three examples from the language of propositional logic: an instance of conjunction elimination, an instance of modus ponens, and an instance of disjunction introduction. Throughout we shall rely on Martin-Löf's meaning explanations, and in particular on the conception of assertoric content as a task. As a fourth example, we shall look at *ex falso quodlibet*, which we shall justify by reference to the pragmatics of assertion.

An instance of conjunction elimination has the following form:

$$\frac{\vdash A \wedge B \text{ true}}{\vdash A \text{ true}}$$

Assume that the premiss, $\vdash A \wedge B$ true, is assertorically known. Executing this knowledge, we perform the task $A \wedge B$ true by producing a proof d of the proposition $A \wedge B$. This proof ushers in a new task, namely the task of evaluating d to canonical form. Carrying out this task, we find a pair $\langle a, b\rangle$, where a is a proof of A, and b is a proof B. We know then how to perform the task A true, namely by asserting $\vdash a : A$.

The form of modus ponens is familiar:

$$\frac{\vdash A \supset B \text{ true} \qquad \vdash A \text{ true}}{\vdash B \text{ true}}$$

Assume that the premisses, $\vdash A \supset B$ true and $\vdash A$ true, are assertorically known. Executing this knowledge, we produce a proof c of the proposition $A \supset B$ and a proof d of the proposition A. These proofs usher in two new tasks, namely to evaluate c and d to canonical form. Carrying out the first task, we find a proof of

[8]Prawitz, in his Rolf Schock lecture, suggested that the circularity might be inevitable but harmless.

$A \supset B$ of canonical form. This is a proof of the form $\lambda(f)$, where f is a function from A to B. That f is a function from A to B just means that $f(a)$ is a proof of B whenever a is a proof of A. In particular, $f(d)$ is a proof of B. We therefore know how to perform the task B true, namely by asserting $\vdash f(d) : B$.

For an instance of an introduction rule, consider the inference

$$\frac{\vdash A \text{ true}}{\vdash A \vee B \text{ true}}$$

Assume that the premiss, $\vdash A$ true, is assertorically known. Executing this knowledge, we produce a proof a of A. By one of the stipulations making up Martin-Löf's meaning explanations, $i(a)$ is a canonical proof of $A \vee B$. We therefore know how to perform the task $A \vee B$, namely by asserting $\vdash i(a) : A \vee B$.

These justifications rely entirely on the meanings of the assertions involved. Let us consider the modus ponens inference in more detail. The first step of the justification relies on how the task $A \supset B$ true has been explained: it is performed by the provision of a proof of $A \supset B$. By assumption, we know how to perform this task. Doing so, we find such a proof, c. The next step relies on the explanation of what it is to be a proof of a proposition: a proof of a proposition is a method which, when executed, yields a canonical proof of the proposition (Martin-Löf, 1984, p. 9). The proof c, in particular, is such a method. Executing it, we obtain a canonical proof of $A \supset B$, which, again by definition, has the form $\lambda(f)$, where f is a function from A to B. From the canonical proof $\lambda(f)$, we extract the function f. At this stage we appeal to the other premiss, $\vdash A$ true. By assumption, we know this premiss assertorically, whence we are able to exhibit a proof of A. Doing so, we obtain a proof d of A. But then we know how to perform the task B true, namely by exhibiting the proof $f(d)$ of B. That $f(d)$ is a proof of B follows immediately from the definition of what a function from A to B is together with the fact that d is a proof of A. Thus we have shown that the conclusion, $\vdash B$ true, is assertorically known under the assumption that the two premisses, $\vdash A \supset B$ true and $\vdash A$ true, are assertorically known.

The justification of an instance of *ex falso quodlibet* has a somewhat different character and must be discussed separately. The proposition $\bot$ has—by definition—no canonical proofs, hence any task of the form $a : \bot$ is undoable, whence so is the task $\bot$ true. For any proposition A, the following inference is valid:

$$\frac{\vdash \bot \text{ true}}{\vdash A \text{ true}}$$

A quick argument proceeds by noting that correctness is indeed preserved from premiss to conclusion here, since there is no correctness to preserve in the first place. If the premiss is correct, then so is the conclusion, since the premiss is not correct. This argument seems, however, to rely on a form of *ex falso* in our metalanguage. The justification would appear to be rule circular, in the terminology of Boghossian (2000): the justification itself instantiates the same form of inference as the concrete inference that is to be justified. Even if rule-circular justification should turn out to be acceptable for certain forms of inference, perhaps including *ex falso*, it is worthwhile looking for an alternative. The explanation of assertion and correctness of assertion outlined in this article contains the seeds of such an alternative.

The assertion $\vdash D$, where D is a task known by both assertor and interlocutors to be undoable, is a breach of a contract of cooperation. The assertor pretends to take on an obligation towards his interlocutors that everyone involved knows he cannot fulfil. He is like a cutler selling a knife that everyone involved knows is blunt and useless. This breach of contract renders the dialogue null and void and any of its participants free to assert $\vdash C$, whatever the content C is.

It should be emphasized that the argument depends on our knowing D to be undoable. All parties involved must know D to be undoable if they are to regard $\vdash D$ as a breach of a contract of cooperation. Since $\perp$ true is such a task, the argument applies, in particular, to *ex falso quodlibet*. Here we thus have the outline of a justification of *ex falso quodlibet* on the basis of principles of interaction between speaker and hearer.

5.3. **Validity of rules of inference.** What has now been explained, and illustrated in these examples, is the validity of what may be called concrete inference. We are considering an inference *in medias res* and asking what it is for it to be valid. In logic, one's interest is usually in forms of inference. A form of inference may be regarded as arising from a concrete inference by means of two separate processes of generalization. Firstly, we abstract from the particular circumstances of a concrete inference. The result may be called an abstract inference. Secondly, we might consider an inference that is, to a larger or smaller extent, formalized or schematized. A good illustration is the distinction between modus ponens in its usual, general presentation, using schematic letters, and a particular instance of it. By a form of inference we shall understand an inference that is abstract and that is, to at least some extent, formalized in this sense. By a rule of inference we shall here simply understand a form of inference. (Whatever more it takes to make a form of inference into a rule need not concern us here.)

A rule of inference is valid if, and only if, all of its instances are valid. That is, a rule of inference is valid if, and only if, an arbitrary instance of it is valid. This account of the validity of a rule of inference agrees with what would seem to be a common way of justifying a rule of inference: one considers an arbitrary instance of the rule and shows that it is a valid inference, that it preserves correctness from premisses to conclusion. Good examples are provided by Martin-Löf's justifications of the various elimination rules of his type theory in (Martin-Löf, 1984): an arbitrary instance is considered and shown to be justified under the meaning explanations of the language.[9] A similar strategy is recognizable in metamathematical investigations when one shows the soundness with respect to a property P of a rule of inference—or rather, the "formal analogue" of a rule of inference (Hilbert and Bernays, 1934, p. 62), since we do not have inference proper in an uninterpreted formal language. To show soundness with respect to a property P, one considers an arbitrary concrete instance of the rule and argues that if the premisses have P, then so does the conclusion.

The notion of validity of a rule of inference is thus conceptually posterior to the notion of validity of a concrete inference. A contrasting view is held by Broome (2013), for whom an inference is correct if, and only if, it is made according to a correct rule (pp. 247, 255). Broome thus takes the correctness of a rule to be conceptually prior to the correctness of its instances. This may indeed be a natural view to take for someone, such as Broome or Boghossian, who characterizes inference in terms of rule following. In one of Broome's many succinct formulations: "in

[9] On this method of justifying elimination rules, see also Klev (2019).

reasoning you operate on the contents of your attitudes, following a rule" (p. 247, cf. p. 252). (The first quotation from Frege in Section 4.1 above also points in this direction.) That a rule is correct means for Broome, in effect, that rationality permits the making of judgements according to it (pp. 247, 255). To work out which rules are correct in this sense, and to explain why they are correct, is a task for epistemology (pp. 190, 248) and not one Broome carries out in any detail in his book. He provides only a few examples, one of which is the modus ponens permission: rationality permits me to judge $\vdash B$ true on the basis of my having judged $\vdash A$ true and $\vdash A \supset B$ true. Why rationality permits this, Broome does not say anything about in his book.

Prawitz's account of the validity of inference in his theory of grounds (Prawitz, 2015) agrees with Broome's account in its prioritizing the general rule to its instances. For the sake of simplifying the presentation, let us concentrate on inferences with just a single premiss. According to the theory of grounds, making an inference from a premiss of the form $\vdash A$ true to a conclusion of the form $\vdash B$ true involves applying a function f to any proof of A. The inference is valid if, and only if, f is indeed a function from A to B, that is, if, and only if, $f(a)$ is a proof of B whenever a is a proof of A. Prawitz speaks of grounds rather than proofs here, and his understanding of grounds differs from Martin-Löf's understanding of proofs, but that difference does not matter for our present purposes.

In the theory of grounds, then, the validity of an inference is explained in terms of the validity of the function it involves, where by the validity of a function I understand its being well-defined and of the appropriate type. In this account of the validity of inference, the notion of function plays the role that the notion of rule plays in Broome's account. In Broome's account, for an inference to be valid means for it to be made according to a valid rule. In the theory of grounds, for an inference to be valid means for the function it involves to be valid in the sense of being well-defined and of the appropriate type.

We may spell out the difference with the approach preferred in the present article by considering the rule of conjunction elimination in the language of Martin-Löf's type theory:

$$\frac{\vdash d : A \wedge B}{\vdash \mathbf{fst}(d) : A}$$

In the theory of grounds, we would say that this rule is valid because **fst** is indeed a function from $A \wedge B$ to A. In Martin-Löf's type theory equipped with his meaning explanations, we argue as in Section 5.2 that the rule is valid: we assume the premiss to be correct and show under that assumption that the conclusion is then also correct. The validity of the rule entails that **fst** is indeed a function from $A \wedge B$ to A.[10] Hence, whereas in the theory of grounds, the functionhood of **fst** is a precondition for the validity of the rule, in Martin-Löf's type theory, the validity of the rule is a precondition for the functionhood of **fst**.

In practice we may, of course, judge an inference to be valid on the grounds that it is an instance of a rule that we know to be valid. When presenting a piece of reasoning in a regimented language such as type theory, we might indeed be

[10] One also needs to show that **fst** satisfies functionality with respect to definitional identity. This is entailed by the validity of the equality part of conjunction elimination:

$$\frac{\vdash d = d' : A \wedge B}{\vdash \mathbf{fst}(d) = \mathbf{fst}(d') : A}$$

required to cite the rules that the individual inferences instantiate. The order of conceptual priority between the validity of a rule and the validity of its instances need, however, not be as suggested by this practice. We have taken the view that the validity of the instances is conceptually prior to the validity of the rule. We have mentioned Broome and Prawitz as advocates of an alternative view, according to which the validity of a rule is conceptually prior to the validity of its instances. It lies outside the scope of this article to assess that alternative in detail.

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