“SIND DIE ZAHLFORMELN BEWEISBAR?”

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Abstract. By a numerical formula, we shall understand an equation, $m = n$, between closed numerical terms, m and n. Assuming with Frege that numerical formulae, when true, are demonstrable, the main question to be considered here is what form such a demonstration takes. On our way to answering the question, we are led to more general questions regarding the proper formalization of arithmetic. In particular, we shall deal with calculation, definition, identity, and inference by induction.

1. Introduction

Frege opens the critical part of his *Grundlagen der Arithmetik* (Frege, 1884) by discussing a very mundane form of mathematical theorem, a form of theorem with which all of his readers could be assumed to be familiar: numerical equations, such as $2 + 2 = 4$ and $7 \times 6 = 42$. Following Kant, Frege calls such equations “numerical formulae”, “Zahlformeln”, and he asks whether they are demonstrable: “sind die Zahlformeln beweisbar?” He quickly reaches the conclusion that they are indeed demonstrable. Precisely how the demonstration of a numerical formula is meant to proceed is, however, not specified in the *Grundlagen*, nor (as far as I know) in any other of Frege’s works. Leibniz had offered a demonstration of the equation $2 + 2 = 4$ relying on the principle of intersubstitutability of equals for equals together with definitions of the individual numbers greater than zero in terms of the successor function. Frege appears to be sympathetic to this method of demonstration, but he notes that Leibniz’s purported demonstration contains a gap. Frege thinks the gap is to be filled by means of the law of the associativity of addition. The demonstration of other numerical formulae might rely on other general laws. Indeed, Frege speaks repeatedly of the general laws—in the plural—on which numerical formulae rest. He fails, however, to specify precisely which laws these are.

Assuming with Frege that numerical formulae are demonstrable, I wish to reflect here on what the demonstration of a numerical formula looks like. Following Frege’s example of studying demonstration with the help of its formalization, I shall pursue the question of how to implement such demonstrations in formal systems of arithmetic. We shall see that a satisfactory answer requires us to deal with the notion of calculation, the theory of definition, and the logic of identity. The answer I shall recommend calls for the supplementation of ordinary logical reasoning by the notion of definitional identity. The theory of definitional identity takes care of both calculation and (nominal) definition, and its presence in a formal system ensures that the logic of the identity predicate is undisturbed by definitional equations, which otherwise have to be regarded as introduction rules for this predicate.

The topic will be introduced by way of a discussion, in Section 2, of Frege’s *Grundlagen* §§ 5–6. Disregarding quotation marks, the title of the present paper has been appropriated from this part of Frege's book. I shall take it for granted that the demonstration of a numerical formula essentially involves calculation. Calculation will be understood as definitional reduction, i.e., as the continued substitution of definiens for definiendum, of defining term for defined term. A demonstration by substitution may, as a consequence, be understood as a demonstration by calculation. This will be the conclusion of Section 3. Nominal definition is the topic of Section 4. A nominal definition is a definition formulated as one or more equations. In a first-order language, definitional identity is the smallest equivalence relation on terms generated by definitional equations. In Section 5, we shall see that by adding a theory of definitional identity to natural deduction, a very natural formalization of arithmetic results that includes a theory of both nominal definition and calculation. This system can be regarded as a translation into predicate logic of a fragment of Martin-Löf type theory, the topic of Section 6. In the concluding Section 7 we shall see that there are—at least in the formal systems presented in the preceding sections—two kinds of demonstration of numerical formulae: one proceeding exclusively by calculation, another involving as well a logical step, namely an application of the law of identity, asserting the truth of the proposition $a = a$, for an arbitrary individual a.

The methodology followed here is very much in the spirit of Göran Sundholm's work: from a combination of historical scholarship and insights from Martin-Löf type theory systematical conclusions are drawn concerning the philosophy of logic and mathematics. The anchoring of the topic in history provides both concreteness and depth. Type theory provides precision. My topic is not one that Göran has dealt with directly in any of his many writings on Frege, but it is closely related to a topic that he has dealt with in considerable detail in recent years, namely the notion of analyticity.[1] Analyticity, for Frege, is an epistemological notion, pertaining to the "epistemological nature", "die erkenntnistheoretische Natur", of a judgement. Being concerned with the demonstrations of numerical formulae, we are in effect concerned here with their epistemological nature. Numerical formulae will indeed come out as analytic in Frege's sense (*Grundlagen* § 3), since their demonstrations, on both accounts offered in Section 7, will be seen to rest only on general logical laws and definitions.

Following Göran's advice (Sundholm, 2016, pp. 206–207), I will use "demonstration" where many authors would prefer "proof". A demonstration is a chain of inferences. Its role is to make a judgement evident, thereby providing it with the status of a theorem. The word "demonstration" has a process/product ambiguity (Sundholm, 2012, p. 947–949), which, however, will play no role in what follows. The word "proof" will be used only in the sense of proof object, or truthmaker. Frege's logical ontology does not know proofs in this sense, but they are found in Martin-Löf type theory. The principle of mathematical induction will be called the principle of *inference by induction.* Over the natural numbers, an inference by induction concludes that $A(n)$ is true, for an arbitrary number n, from the premisses that $A(0)$ is true (base case) and that $A(\mathbf{s}(x))$ is true whenever $A(x)$ is true (induction step).

[1]See Sundholm (2011, 2013, 2023). Of other writings on Frege, mention may be made of (Sundholm, 1994)—which uses Martin-Löf type theory in a significant way—and the historical papers (Sundholm, 2000, 2001). Frege was the main character of Göran's exaugural lecture (Sundholm, 2019).

2. *Grundlagen* §§ 5–6

The term "numerical formula", "Zahlformel", in the sense Frege uses it was introduced by Kant to designate what he calls theorems of numerical relation (Sätze der Zahlverhältnis) (Kant, 1787, A163/B204–A165/B206). They are singular, that is, they concern particular numbers, hence their formulation does not involve variables, but only closed numerical terms. Kant appears to have had primarily the relation of equality in mind. Following his example, we shall therefore say that a numerical formula is a theorem in the form of an equation between closed numerical terms. The equations $2 + 2 = 4$ and $7 + 5 = 12$ are numerical formulae, but the equations $x + y = y + x$ and $(x + y) + z = x + (y + z)$ are not.

Numerical formulae are immediately evident (unmittelbar gewiss) and indemonstrable (indemonstrabilia), according to Kant (A164/B204). They are, moreover, synthetic: they cannot be made evident by a mere analysis of the concepts involved (B15–16). In being synthetic and immediately evident, numerical formulae agree with axioms (A732/B760). Axioms are, however, required to be general (A164/B205), whereas numerical formulae are singular. Numerical formulae are therefore not axioms, whence Kant introduces a special name for them.

Citing passages found in Baumann (1868), Frege notes that also Hobbes, Locke, and Newton held numerical formulae to be immediately evident and indemonstrable. According to Locke (1690, IV.vii.10), "two and two are four" and "three times two are six" are self-evident judgements, and knowledge of them does not depend on knowledge of general principles such as that the whole is equal to its parts taken together. Newton, in a passage Frege cites, says of numerical formulae that they are manifest in themselves (per se manifestae), but only when they are not too complex.

That numerical formulae can plausibly be claimed to be immediately evident only when the terms involved are not too complex is Frege's main argument against their indemonstrability in general. Frege takes it for granted, and assumes that his readers take it for granted, that more complex numerical formulae are indeed not immediately evident (unmittelbar einleuchtend). The equation

$$135664 + 37863 = 173527$$

for instance, is not immediately evident. If less complex numerical formulae, by contrast, are immediately evident, then one should be able to find a threshold that divides numerical formulae into those that are and those that are not immediately evident—but any such threshold must seem arbitrary. Hence, since complex numerical formulae are not immediately evident, nor are the less complex ones, such as $2 + 2 = 4$ and $7 + 5 = 12$. If, however, no numerical formula is immediately evident, then every numerical formula must be demonstrable.

Frege thus agrees with Leibniz that numerical formulae are, in fact, demonstrable. In the *New Essays* IV.vii.10 (Leibniz, 1765), corresponding to the above cited passage from Locke, Leibniz supports this claim of demonstrability by offering a demonstration of $2 + 2 = 4$. Leibniz's demonstration, which Frege quotes, rests on the definition of 2 as $1 + 1$, of 3 as $2 + 1$, and of 4 as $3 + 1$, as well as on the axiom that "if equals be substituted for equals, the equality remains".[2] The demonstration itself is presented as a sequence of equations, which, following Frege, we may write as follows:

$$2+2 \underset{\text{Def of 2}}{=} 2+1+1 \underset{\text{Def of 3}}{=} 3+1 \underset{\text{Def of 4}}{=} 4$$

After the initial term, $2+2$, each term is to be obtained from the foregoing by substitution of a defining term for the corresponding defined term.

Leibniz employed the method of demonstration by substitution not only in this arithmetical example, but also in his logic. The relation in question there, however, is identity rather than equality (e.g. Leibniz, 1686). Unlike Frege, Leibniz did not identify the notions of identity and equality. Whereas identity, for Leibniz, is defined as intersubstitutability *salva veritate*, equality is defined by him as intersubstitutability *salva magnitudine*.[3] Those things are equal, for Leibniz, whose magnitude is the same. This is, in fact, a traditional view going back at least to Aristotle (e.g. *Metaphysics* Δ.15). For Frege, by contrast, identity and equality are one and the same notion, as he emphasized in many of his writings (e.g. Frege, 1892, p. 25, note).[4] Frege's identification of these notions explains his reference in *Grundlagen* § 6 to Leibniz's *salva veritate* account of identity as just an alternative formulation of the axiom that substitution of equals for equals preserves equality (cf. ibid. § 65). Leibniz's distinguishing these notions explains why, in the text of the *New Essays* that follows his demonstration of $2+2=4$ (not quoted by Frege), he remarks that identicals are equal to each other: since 2 is identical, by definition, to $1+1$, it is also equal to $1+1$, hence, by the axiom, equality is preserved if the one is substituted for the other.

Leibniz's demonstration is incomplete, since it takes for granted the equation

$$2+(1+1)=(2+1)+1$$

Frege notes that this missing link is an instance both of the law of associativity and of the law

$$x+(y+1)=(x+y)+1$$

first used by Grassmann (1861) in defining addition. Frege mentions Grassmann's equation only in order to criticize the suggestion that it defines addition (more on this in Section 4 below). He thinks the gap in Leibniz's demonstration is to be filled by means of the law of associativity.

The view Frege has reached at the end of his discussion in *Grundlagen* §§ 5–6 is that numerical formulae are indeed demonstrable, but that the demonstration of every such formula depends on one or more general arithmetical laws. This view is manifest, not only in these sections, but also in summaries of foregoing sections offered in *Grundlagen* §§ 9, 18. In both places, Frege notes that numbers are to be

[2]This translation is from Leibniz (1981). Baumann's German rendering of the axiom (Baumann, 1869, p. 40), which Frege quotes—"Wenn man Gleiches an die Stelle setzt, bleibt die Gleichung bestehen"—is closer to the French original: "Mettant des choses égales à la place, l'égalité demeure".

[3]Both definitions can be found in many places in Leibniz's logical writings. They are found together at Leibniz (1999, p. 406).

[4]Frege's view is dominant today, but its adoption took time: identity and equality are kept apart by, for instance, Hilbert and Bernays (1934, p. 167).

defined in the manner of Leibniz, namely in terms of the addition of one,[5] but that one needs general laws (allgemeine Sätze, allgemeine Gesetze) in order to derive numerical formulae from these definitions. Here, as well as in § 6, Frege speaks of laws in the plural. In spite of the plural, however, Frege only ever mentions just one such general law, namely the law of associativity. It is clear that he thinks that also other laws might be needed in demonstrations of numerical formulae, but he fails to specify which.

We shall see that one can make do with no general laws at all, but only definitions, provided that among definitions one also counts definitions by induction. The terms flanking the identity sign in a numerical formula are what we shall call definitionally identical. If we wish to lift a definitional identity to the level of predicate logic, laws are needed, but only the very basic law of identity and a law that allows for the replacement, within a demonstration, of a proposition by a definitionally identical proposition.

3. Calculation

A numerical formula can be demonstrated by means of calculation. One can demonstrate $2 + 2 = 4$ by calculating $2 + 2$ to see that the result is indeed 4. One can demonstrate

$$135664 + 37863 = 78365 + 95162$$

by calculating both sides to see that one obtains the same result, viz. 173527. The first procedure is a special case of the second, since 4 may be regarded as the result of calculating 4 itself. In general, therefore, a numerical formula can be demonstrated or refuted by calculating both sides: if the calculations have the same result, the equation is demonstrated, if they have different results, the equation is refuted.

It is a psychological fact that we can carry out the calculation of sufficiently simple numerical terms "mentally", or "in the head".[6] That we can do so instantly for especially simple terms, such as $2+2$ or $7+5$, may be what lies behind the view of Hobbes, Locke, Kant, and others that numerical formulae are indemonstrable and immediately evident. Observing that the calculation of more complex terms needs in general to be carried out in more steps, and assuming that numerical formulae are either all indemonstrable or all demonstrable, we are led, with Frege, to the contrary view that numerical formulae are demonstrable. The demonstration, as we have just seen, may be carried out by calculation.

Leibniz seems to have intended his demonstration of $2+2 = 4$ to consist entirely of substitutions of definitionally identical terms for each other. Each step of the demonstration is to rest on the evident principle that such substitutions preserve (definitional) identity. Leibniz succeeds only partially, since he leaves out the equation $2 + (1 + 1) = (2 + 1) + 1$. Also this equation, however, can be justified by such a substitution, provided we dissociate the general operation of addition from the special operation of adding one, namely by identifying the latter operation with the successor function, s. This dissociation was not made by Leibniz, and it would not be made until the late 19th century by Frege (*Grundlagen* § 76) and Dedekind

[5] Frege still endorses this manner of defining numbers in print almost 20 years later (Frege, 1903b, p. 320).

[6] In English and the Romance languages one speaks of mental calculation, in the Germanic languages of calculation-in-the-head (Kopfrechnung, hoofdrekenen, hoderegning), and in Eastern Slavic languages of oral calculation (устный счёт, усні обчислення).

(1888, § 6). By dissociating these two operations, however, we can turn Leibniz's demonstration into one consisting entirely of substitutions of definitionally identical terms.

We redefine 4 as $\mathbf{s}(3)$, 3 as $\mathbf{s}(2)$, and 2 as $\mathbf{s}(1)$, and we define 1 as $\mathbf{s}(0)$. We moreover stipulate that addition is defined by the following two equations:

$$\begin{aligned} x + 0 &= x \\ x + \mathbf{s}(y) &= \mathbf{s}(x + y) \end{aligned}$$

This definition is to be understood so that any instance of either of these equations is a definitional equation with definiendum to the left and definiens to the right. We now demonstrate $2 + 2 = 4$ as follows:

$$(\Lambda) \qquad \begin{aligned} 2 + 2 &\underset{\text{Def of 2}}{=} 2 + \mathbf{s}(1) \\ &\underset{\text{Def of +}}{=} \mathbf{s}(2 + 1) \\ &\underset{\text{Def of 1}}{=} \mathbf{s}(2 + \mathbf{s}(0)) \\ &\underset{\text{Def of +}}{=} \mathbf{s}(\mathbf{s}(2 + 0)) \\ &\underset{\text{Def of +}}{=} \mathbf{s}(\mathbf{s}(2)) \\ &\underset{\text{Def of 3}}{=} \mathbf{s}(3) \\ &\underset{\text{Def of 4}}{=} 4 \end{aligned}$$

The equation $2 + 2 = 4$ has thus been demonstrated, as Leibniz intended, by a sequence of substitutions of definitionally identical terms for each other. It should be clear from this example that any numerical formula, $m = n$, whose terms m and n are built up from numbers in decimal notation and addition, $+$, can be demonstrated in the same way. Indeed, any numerical formula can be so demonstrated that involves only so-called primitive recursive functionals, i.e., functions, possibly higher-order, definable in Gödel's theory T (Gödel, 1958). (A justification of this latter claim will be given in the following section.)

Is the enhanced Leibnizian demonstration (Λ) different from a demonstration by calculation? To my mind, there is no essential difference between demonstration by substitution and demonstration by calculation. A demonstration by calculation proceeds by calculating the two terms flanking the equality sign and observing that both calculations have the same result. A natural way of understanding the calculation of a numerical term, however, is as the spelling out of definitions. The calculation of a numerical term may be understood as the continuous replacement of definiendum by the corresponding definiens, a process I shall call definitional reduction. Aristotle appeals to definitional reduction at various places in the *Topics*, and it was essential to Leibniz's logic. Its affinity with calculation is suggested already by Leibniz's terminology, since he speaks of a definiens as the value (valor) of any term it defines.[7] The affinity was made explicit and spelled out in more detail much later by Curry and Feys (1958), who also introduced the term "definitional reduction" for this process. Although the lambda calculus is not often presented as a theory of definitional reduction, many of its rules may be regarded as belonging to such a theory (cf. Klev, 2019a, §§ 10–12). The beta rule, in particular, viz., $\lambda x.t(u) \to t[u/x]$, may be so regarded, since it is in effect the definition of the λ-operator.

[7]Many places of Leibniz's logical works could be cited; see, e.g., Leibniz (1686, p. 746).

The practice of calculation assumes the idea of an endpoint of calculation. A task of calculation may be understood as a task of rewriting a number into a form that has been agreed upon to serve as such an endpoint. In school arithmetic, natural numbers in decimal notation are the endpoints usually agreed upon, implicitly or explicitly. If calculation is explained as definitional reduction, and the decimal notation is defined by means of a unary notation, say in terms of 0 and $\mathbf{s}$, then the endpoints become numbers in this unary notation, that is, numerals. A pair of calculations in this sense of the numbers $2+2$ and 4, respectively, may thus be portrayed as in the following diagram. To save space, I write $\mathbf{s}^2(m)$ to indicate double application of the function $\mathbf{s}$ to m, and likewise for $\mathbf{s}^3(m)$ and $\mathbf{s}^4(m)$.

$$\begin{array}{r} 2+2 \to 2+\mathbf{s}(1) \to \mathbf{s}(2+1) \to \mathbf{s}(2+\mathbf{s}(0)) \to \mathbf{s}^2(2+0) \to \mathbf{s}^2(2) \to \mathbf{s}^3(1) \to \mathbf{s}^4(0) \\ 4 \to \mathbf{s}(3) \to \mathbf{s}^2(2) \to \mathbf{s}^3(1) \to \mathbf{s}^4(0) \end{array}$$

Both calculations have the same endpoint, that is to say, they yield the same result, hence we have a demonstration of $2+2=4$.

The diagram may thus be glossed as a demonstration of $2+2=4$ in accordance with our characterization of demonstration by calculation: both calculations yield the same result. We can, however, also gloss the diagram in a slightly different way. Within each line of the diagram, each number (except the initial one) is obtained from its left-hand neighbour by substitution of equals for equals. Such substitution preserves identity, hence all numbers in a single line are identical to each other. Since the number $\mathbf{s}^4(0)$ occurs in both lines, all the numbers occurring in the diagram are identical to each other. In particular, $2+2$ is identical to 4. What we have now described is essentially the Leibnizian demonstration (Λ), with the small difference that Leibniz uses the number $\mathbf{s}^2(2)$ to connect the two calculations.

4. Nominal definitions

The calculation of $2+2$ relies on the definitions of 2, of 1, and of addition. These are instances of explicit definition and definition by induction, respectively, which in turn are forms of what we shall call nominal definition, following the terminology of Pascal (1657) and Arnauld and Nicole (1662, I.12–13). According to them, a nominal definition (*définition de nom, definitio nominis*) is an arbitrary convention that lays down the meaning of a term by identifying it with the meaning of another term.

Let t and u be numerical terms containing, besides variables and the constants 0 and $\mathbf{s}$, only constants that have already been defined. Let δ be a symbol not yet in use.

Assume that the (free) variables of t are among those in the list $\overline{x}$ of mutually distinct variables. An *explicit definition* takes the form

$$\delta(\overline{x}) = t \tag{1}$$

The special case is included where the list $\overline{x}$ is empty and δ is an individual constant.

Assume that the (free) variables of t are among $\overline{x}$ and that the (free) variables of u are among those in the list $\overline{x}, y, z$. A *definition by induction* takes the form

$$\begin{aligned} \delta(\overline{x}, 0) &= t \\ \delta(\overline{x}, \mathbf{s}(y)) &= u[\delta(\overline{x}, y)/z] \end{aligned} \tag{2}$$

The formula $u[\delta(\overline{x},y)/z]$ is the result of substituting $\delta(\overline{x},y)$ for z in u. The special case is included where the list $\overline{x}$ is empty.[8]

We call (2) a scheme of definition by induction rather than of recursive definition, since a definition of this form need not be recursive in the sense that $\delta(\overline{x},y)$ occurs in u. A good example of a definition by induction that is not recursive is the definition of the predecessor function:

$$\begin{aligned}\mathrm{pd}(0) &= 0\\ \mathrm{pd}(\mathbf{s}(y)) &= y\end{aligned}$$

The functions definable by means of schemes (1) and (2) are precisely the primitive recursive functions (see Kleene, 1952, § 54). This result suggests a close connection with calculability. Indeed, any closed numerical term built up from 0, $\mathbf{s}$, and nominally defined constants can be calculated by means of definitional reduction. That is, by continued definitional reduction from such a term we shall eventually reach a numeral that, moreover, is uniquely determined by the term we started out with. This Canonicity Theorem, as we may call it, holds as well if we allow variables and constants to be of higher types, in which case the functions so definable are the primitive recursive functionals.[9] This theorem justifies our claim above that numerical equations involving only primitive recursive functionals can be demonstrated by means of substitution, as Leibniz demonstrated $2+2=4$.

The Canonicity Theorem also provides a metamathematical justification of definition by induction. Let us call the numeral reached by the calculation of a closed numerical term its *value*. The Canonicity Theorem assures us that, if an n-ary function δ is defined by induction, and $\overline{m}$ is a sequence of n closed numerical terms, then the value of $\delta(\overline{m})$ exists and is unique. Such assurance with regards to addition is precisely what Frege asked for in *Grundlagen* § 6 when objecting to Grassmann's suggested definition of addition by means of the equation $x+(y+1)=(x+y)+1$: we need a demonstration that, for every pair of numbers k,l, there is a unique number $k+l$ satisfying the equation. In a more general form, the requirement asks for a demonstration that definition by induction is a safe form of definition: that a function so defined yields a uniquely determined value for each list of arguments. Frege later offered such a demonstration in terms of his notion of the ancestral of a relation (see Heck, 2012, ch. 7). Dedekind's more well-known demonstration employs finite approximations to the function to be defined (Dedekind, 1888, nr. 126). Neither of these demonstrations invoke the notion of calculation, or evaluation, in the way the Canonicity Theorem does.

The first of Frege's basic principles of definition in *Grundgesetze* I § 33 (1893) is the requirement of eliminability, which, in a rough formulation, says that every defined term must be eliminable by means of its definition. Symbols defined by induction do not satisfy the requirement of eliminability so formulated. For instance, from the term $0+x$, the addition symbol, $+$, is not eliminable, since neither of the two definitional equations for addition can be used to reduce this term.

[8]For interesting historical remarks on definition by induction (as well as other topics pertinent to the present paper), see von Plato (2016, 2017).

[9]The Canonicity Theorem is a corollary of the strong normalization and confluence of definitional reduction. Strong normalization can be proved by use of a so-called computability predicate, first introduced by Tait (1967). Confluence then follows from weak confluence by Newman's Lemma (Huet, 1980, Lemma 2.4), and weak confluence is established by an easy case analysis.

This failure of eliminability might seem to pose a problem for definition by induction. Frege (ibid.) introduces the requirement of eliminability by saying that "every name correctly formed from defined names must have a reference" and makes it precise by saying that every such name must be shown to be equal in reference (*gleichbedeutend*) with a name—unique up to the shape of the bound variables—constructed entirely from primitive vocabulary. Indeed, it is not clear how we can regard a term involving defined vocabulary as meaningful other than through an anchoring in primitive vocabulary. Failure of eliminability seems to entail that a term lacks such anchoring.

The problem is solved by a general account of the meaningfulness of terms. The account has similarities with the Gödel–Tait definition of a computability predicate, with Prawitz's definition of proof-theoretical validity, and with Martin-Löf's meaning explanations, but it is in fact implicit already in the *Grundgesetze*.[10]

A closed term is meaningful if it definitionally reduces to a term constructed entirely from primitive vocabulary. Definitional reduction preserves reference (however the notion of reference is explained), hence a closed term that reduces to a primitive term is equal in reference to it. The term $2+2$, for instance, is meaningful because it reduces to, hence is equal in reference to, $\mathbf{s}^4(0)$.

Following Frege's explanation of the meaningfulness of functional expressions in *Grundgesetze* I § 29, we say that an open term is meaningful if every closed term that results from substituting meaningful terms for its free variables is meaningful. That $0+x$ is meaningful, for instance, is stipulated to mean that $0+m$ is meaningful, whenever m is closed and meaningful.

Once open terms are understood in this way, it is enough to insist on eliminability only for closed terms: if every closed term is meaningful in the way stipulated, then also every open term will be meaningful. In a language with higher types, it is, for similar reasons, enough to insist on eliminability for closed terms of ground type. Definition by induction does preserve this form of eliminability. Indeed, the Canonicity Theorem assures us that, from a closed numerical term of ground type, all defined constants—including such as are defined by induction—can be eliminated. Also Frege, in his first basic principle of definition, seems to restrict eliminability to this case: he speaks only of closed terms, or proper names, and these are all of ground type in his ideography.

The requirement of conservativeness is not among Frege's principles of definition in *Grundgesetze* I § 33, but it seems clear that he adhered to it.[11] Already in his first methodological discussion of definitions, in *Begriffsschrift* § 24, he says that nothing follows from a nominal definition "that could not be inferred also without it". In a *Nachlass* piece from about 35 years later he makes a similar claim (Frege, 1983, p. 225). In a letter to Hilbert,[12] he offers a sarcastic verse that clearly shows what he thinks of non-conservative definitions:

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[10]The Frege connection has been noted by Martin-Löf (2021) and Pistone (2018).

[11]Non-conservative definitions are often called creative definitions, after the German "schöpferische Definition". Frege uses this term (1893, pp. vi, xiii; 1903a, § 143), but he seems then to have in mind a definition by which one purports to create an object, or to make an object have a certain property. An example mentioned by Husserl (1891, p. 269) is Schröder's definition of $\emptyset$ as a domain satisfying $\emptyset \subset A$, for any domain A (Schröder, 1890, p. 188).

[12]Letter dated 27.12.1899 (Frege, 1976, p. 62). Frege later includes the verse in his published reflections on Hilbert's geometry (Frege, 1903b, p. 321).

Was man nicht recht beweisen kann,
Das sieht man als Erklärung an.

That is, without the rhyme: what you cannot quite demonstrate, you may regard as a definition.

That a definition by induction may fail to be conservative was noted by Hilbert and Bernays (1934, pp. 299-303). A suitably chosen subsystem T of Peano arithmetic has finite models but is such that, by adding to it the definition of the predecessor function, pd, one can derive a formula in the language of T that is true only in infinite models. The definition of pd is therefore not conservative over T. More generally, and perhaps better, we can say that scheme (2) is not conservative over T: by adding scheme (2) to T we can derive a theorem, formulated in the vocabulary of T, that is not derivable in T.

In a *Nachlass* piece, Frege (1983, p. 225) remarks that if a definition allows one to give a demonstration that cannot be given without it, then the definition hides something that has to be either demonstrated as a theorem or laid down as an axiom. Scheme (2) may indeed be said to hide, among other things, the condition that $\mathbf{s}(m) = 0$ is false for all m, since otherwise a function defined according to the scheme will not, in general, be well-defined. In this way, scheme (2) differs from scheme (1), which comes with no such presuppositions. The dilemma noted by Frege gives rise to two conflicting approaches to definition by induction. The first approach, taken by Frege and Dedekind, is to allow a scheme of definition by induction to be added to a theory only if every function defined by means of it can be shown to be extensionally equal to a function that the theory already deems to exist. The other approach, taken for instance in Martin-Löf type theory, is to regard a scheme of definition by induction as a primitive principle whose justification rests, ultimately, on our understanding of the underlying inductively generated domain.

5. Formalizing calculation

Demonstrations by calculation cannot be formalized in formal systems of arithmetic as usually presented, since, as usually presented, such systems do not contain a mechanism for introducing new terms by definition. In order to formalize the above demonstration of $2 + 2 = 4$, for instance, we need the equation $4 = \mathbf{s}(3)$, which contains terms that are not part of the language of Peano arithmetic. From the recursion equations for addition and Leibniz's Law, one can derive $\mathbf{s}^2(0)+\mathbf{s}^2(0) = \mathbf{s}^4(0)$, but the passage from $2+2 = 4$ to this equation already involves a number of steps of calculation. A formalization of a demonstration by calculation of $\mathbf{s}^2(0) + \mathbf{s}^2(0) = \mathbf{s}^4(0)$ is therefore not yet a formalization of a demonstration by calculation of $2 + 2 = 4$.

If calculation is understood as definitional reduction, then the formalization of calculation must somehow be based on a formalization of nominal definition. Since the introduction of new terms by nominal definition is an important part of mathematical method, one could argue that it is anyhow desirable for a formalization of arithmetic to account for that practice.

To that end, one might stipulate that a nominal definition may at any point be added as a novel axiom to the theory. Adding an explicit definition means adding a single equation as an axiom, and adding a definition by induction means adding two equations. This proposal has the advantage of not introducing any new metatheoretical concepts: definitions are just axiomatic equations. It does, however, complicate the explanation of the meaning of the identity predicate.

Shunning second-order quantification, we have few choices when attempting to explain the identity predicate but to rely on the rules that govern its use. If in arithmetic we are allowed at will to add equations as axioms, however, we shall not have a single set of rules that there governs the use of the identity predicate, since every new axiomatic equation will be a new such rule. The identity predicate will therefore be left with an unstable meaning, one that changes each time a new nominal definition is added. A similar problem in fact affects first-order Peano arithmetic as standardly formulated. Among the axioms of first-order Peano arithmetic are the four equations making up the nominal definitions of addition and multiplication:

$$\begin{aligned} x + 0 &= x & \qquad x \times 0 &= 0 \\ x + \mathbf{s}(y) &= \mathbf{s}(x + y) & \qquad x \times \mathbf{s}(y) &= x \times y + x \end{aligned}$$

Being axiomatic equations, these have to be regarded as introduction rules for the identity predicate in arithmetic. Hence, even if we restrict the meaning-determining rules to the introduction rules, we are forced to say that the meaning of the identity predicate in arithmetic differs from its meaning elsewhere, since it there has different introduction rules from what it has elsewhere. Identity then fails to be a topic-neutral notion.

Some other way of dealing with nominal definitions is needed if we want the rules governing the identity predicate to provide it with a stable, topic-neutral meaning. The right approach, to my mind, is to formalize nominal definition and calculation as belonging to a compartment that is in some way separate from the compartment in which proofs are constructed. Rules governing the identity predicate belong to the latter compartment. Nominal definition and calculation are not to be formalized in terms of this predicate, but in terms of a special relation of definitional identity, which we shall write as "$\equiv$".[13]

The theory of definitional identity, which is also a theory of calculation, is an equational theory whose axioms are definitional equations. More precisely, in addition to the usual rules of equational logic, the rules governing the symbol "$\equiv$" include the axiom scheme

(D) $$\textit{definiendum} \equiv \textit{definiens}$$

In a language with higher types, function extensionality may be assumed: if $f(x) \equiv g(x)$, then $f \equiv g$.[14]

Such a theory may be added to a system of natural deduction by means of a rule of *formula conversion*:

$$\frac{A[t]_!}{A[u]_!}\ t \equiv u$$

The notation $A[t]_!$ is used to indicate that t occurs in A and to mark out one such occurrence. The formula $A[u]_!$ results from $A[t]_!$ by replacing its marked occurrence of t by u. Suppose we have a derivation of $A[t]_!$ and, in the theory of

[13] The use of the triple stroke for definitional identity has a long history going back at least to Neurath (1910) and Moore (1910).

[14] A formalization of the theory of definitional identity along these lines, but excluding extensionality, was first given by Martin-Löf (1975). Reasons for including extensionality can be found in (Klev, 2019a).

definitional identity, a derivation of the equation $t \equiv u$. Formula conversion allows us then to rewrite $A[t]_!$ as $A[u]_!$. Since the conclusion here is a mere rewriting, or reformulation, of the premiss, a dashed line is written between them.[15]

We shall assume the following natural-deduction rules for the identity predicate, proposed by Martin-Löf (1971, p. 190):

$$(=\mathrm{I})\quad a = a \qquad\qquad (=\mathrm{E})\ \frac{a = b \qquad A[x,x]}{A[a,b]}$$

In the elimination rule, (=E), the minor premiss $A[x,x]$ is a formula obtained from some $A[y,z]$ by substituting x for both y and z. An application of (=E) binds the variable x in the derivation of $A[x,x]$. The introduction rule, (=I), says that identity is a reflexive relation, and the elimination rule, (=E), says that identity is the smallest reflexive relation. By letting A be $B[y/x] \supset B[z/x]$, one sees that (=E) entails Leibniz's Law, a rule that is often taken to be the elimination rule for the identity predicate.

The identity predicate, $=$, and the symbol of definitional identity, $\equiv$, thus have quite different characterizations: the first in terms of the introduction rule (=I), the second in terms of the purely combinatorial rules of equational logic. The difference is borne out by a metamathematical result which shows that $m = n$ may be derivable although m and n fail to be definitionally identical. The equation $x + y = y + x$, for instance, is a theorem of Peano arithmetic, but the two terms $x + y$ and $y + x$ are not definitionally identical.

By making use of definitional identity we can formalize arithmetic in a way that (i) takes account of calculation, (ii) takes account of nominal definition, and (iii) preserves the topic-neutrality of the identity predicate. The rules of definitional identity together with formula conversion make it possible to derive all the first-order Peano axioms besides induction.

The axioms $x+0 = x$, $x+\mathbf{s}(y) = \mathbf{s}(x+y)$, $x\times 0 = 0$, and $x\times\mathbf{s}(y) = x\times y+x$ follow by formula conversion from the corresponding equations written as definitions, using "$\equiv$" instead of "$=$". The Peano axiom stating the injectivity of the successor function—sometimes known as Peano's third axiom[16]—can be derived by use of the predecessor function:

$$\cfrac{\cfrac{\cfrac{\mathbf{s}(x) = \mathbf{s}(y)^1 \qquad \mathrm{pd}(z) = \mathrm{pd}(z)}{\mathrm{pd}(\mathbf{s}(x)) = \mathrm{pd}(\mathbf{s}(y))}\ (=\mathrm{E})}{x = y}\ \mathrm{pd}(\mathbf{s}(x)) \equiv x,\ \mathrm{pd}(\mathbf{s}(y)) \equiv y}{\mathbf{s}(x) = \mathbf{s}(y) \supset x = y}\ (\supset\mathrm{I}),\ 1$$

The double dashed line is here used to indicate two consecutive applications of formula conversion.

[15]The rule of formula conversion was introduced under that name by Martin-Löf (1998, p. 155). A similar rule was called 'replacement' by Łukasiewicz (1929, pp. 33, 40). An ancient ancestor of formula conversion is the so-called "topos from definition", first formulated by Aristotle in *Rhetoric* II.23 and given a polished formulation more than 1500 years later by Peter of Spain (Copenhaver et al., 2014, ch. 5 §§ 5–9).

[16]See the enumeration of Peano's axioms in (Peano, 1891, 1898) and (Hilbert and Bernays, 1934, p. 218).

For Peano's fourth axiom, $0 \neq \mathbf{s}(x)$, we must extend the scheme of definition by induction so as to allow the following definition:

$$F(0) \equiv 0 = 0$$
$$F(\mathbf{s}(y)) \equiv \bot$$

We thus assume that definitional identity makes sense, not only between numerical terms, but also between propositions (or their formalistic counterparts, formulae). This assumption is in line with mathematical practice, where nominal definitions are made in all types. Using this F, we may derive Peano's fourth axiom as follows:

$$\dfrac{\dfrac{0 = 0 \qquad \dfrac{\dfrac{0 = \mathbf{s}(x)^1 \qquad \dfrac{F(z)^2}{F(z) \supset F(z)}\,(\supset\mathrm{I}),\,2}{F(0) \supset F(\mathbf{s}(x))}\,(=\mathrm{E})}{0 = 0 \supset \bot}\; F(0) \equiv 0 = 0,\ F(\mathbf{s}(x)) \equiv \bot}{\bot}}{0 = \mathbf{s}(x) \supset \bot}\,(\supset\mathrm{I}),\,1$$

The only Peano axiom that has not been seen thus to follow from definitions is the induction scheme. Martin-Löf (1971, p. 190) showed that the same general procedures which render (=E) the elimination rule for a predicate with introduction rule (=I) renders the induction scheme in rule form the elimination rule for a predicate N with the following two introduction rules:

$$N(0) \qquad\qquad \dfrac{N(m)}{N(\mathbf{s}(m))}$$

Since we are only considering numerical terms, the predicate N is, for us, universal and so, in a sense, invisible. The following rule, (Ind), is the elimination rule for this invisible predicate:

$$(\mathrm{Ind})\ \dfrac{A[0] \qquad \begin{array}{c} A[x] \\ | \\ A[\mathbf{s}(x)] \end{array}}{A[n]}$$

An application of (Ind) discharges the assumption $A[x]$ and binds the variable x in the derivation of $A[\mathbf{s}(x)]$. The term n may be any numerical term. An application of (Ind) forms a detour in a derivation if n is definitionally identical either to 0 or to a term of the form $\mathbf{s}(m)$. Such detours may be eliminated by means of suitably formulated reductions.[17]

Our search for a way of formalizing calculation and nominal definition in arithmetic has thus led us to a formalization of Peano arithmetic in which induction is the only axiom, treated as an elimination rule, and all the other Peano axioms are theorems.

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[17]A derivation of the form

$$\dfrac{\begin{array}{c} \mathcal{D} \\ A[0] \end{array} \qquad \begin{array}{c} A[x] \\ \mathcal{D}' \\ A[\mathbf{s}(x)] \end{array}}{A[n]}$$

reduces, if $n \equiv 0$, to the derivation

$$\begin{array}{c} \mathcal{D} \\ \dfrac{A[0]}{A[n]}\ 0 \equiv n \end{array}$$

Frege would of course not have accepted leaving induction as a primitive rule. This would be a principle of reasoning peculiar to arithmetic, and the need to appeal to it in the demonstration of some arithmetical theorem—such as in the demonstration of the commutative law for addition—would render arithmetic synthetic in Frege's sense (*Grundlagen* § 3), rather than analytic, as required by his logicism. Frege is therefore keen to note (ibid. § 80) that the principle of induction can be reduced to "general logical laws" by means of his notion of the ancestral. It will be profitable to postpone discussion of the status of induction as a primitive rule until after we have introduced type theory.

6. Type theory

The formalization of arithmetic sketched in the previous section is an adaptation to natural deduction of the formalization of arithmetic in Martin-Löf type theory. Calculation and definition, on the one hand, and construction of proof, on the other, are there taken care of by means of two different forms of judgement: $a = b : \mathscr{C}$ and $a : \mathscr{C}$, where the predicate $\mathscr{C}$ is either **type** or a type α.[18] A demonstration with a conclusion of the form $a = b : \mathscr{C}$ may be understood as a demonstration in the theory of definitional identity, and therefore as a calculation. A demonstration with a conclusion of the form $a : \mathscr{C}$ may be understood as a construction of the object a of category $\mathscr{C}$. When $\mathscr{C}$ is a proposition, then a may be understood as a proof, or truthmaker, of $\mathscr{C}$, and the demonstration of $a : \mathscr{C}$ as a construction of such a proof.

Formula conversion is a translation into predicate logic of the following rule of type equality:

$$\frac{a : \alpha \qquad \alpha = \beta : \mathbf{type}}{a : \beta}$$

If α and β are propositions, A and B, the rule says, in effect, that if a is a proof of A, and A is definitionally identical to B, then a is a proof of B as well.

Definition by induction is handled in type theory by means of a higher-order function often known as a recursor, $\mathbf{R}$. For every function f definable by means of the schemes (1) and (2) of nominal definition, an extensionally identical function can be explicitly defined by means of $\mathbf{R}$.[19] The definition of $\mathbf{R}$ is itself a definition by induction, hence one does not pretend to have done away with such definitions. Indeed, as already indicated, it is one of the basic postulates of Martin-Löf type theory that definition by induction makes good sense on inductively generated domains.

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and, if $n \equiv \mathbf{s}(m)$, to the derivation

$$\begin{array}{c} \dfrac{\begin{array}{c} \mathcal{D} \\ A[0] \end{array} \qquad \begin{array}{c} A[x] \\ \mathcal{D}' \\ A[\mathbf{s}(x)] \end{array}}{A[m]} \\ \mathcal{D}'[m/x] \\ \dfrac{A[\mathbf{s}(m)]}{A[n]}\ \mathbf{s}(m) \equiv n \end{array}$$

[18]In this presentation we shall not consider hypothetical judgements, which give rise to many further predicates $\mathscr{C}$.

[19]Here extensional identity of functions $f, g : A \to B$ means that the proposition $(\forall x : A)\mathbf{Id}(B, f(x), g(x))$ is true. This is a weaker condition than $f(x) \equiv g(x)$. The notation $\mathbf{Id}$ is explained below.

There is no scheme of explicit definition in type theory as usually formulated, but such a scheme may be added (Martin-Löf, 1993, pp. 68–74). One must then be careful to add a typing clause to the definitional equation. For instance, the definition of 1 as $\mathbf{s}(0)$ must include the judgement $1 : \mathbf{N}$ stating explicitly that 1 is a natural number.

The derivations of Peano’s third and fourth axiom given above were adapted from corresponding derivations in type theory (see Nordström et al., 1990, pp. 66, 86). The domain of the function F used in the demonstration of Peano’s fourth axiom is a so-called universe, a type of individuals whose elements are codes of other types of individuals. It was shown by Smith (1988) that the appeal to a universe in this demonstration is necessary.

The principle of inference by induction in rule form is the elimination rule for the type $\mathbf{N}$ of natural numbers. In Martin-Löf type theory, an elimination rule is, in the first instance, a licence to define functions by induction. That an elimination rule is also a principle of inference by induction follows under the Curry–Howard correspondence. The status of the principle of proof by induction over the natural numbers as synthetic or analytic ought therefore, by analogy, to be just the same as that of the elimination rule for any of the logical operators and quantifiers. If one is prepared to regard the former as synthetic, one should therefore be prepared also to regard the latter as synthetic. Frege, for one, would certainly not be prepared to call the elimination rules for the logical operators and quantifiers synthetic.

The identity predicate of predicate logic corresponds in type theory to a ternary proposition-forming operator $\mathbf{Id}$: applied to a type of individuals A and objects a, b of type A, it yields a proposition $\mathbf{Id}(A, a, b)$. The introduction and elimination rules for $\mathbf{Id}$ generalize (=I) and (=E) to the type-theoretical language. The introduction rule is as follows:

$$(\mathbf{Id}\text{-I})\ \frac{a : A}{\mathbf{refl}(a) : \mathbf{Id}(A, a, a)}$$

This rule generalizes (=I) in the sense that it makes explicit the domain, A, to which the self-identified object a belongs. It also introduces the proof, or truthmaker, $\mathbf{refl}(a)$, of the proposition $\mathbf{Id}(A, a, a)$. The elimination rule for $\mathbf{Id}$ generalizes (=E), but to formulate it, we would have to go into more details of the syntax of type theory than we shall do here (see Klev, 2019b, 2022).

7. Are numerical formulae demonstrable?

In both Martin-Löf type theory and natural deduction extended with definitional identity, the notion of a numerical formula as defined in the opening of this paper may be understood in two ways. In type theory, a numerical formula may be understood either as a judgement of the form $m = n : \mathbf{N}$ or as a judgement that a proposition of the form $\mathbf{Id}(\mathbf{N}, m, n)$ is inhabited, i.e., as a judgement $p : \mathbf{Id}(\mathbf{N}, m, n)$. In natural deduction extended with definitional identity, a numerical formula may be understood either as a formula of definitional identity, $m \equiv n$, or as an ordinary predicate-logical formula, $m = n$. Corresponding to these two ways of understanding what a numerical formula is are two ways of understanding what a demonstration of a numerical formula is: it is either a demonstration in the theory of definitional identity or one also making use of the identity introduction rule and—unless the terms flanking the identity sign are syntactically identical—of a conversion rule. A demonstration of the former kind is the more fundamental of the two, since any use of conversion has to appeal to such a demonstration.

A demonstration of a definitional identity is a demonstration by calculation in the sense of Section 3 above. It is of a kind with the demonstration of $2 + 2 = 4$ offered by Leibniz. Frege claimed that this demonstration, or the demonstration of numerical formulae quite generally, depends on certain general laws, of which he mentions only the associativity of addition. We have seen, to the contrary, that no arithmetical laws are needed for such demonstrations apart from the licence to define functions on the natural numbers by induction. The demonstration itself proceeds in effect by the substitution of definitionally identical numbers for each other and has therefore a purely combinatorial character. Such a demonstration renders the numerical formula analytic, not only in Frege's sense (*Grundlagen* § 3), but also in the more traditional sense that the demonstration is a piece of concept analysis, or more precisely, definitional unfolding.

In § 87 of the *Grundlagen* Frege claims that a consequence of the success of his logicist programme would be that calculation becomes deductive reasoning: "Rechnen wäre Schlussfolgern". I do not know of any illustrations of this idea in Frege's writings, but the explication of calculation offered here, namely as definitional reduction formalized in a theory of definitional identity may perhaps be one. The reasoning carried out in such a theory is, however, not quite logical reasoning, since no laws governing logical constants are involved. On the other hand, neither is it extra-logical reasoning in the sense of reasoning relying on laws pertaining to specific domains of discourse. Perhaps one might call it pre-logical reasoning, since it has such a basic character and can itself be appealed to in logical reasoning.

The demonstration of a judgement of the form $p : \mathbf{Id}(\mathbf{N}, m, n)$ may be assumed to have the following form:

$$(\Delta) \qquad \cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ n : \mathbf{N}\end{array}}{\mathbf{refl}(n) : \mathbf{Id}(\mathbf{N}, n, n)}\,(\mathbf{Id}\text{-I}) \qquad \begin{array}{c}\mathcal{D}'\\ \mathbf{Id}(\mathbf{N}, n, n) = \mathbf{Id}(\mathbf{N}, m, n) : \mathbf{prop}\end{array}}{\mathbf{refl}(n) : \mathbf{Id}(\mathbf{N}, m, n)}$$

The left-hand demonstration, $\mathcal{D}$, establishes $n : \mathbf{N}$. It shows the formation of the natural number n. The right-hand demonstration, $\mathcal{D}'$, is a demonstration by calculation, showing that $\mathbf{Id}(\mathbf{N}, n, n)$ and $\mathbf{Id}(\mathbf{N}, m, n)$ are definitionally identical propositions. Both of these demonstrations are, in a natural sense, pre-logical, involving only the formation of a natural number and some calculation. The demonstration Δ, by contrast, is logical, by virtue of the application of ($\mathbf{Id}$-I), the law of identity. A demonstration of a numerical formula of this second kind does, therefore, rest on logic. The only logical law that must be appealed to in general, however, is the law of identity. The associativity of addition or other similarly advanced arithmetical laws are not needed.

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