

Peano, Frege, and Russell's Logical Influences

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1 Introduction

The names Bertrand Russell and Gottlob Frege will almost certainly always be linked in discussions of the history of analytic philosophy. Both men are among those first to be mentioned when discussing the “founders” of analytic philosophy. On some important issues, they held very similar positions. Both were proponents of the use of precise logical languages in systematic investigations generally, and in understanding the foundations of mathematics in particular. No pair of individuals other than they can claim to be more responsible for the popularization of modern quantifier logic among academic philosophers. The two also both were logicians in the philosophy of mathematics, and argued that mathematical truths, or at least those of arithmetic, were in some sense based in or reducible to logic. Indeed, both engaged in projects attempting to carry out deductive demonstrations of mathematical results within axiomatic systems meant as solely logical. The two names are also linked because of their famous correspondence, in which Russell informed Frege of the inconsistency of his mature logical system due to the antinomy now known as “Russell’s paradox”.

But the relationship between the two men is also often misunderstood. Because Frege is older, and his important works were published earlier, it is widely believed that Russell adopted their overlapping due to Frege’s influence. This is untrue. Russell independently adopted his own form of logicism, and began working on his own symbolic logic prior to having studied Frege’s works in any detail. On these matters, it is more accurate to identify Russell’s chief early inspiration as Giuseppe

Peano and the school of mathematicians and logicians with which Peano is associated. Although Peano's work is not widely appreciated today as being especially philosophically important, Peano did hold views about the fundamental nature of mathematics that were influential on Russell, as was Peano's advocacy of precise logical symbolism. In what follows, we shall explore that influence in more detail.

Those scholars who do appreciate the influence of Peano and his school on Russell have sometimes sought to downplay Frege's influence. For example, Ivor Grattan-Guinness complained that "some commentators grossly exaggerate the extent of Frege's influence on Russell" (Grattan-Guinness 2003, 61). I think it is more accurate to say that some commentators have gotten wrong exactly how Frege influenced Russell. Reflecting on Frege's work did, I believe, greatly influence Russell's thinking about such issues as the nature of classes, functions, logical form, meaning and denotation, but his influence was more subtle and indirect than is generally realized. We shall also explore these topics in what follows.

2 Peano's Symbolic Logic and Approach to Mathematics

Giuseppe Peano (1858–1932) was an important Italian mathematician interested in, among other things, the foundations of mathematics. In 1891 he began working on what became his *Formulario Mathematico*. He enlisted a number of associates—among them Mario Pieri, Alessandro Padoa and Cesare Burali-Forti—for assistance with this project. Today this group of mathematicians is sometimes referred to as the "Peano School" (see e.g., Cantù and Luciano 2021). The goal of the project was to create a kind of universal international encyclopedia of mathematics. To make it truly international, it was decided to write as much as possible of it using newly invented symbolism to represent the logical relationships between various theorems in order to avoid relying on any one natural language. Various works by members of the School, including editions and parts of the *Formulario*, explored the axiomatic foundations of the subject. This is why the five basic principles now often seen as defining natural number theory are referred to as the "Peano axioms", although the respective roles of Peano and Dedekind in formulating these principles is a complicated affair (see Ferreirós 2005).

In the Boolean tradition of symbolic logic which had been dominant before this time, mathematical symbols such as “+”, “×”, and “=” were repurposed with logical meanings. In the algebra of logic, letters or variables were interpreted either to stand for aggregates of objects (classes), on the “primary” interpretation, or for propositions, on the “secondary interpretation”. The two interpretations were thought as mirroring each other in that each operation in the one interpretation was thought to correspond to one on the other. The sign “+” represented the union of classes on the primary interpretation; the corresponding operation on propositions is disjunction. Similarly, “×” could be used for either intersection or conjunction.

Because Peano wanted to use his new symbolism to represent the logic of mathematical proofs, it would have been too confusing to employ mathematical symbols both with their mathematical and their logical uses. However, he wanted to maintain the parallelism between the two logical meanings. Therefore he introduced the symbol “ \cup ” to be used for both disjunction and union, “ \cap ” for both conjunction and intersection, etc. Another symbol, “ \supset ”, was introduced to be used so that “ $a \supset b$ ” could mean either that b is a consequence of a (i.e., $b \subset a$ reversed) on the propositional reading, or that a is a subset (yes, subset) of b on the class reading, as in such a situation, membership in b is a consequence of membership in a . Peano also introduced “ ϵ ” for the class membership relation, but differentiated it from \supset , whereas most previous logics treated an individual being a member of class as a version of one class being a subset of another, i.e., the individual was thought of as a class with one member.

Frege had introduced the modern variable-binding quantifier in his *Begriffsschrift* of 1879, but his works were largely ignored at the time. C. S. Peirce had also worked out something similar not long after, but this work lay unpublished until years later (Beatty 1969). It was thus left to Peano to first popularize something with the power of modern quantification theory to the mathematical world. Peano’s logic had two variable-binding operators. One, written as an upside down epsilon, ε , was used as part of a class abstraction operator, so that “ $x \varepsilon \dots x \dots$ ” was taken to mean the class of all x ’s such that $\dots x \dots$. Together with a predicate “ \exists ”, true of all and only non-empty classes, Peano introduced something very similar to modern existential quantification, with “ $\exists(x \varepsilon \dots x \dots)$ ” meaning that there are x ’s such that $\dots x \dots$. The other variable-binding symbolism Peano used captured something similar to universal quantification except it was always

attached as a subscript to another operator. For example, “ $x \in a \supset_x x \in b$ ” could be taken to mean that $x \in b$ is a consequence of $x \in a$ for every value of x . With these operators, Peano’s notation is able to resolve the kind of scope ambiguities that had been a thorn in the side of earlier treatments of multiple quantification.

Emphasis was placed on the notion of rigorous proof in Peano’s works, and in giving precise definitions to mathematical concepts. Peano treated cardinal numbers as things introduced by definitions by abstraction. This kind of definition was given significant attention by the Peano school, in many ways prefiguring contemporary “abstractionist” approaches to the philosophy of mathematics (see, e.g, Mancosu 2018). In modern notation, definitions by abstraction take this form, where E represents an equivalence relation (symmetric and transitive relation):

$$f(a) = f(b) \leftrightarrow E(a, b)$$

The *something* (whatever f stands for) of a is the same as the *something* of b just in case a and b stand in relation E . On the approach adopted by Peano, expressions for numbers are not defined outright in terms of more basic symbolic, but identity claims between numbers, e.g., “the number of u = the number of v ”, are defined to be equivalent to claims about whether or not the classes that have these numbers (here u and v) can be put into one-one correspondence (essentially, what is now called “Hume’s Principle”). This equivalence is enough to allow most proofs one would make about cardinal numbers to be carried out rigorously.

3 Peano’s Influence on Russell

Russell met Peano at the August 1900 International Congress of Philosophy in Paris, and was deeply impressed with Peano’s presentation and its logical rigor, which Russell attributed to Peano’s use of this new form of symbolic logic. Russell later described this encounter as bringing about a “revolution” in his philosophical work, making his previous work “irrelevant to everything [he] did later” (Russell 1958, 11). Russell had, of course, been exposed to the algebraic tradition of logic through the work of his teacher and collaborator A. N. Whitehead (Whitehead 1898) and elsewhere, but seems not to have found it to be an especially valuable tool for his work. He

describes the months that follow meeting Peano as a kind of “intellectual honeymoon” (Russell 1958, 73; cf. Russell 1997, 13; 1998, 218–19). He quickly mastered the symbolic logic of Peano and his associates, and made significant new progress in his work on the foundations of mathematics, a project he had already begun years earlier. Russell was particularly impressed with the distinction between “ ϵ ” and “ \supset ”. Russell’s first major technical publication, “The Logic of Relations with Some Applications to the Theory of Series” (1993c), was an attempt to expand on the logic of relations in Peano-style symbolic logic. It was published in *Revue de Mathématiques*, the journal edited by Peano, a year later.

Frege is now widely known to have been skeptical about the use of definition by abstraction for defining numbers because of what is now called the “Julius Cæsar problem”. The abstractionist definition makes it possible to determine whether or not a given number, described as the number of one class or concept, is or is not identical to another number, similarly described as the number of some class or concept. But it cannot help us determine whether or not a number is or is not identical to an object not specifically described as the number of some class or concept, such as “Julius Cæsar” (Frege 1950, sec. 66). Russell recognized the importance of definitions by abstraction in Peano’s work, but like Frege, thought this kind of definition was not sufficient to completely pin down the meaning of what was allegedly being “defined”. The expression being defined by abstraction seems to be the functor “the number of ...”, where the argument spot is to be filled with the name of a class or concept. But the abstractionist definition does not seem adequate to specifically identify a unique function as what this functor represents, since there are many functions from concepts or classes to objects that will yield a distinct object when, and only when, the classes or concepts to which they are applied cannot be put in one-one correspondence (see e.g., *PoM*, §110). This led Russell to the view that full explicit definitions were necessary in these cases, and he noted that one could define the things in question as equivalence classes of all those objects bearing the equivalence relation to each other. The number of the class u , for example, could be defined as the class of all classes that can be put in one-one correspondence with u , or the class of all classes alike in cardinality with u . This is how Russell independently discovered the “Frege-Russell” definition of number, and it was advanced already in Russell (1993c), which Russell published before ever reading Frege. Russell sometimes calls the realization that one can

replace definitions by abstraction with explicit definitions involving equivalence classes which all and only entities bearing an equivalence relation to each other co-member, “the principle of abstraction”, but also, later, “the principle of replacing abstraction”, or the “principle which dispenses with abstraction” (Russell 1956, 41).

By replacing Peano’s definitions by abstraction with his own explicit definitions, Russell made it possible to provide a treatment of mathematics requiring no special non-logical vocabulary or additional non-logical axioms. In effect, he came to adopt logicism. While it is probably not accurate to describe Peano himself as a logicist, Peano’s influence on Russell’s logicism cannot be overstated (see also Cantù 2021).

Russell also made refinements to Peano’s logical symbolism and terminology. Russell thought it important to differentiate the propositional symbols from the class-theoretic ones, and thus, e.g., replaced “ \cup ” with “ \vee ” when the propositional meaning was meant, and reversed “ \supset ” to “ \subset ” when used for subset. Russell introduced the terms “material implication” and “formal implication” to differentiate respectively the usage of “ \supset ” without a subscript, flanked by closed formulas, from “ \supset_x ”, flanked by open formulas, used to assert a conditional for all values of a variable. In Peano’s logic, the simplest propositions typically took the form “ $a \in u$ ”, i.e., some individual is a member of some class, and he read the “ ϵ ” as shorthand for the Greek *ἐστι* (is-a); it essentially is a copula. Since Russell and most other logicians understood the copula as a relation between an individual and a property or intension or concept, Russell (*PoM*, §69) accused Peano as having conflated classes (extensions) with “class-concepts” (intensions of classes). Russell thought there were two separate relations, one between individuals and concepts, and one between individuals and classes, which needed to be kept separate, though he claimed it was “to some extent optional” which one we interpret “ ϵ ” to mean (*PoM*, §76).

Russell’s “intellectual honeymoon” came to an end during the Spring of 1901 when he discovered that the Peano-esque symbolic logic he had been employing led to contradictions. Peano did not quite formulate “inference rules” for his logical symbolism, at least not in a form living up to modern standards of rigor. However, he seems always to endorse transitions back and forth between statements of the form “ $a \in (x \ni \dots x \dots)$ ” using an expression for a class and the

corresponding statement directly about the individual ... a ..., which is tantamount to naïve class theory. One can then formulate Russell’s paradox, the contradiction that it appears to be true that

$$(x \ni x \sim \epsilon x) \in (x \ni x \sim \epsilon x) \equiv (x \ni x \sim \epsilon x) \sim \epsilon (x \ni x \sim \epsilon x)$$

Depending on how you interpret the “ ϵ ”, this could be taken as asserting that the class of all classes not members of themselves is a member of itself if and only if it is not, or that the concept of non-self-instantiation instantiates itself if and only if it does not.¹ Both these antinomies are now called “Russell’s paradox”.

4 The Frege-Russell Correspondence and Russell’s Appendix on Frege

Russell did not study Frege’s work until mid-1902, after he had already completed a draft of his *Principles of Mathematics* (*PoM*). The notes he took have been preserved, and consist largely of translations of Frege’s logical notation into Peano’s (Linsky 2005, 2006). Realizing the overlap between their views, and the importance and underappreciation of Frege’s works, Russell made some small amendments to the body of the draft to make mention of Frege, and began work on what was to become Appendix A of *PoM*, “The Logical and Arithmetical Doctrines of Frege”. Although the “cumbrous” nature of Frege’s symbolism is criticized, the appendix opens with fulsome praise:

Frege’s work abounds in subtle distinctions, and avoids all the usual fallacies which beset writers on Logic. His symbolism ... is based upon an analysis of logical notions much more profound than Peano’s, and is philosophically very superior to its more convenient rival. (Russell (1931), §475)

One gets the impression that Russell himself estimates that had he discovered Frege prior to Peano,

¹Russell seems to have discovered the two versions at the same time, though the first explicit description of the antinomy to be found in his extant manuscripts mentioned “predicates” not predicable of themselves; by “predicate” Russell means the concept or universal itself, not anything linguistic (Russell 1993a, 195).

it would have been Frege who would have shaped his approach to symbolic logic. The appendix does identify three main sources of difference between them, having to do with (1) whether or not there are any kinds of things, such as concepts, that cannot be made into logical subjects, (2) whether or not every judgment involving an individual can be divided into the individual and the remainder, understood as an “assertion” or “function” applied to the individual, and (3) the fact that Frege was unaware of the contradiction derived from Russell’s paradox.

These topics are also a major theme of the correspondence between them, which began in June 1902. Russell’s opening letter contains similar praise but goes on to mention that he had encountered “a difficulty only on one point”, arguing that there can be no such thing as “the predicate of being a predicate which cannot be predicated of itself” or the “class (as a whole) of those classes which, as wholes, are not members of themselves” (Frege 1980, 130–31). (Russell also mentions in passing that he had written to Peano about the same issue, but had not gotten a reply.) Frege’s terminology did not speak often of either predicates or classes, but rather of “concepts”, understood as a species of function, and of their “extensions”. In Frege’s theory of functions and objects, a function is never of the right sort to take itself as an argument, so one cannot generate a version of the paradox directly in terms of functions. But Frege did think that every function had a “value-range”, a sort of complete mapping of arguments to values, which could be considered as an object, and could go proxy for the function when considered as an object. The value-ranges of co-extensional concepts would be the same, and so Frege identified their value-ranges with their extensions. In Frege’s views, the paradox only took a single form: that of the extension of the concept *being the extension of a concept not falling under its defining concept*, and whether or not this very extension falls under its defining concept.² In any case, after receiving Russell’s letter, Frege realized that the logical system of his magnum opus, *Grundgesetze der Arithmetik* was inconsistent and in need of emendation. In his first reply, he described himself as having been “surprised . . . beyond words” and “thunderstruck”, though he expressed some optimism too, adding the “discovery is at any rate a remarkable one, and it may perhaps lead to a great advance in logic” (Frege 1980, 132)

²It is sometimes said that Frege’s views were immune to the versions involving predicates or functions, but not to the version involving classes. I think it is more accurate to say that given that Frege thinks of value-ranges as what goes proxy for functions when thought of as subjects, the two versions of the paradox collapse into one form for him; see Klement (2012).

They continued to discuss the issue, along with several related ones in philosophical logic, roughly over the next two years. Years later, in correspondence with Michael Dummett about the possibility of publishing his correspondence with Frege, Russell wrote that the letters “are mainly concerned in refuting suggestions of my own, which turned out to be inadequate” (quoted in Klement 2014b, 25). While it is true that a large part of the correspondence does concern itself with suggestions made by Russell, none of which Frege found satisfactory, this is definitely not the whole story.

The issue, also as noted, mentioned in Russell’s Appendix, whether or not concepts or functions generally can serve as logical subjects or take themselves as argument, is one major theme. This was of course occasioned by Russell’s initial description of the paradox as involving a predicate that is or is not predicable of “itself”. Russell argues in *PoM* (§47) that there is a logical kind, that of *term, individual, or logical subject*, that subsumes all entities. He took the view that there was any entity that could not be made into a logical subject as self-undermining. To express the view, one would have to advocate a proposition of the form *E cannot be a logical subject* in which *E* occurs as logical subject. Even concepts and predicates that are capable of occurring in ways other than logical subject are the very same entities as those that can also occur as subjects when we predicate things about them. Thus the same entity (Humanity) occurs one way in the proposition expressed by “Xanthippe is human”, and the other way in “Humanity is eternal”. In Frege’s views, however, there is a deep gulf between objects, which proper names refer to, and functions, which predicate expressions and other expressions with argument places, refer to. Proper names are complete expressions, whereas function expressions have a gap in them, they are “unsaturated”, and are not of the right form to fit their own argument spots. E.g., for Frege, “Xanthippe” names a person, an object, and fits in the argument spot of “... is human”, but “... is human” itself does not fit there. Frege thinks these linguistic facts are mirrored in the references of these expressions: a concept or other function is essentially unsaturated, and a first level function, a function that takes objects as argument, cannot take itself or another function as argument. There are “higher level” functions that could take the reference of “... is human” as argument, e.g., the reference of “there is an *x* such that ... *x* ...”, but this “second level” function could only itself be an argument to an even higher, third-level function, and so on.

Russell’s sloppy early habit of using linguistic-sounding terminology to mean something non-

linguistic, e.g., “term” for individual, “predicate” for monadic universal, “proposition” for state of affairs, etc., unfortunately delayed the two men making it clear where their disagreement stood. Russell originally expressed the argument that a position like Frege’s was self-undermining by discussing propositions of the form “ ξ can never take the place of a proper name [*Eigenname*]”. Russell means here propositions that state that an *object* has the property something has when it can be logical subject, not a property some linguistic expressions have but others do not. Frege naturally, however, misreads him as talking about expressions and gives him a beautiful lesson about the use/mention distinction as a response. Thankfully, Russell clarifies in a later letter:

If we leave aside names altogether and speak merely of what they mean, then [on Frege’s view] we must admit that there is no proposition in which a function takes the place of a subject. But the proposition ‘A function never takes the place of a subject’ is self-contradictory; and it seems to me that this contradiction does not rest on a confusion of a name with what it means. (in Frege 1980, 138)

In response, Frege admits that on his view, there is no valid interpretation of expressions like “... is a function” or “... is an object/logical subject” that will be true for some arguments and false for others. He claims that strictly speaking such predicates as “function”, “object” or “concept” should be rejected (Frege 1980, 141). We cannot say without speaking inexactly that an object is not a function, or a function is not an object, directly. This is the “grain of salt” Frege asks for with regard to his own writing in “On Concept and Object” (Frege 1984b). Frege suggests we can state our views exactly only by speaking about parts of language. We can say that a certain name or expression is a proper name, or a function expression, meaningfully. This response is inadequate to Russell who takes the issue not to be one of language at all. It is also probably the source of what appears to be a misreading of Frege Russell makes in his appendix where he claims that Frege thinks that when we speak of, e.g., “the concept horse” as in “the concept horse is instantiated”, using what appears to be a proper name of a function, we are speaking about the *name* itself, and not the concept (*PoM*, §481). Frege’s actual view is that such expressions refer to special non-linguistic proxy objects which serve as correlates of concepts when used as logical subjects, which might even be their value-ranges.

Interestingly, Russell claims in his first letter to Frege that “[o]n functions in particular ... I have been led independently to the same views even in detail,” (in Frege 1980, 130), but on examination, apart from both men’s insistence that (propositional) functions are important for logic and mathematics, their views on functions do seem quite different, at least at the start of their correspondence. Frege often describes functions as what is got by removing a part from a whole. One begins with *if Socrates is human, then Socrates is mortal*, and one may remove the first occurrence of Socrates and get the function *if... is human, then Socrates is mortal*, or one may remove the second occurrence and get *if Socrates is human, then ... is mortal*, or remove both and get *if... is human, then ... is mortal*. But especially on Frege’s mature views, in which the values of concepts are the two truth-values, the True and the False, it unclear how they could be understood as “wholes” from which parts might be removed. In any case, Russell did not think it was an adequate view of functions to regard them as what is obtained by simply removing constituents from propositions. In his discussion, he called such would-be entities “assertions”. Firstly, he thought that unless one began with a very simple relational proposition, removing a constituent would not allow the remainder to preserve the unity required to make up a single thing. Russell thought it was relations occurring relationally that united the constituents of a proposition. If a relatum of a relation is removed, the relation cannot relate, and no unity can remain. Secondly, if the argument spots of functions are just considered as gaps, as with *if... is human, then ... is mortal*, there is no way to differentiate the one-argument function of this form that requires both argument spots to be the same entity (in contemporary notation “ $\lambda x(x \text{ is human} \supset x \text{ is mortal})$ ”) from the two-argument function in which they can differ (“ $\lambda x \lambda y(x \text{ is human} \supset x \text{ is mortal})$ ”) (*PoM*, §§82, 482). Lastly, if every proposition can be analyzed into a subject an assertion as remainder, Russell feared we would be left with an assertion “ $\sim \dots(\dots)$ ” (or “ $\sim \varphi(\varphi)$ ”), non-self-assertibility, which would give rise to yet another form of Russell’s paradox (*PoM*, §85). He concludes that “[t]hus what Frege calls a function, if our conclusion was sound, is in general a non-entity” (*PoM*, §482). Russell thinks instead that there are a variety of things we could be talking about instead when we speak of a propositional function (*PoM*, §482). In his own writing, he speaks of propositional functions as proposition-like structures except containing variables in place of certain objects, where variables are ontologized, and understood as special kinds of logical entities which are one thing but no one thing in particular (see *PoM*, §§93, 106). At other

places, he seems to think of propositional functions, considered as “single entities”, as relations between entities and propositions (*PoM* §482). E.g., the function *x is human* would be thought of as the relation Socrates has to *Socrates is human*, Xanthippe has to *Xanthippe is human* and so on.

In any case, early Russell did not seem to think it impossible for us to name functions with complete expressions, or consider them as arguments to themselves. Frege used the notation $\hat{\epsilon}F(\epsilon)$ for the value-range of $F(\)$. In a May 1903 letter, Russell proposes repurposing that notation for the function itself, thus, e.g., “ $\hat{\epsilon}(\epsilon \neq \epsilon)$ ” would stand for the non-self-identity function.³ Such function abstracts could appear without or without their arguments, and Russell proposed to replace talk of classes in symbolic logic and the foundations of mathematics with functions instead. He wrote such things as “ $(1 \rightarrow 1)(f)$ ” to mean that f was a one–one function, or “ $\varphi \text{ sim } \psi$ ” to mean that φ and ψ applied to the same number of things. In such constructions, the “ f ”, “ ψ ” or “ ψ ” in subject position could be replaced by a function abstract such as “ $\hat{\epsilon}(\epsilon \neq \epsilon)$ ” to make a claim about a complexly defined function. In a belated reply, Frege objected strenuously to the notation, since in his mind it obscured the incomplete or unsaturated nature of functions, and ran together functions of different levels. He also pointed out that the notation could be used to define a function $\hat{\epsilon}(\sim\epsilon(\epsilon))$, which could be used to derive a paradox. By the time Russell received these objections, he claims he had already abandoned that precise approach, roughly on the grounds Frege mentioned (see Frege 1980, 166), but he continued to explore methods of replacing class-talk and quantification over classes, with talk of functions and quantification over them in the years that followed.

During the correspondence, Russell also mentioned some other versions of “Russell-like” paradoxes he had also struggled with. One involved relations. Suppose T is the relation that holds between relations R and S just in case R does not hold between itself and S . Now let us consider whether or not T holds between T and T ; it will hold just in case it does not. Frege was at first unimpressed as he also held relations to be functions which cannot take themselves as arguments. Russell however successfully adapts the paradox to involve the double value-ranges Frege had used for the extensions of relations, and Frege eventually sees the point.

³In suggesting this notation, both in this letter and his manuscripts, Russell was anticipating much later systems involving function abstracts such as Church’s lambda calculus; see Klement (2003).

Another version of the paradox involves propositions. By Cantor’s powerclass theorem, the class of all propositions should have more subclasses than members, i.e., there should be more classes of propositions than propositions. However, it seems possible to generate a distinct proposition for every class of propositions, m , e.g., the proposition stating that all members of the class are true, $p \in m \supset_p p$ in Russell’s adaptation Peano’s notation. But now let w be the class of all propositions of the form $p \in m \supset_p p$ which are not members of the class m they are about. Further consider the proposition stating that all members of w are true, viz., $p \in w \supset_p p$, and let us consider whether or not that proposition is itself a member of w . Once again, from either answer, the opposite follows. Russell also discusses this paradox in appendix B of *PoM*. This has come to be known as the “Russell-Myhill antinomy”.

Here, it was more difficult for Russell to convince Frege that this was a problem he should have been concerned about. Frege objected to Peano’s notation, Russell’s assumption that propositions are what formulas directly stand for, and Russell’s assumption that classes themselves can be constituents of propositions. Frege understood closed formulas as referring to their truth-values. The closest thing to a Russellian proposition countenanced by Frege is what he called a “thought”, which he understood as the *sense* of a complete sentence. Frege further held that only senses can be constituents of other senses. The discussion of the paradox was thus derailed by discussion of broader philosophical and semantic issues.⁴ Frege argued that it must be senses, not the objects themselves, which are constituents of thoughts, since the thought expressed by a sentence seems to change when we replace a component by something with a different sense but same reference. “Mont Blanc is more than 4000 metres high” expresses a different thought from “The highest mountain in the Alps is more than 4000 metres high”. Thus he writes, “Mont Blanc with its snowfields is not itself a component part of the thought that Mont Blanc is more than 4000 metres high” (Frege 1980, 163). Russell held a staunch realism and thought it must be possible, at least sometimes, for our consciousnesses to be in direct acquaintance with external objects without the mediation of something like a Fregean sense. He thus replies, “I believe that in spite of all its snowfields Mont Blanc itself is a component part of what is actually asserted in the proposition ‘Mont Blanc is more than 4000 metres high’. . . . If we do not admit this, then we get the conclusion

⁴For more on the lost opportunity for Frege to realize that the paradox was something he should have been concerned with, see (Klement 2001, 2002).

that we know nothing at all about Mont Blanc” (in Frege 1980, 169). We shall discuss Russell’s reaction to Frege’s sense/reference distinction in more detail below.

5 Russell’s Fregean Period and Developing Views on Functions

Frege learned of the inconsistency of his system due to Russell’s paradox while the second volume of his *Grundgesetze der Arithmetik* was already in the process of being published. He hastily added an appendix in which he proposed a revision to his theory of value-ranges according to which two different functions could be considered as having the same value-range if they differ only with regard to their value for that very value-range as argument. (He also mentioned this proposal in a letter to Russell; see Frege (1980), p. 150.) This maneuver was motivated by a more general result that there can be no function from functions to objects that always yields a distinct object for each function as argument.⁵ Russell received his copy of the new volume of *Grundgesetze* in late 1902 just as his own *PoM* was being prepared for publication. His response was very deferential. He added a short note to the end of the Appendix on Frege claiming that Frege’s proposed solution “seems very likely . . . true solution, [so] the reader is strongly recommended to examine Frege’s argument on the point”. What followed—from the end of 1902 through the discovery of the theory of descriptions in 1905—is the period in which I think Russell was most influenced by Frege, and took Frege’s and Fregean views extremely seriously. Indeed, the influence is evident even in the titles of Russell’s manuscripts of the period, which include things such as “Functions and Objects”, “On the Meaning and Denotation of Phrases”, “On Meaning and Denotation”, etc., echoing Frege’s “On Sense and Reference”, “Function and Concept”, “On Concept and Object”, and so on (see Russell 1994, 50–53, 283–296, 314–358).

What seemed to most interest Russell was Frege’s suggestion that two non-equivalent functions might “determine the same class” (i.e., share a value-range); a significant amount of Russell’s manuscripts from early 1903 explore this position. He even considers something analogous for

⁵Although Frege seems not to have noticed the connection, the reasoning here is essentially just the Cantorian argument to the effect that there must be more functions than objects, or generally that $m^n > n$ for any $m \geq 2$.

the propositional paradox when altered to involve propositional functions rather than classes (see, e.g., Frege 1980, 159–60). Consider propositions of the form $\varphi p \supset_p p$ that do not satisfy the function φ they are “about”. Let ψ be the function satisfied by all and only such propositions.

$$\psi q \equiv_q (\exists \varphi)[q = \{\varphi p \supset_p p\} \cdot \sim \varphi q]$$

Does asking whether or not $\psi p \supset_p p$ satisfies ψ lead to a contradiction? It does only if we assume that $\psi p \supset_p p$ cannot be identical to any other $\varphi p \supset_p p$ where ψ holds of it but φ does not. This assumption is not obviously correct, and Russell considers abandoning it. Although it is worth noting it seems to obscure his own theory of the constituents of propositions, as it seems that identical propositions must have identical constituents, and it is not clear how $\psi p \supset_p p$ could be the same proposition as any $\varphi p \supset_p p$ where φ differs from ψ .

It is now known that Frege’s proposed solution, in its details, is unsuccessful, and leads to a more complicated contradiction.⁶ It is not known for sure whether or not Russell discovered this contradictions, or whether he just found the approach too limiting or philosophically *ad hoc*. He soon moved on to other proposals. But throughout the next few years he was constantly focused on the notion of a function and whether or not functions, or complexes, are more fundamental: a very Fregean theme. He briefly considers a view according to which functions are more basic than complexes; that what we think of as concepts and relations are functions, and are simpler than their values:

In common language, verbs, prepositions and in a sense adjectives, express functions; the words that do not express functions may all be called, by a slight extension, proper names. Thus functions . . . are simpler than their values: their values are complexes formed of themselves together with a term. (Russell 1994, 50, cf. p. 251f., 265ff.)

At other times, however, he seems to adopt a different view, according to which concepts and relations are distinct from functions and provide the basis for complexity, as with complex propositions, and functions are rather obtained by analysing or decomposing a complex and

⁶For further discussion of Frege’s “Way Out”, see Quine (1955); Geach (1956); Landini (2006).

viewing parts as replaceable by others (Russell 1994, 69ff, 98ff., 338ff). He eventually returns a position a bit more like his position about assertions in *PoM*, according to which it is not always possible to analyze a complex containing a given constituent into that constituent and the remainder, where the remainder is considered a distinct object. Similarly, some expressions containing variables could not be considered “functions of” those variables, and it was fair game to doubt that such constructions defined a class as well. Russell eventually grew unhappy with this view as well, mainly because he could not find a good answer to the question as to when it should be, and when it should not be, possible to so analyze a proposition that was no *ad hoc* or overly restrictive.

As noted earlier, Russell often thought of a propositional function as a function-like object except containing a variable or variables instead of a normal object at one or more places. His understanding of variables was tied up with his theory of *denoting*, i.e., that a proposition can contain special entities, which, when they occur in the proposition, make the proposition not about them, but about some entities to which they are related. Thus, e.g., “the present King of England” means a certain denoting concept, and when this concept occurs in a proposition, the proposition is not about this concept, but rather about Charles III (at least when expressed in late 2022). Similarly, propositional functions are not *about* their variables, but about the values of these variables. But as Russell grew more dissatisfied with this earlier theory of denoting, he became more suspicious of variables and functions considered as entities in their own right at all.

Right around the same time he abandoned his earlier theory of denoting, Russell took up a position about the nature of functions that was in a way a version of the Fregean notion that functions are to be understood by analyzing wholes stronger than any Frege himself ever adopted. On this view, functions should not be considered basic or fundamental entities in their own right, but as just a roundabout way of talking about a proposition and various substitutions within it. Thus rather than talking about, e.g, a function, *x is human*, we could talk about the substitutional matrix consisting of a proposition such as *Socratesishuman*, and a constituent, *Socrates*, to be substituted-for within it. Since a substitutional matrix is two entities, not one, there is no sense to be made of a substitutional matrix taking “itself” as argument. This view had its own problems, however, primarily involving versions of the paradox involving propositions, and eventually it

too gave way to Russell's mature position of *Principia Mathematica* (*PM*),⁷ but one can see the seeds of all this development in Russell's consideration of Frege's views on functions.

Indeed, the mature "theory of types" of *PM* is in some ways very Fregean, and in other ways, not Fregean at all. The type system of the work differentiates variables for individuals from variables for functions of the type that take individuals as argument, and these from variables for functions of the type that take such functions as argument. A function cannot take itself as argument, and this dissolves the "functions" version of Russell's paradox. This is all reminiscent of Frege's theory of levels of functions. But unlike Frege, who views such distinctions as grounded deep "in the nature of things" (Frege 1984a, 148), mature Russell instead describes a function as a "nothing" (*PLA*, 234) or "nothing but an expression ... [that] does not, by itself, represent everything" (*MPD*, 53). The idea seems to be that higher-order variables, considered as parts of language, can be given values by being replaced by expressions of different logical forms for different types, but the resulting expressions should not be understood as referring to some entity that is the value of the variable, but as potentially complex quantified constructions getting their truth conditions defined recursively in terms of their instances (Klement 2013); so on a philosophical level, the positions are at least apparently different.

6 Meaning and Denotation

Frege is well known for his theory that every independently meaningful expression has both a sense (*Sinn*) and a referent (*Bedeutung*), which must be differentiated (Frege 1984c). Expressions such as "the morning star" and "the evening star" can co-refer without having the same sense. Frege describes a sense as a "mode of presentation" of a referent, and thinks that all referring takes place with the mediation of a sense. Frege also uses this view to explain why identity statements are sometimes informative; while knowing that "the evening star is the evening star" is a virtual triviality, knowing that "the evening star is the morning star" requires recognizing that the same object is presented in different ways.

Prior to encountering Frege, the closest thing in Russell's philosophy is the distinction between

⁷For an excellent breakdown of Russell's views during this period, and how they have rise to the views of *PM*, see Landini (1998). See also Klement (forthcoming), Klement (2010).

what he called a “denoting concept” and what that concept denotes (*PoM*, §56–57). A denoting concept was understood as something which, when it occurs in a proposition, makes the proposition not about it but about some entity to which that concept relates. The meaning of the phrase “the morning star” could be considered such a concept: in itself, it is not the planet, but an entity having a special relation to the concept *morning star*. But when it occurs in a proposition, the proposition is about the planet Venus, which is its denotation. Thus, “the morning star is a planet” and “the evening star is a planet” would be distinct propositions about the same thing.

However, early Russell, unlike Frege, seems to think that the only phrases that represent denoting concepts are quantifier phrases of forms such as “all *X*”, “some *X*”, “any *X*”, “the *X*”, etc., where the “*X*” is replaced by an expression for a concept. Early on, Russell does not recognize a distinction between denoting concept and denoted object in the case of a simple proper name like “Socrates” or “Mont Blanc”. As we have seen, in such cases, Russell think the object itself is a direct constituent of the proposition expressed. Thus, Russell’s initial reaction to Frege’s sense/referent distinction is to see it as a more sweeping version of this distinction in his early philosophy. In his writings on Frege, Russell usually translates Frege’s “*Sinn*” (sense) as “meaning”, and his “*Bedeutung*” (referent) as either “denotation” or “indication”. Thus, Russell writes:

This theory of indication is more sweeping and general than mine, as appears from the fact that *every* proper name is supposed to have the two sides. It seems to me that only such proper names as are derived from concepts by means of *the* can be said to have meaning, and that such words as *John* merely indicate without meaning. (*PoM* §476)

It seems important to Russell’s overall philosophy at the time that we are able to think directly about at least *some* objects and concepts (universals), without our grasp of them being mediated by another entity, a meaning or denoting concept. He seems to associate the alternative view with the kind of idealism he had abandoned around the turn of the century (see e.g., Hylton 1990), and worries it’ll lead to the result that we have no knowledge about the world independent of our thought. This is perhaps why he writes, as we have seen, that it follows that “we know nothing at all about Mont Blanc” if we don’t grant that the mountain itself can occur in propositions we can

entertain.

But this is not to say that Russell was not influenced at all by Frege's sense/reference distinction. His engagement with it seems to have led him to expand his conception of which expressions represent meanings or denoting concepts in disguise. It is a bit of a controversy whether or not Russell's early views necessarily led to the conclusion that unreal objects like golden mountains or Olympian gods must be taken to have some reality, or have being in some sense (see e.g., Griffin 1996; Stevens 2011; Cohen 2022), but in at least one passage of *PoM* he does write as if they do (§427). At the very least, Russell notices during his "Fregean period" that this conclusion can be avoided if we regard names like "Apollo" to work like descriptions in disguise. In the 1903 manuscript, "On the Meaning and Denotation of Phrases", Russell writes:

There is, however, plainly a proper and an improper use of the word *Apollo*, from which it follows that, since nothing is denoted by it, something must be meant. This is, in fact, a general principle with imaginary persons or events ... they are described by means of a collection of characteristics, of the combination of which they are conceived to be the only instance. Thus when we look up *Apollo* (if we ever do) in a classical dictionary, we find a description which is really a definition ... Thus *Apollo* is not a proper name like Aeschylus; and even genuine proper names, when they belong to interesting people, tend to become names which have meaning. If we ask: "Was there such a person as Homer?", the *meaning* of the word *Homer* is fixed, and the question is: Does this meaning denote anything? (Russell 1994, 285)

Russell maintains that *some* ordinary proper names (here, "Aeschylus") only have denotation, but expands the scope of that to which he would assign meaning, and Frege is almost certainly an influence.

Indeed, Russell is explicit about that influence in his correspondence with Meinong, whom Russell of course is well known for accusing as having an overly large ontology of unreal objects. Thus in a December 1904 letter to Meinong, Russell writes:

I have always believed until now that every object must in some sense have *being*, and I

find it difficult to admit unreal objects, In such a case that of the golden mountain or the round square one must distinguish between *sense* and *reference* (to use Frege's terms): the sense is an object, and has being; the reference, however, is not an object. The difference between sense and reference is best illustrated by mathematical examples: "the square root of 4" is a complex sense, the reference of which is the number 2. (Russell 2003, 82)

Here it almost seems as if Russell is adopting the sense/reference distinction wholesale. This is not entirely true, but the influence of Frege is unmistakable.

Russell's views on these matters changed significantly in 1905. As we have just seen, Russell understood the senses or meanings or phrases like "the square root of 4", "the round square" or "the morning star" as *complexes*. Russell's work on attempting to solve the logical paradoxes led him a view according to which it was difficult to maintain the view that a complex—something with a logical form—can occur in proposition in a position or way similar to how a simple individual can occur, and adequately differentiate the complex considered as one thing from it considered as many Ito (n.d.). This led him to adopt his new theory of descriptions. According to that theory, a description such as "the morning star" does not contribute *one* complex constituent to the proposition expressed, but the proposition has a more complex logical form involving quantification, and the property *morning star* will be only one of a number of constituents the description contributes to that form.

Along with the new theory of descriptions, Russell adopted an epistemological distinction between knowledge by acquaintance and knowledge by description (see esp., Russell 1992). On this view, if I am directly acquainted with something, I can entertain propositions directly about it without making use of a description. However, if I am not directly acquainted with something, I can only think of it indirectly by means of a description that picks it out, and the thing in question is not itself a constituent of the propositions I entertain. Since I personally have never been to the Alps, I am not acquainted with Mont Blanc. I cannot entertain propositions directly about, or containing, the actual mountain. However, I can still have knowledge by description of it, since there are properties I associate with it which it alone possesses, e.g., being the highest mountain in the

mountain range overlapping the French, Swiss and Italian borders. Of course, it is possible that I could have knowledge by description of the same object without realizing it, if, e.g., I make use of different descriptions that happen to pick out the same thing as uniquely possessing the properties utilized in the description. Identity statements can be informative if one or two descriptions are utilized, as they require that the same thing identified another way also uniquely satisfies the description.

This maintains something similar to Frege's sense/reference distinction, but without postulating "senses" as additional objects in addition to concepts/properties and the other elements (e.g., quantifiers, identity, etc.) a description contributes to the logical form of propositions making use of them. In his mature work, Russell puts the point like this: "I believe that the duality of meaning and denotation, though capable of a true interpretation, is misleading if taken as fundamental" (Russell 1992, 157).

7 Conclusion

It was not Frege's influence that led Russell to logicism, nor to his love or fascination with symbolic logic and its applications in philosophy. These Russell came to largely due to the influence of Giuseppe Peano and his school. One can see Peano's influence writ large in Russell's logical notation, and even in the notation widely used in logic and mathematics today. Nevertheless, with regard to important topics such as the nature of (propositional) functions, the distinction between meaning and denotation, and so on, Frege was an influence on Russell's symbolic logic. Frege did even influence details of Russell's logic, such as the adoption of quantifiers in the modern sense.⁸ And most likely, Frege would have been an even bigger influence on Russell had he read him before reading Peano. Russell took every opportunity later in his career to praise Frege. Such remarks, as well as the Appendix on Frege in *PoM*, are probably largely responsible for popularizing Frege among later analytic philosophers. Here are Russell's own words used when he was asked about publishing part of their correspondence in a November 1962 letter to Jean van Heijenoort:

As I think about acts of integrity and grace, I realise that there is nothing in my

⁸Manuscripts have survived showing Russell in essence changing Peano's notation $\exists(x \ni \dots x \dots)$ by hand to $(\exists x)(\dots x \dots)$ as a way of expressing that there are x 's such that \dots ; see Russell (1994), plate iv.

knowledge to compare with Frege's dedication to truth. His entire life's work was on the verge of completion, much of his work had been ignored to the benefit of men infinitely less capable, his second volume was about to be published, and upon finding that his fundamental assumption was in error, he responded with intellectual pleasure clearly submerging any feelings of personal disappointment. It was almost superhuman, and a telling indication of that of which men are capable if their dedication is to creative work and knowledge instead of cruder efforts to dominate and be known. (quoted in van Heijenoort 1967, 127)

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