The Logic of Hyperlogic
Part B: Extensions and Restrictions

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Abstract. This is the second part of a two-part series on the logic of hyperlogic, a formal system for regimenting metalogical claims in the object language (even within embedded environments). Part A provided a minimal logic for hyperlogic that is sound and complete over the class of all models. In this part, we extend these completeness results to stronger logics that are sound and complete over restricted classes of models. We also investigate the logic of hyperlogic when the language is enriched with hyperintensional operators such as counterfactual conditionals and belief operators.

B1 Introduction

This is the second part of a two-part series on the logic of *hyperlogic*, a hyperintensional semantics designed to regiment metalogical claims (e.g., "Intuitionistic logic is correct" or "The law of excluded middle holds") in the object language. To recap, this regimentation is achieved using:

- a multigrade entailment operator $\rhd$
- propositional quantifiers $\forall p$ and $\exists p$
- interpretation terms ι that double as atomic formulas ("ι is correct")
- hybrid operators $@_\iota$ ("according to ι") and $\downarrow i$ ("where i is the current interpretation").

The semantics of hyperlogic introduces the notion of a "hyperconvention", i.e., a complete interpretation of the propositional variables, boolean connectives, and $\rhd$ over some space of possible worlds propositions. Interpretation terms denote "conventions", modeled as sets of hyperconventions. Propositional quantifiers range over (special kinds of) index propositions, i.e., sets of world-hyperconvention pairs. Models in this semantics determine (i) a set of worlds W; (ii) a domain of admissible conventions $D_{\mathbb{C}}$; (iii) a domain of admissible index propositions $D_{\mathbb{P}}$; and (iv) a valuation V. Truth-in-a-model is evaluated relative to worlds and hyperconventions. Operators like $@_\iota$ can shift the hyperconvention parameter. This allows formulas to be assessed on alternative interpretations of the connectives and entailment. Hyperintensionality is thus captured through shifting these interpretations.

In Part A (Kocurek, 2022), a complete axiomatization for this semantics was given. The axiomatization in Part A captures consequence over the class of *all* models. Almost no constraints are placed on either a model's convention or proposition domain. The resulting logic for hyperlogic is, therefore, fairly minimal. For example, no constraints are placed on the interpretations the entailment operator $\rhd$ can receive. Yet, intuitively, it would be a stretch to say $\rhd$ really represents a notion of "entailment" if, say, it wasn't factive (i.e., if $\rhd\phi$ did not imply ϕ), or if it wasn't reflexive or transitive. It would then be natural to inquire into how imposing such constraints affects the underlying logic of hyperlogic.

Furthermore, hyperlogic was initially motivated by concerns with the interaction between metalogical claims and hyperintensional operators such as attitude verbs, counterfactuals, and so on. Yet the language of hyperlogic introduced in Part A does not contain any of such operators.

In Part B of this series, we take initial steps to filling these gaps. We start by studying stronger logics for hyperlogic that can be obtained by adding additional rules and axioms in §B2. These stronger logics can be shown to be sound and complete over classes of models whose convention and proposition domains satisfy certain natural constraints. In §B3, we examine how the completeness results from Part A carry over to languages with hyperintensional operators. We conclude in §B4 with some questions left open by this two-part investigation into the logic of hyperlogic. §B5 is a technical appendix containing proofs of completeness for various classes of hypermodels.

Note: as this is a continuation of a two-part series, I will freely refer back to the definitions, notation, and results from Part A (Kocurek, 2022), rather than repeat them. Labels for sections, definitions, theorems, and tables are prefixed with the part that they refer to (e.g., '§A3' refers to §3 of Part A).

B2 Restrictions on Hypermodels

Let us start by exploring constraints we may impose on the class of hypermodels and how that affects the logic of hyperlogic. In §B2.1, we look at general constraints on the convention domain and present axiomatizations in the quantifier-free fragment over those hypermodels. In §B2.2, we extend some of these results to languages with propositional quantifiers. Finally, in §B2.3, we examine constraints on the proposition domain.

B2.1 Quantifier-Free Fragment

Table B1 contains a sample of constraints one may want to impose on the convention domain. For the analyticity constraint, we write $c \approx c'$ to mean c and c' are exactly alike except possibly in how they interpret propositional variables (i.e., $c(\triangle) = c'(\triangle)$ for all $\triangle$). The intersection of each class in Table B1 is denoted by concatenation (e.g., the class of analytic and full hypermodels is AnF). Where X is a class of hypermodels, we define classical and universal entailment **over X**, written $\Gamma \vDash_{\mathsf{X}} \phi$ and $\Gamma \Vdash_{\mathsf{X}} \phi$ respectively, as in Definition A2.16 except restricting to hypermodels in X.

Name	*Class*	*Constraint* (on all $C \in D_{\mathbb{C}}$)
full	F	$\pi_c = \wp W$ for each $c \in C$
atomic	At	$\{w\} \in \pi_c$ for each $c \in C$ and $w \in W$
boolean	B	π_c is closed under complement and finite intersection for each $c \in C$
quantification uniform	$\mathsf{U_q}$	$\pi_c = \pi_{c'}$ for each $c, c' \in C$
operation uniform	$\mathsf{U_o}$	$c(\triangle) = c'(\triangle)$ for each $c, c' \in C$ and each $\triangle$
singular	Si	$\lvert C \rvert = 1$
analytic	An	for any $c, c' \in D_{\mathbb{H}}$, if $c \in C$ and $c \approx c'$, then $c' \in C$
S5-universal	$\mathsf{S_5}$	each $c \in C$ is classical
classically complete	Co_{cl}	$V(cl) = \{c \in \mathbb{H}_W \mid c \text{ is classical}\}$

Table B1: Some constraints on convention domains.

Table B2 contains axiomatizations of consequence over various classes. Some of the axioms make use of the following abbreviations:

$$\widehat{@}_\iota\, \phi := \sim @_\iota \sim \phi \qquad\qquad (\phi)^\iota = (\psi)^\kappa := (@_\iota\, \phi = @_\kappa\, \psi).$$

Here are their truth conditions (where $[\![\phi]\!]^C = \bigcap_{c \in C} [\![\phi]\!]^c$):

$$\begin{aligned} w, c \Vdash \widehat{@}_\iota\, \phi \quad &\Leftrightarrow \quad \text{for some } c' \in V(\iota)\text{: } w, c \Vdash \phi \\ w, c \Vdash (\phi)^\iota = (\psi)^\kappa \quad &\Leftrightarrow \quad [\![\phi]\!]^{V(\iota)} = [\![\psi]\!]^{V(\kappa)}. \end{aligned}$$

In addition, we write $\overline{\phi}$ for $\phi_1, \ldots, \phi_n$, and $\overline{\phi}^\kappa = \overline{\psi}^\lambda$ for $\&_{i=1}^n (\phi_i^\kappa = \psi_i^\lambda)$. Where **L** is a logic and A is an axiom, **L** + A is the result of extending **L** with A (i.e., the rules still apply to formulas derived using A). If R is a rule, **L** + R is the result of closing **L** under R along with the other rules. Given this, we have the following:

Name	*Axiom/Rule*
$\mathrm{Uni}_{\circ}$	$\kappa \in \iota, \lambda \in \iota, \overline{\phi}^{\kappa} = \overline{\psi}^{\lambda} \Vdash (\triangle(\overline{\phi}))^{\kappa} = (\triangle(\overline{\psi}))^{\lambda}$
Sing	$\Vdash \lvert\iota\rvert_1$
Self-Dual$_{@}$	$@_{\iota}\, \phi \dashv\Vdash \widehat{@}_{\iota}\, \phi$
$\mathrm{Bool}_{\Vdash}$	$\star\, \phi \dashv\Vdash \bigstar\, \phi$
	$(\phi \circ \psi) \dashv\Vdash (\phi \bullet \psi)$
RAn	if $\overline{\alpha}, \lvert\kappa\rvert_1, \lvert\lambda\rvert_1, (\overline{p})^{\kappa} = (\overline{q})^{\lambda} \Vdash (\triangle(\overline{p}))^{\kappa} = (\triangle(\overline{q}))^{\lambda}$ for each $\triangle$ where none of $\overline{p}, \overline{q}$ are in $\overline{\alpha}$, then $\overline{\alpha}, \lvert\kappa\rvert_1, \lvert\lambda\rvert_1 \Vdash (\kappa \in \iota) \equiv (\lambda \in \iota)$
Class	*Axiomatization*
F, $\mathsf{U_q}$, At, B	**H**
$\mathsf{U_o}$	**H** + $\mathrm{Uni}_{\circ}$
Si	**H** + Sing = **H** + Self-Dual$_{@}$
AnF, $\mathsf{AnU_q}$	**H** + RAn
$\mathsf{S_5}$	**H** + $\mathrm{Bool}_{\Vdash}$

Table B2: Axiomatizations in $\mathcal{L}^{\mathsf{H}}$ for various classes from Table B1. Axiomatizations in $\mathcal{L}^{\mathsf{HE}}$ (except those appealing to RAn, which becomes infinitary when add $\triangleright$) are obtained by replacing **H** with $\mathbf{H}_{\triangleright}$ and generalizing the corresponding axioms accordingly.

Theorem B2.1 (Relative Completeness in $\mathcal{L}^{\mathsf{H}}$ and $\mathcal{L}^{\mathsf{HE}}$). The axiomatic systems in Table B2 are sound and complete for (consequence over) the relevant class of hypermodels.[1] (See §B5.1 for the proof.)

In addition, we can consider imposing restrictions specifically on the interpretation of $\triangleright$. Usual suspects include reflexivity, transitivity, monotonicity, etc. But there are also "unusual" suspects to consider (e.g., factivity) since $\triangleright$ is an object language operator. Table B3 contains examples of such constraints, with their corresponding axioms stated in Table B4. Following our earlier convention, we write $\overline{X}$ for $X_1, \ldots, X_n$. (If $n = 0$, then $\overline{X} = \langle\rangle$.) We also write $(\overline{X} \triangleright_c \overline{Y})$ for $\bigcap_i (\overline{X} \triangleright_c Y_i)$.

[1] It is an open question whether consequence in $\mathcal{L}^{\mathsf{H}}$ over An or Co_{cl} can be axiomatized. An axiomatization for An in $\mathcal{L}^{\mathsf{QH}}$ is given in §B2.2. (Interestingly, the key axiom invokes $\forall\exists$-quantification, which cannot be directly expressed in $\mathcal{L}^{\mathsf{H}}$.) By contrast, consequence for Co_{cl} in $\mathcal{L}^{\mathsf{QH}}$ is provably unaxiomatizable (Corollary B2.5).

Name	*Constraint* (on all $c, c' \in D_{\mathbb{H}}$)
reflexive	$(X \rhd_c X) = W$
transitive	$(\overline{X} \rhd_c \overline{Y}) \cap (\overline{Y} \rhd_c Z) \subseteq (\overline{X} \rhd_c Z)$
monotonic	$(\overline{X}, \overline{Y} \rhd_c Z) \subseteq (\overline{X}, U, \overline{Y} \rhd_c Z)$
contractive	$(\overline{X}, U, U, \overline{Y} \rhd_c Z) \subseteq (\overline{X}, U, \overline{Y} \rhd_c Z)$
commutative	$(\overline{X}, U_1, U_2, \overline{Y} \rhd_c Z) = (\overline{X}, U_2, U_1, \overline{Y} \rhd_c Z)$
congruential	$((\overline{X} \rhd_c \overline{Y}) \cap (\overline{Y} \rhd_c \overline{X}) \cap (\overline{X} \rhd_c Z)) \subseteq (\overline{Y} \rhd_c Z)$
self-aware	$(\overline{X} \rhd_c (\overline{Y} \rhd_c Z)) = (\overline{Y} \rhd_c Z)$
fully aware	$(\overline{X} \rhd_c (\overline{Y} \rhd_{c'} Z)) = (\overline{Y} \rhd_{c'} Z)$
import-export	$(\overline{X} \rhd_c (\overline{Y} \rhd_c Z)) = (\overline{X}, \overline{Y} \rhd_c Z)$
$\supset$-residuation	$(\overline{X}, Y \rhd_c Z) = (\overline{X} \rhd_c (\overline{Y} \cup Z))$
$\rightarrow$-residuation	$(\overline{X}, Y \rhd_c Z) = (\overline{X} \rhd_c (Y \rightarrow_c Z))$
&-fusion	$(\overline{X}, U_1, U_2, \overline{Y} \rhd_c Z) = (\overline{X}, U_1 \cap U_2, \overline{Y} \rhd_c Z)$
$\wedge$-fusion	$(\overline{X}, U_1, U_2, \overline{Y} \rhd_c Z) = (\overline{X}, U_1 \wedge_c U_2, \overline{Y} \rhd_c Z)$
factive	$((\overline{X} \rhd_c Y) \cap X_1 \cap \cdots \cap X_n) \subseteq Y$
noncontingent	either $(\overline{X} \rhd_c Y) = W$ or $(\overline{X} \rhd_c Y) = \varnothing$
strict	$(\overline{X} \rhd_c Y) = \{w \in W \mid X_1 \cap \cdots \cap X_n \subseteq Y\}$

Table B3: Some constraints on the interpretation of $\rhd$.

Theorem B2.2 (Relative Completeness for $\rhd$). The axiomatic systems resulting from adding the relevant axioms in Table B4 to $\mathbf{H}_{\rhd}$ are sound and complete for the relevant class of hypermodels.

Proof (*Sketch*): We revise the definition of the proposition space for canonical hyperconventions (Definition A3.13) so that $\pi_{c_\kappa} = \{X \mid [X]_\kappa \neq \varnothing\}$.[2] The completeness proof in §A3.2 remains in tact. We just need to verify that if we impose an axiom, the canonical model satisfies the corresponding constraint. The proof is more-or-less the same for each case. We illustrate with the transitivity case. Suppose $\Delta \in (\overline{X} \rhd_{c_\kappa} \overline{Y}) \cap (\overline{Y} \rhd_{c_\kappa} Z)$. Since $\Delta \in (\overline{X} \rhd_{c_\kappa} \overline{Y})$, there are some $\overline{\phi} \in [\overline{X}]_\kappa$ and $\overline{\psi} \in [\overline{Y}]_\kappa$ such that $@_\kappa(\overline{\phi} \rhd \psi_i) \in \Delta$ for each i (note: we can let $\overline{\phi}$ be the same for each ψ_i by Lemma A3.12 and $\text{Rep}_{\rhd}$). Since

[2] Observe that this revised definition of π_{c_κ} is not guaranteed to be full or atomic, so this proof does not automatically carry over when these constraints are also imposed.

Name	*Axiom*	*Corresponding Constraint*
Id	$\Vdash (\phi \rhd \phi)$	reflexive
Tr	$(\overline{\phi} \rhd \overline{\psi}), (\overline{\psi} \rhd \chi) \Vdash (\overline{\phi} \rhd \chi)$	transitive
Weak	$(\overline{\alpha}, \overline{\beta} \rhd \chi) \Vdash (\overline{\alpha}, \phi, \overline{\beta} \rhd \chi)$	monotonic
Contr	$(\overline{\alpha}, \phi, \phi, \overline{\beta} \rhd \chi) \Vdash (\overline{\alpha}, \phi, \overline{\beta} \rhd \chi)$	contractive
Perm	$(\overline{\alpha}, \phi, \psi, \overline{\beta} \rhd \chi) \Vdash (\overline{\alpha}, \psi, \phi, \overline{\beta} \rhd \chi)$	commutative
Cong	$(\overline{\phi} \rhd \overline{\psi}), (\overline{\psi} \rhd \overline{\phi}), (\overline{\phi} \rhd \chi) \Vdash (\overline{\psi} \rhd \chi)$	congruential
Self-Aware	$(\overline{\phi} \rhd (\overline{\psi} \rhd \chi)) \dashv\Vdash (\overline{\psi} \rhd \chi)$	self-aware
Aware	$(\overline{\phi} \rhd (\overline{\psi} \rhd_\iota \chi)) \dashv\Vdash (\overline{\psi} \rhd_\iota \chi)$	fully aware
IE	$(\overline{\phi} \rhd (\overline{\psi} \rhd \chi)) \dashv\Vdash (\overline{\phi}, \overline{\psi} \rhd \chi)$	import-export
$\text{Res}_\supset$	$(\overline{\phi} \rhd (\psi \supset \chi)) \dashv\Vdash (\overline{\phi}, \psi \rhd \chi)$	$\supset$-residuation
$\text{Res}_\rightarrow$	$(\overline{\phi} \rhd (\psi \rightarrow \chi)) \dashv\Vdash (\overline{\phi}, \psi \rhd \chi)$	$\rightarrow$-residuation
$\text{Fus}_\cap$	$(\overline{\alpha}, \phi, \psi, \overline{\beta} \rhd \chi) \dashv\Vdash (\overline{\alpha}, \phi \,\&\, \psi, \overline{\beta} \rhd \chi)$	&-fusion
$\text{Fus}_\wedge$	$(\overline{\alpha}, \phi, \psi, \overline{\beta} \rhd \chi) \dashv\Vdash (\overline{\alpha}, \phi \wedge \psi, \overline{\beta} \rhd \chi)$	$\wedge$-fusion
$\text{T}_\rhd$	$(\overline{\phi} \rhd \psi), \overline{\phi} \Vdash \psi$	factive
$\text{Rigid}_\rhd$	$(\overline{\phi} \rhd \psi) \Vdash \blacksquare(\overline{\phi} \rhd \psi)$	noncontingent
$\text{Strict}_\rhd$	$(\overline{\phi} \rhd \psi) \dashv\Vdash \blacksquare(\widehat{\phi} \supset \psi)$	strict

Table B4: Axiomatizations in $\mathcal{L}^{\mathsf{HE}}$ for various classes from Table B3.

$\Delta \in (\overline{Y} \rhd_{c_\kappa} Z)$, there is a $\chi \in [Z]_\kappa$ such that $@_\kappa(\overline{\psi} \rhd \chi) \in \Delta$. By Tr, $@_\kappa(\overline{\phi} \rhd \chi) \in \Delta$. Hence, $\Delta \in (\overline{X} \rhd_{c_\kappa} Z)$. ■

B2.2 Adding Quantifiers

Adding propositional quantifiers to the language allows us the ability to distinguish between classes of models that previously generated the same logic. Notably, the consequence relations over F, $\mathsf{U_q}$, At, and B are now all distinguishable. In addition, we can now present an axiomatization for An, which was absent from §B2.1 (see footnote 1).

Axiomatizations for some of those classes are given in Table B5. Where Σ is a set of axioms of the form $\Vdash \sigma$, we let $\mathbf{L} \cup \Sigma$ be the proof system defined as follows: $\Gamma \Vdash_{\mathbf{L}\cup\Sigma} \phi$ iff $\Gamma \cup \{\sigma \mid (\Vdash \sigma) \in \Sigma\} \Vdash_{\mathbf{L}} \phi$ (in other words, Σ are treated as premises, not axioms; this means, among other things, that one cannot necessarily derive the universal generalization of members of Σ).

The axiomatizations in Table B5 make use of the following abbreviations:

$$\pi_\kappa \subseteq \pi_\lambda := \forall p\, \exists q (p^\kappa = q^\lambda) \qquad \pi_\kappa = \pi_\lambda := (\pi_\kappa \subseteq \pi_\lambda) \,\&\, (\pi_\lambda \subseteq \pi_\kappa)$$
$$|\pi_\kappa|_1 := \forall p\, \forall q (p =_\kappa q) \qquad \kappa \approx \lambda := \&\{\triangle_\kappa = \triangle_\lambda\}_\triangle \,\&\, (\pi_\kappa = \pi_\lambda).$$

These have the obvious truth conditions assuming $|V(\kappa)| = |V(\lambda)| = 1$ (which is the only relevant case for the axiomatizations below).

Name	*Axiom/Rule*
Atom	$\Vdash \exists p(p \,\&\, \forall q(\blacksquare(p \supset @_\iota\, q) + \blacksquare(p \supset {\sim} @_\iota\, q)))$
BoolEx	$\Vdash \mathsf{E} {\sim} p$
	$\Vdash \mathsf{E}(p \,\&\, q)$
Ex	$\Vdash \mathsf{E}\phi$
An	${\sim}|\iota|_1, \kappa \in \iota, |\lambda|_1, \kappa \approx \lambda \Vdash \lambda \in \iota$
Many$_{\mathsf{INom}}$	$\Vdash (|l|_1 \,\&\, |\iota|_1 \,\&\, \iota \approx l) \supset (\iota = l)$ where $l \in \mathsf{INom}$
Uni$_\mathsf{q}$	$\kappa \in \iota, \lambda \in \iota \Vdash \pi_\kappa = \pi_\lambda$
Uni$_{\mathsf{o}\forall}$	$\kappa \in \iota, \lambda \in \iota \Vdash \kappa \approx \lambda$
Class	*Axiomatization*
At	**QH** + Atom
B	**QH** + BoolEx
$\mathsf{U_q}$	**QH** + Uni$_\mathsf{q}$
$\mathsf{U_o}$	**QH** + Uni$_{\mathsf{o}\forall}$
An	(**QH** + An) $\cup$ Many$_{\mathsf{INom}}$
Si	**QH** + Sing = **QH** + Self-Dual$_@$
$\mathsf{S_5}$	**QH** + Bool$_\Vdash$ + Ex

Table B5: Axiomatizations in $\mathcal{L}^{\mathsf{QH}}$ for various classes from Table B1.

Theorem B2.3 (Relative Completeness in $\mathcal{L}^{\mathsf{QH}}$). The proof systems in Table B5 are sound and complete for the relevant class of hypermodels. (See §B5.2.)

Notice that no axiomatization for F is stated. This is because consequence over F is unaxiomatizable.

Theorem B2.4 (Unaxiomatizability of Full Consequence in $\mathcal{L}^{\mathsf{QH}}$). $\vDash_\mathsf{F}$ in $\mathcal{L}^{\mathsf{QH}}$ is unaxiomatizable. Moreover, where X is the intersection of any of the classes mentioned in Table B1 and Table B6, if $\mathsf{FX} \neq \varnothing$, then $\vDash_{\mathsf{FX}}$ in $\mathcal{L}^{\mathsf{QH}}$ is unaxiomatizable.

Proof: Let $\mathsf{At}(p) := \blacklozenge\, p \;\&\; \forall q(\blacksquare(p \supset q) + \blacksquare(p \supset {\sim} q))$. It is easy to verify that if c is full, then $\mathcal{M}, w, c \Vdash \mathsf{At}(p)$ iff $|V(p)(c)| = 1$. Let Δ consist of the following formulas:

$$\begin{array}{ll}
\forall p\, @_k(\neg\, p = {\sim}\, p) & \forall p\, @_k(\Diamond\, p = \neg \Box \neg\, p) \\
\forall p\, \forall q\, @_k((p \wedge q) = (p \,\&\, q)) & \forall p\, \forall q\, @_k\, \blacksquare(\Box(p \to q) \to (\Box\, p \to \Box\, q)) \\
\forall p\, \forall q\, @_k((p \vee q) = (p + q)) & \forall p\, @_k(\blacksquare\, p \supset \blacksquare\, \Box\, p) \\
\forall p\, \forall q\, @_k((p \to q) = (p \supset q)) & \forall p\, @_k\, \blacksquare(\forall q(\mathsf{At}(q) \supset \Box(q \supset p)) \supset \Box\, p).
\end{array}$$

Since the propositionally quantified modal logic $\mathbf{K}\pi+$ is unaxiomatizable (Fine, 1970), it suffices to show that for any $\phi \in \mathcal{L}^{\mathsf{Q}}$ (the language of propositionally quantified modal logic), $\vDash_{\mathbf{K}\pi+} \phi$ iff $\Delta, |k|_1 \vDash_{\mathsf{F}} @_k\, \phi$. We do this by constructing, for each $\mathbf{K}\pi+$-model, an equivalent full hypermodel of Δ and vice versa.

First, let $\mathcal{K} = \langle W, R, V\rangle$ be a $\mathbf{K}\pi+$-model. Let c_k be defined as follows:

$$\begin{array}{rlrl}
\pi_{c_k} &= \wp\, W & \neg_{c_k} X &= \overline{X} \\
c_k(p) &= V(p) & X \wedge_{c_k} Y &= X \cap Y \\
\Box_{c_k} X &= \{w \in W \mid R[w] \subseteq X\} & X \vee_{c_k} Y &= X \cup Y \\
\Diamond_{c_k} X &= \{w \in W \mid R[w] \cap X \neq \varnothing\} & X \to_{c_k} Y &= \overline{X} \cup Y.
\end{array}$$

Define $\mathcal{M}^{\mathcal{K}} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, V^{\mathcal{K}}\rangle$ so that (i) $c_k \in D_{\mathbb{H}}$, (ii) each $c \in D_{\mathbb{H}}$ is full, (iii) $V^{\mathcal{K}}(p) = P_p$, and (iv) $V^{\mathcal{K}}(k) = \{c_k\}$. Clearly, $\mathcal{M}^{\mathcal{K}}, w, c_k \Vdash \Delta \cup \{|k|_1\}$. Moreover, by induction, for all $\phi \in \mathcal{L}^{\mathsf{Q}}$ and all $Q_1, \ldots, Q_n$ where $Q_i(c_k) = X_i$, we have $\mathcal{K}^{q_1,\ldots,q_n}_{X_1,\ldots,X_n}, w \Vdash \phi$ iff $(\mathcal{M}^{\mathcal{K}})^{q_1,\ldots,q_n}_{Q_1,\ldots,Q_n}, w, c_k \Vdash \phi$. Hence, $\mathcal{K}, w \Vdash \phi$ iff $\mathcal{M}^{\mathcal{K}}, w, c \Vdash @_k\, \phi$.

Next, let $\mathcal{M} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, V\rangle$ be a full hypermodel satisfying $\Delta \cup \{|k|_1\}$. Let c_k be such that $V(k) = \{c_k\}$. Define $\mathcal{K}^{\mathcal{M}} = \langle W, R, V^{\mathcal{M}}\rangle$ so that (i) wRv iff for all $X \subseteq W$, if $w \in \Box_{c_k} X$, then $v \in X$, and (ii) $V^{\mathcal{M}}(p) = c_k(p)$. We establish by induction that for all $\phi \in \mathcal{L}^{\mathsf{Q}}$ and all $Q_1, \ldots, Q_n$ where $Q_i(c_k) = X_i$, we have $\mathcal{M}^{q_1,\ldots,q_n}_{Q_1,\ldots,Q_n}, w, c_k \Vdash \phi$ iff $(\mathcal{K}^{\mathcal{M}})^{q_1,\ldots,q_n}_{X_1,\ldots,X_n}, w \Vdash \phi$. The only interesting case is the $\Box$-clause.

Observe that $R[w] = \{v \in W \mid w \in \Diamond_{c_k} \{v\}\}$.[3] For notational ease, let $\mathcal{M}^* = \mathcal{M}^{q_1,\ldots,q_n}_{Q_1,\ldots,Q_n}$ and $\mathcal{K}^* = (\mathcal{K}^{\mathcal{M}})^{q_1,\ldots,q_n}_{X_1,\ldots,X_n}$.

(⇒) Suppose $\mathcal{M}^*, w, c_k \Vdash \Box\phi$. Thus, $w \in \Box_{c_k} [\![\phi]\!]^{\mathcal{M}^*,c_k}$. Let $v \in R[w]$. Then for all $X \subseteq W$, if $w \in \Box_{c_k} X$, then $v \in X$. Hence, $v \in [\![\phi]\!]^{\mathcal{M}^*,c_k}$, which by IH means $v \in [\![\phi]\!]^{\mathcal{K}^*}$. Hence, $\mathcal{K}^*, w \Vdash \Box\phi$.

(⇐) Suppose $\mathcal{M}^*, w, c_k \nVdash \Box\phi$. Thus, $w \notin \Box_{c_k} [\![\phi]\!]^{\mathcal{M}^*,c_k}$. Since c_k is full, by Definition A2.11 (constraint ii on $D_{\mathbb{P}}$), there exists a P such that $P(c_k) = [\![\phi]\!]^{\mathcal{M}^*,c_k}$. By the definition of Δ, $(\mathcal{M}^*)^p_P, w, c_k \Vdash \exists q(\mathsf{At}(q) \wedge \Diamond(q \wedge \neg p))$. Let Q be such that $(\mathcal{M}^*)^{p,q}_{P,Q}, w, c_k \Vdash \mathsf{At}(q) \wedge \Diamond(q \wedge \neg p)$. Thus, $Q(c_k) = \{v\}$ for some $v \notin P(c_k) = [\![\phi]\!]^{\mathcal{M}^*,c_k}$. By IH, $v \notin [\![\phi]\!]^{\mathcal{K}^*}$, i.e., $\mathcal{K}^*, v \nVdash \phi$. And since $w \in \Diamond_{c_k}(Q(c_k) \cap \overline{P(c_k)}) = \Diamond_{c_k}\{v\}$, that means $v \in R[w]$, and so $\mathcal{K}^*, w \nVdash \Box\phi$. ■

Corollary B2.5 (Unaxiomatizability of Classically Complete Consequence in $\mathcal{L}^{\mathsf{QH}}$). $\vDash_{\mathsf{Co}_{cl}}$ in $\mathcal{L}^{\mathsf{QH}}$ is unaxiomatizable, as is $\vDash_{\mathsf{Co}_{cl}\mathsf{X}}$ for any X that is the intersection of any of the classes mentioned in Table B1 and Table B6 where $\mathsf{Co}_{cl}\mathsf{X} \neq \varnothing$.

Proof: Since $V(cl) = \{c \in \mathbb{H}_W \mid c \text{ is classical}\}$, there is a $c \in V(cl)$ such that c is full. So adding $@_{cl}\,\forall p\,\exists q(p = @_k\, q)$ to Δ is enough to ensure that c_k is full. ■

B2.3 Constraints on Propositions

Let's now turn to constraints on the proposition domain. A sample of such constraints is given in Table B6. For strong closure, we write $\mathcal{M} \approx \mathcal{M}'$ to mean $\mathcal{M}$ and $\mathcal{M}'$ are based on the same hyperframe (i.e., only differ in valuation). Axiomatizations for consequence over some classes are stated in Table B7. Some of the axioms use the following abbreviation: $(\phi =_\iota \psi) := @_\iota(\phi = \psi)$. Completeness for the intersections of these classes can be gotten

[3] For the $\subseteq$-direction: If $v \in R[w] = \{u \in W \mid \forall X \subseteq W\colon w \in \Box_{c_k} X \Rightarrow u \in X\}$, then $w \notin \Box_{c_k} \overline{\{v\}} = \Box_{c_k} \neg_{c_k} \{v\}$, and so $w \in \Diamond_{c_k}\{v\}$. For the $\supseteq$-direction: If $w \in \Diamond_{c_k}\{v\}$, then $w \notin \Box_{c_k}\overline{\{v\}}$. So let $X \subseteq W$ where $w \in \Box_{c_k} X$. If $v \notin X$, then $X \subseteq \overline{\{v\}}$. Thus, $(X \rightarrow_{c_k} \overline{\{v\}}) = W$. By the necessitation formula, $\Box_{c_k} W = W$. Hence, $w \in \Box_{c_k}(X \rightarrow_{c_k} \overline{\{v\}})$. By the K axiom formula, $w \in \Box_{c_k}\overline{\{v\}}$, ↯. Hence, $v \in X$.

from combining the relevant axiomatizations, with the exception of $\mathsf{Cl}_\Phi\mathsf{Df}_\Phi$ and $\mathsf{Cl}^+_\Phi\mathsf{Df}_\Phi$, which are mentioned explicitly in Table B7.

Name	*Class*	*Constraint* (on all $P \in D_\mathbb{P}$)
complete	Cp	$D_\mathbb{P} = \mathbb{P}_{D_\mathbb{H}}$
correlated	Cr	$P(c) = P(c')$ whenever $c, c' \in C$
closed under Φ	Cl_Φ	$[\![\phi]\!]^\mathcal{M} \in D_\mathbb{P}$ for all $\phi \in \Phi$
strongly closed under Φ	Cl^+_Φ	$[\![\phi]\!]^{\mathcal{M}'} \in D_\mathbb{P}$ for all $\phi \in \Phi$ and all $\mathcal{M}' \approx \mathcal{M}$
definable in Φ	Df_Φ	if $P \in D_\mathbb{P}$, then there is a $\phi \in \Phi$ such that $[\![\phi]\!]^\mathcal{M} = P$
discerning	Di	for all $c, c' \in D_\mathbb{H}$, if $c \neq c'$, then for some $P \in D_\mathbb{P}$, $P(c) \neq P(c')$
combinatorial	Cb	if $X_1 \in \pi_{c_1}, \ldots, X_n \in \pi_{c_n}$ for some distinct $c_1, \ldots, c_n \in D_\mathbb{H}$, then for some $P \in D_\mathbb{P}$ such that $P(c_i) = X_i$ for $i \leqslant n$

Table B6: Some constraints on proposition domains.

Theorem B2.6 (Relative Completeness in $\mathcal{L}^{\mathsf{QH}}$). The proof systems in Table B7 are sound and complete over the relevant class. (See §B5.3.)

B3 Hyperintensional Operators

In this section, we enrich the language of hyperlogic with hyperintensional operators and explore their logic(s). We start by adding a counterfactual conditional and then show how a similar approach can apply to belief and knowledge operators. In §B3.1, we expand the syntax and semantics from §A2 to include a counterfactual conditional (following Kocurek (2021b)). In §B3.2, we axiomatize the minimal counterfactual hyperlogic on this semantics. In §B3.3, we extend this axiomatization to include an entailment operator/propositional quantifiers. In §B3.4, we explore stronger counterfactual hyperlogics obtained by imposing restrictions on the selection function. Finally, in §B3.5, we show how a similar approach can address the hyperlogic of belief/knowledge.

Name	*Axiom/Rule*
Corr	$\kappa \in \iota, \lambda \in \iota \Vdash \forall p(p^\kappa = p^\lambda)$
$\text{Elim}_{\forall\Phi}$	$\forall p\, \phi \Vdash \phi[\chi/p]$ where $\chi \in \Phi$ and χ is free for p
Ex_Φ	$\Vdash \exists p \,\&_{i=1}^n (p =_{\iota_i} \chi)$ where $\chi \in \Phi$ and p does not occur free in χ
Ex_Φ^-	$\Vdash \mathsf{E}\chi$ where $\chi \in \Phi$
PII^+	$\lvert\iota\rvert_1, \lvert\kappa\rvert_1, \forall p(p^\iota = p^\kappa) \Vdash (\iota = \kappa)$
PII_1^+	$\lvert\iota\rvert_1, \lvert\kappa\rvert_1, \forall p(p^\iota = p^\kappa), \lvert\pi_\iota\rvert_1 \Vdash (\iota = \kappa)$
Split	$\{\lvert\iota_i\rvert_1\}_{i=1}^n, \{(\iota_i \neq \iota_j)\}_{i \neq j} \Vdash \exists p \,\&_{i=1}^n (p =_{\iota_i} q_i)$ where $p \notin \{q_1, \ldots, q_n\}$
Hom_Φ	$\Vdash \forall p(p =_\iota \chi \supset p =_\kappa \chi)$ where $\chi \in \Phi$ and p does not occur free in χ
$\text{Gen}_{\forall\Phi}$	if each $\chi \in \Phi$ is free for p in ψ and $\vec{\alpha} \Vdash \psi[\chi/p]$ for each $\chi \in \Phi$, then $\vec{\alpha} \Vdash \forall p\, \psi$
Class	*Axiomatization*
Cr	**QH** + Corr
Cl_Φ	**QH** $\cup$ Ex_Φ
Cl_Φ^+	**QH** + Ex_Φ = **QH** + $\text{Elim}_{\forall\Phi}$
Df_Φ	**QH** + $\text{Gen}_{\forall\Phi}$ (only weakly complete if Φ is infinite)
$\mathsf{Cl}_\Phi\mathsf{Df}_\Phi$	(**QH** $\cup$ Ex_Φ) $\cup$ Hom_Φ = (**QH** $\cup$ Ex_Φ^-) $\cup$ Hom_Φ
$\mathsf{Cl}_\Phi^+\mathsf{Df}_\Phi$	(**QH** + Ex_Φ) $\cup$ Hom_Φ
Di	**QH** + PII^+
Cb	**QH** + PII^+ + Split = **QH** + PII_1^+ + Split
CpSi	**QH** + Split + Sing = **QH** + Split + $\text{Self-Dual}_@$

Table B7: Axiomatizations in $\mathcal{L}^{\mathsf{QH}}$ for various classes from Table B6.

B3.1 Selection Semantics

For any language $\mathcal{L}^*$ mentioned in Part A, we can consider the langauge $\mathcal{L}^*_{\Box\!\!\to}$ that results from extending $\mathcal{L}^*$ with a counterfactual conditional $\Box\!\!\to$. For instance, $\mathcal{L}^0_{\Box\!\!\to}$ is the result of extending $\mathcal{L}^0$ with $\Box\!\!\to$, $\mathcal{L}^{\mathsf{H}}_{\Box\!\!\to}$ the result of extending $\mathcal{L}^{\mathsf{H}}$ with $\Box\!\!\to$, and so on. To extend hyperlogic with a counterfactual conditional, Kocurek (2021b) proposes we allow counterfactuals to shift the hyperconvention parameter of an index. This can be achieved by simply replacing worlds in the standard (intensional) selection semantics for counterfactuals (Stalnaker, 1968; Lewis, 1973) with world-hyperconvention pairs. Thus, we revise Definitions A2.11 and A2.12 as follows:

Definition B3.1 (Selection Hypermodel). A **selection hyperframe** is a tuple $\mathcal{F} = \langle W, D_{\mathbb{C}}, D_{\mathbb{P}}, f \rangle$ where $\langle W, D_{\mathbb{C}}, D_{\mathbb{P}} \rangle$ is a hyperframe and $f : \wp\, \mathbb{I}_{D_{\mathbb{H}}} \times \mathbb{I}_{D_{\mathbb{H}}} \to \wp\, \mathbb{I}_{D_{\mathbb{H}}}$ is a **selection function**. A **selection hypermodel** over $\mathcal{F}$ is a selection hyperframe paired with a valuation for $\mathcal{F}$. Satisfaction is defined as in Definition A2.12 with the following addition, where $[\![\phi]\!]^{\mathcal{M}} = \{\langle v, d \rangle \in \mathbb{I}_{D_{\mathbb{H}}} \mid \mathcal{M}, v, d \Vdash \phi\}$:

$$\mathcal{M}, w, c \Vdash \phi \mathrel{\Box\!\!\rightarrow} \psi \quad \Leftrightarrow \quad f([\![\phi]\!]^{\mathcal{M}}, w, c) \subseteq [\![\psi]\!]^{\mathcal{M}}.$$

At the outset, we impose no restrictions on the selection function. Some theorists (e.g., Cohen 1990; Nolan 1997) argue that if counter(meta)logicals are nonvacuous, then the logic of counterfactuals is trivial. For example, it is nearly universally accepted that $\phi \mathrel{\Box\!\!\rightarrow} (\psi \wedge \chi) \vDash \phi \mathrel{\Box\!\!\rightarrow} \psi$. Yet, here is an alleged counterexample:

(1) a. If every instance of conjunction elimination had failed, Alice and Beth would be sad.
    b. $\overset{?}{\not\Rightarrow}$ If every instance of conjunction elimination had failed, Alice would be sad.

Similar "counterexamples" can be constructed to nearly every principle of counterfactual reasoning.[4] Even principles such as $\vDash \phi \mathrel{\Box\!\!\rightarrow} \phi$ have been called into question (Nolan, 1997, p. 555).[5]

[4] Nolan (1997) makes an exception for modus ponens ($\phi \mathrel{\Box\!\!\rightarrow} \psi, \phi \vDash \psi$), which is immune to counterexamples of this sort.

[5] We might try to save the standard logic for counterfactuals with possible antecedents (Brogaard and Salerno, 2013; Berto et al., 2018). It is not obvious this will work, though. Imagine Alice endorses a strange logic on which every instance of conjunction elimination fails. Then (i) is as problematic as (1) despite only having counterfactuals with possible antecedents (Alice could have had the right views about logic).

(i) a. If Alice were right about logic, every instance of conjunction elimination would fail.
    b. If Alice were right about logic, Beth and Cher would be sad.
    c. Therefore, if Alice were right about logic, Beth would be sad.

One may try to block this counterexample by denying the first premise on the grounds that the antecedent is possible and "nothing impossible would obtain were something that's possible to obtain". This reasoning appeals to what Nolan (1997) calls the "Strangeness of Impossibility Condition": no impossible world can occur closer to the actual world than any possible world. But this principle has been called into question (Nolan, 1997; Vander Laan, 2004; Bernstein, 2016; Clarke-Doane, 2019). Hyperlogic, by contrast, can explain what's

Hyperlogic offers refuge to those who find this disheartening. As we'll see, even though counter(meta)logicals are nonvacuous in hyperlogic, its counterfactual logic is nontrivial: the standard counterfactual principles can be salvaged when the connectives used to state those principles are classically rigidified. This means, among other things, that imposing constraints on the selection function is not incompatible with the nonnvacuity of counter(meta)logicals, such as those in (1).

B3.2 Completeness

Let's turn to the logic of counterfactual hyperlogic. Given that we are not placing any constraints on the selection function, what counterfactual principles, if any, are valid?

Kocurek and Jerzak (2021, Appendix) show that the logic of classical consequence in $\mathcal{L}^{0}_{\Box\!\!\to}$ is the same as the logic of the standard "impossible worlds" semantics for counterfactuals, where we can model an impossible world as an arbitrary set of formulas. But this is only because (as Cohen (1990); Nolan (1997) suggest) there are no valid principles of counterfactual reasoning that aren't already substitution instances of **S5**-theorems. Thus, without further constraints, (1) is invalid in the hyperconvention semantics when regimented so:

$$(\forall p\, \forall q\, {\sim}((p \wedge q) \rhd p) \Box\!\!\to (a \wedge b)) \therefore (\forall p\, \forall q\, {\sim}((p \wedge q) \rhd p) \Box\!\!\to a).$$

Fortunately, counterfactual hyperlogic in $\mathcal{L}^{\mathsf{H}}_{\Box\!\!\to}$ is more interesting, since it has the expressive resources to "hold fixed" the interpretation of a certain connective within the scope of a counterfactual (Kocurek and Jerzak, 2021, p. 21). If we require a certain formula within a counterfactual to be interpreted according to, say, a classical hyperconvention, then any entailments that formula generates in classical logic must be preserved. For example, the reason (1) seems to invalidate conjunction elimination in the consequent is that the word 'and' in the consequent is being interpreted relative to a logic where conjunction elimination fails. If we force that 'and' to be interpreted classically, however, then the argument is valid. That is, (1) is valid when

going on in examples like (1) and (i) without taking a stand on this issue.

regimented so:[6]

$$(\forall p\, \forall q \sim((p \wedge q) \rhd p) \mathrel{\Box\!\!\rightarrow} (a \,\&\, b)) \mathrel{.\,\cdot\,.} (\forall p\, \forall q \sim((p \wedge q) \rhd p) \mathrel{\Box\!\!\rightarrow} a).$$

This could explain why (1) has a ring of plausibility to it. Even though the counterlogical supposition is asking us to interpret conjunction so that conjunction elimination fails, it's nevertheless tempting to hold on to our "standard" classical way of interpreting 'and' when evaluating the consequent.[7]

We can generalize this observation by axiomatizing consequence in $\mathcal{L}^{\mathsf{H}}_{\Box\!\rightarrow}$. The axiomatic system $\mathbf{H}_{\Box\!\rightarrow}$ is given in Table B8. Some notation:

$$\begin{array}{rcl@{\qquad}rcl}
\phi \mathrel{\Box\!\!\rightarrow_\iota} \psi & := & @_\iota(\phi \mathrel{\Box\!\!\rightarrow} \psi) & \phi \mathrel{\Diamond\!\!\rightarrow_\iota} \psi & := & \sim @_\iota(\phi \mathrel{\Box\!\!\rightarrow} \sim \psi) \\
\Box_{\phi,\iota}\, \psi & := & \phi \mathrel{\Box\!\!\rightarrow_\iota} \psi & \Diamond_{\phi,\iota}\, \psi & := & \phi \mathrel{\Diamond\!\!\rightarrow_\iota} \psi \\
\Box^{\alpha}_{\phi,\iota}\, \psi & := & \alpha \supset \Box_{\phi,\iota}\, \psi & \Diamond^{\alpha}_{\phi,\iota}\, \psi & := & \alpha \,\&\, \Diamond_{\phi,\iota}\, \psi \\
\boxdot^{\alpha}_{\phi,\iota}\, \psi & := & \Box^{\alpha_1}_{\phi_1,\iota_1} \cdots \Box^{\alpha_n}_{\phi_n,\iota_n}\, \psi & \odot^{\alpha}_{\phi,\iota}\, \psi & := & \Diamond^{\alpha_1}_{\phi_1,\iota_1} \cdots \Diamond^{\alpha_n}_{\phi_n,\iota_n}\, \psi.
\end{array}$$

[6] As an anonymous referee points out, hyperlogic predicts the following inference is still (universally) valid:

$$(\forall p\, \forall q \sim((p \,\&\, q) \rhd p) \mathrel{\Box\!\!\rightarrow} (a \,\&\, b)) \mathrel{.\,\cdot\,.} (\forall p\, \forall q \sim((p \,\&\, q) \rhd p) \mathrel{\Box\!\!\rightarrow} a).$$

Here, '(the law of) conjunction elimination' is regimented using & rather than $\wedge$. I am unsure whether this is an unwelcome result (we are, after all, still using our actual notion of entailment to reason about these counterfactuals, not the notion of entailment denoted by $\rhd$ in the antecedent). However, if we want to avoid this result, we could revise the semantics of hyperlogic, following a suggestion from Kocurek (2021b, p. 13683), so that counterfactuals can shift the denotation of interpretation nominals (though not interpretation variables). Since & is defined in terms of cl, this revision would allow that & no longer has its classical meaning in the consequent. The resulting counterfactual logic would still be nontrivial, since the inference would hold if we regiment the premise as follows (given interpretation variables have rigid denotation):

$$\downarrow i.\, @_{cl} \downarrow k.\, @_i(\forall p\, \forall q \sim((p \wedge q) \rhd p) \mathrel{\Box\!\!\rightarrow} \downarrow j.\, @_k(@_j\, a \wedge @_j\, b)).$$

It is an open question how this revision would affect the resulting logic of hyperlogic.

[7] This strategy requires we interpret 'and' in the consequent of (1a) in terms of & even though we interpret '(the law of) conjunction elimination' in terms of $\wedge$. We see a similar phenomena with in-scope de re readings of counterfactuals. Consider:

(i) If I hadn't gone to college, my professor would find the class easier to teach.

Here, 'my professor' in the consequent picks out the speaker's professor in the actual world even though we are entertaining the speaker never going to college. The claim that 'and' in the consequent of (1a) can be interpreted according to our actual (classical) conventions even though we are entertaining an alternative convention is similar.

$\mathbf{H}_{\Box\!\!\rightarrow}$	
All the axioms and rules in **H**, plus:	
$\mathrm{K}_{\Box\!\!\rightarrow}$	$\phi \Box\!\!\rightarrow (\psi \supset \chi), \phi \Box\!\!\rightarrow \psi \Vdash \phi \Box\!\!\rightarrow \chi$
$\mathrm{Nec}_{\Box\!\!\rightarrow}$	$@_\iota \blacksquare \psi \Vdash \phi \Box\!\!\rightarrow (\iota \supset \psi)$
$\mathrm{Gen}_{\Box\!\!\rightarrow}$	if $\vec{\alpha}, \lvert\kappa\rvert_1 \Vdash \phi \Box\!\!\rightarrow (\kappa \supset \psi)$ where κ does not occur free in $\vec{\alpha}$, ϕ, or ψ, then $\vec{\alpha} \Vdash \phi \Box\!\!\rightarrow \psi$
REA	if $\vec{\alpha} \Vdash \phi =_\kappa \phi'$ where κ do not occur free in $\vec{\alpha}$, ϕ, or ϕ', then $\vec{\alpha} \Vdash (\phi \Box\!\!\rightarrow \psi) = (\phi' \Box\!\!\rightarrow \psi)$
Derivable rules:	
$\mathrm{Gen}_{\Box\!\!\rightarrow_\iota}$	if $\vec{\alpha}, \lvert\kappa\rvert_1 \Vdash \phi \Box\!\!\rightarrow_\iota (\kappa \supset \psi)$ where κ does not occur free in ι, $\vec{\alpha}$, ϕ, or ψ, then $\vec{\alpha} \Vdash \phi \Box\!\!\rightarrow_\iota \psi$
$\mathrm{RK}_{\Box\!\!\rightarrow_{(\iota)}}$	if $\psi_1, \ldots, \psi_n \Vdash \chi$, then $\phi \Box\!\!\rightarrow_{(\iota)} \psi_1, \ldots, \phi \Box\!\!\rightarrow_{(\iota)} \psi_n \Vdash \phi \Box\!\!\rightarrow_{(\iota)} \chi$

Table B8: Axioms and rules for provability in $\mathcal{L}^{\mathsf{H}}_{\Box\!\!\rightarrow}$ (with some derivable rules). The rules for $\Vdash$ can be converted into rules for $\vdash$ (given κ isn't cl) by applying C2U, U2C, and Cl.

As before, let $\mathcal{L}^{\mathsf{H}+}_{\Box\!\!\rightarrow}$ be the expansion of $\mathcal{L}^{\mathsf{H}}$ with Prop^+ and INom^+.

Definition B3.2 (Lindenbaum Set). A set $\Gamma \subseteq \mathcal{L}^{\mathsf{H}+}_{\Box\!\!\rightarrow}$ is **Lindenbaum** if it is a $\mathcal{L}^{\mathsf{H}+}_{\Box\!\!\rightarrow}$-maximal consistent set that satisfies constraints (i)–(iii) from Definition A3.6 (nominalized, witnesses $\neg$ @s, differentiates terms) as well as the following:

(iv) Γ^+ **differentiates antecedents**: if $(\Box_{\phi,\iota}\, \psi \neq \Box_{\phi',\iota}\, \psi) \in \Gamma^+$, then $\lvert l^+\rvert_1 \in \Gamma^+$ and $(\phi \neq_{l^+} \phi') \in \Gamma^+$ for some fresh $l^+ \in \mathsf{INom}^+$.

(v) Γ^+ **witnesses actual $\diamonddot$s**: if $\diamonddot^{\alpha}_{\phi,\iota}\, \psi \in \Gamma^+$, then $\lvert l^+\rvert_1 \in \Gamma^+$ and $\diamonddot^{\alpha}_{\phi,\iota}(l^+ \,\&\, \psi) \in \Gamma^+$ for some fresh $l^+ \in \mathsf{INom}^+$.

(vi) Γ^+ **witnesses possible $\diamonddot$s**: if $\Diamond(\alpha_0 \wedge \diamonddot^{\alpha}_{\phi,\iota}\, \psi) \in \Gamma^+$, then $\lvert l^+\rvert_1 \in \Gamma^+$ and $\Diamond(\alpha_0 \wedge \diamonddot^{\alpha}_{\phi,\iota}(l^+ \,\&\, \psi)) \in \Gamma^+$ for some fresh $l^+ \in \mathsf{INom}^+$.

Lemma B3.3 (Counterfactual Lindenbaum). If $\Gamma \subseteq \mathcal{L}^{\mathsf{H}}_{\Box\!\!\rightarrow}$ is consistent, then there is a Lindenbaum set $\Gamma^+ \subseteq \mathcal{L}^{\mathsf{H}+}_{\Box\!\!\rightarrow}$ such that $\Gamma \subseteq \Gamma^+$.

Proof: The construction is the same as that in Lemma A3.7 except for how we define Γ_{k+1} from Γ'_k (both l^+ and p^+ are unused throughout):

$$\Gamma_{k+1} = \begin{cases} \Gamma'_k \cup \{l^+ \in \iota, \neg @_{l^+} \psi\} & \text{if } \phi_k \in \Gamma'_k \text{ where } \phi_k = \neg @_\iota \psi \\ \Gamma'_k \cup \{@_\iota p^+ \neq @_\kappa p^+\} & \text{if } \phi_k \in \Gamma'_k \text{ where } \phi_k = (\iota \neq \kappa) \wedge |\iota|_1 \wedge |\kappa|_1 \\ \Gamma'_k \cup \{|l^+|_1, \phi \neq_{l^+} \phi'\} & \text{if } \phi_k \in \Gamma'_k \text{ where } \phi_k = (\Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi) \\ \Gamma'_k \cup \{\Diamond^\alpha_{\phi,\iota}(l^+ \,\&\, \psi), |l^+|_1\} & \text{if } \phi_k \in \Gamma'_k \text{ where } \phi_k = \Diamond^\alpha_{\phi,\iota} \psi \\ \Gamma'_k \cup \{\Diamond(\alpha_0 \wedge \Diamond^\alpha_{\phi,\iota}(l^+ \,\&\, \psi)), |l^+|_1\} & \text{if } \phi_k \in \Gamma'_k \text{ where } \phi_k = \Diamond(\alpha_0 \wedge \Diamond^\alpha_{\phi,\iota} \psi) \\ \Gamma'_k & \text{otherwise} \end{cases}$$

Suppose for reductio that Γ_{k+1} is inconsistent. The only cases we need to check are where $\phi_k = (\Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi)$, where $\phi_k = \Diamond^\alpha_{\phi,\iota} \psi$, and where $\phi_k = \Diamond(\alpha_0 \wedge \Diamond^\alpha_{\phi,\iota} \psi)$. Assume throughout the contradiction is derivable from $\gamma_1, \ldots, \gamma_n \in \Gamma_k$.

Suppose $\phi_k = (\Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi)$. Thus:

$$\begin{array}{rl} \widehat{\gamma}, \Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi, |l^+|_1 \vdash \phi =_{l^+} \phi' & \\ cl, l_\Gamma, \widehat{\gamma}, \Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi, |i|_1 \Vdash \phi =_i \phi' & \text{Lemma A3.2, C2U} \\ @_{l_\Gamma} cl, @_{l_\Gamma} \widehat{\gamma}, \Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi, |i|_1 \Vdash \phi =_i \phi' & \text{Gen}_@\text{, Red, Red} \\ @_{l_\Gamma} cl, @_{l_\Gamma} \widehat{\gamma}, \Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi \Vdash \phi = \phi' & \text{Gen}_\downarrow\text{, Vac}_\downarrow\text{, Idle}_\downarrow \\ @_{l_\Gamma} cl, @_{l_\Gamma} \widehat{\gamma}, \Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi \Vdash \phi =_{l^+} \phi' & \text{Gen}_@\text{, Red} \\ l_\Gamma, |l_\Gamma|_1, \widehat{\gamma}, \Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi \Vdash \phi =_{l^+} \phi' & \text{Intro}_@\text{, Cl} \\ l_\Gamma, |l_\Gamma|_1, \widehat{\gamma}, \Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi \vdash \phi =_{l^+} \phi' & \text{U2C} \\ l_\Gamma, |l_\Gamma|_1, \widehat{\gamma}, \Box_{\phi,\iota} \psi \neq \Box_{\phi',\iota} \psi \vdash \Box_{\phi,\iota} \psi = \Box_{\phi',\iota} \psi & \text{REA}, \lightning\ (l_\Gamma, |l_\Gamma|_1 \in \Gamma_k). \end{array}$$

Suppose $\phi_k = \Diamond^\alpha_{\phi,\iota} \psi$. Thus:

$$\begin{array}{rl} \widehat{\gamma}, \Diamond^\alpha_{\phi,\iota} \psi, |l^+|_1 \vdash \neg \Diamond^\alpha_{\phi,\iota}(l^+ \,\&\, \psi) & \\ \widehat{\gamma}, \Diamond^\alpha_{\phi,\iota} \psi, |l^+|_1 \vdash \boxdot^\alpha_{\phi,\iota} \sim(l^+ \,\&\, \psi) & \text{def. of } \Diamond^\alpha_{\phi,\iota} \\ \widehat{\gamma}, \Diamond^\alpha_{\phi,\iota} \psi, |l^+|_1 \vdash \boxdot^\alpha_{\phi,\iota}(l^+ \supset \sim \psi) & \text{RK}_{\Box\!\!\to_\iota} \\ \widehat{\gamma}, \Diamond^\alpha_{\phi,\iota} \psi \vdash \boxdot^\alpha_{\phi,\iota} \sim \psi & \text{Gen}_{\Box\!\!\to_\iota} \\ \widehat{\gamma} \vdash \neg \Diamond^\alpha_{\phi,\iota} \psi & \text{def. of } \Diamond^\alpha_{\phi,\iota}, \lightning. \end{array}$$

Suppose $\phi_k = \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi)$. Thus:

$$\begin{array}{rll}
\widehat{\gamma}, \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi), |l^+|_1 \vdash \Box(\alpha_0 \to \boxdot^{\alpha}_{\phi,\iota}(l^+ \supset \sim\psi)) & & \\
\Diamond\,\widehat{\gamma}, \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi), |l^+|_1, \alpha_0 \vdash \boxdot^{\alpha}_{\phi,\iota}(l^+ \supset \sim\psi) & & \text{S5, Rigid} \\
\Diamond\,\widehat{\gamma}, \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi), \alpha_0 \vdash \boxdot^{\alpha}_{\phi,\iota} \sim\psi & & \text{Gen}_{\Box\!\!\to_\iota} \\
\Diamond\,\widehat{\gamma}, \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi) \vdash \Box(\alpha_0 \to \boxdot^{\alpha}_{\phi,\iota} \sim\psi) & & \text{RK, S5} \\
\widehat{\gamma} \vdash \Box(\alpha_0 \to \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota} \sim\psi) & & \text{S5, } \lightning.
\end{array}$$

■

Lemma B3.4 (Counterfactual Existence). Where $\Delta \in W_\Gamma$:
(a) If $\Box\,\phi \notin \Delta$, then there is a $\Delta' \in W_\Gamma$ such that $\phi \notin \Delta'$.
(b) If $\Diamond_{\phi,\iota}(l \;\&\; \theta) \in \Delta$ where $|l|_1 \in \Delta$, then there is a $\Delta' \in W_\Gamma$ extending $\{l_\Delta, |l_\Delta|_1, |l|_1, @_l\,\theta\} \cup \{@_l\,\chi \mid \Box_{\phi,\iota}(l \supset \chi) \in \Delta\}$.

Proof: Start with (a). By the proof in Lemma A3.9, $\Delta^{-\Box} \cup \{\neg\,\phi\}$ is consistent. Enumerate all formulas of the form $\neg\,@_\iota\,\psi$, of the form $\Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi$, or of the form $\Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi)$ as $\chi_1, \chi_2, \chi_3, \ldots$. We define a sequence of formulas $\delta_0, \delta_1, \delta_2, \ldots$ depending on the form of χ_{n+1}. As before, $\delta_0 := \neg\,\phi$. If $\chi_{n+1} = \neg\,@_\iota\,\psi$, then define δ_{n+1} as in Lemma A3.9. If $\chi_{n+1} = \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi$, define $\delta_{n+1} := \chi_{n+1} \to (|l^+|_1 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}(l^+ \;\&\; \psi))$, where l^+ is the first nominal such that $\Delta^{-\Box}, \delta_0, \ldots, \delta_n, \chi_{n+1} \to (|l^+|_1 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}(l^+ \;\&\; \psi)) \nvdash \bot$. Suppose for reductio there were no such l^+. Reasoning as in Lemma A3.9, we can conclude that $\Box(\widehat{\delta} \to \chi_{n+1}) \in \Delta$ and $\Box(\widehat{\delta} \to \neg(|l^+|_1 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}(l^+ \;\&\; \psi))) \in \Delta$ for all l^+, and that $\Diamond(\widehat{\delta} \wedge \chi_{n+1}) \in \Delta$. Since Δ witnesses possible $\Diamond\!\!\!\cdot$s, there is a l^+ such that $\Diamond(\widehat{\delta} \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}(l^+ \;\&\; \psi)) \in \Delta$, $\lightning$.

If $\chi_{n+1} = \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi)$, define $\delta_{n+1} := \chi_{n+1} \to (|l^+|_1 \wedge \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}(l^+ \,\&\, \psi)))$, where l^+ is the first such that $\Delta^{-\Box}, \delta_0, \ldots, \delta_n, \chi_{n+1} \to (|l^+|_1 \wedge \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}(l^+ \& \psi))) \nvdash \bot$. Suppose there is no such l^+. Then $\Box(\widehat{\delta} \to \chi_{n+1}) \in \Delta$ and $\Box(\widehat{\delta} \to \neg(|l^+|_1 \wedge \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}(l^+ \,\&\, \psi)))) \in \Delta$ for all l^+. As before, $\Diamond(\widehat{\delta} \wedge \chi_{n+1}) \in \Delta$, i.e., $\Diamond(\widehat{\delta} \wedge \Diamond(\alpha_0 \wedge \Diamond\!\!\!\cdot^{\alpha}_{\phi,\iota}\,\psi)) \in \Delta$.

By S5, $\Diamond\widehat{\delta} \wedge \Diamond(\alpha_0 \wedge \diamonddot^{\alpha}_{\phi,\iota}\psi) \in \Delta$. Since Δ witnesses possible $\diamonddot$s, there is an l^+ such that $\Diamond(\alpha_0 \wedge \diamonddot^{\alpha}_{\phi,\iota}(l^+ \,\&\, \psi)) \in \Delta$. By S5 again, $\Diamond(\widehat{\delta} \wedge \Diamond(\alpha_0 \wedge \diamonddot^{\alpha}_{\phi,\iota}(l^+ \,\&\, \psi))) \in \Delta$, $\lightning$.

Now for (b). Let $\Delta^{-\Box_{\phi,\iota}} = \{@_l\,\chi \mid \Box_{\phi,\iota}(l \supset \chi) \in \Delta\}$. Then $\Delta^{-\Box_{\phi,\iota}} \cup \{l_\Delta, |l_\Delta|_1, |l|_1, @_l\,\theta\}$ is consistent. For suppose not. Then for some $@_l\,\chi_1, \ldots, @_l\,\chi_n \in \Delta^{-\Box_{\phi,\iota}}$, we have:

$$\begin{array}{rl}
l_\Delta, |l_\Delta|_1, |l|_1, @_l\,\widehat{\chi} \vdash {\sim}@_l\,\theta & \text{Bool} \\
l_\Delta, |l_\Delta|_1, |l|_1, @_l\,\widehat{\chi} \vdash @_l \sim\theta & \text{Dist}_@ \\
cl, l_\Delta, |l_\Delta|_1, |l|_1, @_l\,\widehat{\chi} \Vdash @_l \sim\theta & \text{C2U} \\
@_{l_\Delta}\,cl, |l_\Delta|_1, |l|_1, @_l\,\widehat{\chi} \Vdash @_l \sim\theta & \text{Gen}_@\text{, Ref, Red} \\
l_\Delta \in cl, |l|_1, @_l\,\widehat{\chi} \Vdash @_l \sim\theta & \text{Elim}_\& \\
l_\Delta \in cl, |l|_1, l, \widehat{\chi} \Vdash \sim\theta & \text{Intro}_@\text{, Elim}_@ \\
l_\Delta \in cl, |l|_1, l \supset \widehat{\chi} \Vdash l \supset \sim\theta & \text{S5} \\
\Box_{\phi,\iota}(l_\Delta \in cl), \Box_{\phi,\iota}|l|_1, \Box^l_{\phi,\iota}\,\widehat{\chi} \Vdash \Box^l_{\phi,\iota} \sim\theta & \text{RK}_{\Box\!\!\rightarrow_\iota} \\
l_\Delta \in cl, |l|_1, \Box^l_{\phi,\iota}\,\widehat{\chi} \Vdash \Box^l_{\phi,\iota} \sim\theta & \text{Rigid, Nec}_{\Box\!\!\rightarrow}\text{, Gen}_{\Box\!\!\rightarrow_\iota}.
\end{array}$$

Since $\Box^l_{\phi,\iota}\,\widehat{\chi}, (l_\Delta \in cl), |l|_1 \in \Delta$, that means $\Box^l_{\phi,\iota} \sim\theta \in \Delta$, contrary to our initial assumption that $\Diamond^l_{\phi,\iota}\,\theta \in \Delta$, $\lightning$.

Now, suppose $\Box\,\chi \in \Delta$. Thus, $@_l \blacksquare @_{l_\Delta}\,\chi \in \Delta$ (by Rigid, Intro$_@$, Red, Bool, and Dist$^+_@$). By Nec$_{\Box\!\!\rightarrow}$, $\Box_{\phi,\iota}(l \supset @_{l_\Delta}\,\chi) \in \Delta$. Hence, $@_l\,@_{l_\Delta}\,\chi \in \Delta^{-\Box_{\phi,\iota}}$. Since $l_\Delta, |l_\Delta|_1 \vdash @_l\,@_{l_\Delta}\,\chi \leftrightarrow \chi$, the set $\Sigma := \Delta^{-\Box_{\phi,\iota}} \cup \{l_\Delta, |l_\Delta|_1, |l|_1, @_l\,\theta\} \cup \{\chi \mid \Box\,\chi \in \Delta\}$ is consistent. The proof strategy from here is essentially the same as in part (a), though some changes need to be made to ensure the steps go through. The main change is that we need to replace $\Box(\widehat{\delta} \to \cdots)$ and $\Diamond(\widehat{\delta} \wedge \cdots)$ with $\Box_{\phi,\iota}(\widehat{\delta} \supset \cdots)$ and $\Diamond_{\phi,\iota}(\widehat{\delta} \,\&\, \cdots)$. To illustrate, I'll use the case where $\chi_{n+1} = \Diamond(\alpha_0 \wedge \diamonddot^{\alpha}_{\beta,\kappa}\psi)$. As before, we define $\delta_{n+1} := \chi_{n+1} \to (|l^+|_1 \wedge \Diamond(\alpha_0 \wedge \diamonddot^{\alpha}_{\beta,\kappa}(l^+ \,\&\, \psi)))$, where l^+ is the first such that: $\Sigma, \delta_0, \ldots, \delta_n, \chi_{n+1} \to (|l^+|_1 \wedge \Diamond(\alpha_0 \wedge \diamonddot^{\alpha}_{\beta,\kappa}(l^+ \,\&\, \psi))) \nvdash \bot$. By Bool and the fact that $l_\Delta, |l_\Delta|_1 \in \Sigma$, this condition is equivalent to $\Sigma, \delta_0, \ldots, \delta_n, \chi_{n+1} \supset (|l^+|_1 \,\&\, @_{l_\Delta} \blacklozenge(\alpha_0 \,\&\, \diamonddot^{\alpha}_{\beta,\kappa}(l^+ \,\&\, \psi))) \nvdash \bot$. Suppose, for reductio, there's no such l^+. So for all l^+, there are some $\gamma_1, \ldots, \gamma_n \in \Sigma$ such that $\widehat{\gamma} \vdash \widehat{\delta} \supset {\sim}(\chi_{n+1} \supset @_{l_\Delta} \blacklozenge(\alpha_0 \,\&\, \diamonddot^{\alpha}_{\beta,\kappa}(l^+ \,\&\, \psi)))$. By RK$_{\Box\!\!\rightarrow_\iota}$, $\Box_{\phi,\iota}\,\widehat{\gamma} \vdash$

$\Box_{\phi,\iota}(\widehat{\delta} \supset \sim(\chi_{n+1} \supset @_{l_\Delta} \blacklozenge(\alpha_0 \,\&\, \odot^{\alpha}_{\beta,\kappa}(l^+ \,\&\, \psi))))$. Since $\Box_{\phi,\iota}\,\widehat{\gamma} \in \Delta$,[8] this means $\Box_{\phi,\iota}(\widehat{\delta}\supset(\chi_{n+1}\&\sim(|l^+|_1\&@_{l_\Delta}\blacklozenge(\alpha_0\&\odot^{\alpha}_{\beta,\kappa}(l^+\&\psi))))) \in \Delta$ for all l^+. As before, $\Diamond_{\phi,\iota}(\widehat{\delta} \,\&\, \chi_{n+1}) \in \Delta$, i.e., $\Diamond_{\phi,\iota}(\widehat{\delta} \,\&\, @_{l_\Delta} \blacklozenge(\alpha_0 \,\&\, \odot^{\alpha}_{\beta,\kappa}\psi)) \in \Delta$. By $\mathrm{RK}_{\Box\!\!\to_\iota}$, $\Diamond_{\phi,\iota}\,\widehat{\delta} \in \Delta$ and $\Diamond_{\phi,\iota}\,@_{l_\Delta}\blacklozenge(\alpha_0 \,\&\, \odot^{\alpha}_{\beta,\kappa}\psi) \in \Delta$. By $\mathrm{Nec}_{\Box\!\!\to}$ and $\mathrm{Gen}_{\Box\!\!\to_\iota}$, $@_{l_\Delta}\blacksquare(\alpha_0 \,\&\, \odot^{\alpha}_{\beta,\kappa}\psi) \Vdash \Box_{\phi,\iota}\,@_{l_\Delta}\blacksquare(\alpha_0 \,\&\, \odot^{\alpha}_{\beta,\kappa}\psi)$. Since $|l_\Delta|_1 \in \Delta$, that means $@_{l_\Delta}\blacklozenge(\alpha_0 \,\&\, \odot^{\alpha}_{\beta,\kappa}\psi) \in \Delta$. Since $l_\Delta \in \Delta$, that means $\Diamond(\alpha_0 \wedge \odot^{\alpha}_{\beta,\kappa}\psi)$ by Bool. Since Δ witnesses possible $\odot$s, there is an l^+ such that $|l^+|_1 \,\&\, @_{l_\Delta}\blacklozenge(\alpha_0 \,\&\, \odot^{\alpha}_{\phi,\iota}(l^+ \,\&\, \psi)) \in \Delta$. By $\mathrm{Nec}_{\Box\!\!\to}$, $\Box_{\phi,\iota}(|l^+|_1 \,\&\, @_{l_\Delta}\blacklozenge(\alpha_0 \,\&\, \odot^{\alpha}_{\phi,\iota}(l^+ \,\&\, \psi))) \in \Delta$. By $\mathrm{RK}_{\Box\!\!\to_\iota}$, $\Diamond_{\phi,\iota}(\widehat{\delta} \,\&\, (|l^+|_1 \,\&\, @_{l_\Delta}\blacklozenge(\alpha_0 \,\&\, \odot^{\alpha}_{\beta,\kappa}(l^+ \,\&\, \psi)))) \in \Delta$, $\lightning$. ■

The proofs of the other intermediate lemmas are all as before. To finish the proof, we need to define the selection function for our canonical model.

Definition B3.5 (Defining Formula). Where $A \subseteq \mathbb{I}_{W_\Gamma}$, we define the set $[A] := \{\phi \in \mathcal{L}^{\mathsf{H}+}_{\Box\!\!\to} \mid A = \{\langle \Delta, c_\kappa\rangle \mid @_\kappa\,\phi \in \Delta\}\}$.

Lemma B3.6 (Replacement of Definitions). For all $A \subseteq \mathbb{I}_{W_\Gamma}$, all c_κ, all $\phi, \phi' \in [A]$, and all ψ, we have $((\phi \Box\!\!\to_\kappa \psi) = (\phi' \Box\!\!\to_\kappa \psi)) \in \Gamma$.

Proof: Suppose for reductio that $((\phi \Box\!\!\to_\kappa \psi) = (\phi' \Box\!\!\to_\kappa \psi)) \notin \Gamma$. Since Γ differentiates antecedents, there are some l^+ such that (by $\mathrm{Dist}_@$) $(@_{l^+}\phi \neq @_{l^+}\phi') \in \Gamma$. Since $\phi, \phi' \in [A]$, $(@_{l^+}\phi = @_{l^+}\phi') \in \Gamma_n$, $\lightning$. ■

Definition B3.7 (Canonical Selection Function). We define f_Γ, the **canonical selection function** for Γ, as follows for all $A \subseteq \mathbb{I}_{W_\Gamma}$, all $\Delta \in W_\Gamma$, and all c_κ. First, if $[A] = \varnothing$, then $f_\Gamma(A, \Delta, c_\kappa) = \{\langle \Delta, c_\kappa\rangle\} \cap A$. Second, if $\phi \in [A]$, then $\langle \Delta', c_\lambda\rangle \in f_\Gamma(A, \Delta, c_\kappa)$ iff for all $\psi \in \mathcal{L}^{\mathsf{H}+}_{\Box\!\!\to}$, if $(\phi \Box\!\!\to_\kappa (\lambda \supset \psi)) \in \Delta$, then $@_\lambda\,\psi \in \Delta'$.

By Lemma B3.6, if $\phi, \phi' \in [A]$, then $(\phi \Box\!\!\to_\kappa (\lambda \supset \psi)) \in \Delta$ iff $(\phi' \Box\!\!\to_\kappa (\lambda \supset \psi)) \in \Delta$, so this definition for f_Γ is well-defined.

[8] This is the step that would not have gone through without the relevant change, since we do not have $\Box\,\widehat{\gamma} \in \Delta$.

Definition B3.8 (Canonical Model). The **canonical selection hypermodel** of Γ is the selection hypermodel $\mathcal{M}_\Gamma = \langle W_\Gamma, D_{\mathbb{C}\Gamma}, D_{\mathbb{P}\Gamma}, f_\Gamma, V_\Gamma \rangle$ where $\langle W_\Gamma, D_{\mathbb{C}\Gamma}, D_{\mathbb{P}\Gamma}, V_\Gamma \rangle$ is defined as in Definition A3.15 and f_Γ is defined as in Definition B3.7.

Lemma B3.9 (Truth). $\mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \phi$ iff $@_\kappa\, \phi \in \Delta$.

Proof: The inductive steps are all the same as before. We just need to check the $\Box\!\!\rightarrow$ step goes through. First, $\Delta, c_\kappa \Vdash \phi \Box\!\!\rightarrow \psi$ iff $f_\Gamma(\llbracket \phi \rrbracket, \Delta, c_\kappa) \subseteq \llbracket \psi \rrbracket$. By Lemma B3.6 and by IH ($\phi \in [\llbracket \phi \rrbracket]$), this holds iff the following holds for all Δ' and c_λ:

$$\text{if } \forall \chi \in \mathcal{L}^{\mathsf{H}+}_{\Box\!\rightarrow} : (\phi \Box\!\!\rightarrow_\kappa (\lambda \supset \chi)) \in \Delta \Rightarrow @_\lambda\, \chi \in \Delta', \text{then } @_\lambda\, \psi \in \Delta'.$$

We now show this condition holds for all Δ' and c_λ iff $@_\kappa(\phi \Box\!\!\rightarrow \psi) \in \Delta$.

($\Leftarrow$) Suppose $@_\kappa(\phi \Box\!\!\rightarrow \psi) \in \Delta$. Let Δ' and c_λ be such that for all $\chi \in \mathcal{L}^{\mathsf{H}+}_{\Box\!\rightarrow}$, if $(\phi \Box\!\!\rightarrow_\kappa (\lambda \supset \chi)) \in \Delta$, then $@_\lambda\, \chi \in \Delta'$. Since $@_\kappa(\phi \Box\!\!\rightarrow \psi) \in \Delta$, we have by $\mathrm{RK}_{\Box\!\rightarrow_\kappa}$ that $@_\kappa(\phi \Box\!\!\rightarrow (\lambda \supset \psi)) \in \Delta$. Hence, $@_\lambda\, \psi \in \Delta'$.

($\Rightarrow$) Suppose $@_\kappa(\phi \Box\!\!\rightarrow \psi) \notin \Delta$. Thus, $\phi \Diamond\!\!\rightarrow_\kappa \sim\psi \in \Delta$. Since Δ witnesses actual $\Diamond$s, there is an l^+ such that $\phi \Diamond\!\!\rightarrow_\kappa (l^+ \,\&\, \sim\psi)$. By Lemma B3.4, there is a $\Delta' \in W_\Gamma$ such that $\Delta' \supseteq \{\neg @_{l^+}\, \psi\} \cup \{@_{l^+}\, \chi \mid \phi \Box\!\!\rightarrow_\kappa (l^+ \supset \chi) \in \Delta\}$. Hence, $\langle \Delta', l^+ \rangle$ is the counterexample we need. ■

B3.3 Adding $\rhd$ and Quantifiers

What changes if we add an entailment operator or propositional quantifiers to $\mathcal{L}^{\mathsf{H}}_{\Box\!\rightarrow}$? In the former case, no additional axioms are required apart from those in $\mathbf{H}_\rhd$ and $\mathbf{H}_{\Box\!\rightarrow}$: all the proofs in §B3.2 go through in the presence of $\rhd$. In the latter case, we do need one additional axiom. Observe that the Barcan formula and its converse are universally valid for counterfactuals (where p does not occur free in ϕ):

$$\forall p(\phi \Box\!\!\rightarrow \psi) \dashv\Vdash \phi \Box\!\!\rightarrow \forall p\, \psi.$$

The converse Barcan formula is derived just by combining **QH** and $\mathbf{H}_{\Box\!\rightarrow}$:

$$\forall p\, \psi \Vdash \psi \qquad\qquad \mathrm{Elim}_\forall$$

$$\phi \,\Box\!\!\rightarrow \forall p\, \psi \Vdash \phi \,\Box\!\!\rightarrow \psi \qquad \mathrm{RK}_{\Box\rightarrow}$$
$$\forall p(\phi \,\Box\!\!\rightarrow \forall p\, \psi) \Vdash \forall p(\phi \,\Box\!\!\rightarrow \psi) \qquad \mathrm{RK}_{\forall}$$
$$\phi \,\Box\!\!\rightarrow \forall p\, \psi \Vdash \forall p(\phi \,\Box\!\!\rightarrow \psi) \qquad \mathrm{Vac}_{\forall}.$$

However, the Barcan formula, which is needed to prove the analogue of Lemma A4.6, must be added separately. Other than that, the proofs of completeness for $\mathcal{L}^{\mathsf{QH}}$ and $\mathcal{L}^{\mathsf{H}}_{\Box\rightarrow}$ can be straightforwardly combined to yield a proof of completeness for $\mathcal{L}^{\mathsf{QH}}_{\Box\rightarrow}$.

$\mathbf{QH}_{\Box\rightarrow}$
All the axioms and rules in **QH** and $\mathbf{H}_{\Box\rightarrow}$, plus:

$\mathrm{BF}_{\Box\rightarrow}$ $\quad \forall p(\phi \,\Box\!\!\rightarrow \psi) \Vdash \phi \,\Box\!\!\rightarrow \forall p\, \psi$ where p does not occur free in ϕ

Table B9: Axioms and rules for provability in $\mathcal{L}^{\mathsf{QH}}_{\Box\rightarrow}$

B3.4 Constraints on Selection Function

Let's now look at some constraints on the selection function. Table B10 contains several such constraints. We can extend the completeness result to include such constraints by adding the relevant axioms from Table B11.

Name	*Class*	*Constraint* (on all $A, B \subseteq \mathbb{I}_{D_{\mathbb{H}}}$)
success	Suc	$f(A, w, c) \subseteq A$
weak centering	W	if $\langle w, c\rangle \in A$, then $\langle w, c\rangle \in f(A, w, c)$
strong centering	C	if $\langle w, c\rangle \in A$, then $f(A, w, c) = \{\langle w, c\rangle\}$
Stalnaker's assumption	Stal	$\lvert f(A, w, c)\rvert \leqslant 1$
vacuism	Vac	if $A(c) = \varnothing$, then $f(A, w, c) = \varnothing$
necessary consequent	NC	$f(A, w, c) \subseteq W \times \{c\}$
necessary entailment	NEC	$f(A, w, c) \subseteq A(c) \times \{c\}$
strangeness of impossibility	SIC	if $A(c) \neq \varnothing$, then $f(A, w, c) \subseteq W \times \{c\}$
operational rigidity	R_{o}	$f(A, w, c) \subseteq W \times \{c' \in D_{\mathbb{H}} \mid c \approx c'\}$

Table B10: Some constraints on selection functions.

Theorem B3.10 (Relative Completeness in $\mathcal{L}^{\mathsf{H}}_{\Box\rightarrow}$). The proof systems in Table B11 are sound and complete for the relevant class of selection hypermodels. (See §B5.4.)

Name	*Axiom(s)*	*Corresponding Constraint*
$\mathrm{Id}_{\Box\!\!\rightarrow}$	$\Vdash \phi \Box\!\!\rightarrow \phi$	success
$\mathrm{MP}_{\Box\!\!\rightarrow}$	$\phi, \phi \Box\!\!\rightarrow \psi \Vdash \psi$	weak centering
Cen	$\phi \Vdash (\phi \Box\!\!\rightarrow \psi) \equiv \psi$	strong centering
CEM	$\Vdash (\phi \Box\!\!\rightarrow \psi) + (\phi \Box\!\!\rightarrow \sim\psi)$	Stalnaker's assumption
Vac	$\sim\blacklozenge\,\phi \Vdash \phi \Box\!\!\rightarrow \psi$	vacuism
NC	$\blacksquare\,\psi \Vdash \phi \Box\!\!\rightarrow \psi$	necessary consequent
	$\Vdash \downarrow i.(\phi \Box\!\!\rightarrow i)$	
NEC	$\blacksquare(\phi \supset \psi) \Vdash \phi \Box\!\!\rightarrow \psi$	necessary entailment
	$\Vdash \downarrow i.(\phi \Box\!\!\rightarrow (i \,\&\, \phi))$	
SIC	$\blacklozenge\,\phi, \blacksquare\,\psi \Vdash \phi \Box\!\!\rightarrow \psi$	strangeness of impossibility
	$\blacklozenge\,\phi \Vdash \downarrow i.(\phi \Box\!\!\rightarrow i)$	
$\mathrm{R_o}$	$\Vdash \downarrow i.\,\Box_{\phi} \downarrow j.[\triangle(\overline{\phi}) = @_i \triangle(@_j\,\overline{\phi})]$	operational rigidity

Table B11: Axiomatizations in $\mathcal{L}^0_{\Box\!\!\rightarrow}$ for various classes from Table B10.

Let me briefly explain the motivation behind some of these constraints. Vacuism is the view that all counterpossibles (counterfactuals with impossible antecedents) are vacuously true.[9] Often, vacuists also endorse the necessary consequent and necessary entailment principles, which are all coderivable given success (the labels come from French et al. 2020). Some of these principles have equivalent "hybrid" formulations. In the hyperconvention semantics (with success), counterpossibles are vacuous when we hold fixed the interpretation of the antecedent. This goes back to one of the main motivations for considering hyperlogic as a semantics for metalogical claims, viz., it can formalize "conventionalist" approaches to hyperintensionality, which explain hyperintensionality in terms of convention-shifting (§A1). We can regiment this idea of "holding fixed" an interpretation using the hybrid binder ↓, which is what allows alternative axiomatizations for some of these principles.

Second, the "strangeness of impossibility condition" was introduced by Nolan (1997, p. 550). If we think of selection functions as selecting the "closest" or "most similar" antecedent-worlds, then the condition effectively says that impossible worlds are always "far out" in that they're less similar than

[9] For a defense of vacuism, see Stalnaker 1968, 1996; Lewis 1973; Kratzer 1979; Bennett 2003; Williamson 2007, 2017; Emery and Hill 2017. For criticism, see Cohen 1987, 1990; Zagzebski 1990; Mares 1997; Nolan 1997; Merricks 2001; Goodman 2004; Vander Laan 2004; Kim and Maslen 2006; Krakauer 2012; Brogaard and Salerno 2013; Kment 2014; Bernstein 2016; Berto et al. 2018; Jenny 2018; Tan 2019. See Berto and Nolan 2021; Kocurek 2021a for an overview.

any possible world.[10] French et al. (2020) present an impossible worlds semantics where this corresponds to the axiom $(\Diamond \phi \wedge \Box \psi) \rightarrow (\phi \Box\!\!\rightarrow \psi)$, which has an analogue in Table B11. Again, this has an equivalent formulation in terms of convention-shifting: counterconventional readings only arise when the antecedent in question is impossible (on its actual interpretation).

Finally, operational rigidity, in effect, states *counterlogical* vacuism, i.e., the view that all counterlogicals (counterfactuals with logically impossible antecedents) are vacuously true. Some nonvacuists have held that even if counterpossibles are generally nonvacuous, counterlogicals are a special exception, while others have argued there's no relevant difference between counterlogicals and other counterpossibles.[11] In hyperlogic, this turns on whether counterfactuals are allowed to shift the interpretation of the connectives. Thus, those who maintain that counterlogicals are a special exception can hold that counterfactuals are only allowed to shift the interpretation of nonlogical vocabulary.

B3.5 Belief and Knowledge

Thus far, we have focused on counterfactual hyperlogic. But the selection semantics (or something like it) is also often employed as a semantics for dyadic belief and knowledge, where $\mathsf{B}^{\phi}\psi$ says the agent believes that ψ given ϕ and likewise for $\mathsf{K}^{\phi}\psi$.[12] It is standard to define the monadic belief operator as $\mathsf{B}\phi := \mathsf{B}^{\top}\phi$ (here, we can define $\top := (p + {\sim} p)$). Letting $R(w, c) := f(\llbracket \top \rrbracket, w, c)$, we then have the following semantics for monadic belief:

$$\mathcal{M}, w, c \Vdash \mathsf{B}\phi \quad \Leftrightarrow \quad \text{for all } \langle v, d \rangle \in R(w, c)\text{: } \mathcal{M}, v, d \Vdash \phi.$$

Thus, we can import the results in §B3.2 to give a logic of belief and knowledge within hyperlogic. As in §B3.4, one could consider imposing any of the usual restrictions on R to obtain stronger logics.

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10 See Mares 1997; Nolan 1997; Vander Laan 2004; Krakauer 2012; Jago 2014; Kment 2014; Bernstein 2016; Clarke-Doane 2019 for discussion of this principle. See Kocurek 2021a for an overview.

11 For defenses of counterlogical vacuism, see Goodman 2004; Kment 2014. For defenses of counterlogical nonvacuism, see Cohen 1990; Mares 1997; Nolan 1997; Vander Laan 2004; Kim and Maslen 2006; Krakauer 2012; Brogaard and Salerno 2013; Berto et al. 2018. Kocurek and Jerzak (2021) defend an intermediate position, viz., counterlogicals are only nonvacuous on counterconventional readings.

12 See, e.g., Boutilier 1992; Moses and Shoham 1993; Lamarre and Shoham 1994; Friedman and Halpern 1997; van Ditmarsch 2005; Baltag and Smets 2006; van Benthem 2007.

One application of doxastic/epistemic hyperlogic is to the problem of logical omniscience. It is well known that on the standard intensional semantics, belief is closed under classical entailment: if $\phi \vDash \psi$, then $\mathsf{B}\phi \vDash \mathsf{B}\psi$.[13] Attempts to avoid this result generally often appeal to limitations or defects in cognitive states, e.g., lack of computational resources, awareness, or informational access. However, another (less discussed) way logical omniscience can fail is via different views on logic. If Inej believes intuitionistic logic is correct, then her beliefs will not generally be closed under classical consequence even if she is a perfect reasoner.

Doxastic hyperlogic is well-equipped to handle such cases. While it does not require that beliefs are closed under classical consequence, it does validate a more modest closure principle: $@_\iota \blacksquare(\phi \supset \psi), \mathsf{B}\iota, \mathsf{B}\phi \Vdash \mathsf{B}\psi$. Restricting to hypermodels where $\triangleright_c$ is factive and noncontingent (Table B3), we can simplify this principle: $@_\iota(\phi \triangleright \psi), \mathsf{B}\iota, \mathsf{B}\phi \Vdash \mathsf{B}\psi$. In other words, beliefs are closed under whatever logic the agent adopts (if there is one, assuming it's reasonable). We obtain the "classical" picture only when we assume $\mathsf{B}cl$ holds.[14]

Of course, doxastic hyperlogic is not a complete solution to the problem of logical omniscience. For one, it still assumes agents are perfect reasoners within their own logic, and is thus not a good model of logical error. Moreover, beliefs are still assumed to be closed under universal consequence: if $\phi \Vdash \psi$, then $\mathsf{B}\phi \Vdash \mathsf{B}\psi$. The moral, rather, is that there are several different problems of logical omniscience that likely need to be addressed with different tools. Appeals to computation, awareness, fragmentation, etc. are better equipped for modeling logical error, whereas doxastic hyperlogic is better equipped for modeling ideal yet nonclassical agents.

B4 Conclusion

This concludes the two-part series exploring the logic of hyperlogic. In Part A of this series, we axiomatized a minimal logic of hyperlogic. In Part B, we extended these results to stronger logics over a restricted class

[13] For discussion of this problem, see Hintikka 1975; Stalnaker 1976a,b, 1984; Duc 1997; Alechina et al. 2004; Berto 2010; Ripley 2012; Bjerring 2013; Jago 2007, 2014, 2015; Bjerring and Schwarz 2017; Yalcin 2018; Bjerring and Skipper 2019; Hawke et al. 2019; Skipper and Bjerring 2020; Elga and Rayo 2021; Hoek 2022; Soysal 2022.

[14] Sedlár (2015) likewise explores a doxastic logic where the belief operator is nonclassical, though the base logic is classical. In some ways, Sedlár's system is similar to doxastic hyperlogic, although the latter is more flexible in the range of logics an agent's beliefs may be sensitive to. Thanks to an anonymous referee for noting this parallel.

of models as well as to languages with hyperintensional operators. In this final section, I wish to sketch a few possible directions for future research that would be worth pursuing.

First, it is an open question how best to extend hyperlogic with first-order quantifiers. We could have hyperconventions specify a domain of individuals, but this might bring technical complications with tracking the denotations of variables across shifts in hyperconvention. Another option would be to have hypermodels directly specify a single domain across all hyperconventions. This might be more manageable, though it builds in substantive metaontological assumptions.

Second, there are many questions remaining for the model theory of hyperlogic, especially concerning "finite" hypermodels. For example, it is easy to see that any satisfiable $\mathcal{L}^{\mathsf{QH}}$-formula is satisfiable in a convention-finite model (i.e., one where $D_{\mathbb{C}}$ is finite): just reduce the hypermodel to the denotations of the free terms in the formula. Yet, there are satisfiable (quantified) $\mathcal{L}^{\mathsf{QH}}$-formulas that not satisfiable in a *hyper*convention-finite model (i.e., one where $D_{\mathbb{H}}$ is finite). What about any satisfiable $\mathcal{L}^{\mathsf{H}}$-formula, though? Does **H** satisfy the finite model property?

Third, we made a number of choice points regarding the initial setup of the hyperconvention semantics that could be revised. One is that we required the "classical" hyperconventions to all interpret $\Box$ and $\Diamond$ as universal modals. It would be natural to weaken this requirement so that $\Box$ and $\Diamond$ only obey weaker normal modal logics. Another choice point concerned whether to treat iterated "according to" operators as redundant. I suspect there is more than one way to naturally generalize the semantics for "according to" so that iteration matters.

Finally, the hyperconvention semantics only concerns claims about logics for the propositional modal language. It does not have a way of capturing metalogical claims concerning alternative logics for *hyperlogic*—specifically, for the propositional quantifiers, hybrid operators, or counterfactuals (except insofar as they also concern alternative logics for the connectives). While Kocurek (2021b, §6) sketches a possible extension to such a language, it is unclear what the resulting logic of this proposed solution is or whether there might be more elegant solutions waiting to be explored.

B5 Appendix: Proofs of Relative Completeness

In this appendix, we give the proofs of various completeness theorems relative to restricted classes of models (Theorems B2.1, B2.3, B2.6 and B3.10).

First, we state a helpful lemma, which follows straightforwardly from Corollary A3.10 and Definition A3.13:

> **Lemma B5.1 (Canonical Operations).** Let $|\kappa|_1, |\lambda|_1 \in \Gamma$. Where $\phi_i \in [X_i]_\kappa$ and $\psi_i \in [Y_i]_\lambda$, $\triangle_{c_\kappa}(\overline{X}) = \triangle_{c_\lambda}(\overline{Y})$ iff $(\triangle(\overline{\phi})^\kappa = \triangle(\overline{\psi})^\lambda) \in \Gamma$.

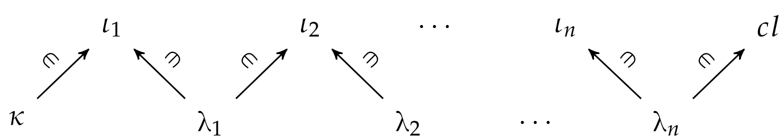
In each case, the proof of soundness is straightforward and left to the reader. Completeness requires showing the canonical model is in the relevant class. In some cases, we must revise the canonical model construction and/or the Lindenbaum construction.

B5.1 Theorem B2.1

The proofs of completeness for F, $\mathsf{U_q}$, At, and B are immediate since the canonical hypermodel (Definition A3.15) is full (and thus, quantification uniform, atomic, and boolean).

$\mathsf{U_o}$: We need to make a slight revision to the definition of a canonical hyperconvention. In particular, we need to revise the third clause to say that c_κ interprets the connectives classically if the following is in Γ for some $\iota_1, \dots, \iota_n$ and $\lambda_1, \dots, \lambda_n$:

$$(\kappa \in \iota_1) \wedge (\lambda_1 \in \iota_1) \wedge (\lambda_1 \in \iota_2) \wedge (\lambda_2 \in \iota_2) \wedge \cdots \wedge (\lambda_n \in \iota_n) \wedge (\lambda_n \in cl)$$



So unlike Definition A3.13, c_κ can be classical even if $@_\kappa\, cl \notin \Gamma$, so long as it satisfies this "zigzag" condition. Now, Lemma A3.16 needs to be restated as the following:

> **Claim**: If $(\kappa \in \iota), (\lambda \in \iota) \in \Gamma$ and c_κ is classical, then c_λ is classical.

> *Proof*: Suppose first that κ satisfies the zigzag condition. Then the zigzag can be extended to λ via ι, and thus c_λ is classical. Suppose instead that κ does not satisfy the zigzag condition. Then $c_\kappa(\neg)(X) = \{\Delta \in W_\Gamma \mid \exists \phi \in [X]_\kappa \colon @_\kappa \neg \phi \in \Delta\}$. Suppose $[X]_\kappa = \varnothing$.

Then $c_\kappa(\neg)(X) = \varnothing$. But since c_κ is classical, $c_\kappa(\neg)(X) = \overline{X}$. So $X = W$, even though $(p + \sim p) \in [W]_\kappa$, $\lightning$. Hence, there is no X where $[X]_\kappa = \varnothing$. This can only happen if W_Γ is finite. List the members of W_Γ as $\Delta_1, \ldots, \Delta_n$. Since these are all distinct maximal consistent sets, there must be some $\delta_1, \ldots, \delta_n$ such that $\delta_i \in \Delta_j$ iff $i = j$. Each $X \subseteq W_\Gamma$ is then definable by a disjunction of these δ_is (if $X = \varnothing$, then it's definable by $\bot$). Now, let $l_\Gamma \in \Gamma$ and for each $X \subseteq W_\Gamma$, let $\delta_X = @_{l_\Gamma}\, \delta_{i_1} + \cdots + @_{l_\Gamma}\, \delta_{i_k}$, where the disjunction of $\delta_{i_1}, \ldots, \delta_{i_k}$ defines X. By Red and Dist$_@$, $((\delta_X)^\kappa = (\delta_X)^\lambda) \in \Gamma$ for every X. By Uni$_\circ$, $((\star\, \delta_X)^\kappa = (\star\, \delta_X)^\lambda) \in \Gamma$. Thus, $c_\kappa(\star) = c_\lambda(\star)$ by Lemma B5.1, and so c_λ is classical. ■

Using this claim in place of Lemma A3.16 in the inductive step for the connectives in Lemma A3.17, the completeness proof goes through as before. We just need to check that $D_{\mathbb{C}\Gamma}$ is operationally uniform. Let $c_\kappa, c_\lambda \in C_\iota$. By the above claim, c_κ is classical iff c_λ is classical. If both are classical, then we're done. So suppose otherwise. I just prove the $\star$-case for illustration. If $[X]_\kappa = \varnothing$, then $[X]_\lambda = \varnothing$ (otherwise, if $\phi \in [X]_\lambda$, then $@_\lambda\, \phi \in [X]_\kappa$). If $[X]_\kappa = [X]_\lambda = \varnothing$, then $c_\kappa(\star)(X) = c_\lambda(\star)(X) = \varnothing$. So suppose $\phi \in [X]_\kappa$ and $\psi \in [X]_\lambda$. Then $@_\kappa\, \phi \in \Delta$ iff $@_\lambda\, \psi \in \Delta$. By Corollary A3.10 and Bool, $@_\kappa\, \phi = @_\lambda\, \psi \in \Gamma$. By Uni$_\circ$, $@_\kappa \star \phi = @_\lambda \star \psi \in \Gamma$. Hence, $c_\kappa(\star)(X) = c_\lambda(\star)(X)$ by Lemma B5.1.

Si: Completeness is straightforward. To establish that $\mathbf{H}$ + Sing = $\mathbf{H}$ + Self-Dual$^+$, we just need to show that Sing is coderivable with Self-Dual$_@$ in $\mathbf{H}$. Self-Dual$_@$ trivially follows from Dist$_@$ and Sing. Here's the other direction:

$\iota, i \Vdash \sim @_\iota \sim i$	Elim$_@$
$\iota, i \Vdash @_\iota\, i$	Self-Dual$_@$
$\iota \Vdash \downarrow i.\, @_\iota\, i$	Gen$_\downarrow$, Vac$_\downarrow$
$\Vdash \lvert\iota\rvert_1$	Gen$_@$, Ref, def. of $\lvert\iota\rvert_1$.

AnF, AnU$_q$: We revise the Lindenbaum construction, specifically the definition of Γ_{k+1}. Let $\kappa \not\sim \lambda$ abbreviate $((\overline{p^+})^\kappa = (\overline{q^+})^\lambda) \wedge \bigvee \{(\triangle(\overline{p^+}))^\kappa \neq (\triangle(\overline{q^+}))^\lambda\}_\triangle$, where $\overline{p^+}$ and $\overline{q^+}$ are unused at this point in the construction. Then we revise the definition of Γ_{k+1} so that $\Gamma_{k+1} = \Gamma'_k \cup \{\kappa \not\sim \lambda\}$ if $\phi_k \in \Gamma'_k$ where $\phi_k = (\kappa \in \iota) \wedge \neg(\lambda \in \iota) \wedge \lvert\lambda\rvert_1$. Suppose Γ_{k+1} is inconsistent in this case.

Then for some $\gamma_1, \ldots, \gamma_n \in \Gamma'_k$, we have $\widehat{\gamma}, \kappa \in \iota, \lambda \notin \iota, |\lambda|_1, (\overline{p^+})^\kappa = (\overline{q^+})^\lambda \Vdash \triangle(\overline{p^+})^\kappa = \triangle(\overline{q^+})^\lambda$ for each $\triangle$. By RAn, $\widehat{\gamma}, \kappa \in \iota, \lambda \notin \iota, |\lambda|_1 \Vdash (\kappa \in \iota) \equiv (\lambda \in \iota)$. Hence, Γ'_k is inconsistent, $\lightning$.

It suffices to show that the canonical hypermodel is analytic. Suppose $c_\kappa \in C_\iota$ and $c_\kappa \approx c_\lambda$. So $(\kappa \in \iota) \in \Gamma$ and $|\lambda|_1 \in \Gamma$. Moreover, if $(\lambda \in \iota) \notin \Gamma$, then by the revised Lindenbaum construction, $\kappa \not\approx \lambda \in \Gamma$, contrary to $c_\kappa \approx c_\lambda$, $\lightning$. Hence, $(\lambda \in \iota) \in \Gamma$.

$\mathsf{S_5}$: We revise Definition A3.13 so that $c(\triangle)$ is always defined classically. The only revision needed to the proofs is to verify the connective case in the truth lemma (Lemma A3.17). This follows from the fact that $|\kappa|_1 \Vdash @_\kappa \triangle(\overline{\phi}) \equiv \triangle(@_\kappa \overline{\phi})$ is $(\mathbf{H} + \mathrm{Bool}_{\Vdash})$-derivable (by $\mathrm{Bool}_{\Vdash}$, $\mathrm{Gen}_@$, and $\mathrm{Dist}_@$ (for $\Vdash$)).

B5.2 Theorem B2.3

For some of these proofs, we use the lemma below, which follows from Definition A4.7 and $\exists$-witnessing.

Lemma B5.2 (Canonical Proposition Space). Let $|\kappa|_1, |\lambda|_1 \in \Gamma$. Then $\pi_{c_\kappa} \subseteq \pi_{c_\lambda}$ iff $(\pi_\kappa \subseteq \pi_\lambda) \in \Gamma$, and $|\pi_{c_\kappa}| = 1$ iff $|\pi_\kappa|_1 \in \Gamma$.

We omit the proofs for B, $\mathsf{U_q}$, $\mathsf{U_o}$, Si, and $\mathsf{S_5}$, which are routine.

At: Let $c_\kappa \in D_{\mathbb{H}\Gamma}$ and $\Delta \in W_\Gamma$. First, observe that $\phi \to \Box(@_\kappa p^+ \to \phi) \in \Delta$. For by Atom, Bool, and $\mathrm{Dist}_@$, $\exists p(@_\kappa p \wedge \forall q(\Box(@_\kappa p \to @_{l_\Delta} q) \vee \Box(@_\kappa p \to \neg @_{l_\Delta} q))) \in \Delta$. Since $l_\Delta \in \Delta$, we have $\exists p(@_\kappa p \wedge \forall q(\Box(@_\kappa p \to q) \vee \Box(@_\kappa p \to \neg q))) \in \Delta$. By $\exists$-witnessing, $@_\kappa p^+ \wedge \forall q(\Box(@_\kappa p^+ \to q) \vee \Box(@_\kappa p^+ \to \neg q)) \in \Delta$ for some p^+. By $\mathrm{Elim}_\forall$, ClEx, and $\exists$-witnessing, $\Box(@_\kappa p^+ \to \phi) \vee \Box(@_\kappa p^+ \to \neg \phi) \in \Delta$. By S5, $\phi \to \Box(@_\kappa p^+ \to \phi) \in \Delta$.

So suppose $@_\kappa p^+ \in \Delta'$ and suppose $\phi \in \Delta$. Thus, $\Box(@_\kappa p^+ \to \phi) \in \Delta$. By Corollary A3.10, $\phi \in \Delta'$. Hence, $\Delta' = \Delta$. So $p^+ \in [\{\Delta\}]_\kappa$, i.e., $\{\Delta\} \in \pi_{c_\kappa}$.

An: Since members of $\mathsf{INom}^+ = \{l_1^+, l_2^+, l_3^+, \ldots\}$ might not be allowed to denote singletons (since conventions must be closed under $\approx$), the Henkin construction needs to be revised so that INom^+ is replaced with $\mathsf{IVar}^+ = \{i_1^+, i_2^+, i_3^+, \ldots\}$ (though we don't allow formulas in $\mathcal{L}^{\mathsf{QH}+}$ to bind members of IVar^+). We also need to make the following amendments to the definition of the canonical model:

$$D_{\mathbb{C}\Gamma} = \{C_\iota \mid \neg\, |\iota|_1 \in \Gamma\} \cup \{C_l \mid |l|_1 \in \Gamma\} \cup \{\{c_\kappa \mid c_\kappa \approx c_i\} \mid |i|_1 \in \Gamma\}$$

$$V_\Gamma(\iota) = \begin{cases} \{c_\iota\} & \text{if } |\iota|_1 \in \Gamma \text{ and } \iota \in \mathsf{IVar} \cup \mathsf{IVar}^+ \\ C_\iota & \text{otherwise} \end{cases}$$

The proof of the truth lemma (Lemma A3.17) remains intact (the only difference is the $@_\iota$-case where ι is i and $|i|_1 \in \Gamma$; in that case, $\Delta, c_\kappa \Vdash @_i\, \phi$ iff $\Delta, c_i \Vdash \phi$ iff $@_i\, \phi \in \Delta$ iff $@_\kappa\, @_i\, \phi \in \Delta$.) Trivially, $\{c_\kappa \mid c \approx c_i\}$ is analytic. So we need to show that C_ι is analytic if $\neg\, |\iota|_1 \in \Gamma$, and that C_l is analytic if $|l|_1 \in \Gamma$.

First, suppose $\neg\, |\iota|_1 \in \Gamma$. Let $c_\kappa \in C_\iota$ and let $c_\lambda \neq c_\kappa$ be such that $c_\kappa \approx c_\lambda$. Since $(\kappa \in \iota), |\lambda|_1 \in \Gamma$, it suffices to show that $(\kappa \approx \lambda) \in \Gamma$; for then by An, $(\lambda \in \iota) \in \Gamma$, and so $c_\lambda \in C_\iota$. By Lemma B5.2, $(\pi_\kappa = \pi_\lambda) \in \Gamma$ since $\pi_{c_\kappa} = \pi_{c_\lambda}$. Moreover, if $((p^+)^\kappa = (q^+)^\kappa) \in \Gamma$, then $\{\Delta \in W_\Gamma \mid @_\kappa\, p^+ \in \Delta\} = \{\Delta \in W_\Gamma \mid @_\lambda\, q^+ \in \Delta\}$. Since $c_\kappa \approx c_\lambda$, that means $c_\kappa(\star)(X) = c_\lambda(\star)(X)$. So by Lemma B5.1, $((\star\, p^+)^\kappa = (\star\, q^+)^\lambda) \in \Gamma$. Therefore, $((p^+)^\kappa = (q^+)^\lambda) \supset ((\star\, p^+)^\kappa = (\star\, q^+)^\lambda) \in \Gamma$. Since Γ witnesses $\exists s, \forall p\, \forall q((p^\kappa = q^\lambda) \supset ((\star\, p)^\kappa = (\star\, q)^\lambda)) \in \Gamma$, i.e., $(\star_\kappa = \star_\lambda) \in \Gamma$. Similarly, $(\bigcirc_\kappa = \bigcirc_\lambda) \in \Gamma$. Hence, $(\kappa \approx \lambda) \in \Gamma$.

Next, suppose $|l|_1 \in \Gamma$. Let $c_\kappa \approx c_l$. By the reasoning above, $(\kappa \approx l) \in \Gamma$. Since $|\kappa|_1, |l|_1 \in \Gamma$, it follows by $\mathsf{Many}_{\mathsf{INom}}$ that $(\kappa = l) \in \Gamma$. Hence, by Lemma A4.9, $c_\kappa = c_l$, and thus $c_\kappa \in C_l$.

B5.3 Theorem B2.6

We omit the proofs for Cr and Di, which are routine.

Cl_Φ: We revise the Henkin construction. Let $\mathsf{Prop}^\circ = \{p_1^\circ, p_2^\circ, p_3^\circ, \ldots\}$ be new propositional variables, and let $\mathcal{L}^{\mathsf{QH}+\circ}$ be the result of expanding $\mathcal{L}^{\mathsf{QH}+}$ with Prop° (again, not including formulas with quantifiers binding these variables). Enumerate the members of Φ as $\chi_1, \chi_2, \chi_3, \ldots$. Let Δ be the set of all formulas of the form $p_k^\circ =_l \chi_k$, where $l \in \mathsf{INom}^+$ and $\chi_k \in \Phi$. The Henkin construction is the same except we redefine Γ'_k so that $\Gamma'_k = \Gamma_k \cup \{\phi_k\}$ if $\Gamma_k, \Delta, \phi_k \nvdash_{\mathbf{QH} \cup \mathrm{Ex}_\Phi} \bot$ (and $= \Gamma_k$ otherwise). Clearly, if $\Gamma_k \cup \Delta$ is $(\mathbf{QH} \cup \mathrm{Ex}_\Phi)$-consistent, then so is $\Gamma'_k \cup \Delta$. The proof that $\Gamma_{k+1} \cup \Delta$ is consistent if $\Gamma'_k \cup \Delta$ is consistent is essentially the same. Thus, we just need to show that $\Gamma_1 \cup \Delta$ is $(\mathbf{QH} \cup \mathrm{Ex}_\Phi)$-consistent. Suppose it's not. Since l_Γ occurs nowhere in Δ, we can eliminate l_Γ by the same reasoning as in Lemma A3.7. Thus, there are some $\alpha_1, \ldots, \alpha_k$ that are instances of Ex_Φ, some $\delta_1, \ldots, \delta_n \in \Delta$ where δ_i is of the form $q_i^\circ =_{k_i} \psi_i$ for some $\psi_i \in \Phi$ and $k_i \in \mathsf{INom}^+$, and some $\gamma_1, \ldots, \gamma_m \in \Gamma$ such that $\widehat{\alpha}, \widehat{\delta} \vdash \neg\, \widehat{\gamma}$ (throughout, I'll use $\vdash$ for provability in **QH** and $\vdash_{\mathrm{Ex}_\Phi}$ for provability in $\mathbf{QH} \cup \mathrm{Ex}_\Phi$). Now, it may be that $q_i^\circ = q_j^\circ$ for some i and j. So let $q_i^\circ \approx \psi_i$ be the conjunction all δ_js such that $q_j^\circ = q_i^\circ$—that is, $q_i^\circ \approx \psi_i$ has the form $(q_i^\circ =_{k_{i_1}} \psi_i) \wedge \cdots \wedge (q_i^\circ =_{k_{i_j}} \psi_i)$. (Given how Δ is defined and how Φ is enumerated, it is never the case that $q_i^\circ = q_j^\circ$ but $\psi_i \neq \psi_j$; so this definition is well-defined.) Thus, $\widehat{\alpha}, q_1^\circ \approx \psi_1, \ldots, q_n^\circ \approx \psi_n \vdash \neg\, \widehat{\gamma}$. By Lemma A4.3, $\widehat{\alpha}, r_1 \approx \psi_1, \ldots, r_n \approx \psi_n \vdash \neg\, \widehat{\gamma}$ where $r_1, \ldots, r_n \in \mathsf{Prop}$ are fresh. By $\mathrm{RK}_\exists$, $\mathrm{Vac}_\exists$, and $\mathrm{VDist}_\exists$, $\widehat{\alpha}, \exists r_1(r_1 \approx \psi_1), \ldots, \exists r_n(r_n \approx \psi_n) \vdash \neg\, \widehat{\gamma}$. So by Ex_Φ, $\vdash_{\mathrm{Ex}_\Phi} \neg\, \widehat{\gamma}$, $\lightning$.

The rest of the proof of the Henkin lemma (Lemma A4.5) goes through as before. And since $\Gamma_k \cup \Delta$ is $(\mathbf{QH} \cup \mathrm{Ex}_\Phi)$-consistent for each k, $\Gamma^+ \cup \Delta$ is $(\mathbf{QH} \cup \mathrm{Ex}_\Phi)$-consistent, which by maximality means $\Delta \subseteq \Gamma^+$. Hence, Γ^+ has the following property: for each $\chi \in \Phi$, there is a p° such that for all $\iota \in \mathsf{ITerm}^+$, $(p^\circ =_\iota \chi) \in \Gamma^+$.

To complete the proof, we revise the definition of π_{c_κ} (when $@_\kappa\, cl \notin \Gamma$) and $D_{\mathbb{P}\Gamma}$:

$$\pi_{c_\kappa} = \{X \mid \exists p \in \mathsf{Prop}^+ \cup \mathsf{Prop}^\circ\colon\ p \in [X]_\kappa\}$$
$$D_{\mathbb{P}\Gamma} = \{P \in \mathbb{P}_{D_{\mathbb{H}\Gamma}} \mid \exists p \in \mathsf{Prop}^* \cup \mathsf{Prop}^\circ\, \forall c_\kappa \in D_{\mathbb{H}\Gamma}\colon\ p \in [P(c_\kappa)]_\kappa\}\,.$$

The rest of the proof goes through as before. So by Lemma A4.12, $[\![\chi_i]\!]^{\mathcal{M}_\Gamma, c_\kappa} = \{\Delta \in W_\Gamma \mid @_\kappa\, \chi_i \in \Delta\} = \{\Delta \in W_\Gamma \mid @_\kappa\, p_i^\circ \in \Delta\}$, so $P_{p_i^\circ}$ can be our witness for $\chi_i \in \Phi$. Hence, $D_{\mathbb{P}\Gamma}$ is closed under Φ.

Cl_Φ^+: The proof is roughly the same as Cl_Φ, but we need to make some revisions. Let $\Phi' = \{\chi[q_1'/q_1, \ldots, q_n'/q_n] \mid q_1', \ldots, q_n' \in \mathsf{Prop}^* \cup \mathsf{Prop}^\circ\}$. Enumerate the members of Φ' as $\chi_1, \chi_2, \chi_3, \ldots$ in such a way that p_k' never occurs in $\chi_1, \ldots, \chi_k$. Proceed with the Henkin construction in the same manner as before, replacing Φ throughout with Φ'. To establish that $\Gamma_1 \cup \Delta$ is $(\mathbf{QH} + \mathrm{Ex}_\Phi)$-consistent, we use the same reasoning, except the last step needs further justification, since ψ_i may not be in Φ but rather of the form $\psi_i = \chi[q_1'/q_1, \ldots, q_n'/q_n]$ for some $\chi \in \Phi$. However, since Ex_Φ is now an axiom, that means if $\chi \in \Phi$, then $\vdash \exists r(r \approx \chi)$. So by $\mathrm{Gen}_\forall$, $\vdash \forall q_1 \cdots \forall q_n\, \exists r(r \approx \chi)$. Hence, by $\mathrm{Elim}_\forall$, $\vdash \exists r(r \approx \psi_i)$.

Making the same revisions as before, the rest of the completeness proof goes through. So we just need to show now that $D_{\mathbb{P}\Gamma}$ is strongly closed under Φ. Let $\mathcal{M} = \langle W_\Gamma, D_{\mathbb{C}\Gamma}, D_{\mathbb{P}\Gamma}, V\rangle$. Then $V(q_i) = P_{q_i'}$ for some q_i'. Hence, by Lemma A4.1, $[\![\phi]\!]^{\mathcal{M}} = [\![\phi[q_1'/q_1, \ldots, q_n'/q_n]]\!]^{\mathcal{M}_\Gamma}$. By how Γ was constructed, there is a p° such that for all ι, $p^\circ =_\iota \phi[q'_1/q_1, \ldots, q'_n/q_n] \in \Gamma$. By Lemma A4.12, $[\![\phi[q'_1/q_1, \ldots, q'_n/q_n]]\!]^{\mathcal{M}_\Gamma} = [\![p^\circ]\!]^{\mathcal{M}_\Gamma} \in D_{\mathbb{P}\Gamma}$. Hence, $D_{\mathbb{P}\Gamma}$ is strongly closed under Φ.

To establish that $\mathbf{QH} + \mathrm{Ex}_\Phi = \mathbf{QH} + \mathrm{Elim}_{\forall\Phi}$, it suffices to show that Ex_Φ is coderivable with $\mathrm{Elim}_{\forall\Phi}$. Deriving Ex_Φ from $\mathrm{Elim}_{\forall\Phi}$ is straightforward by S5 and $\mathrm{Dual}_\forall$. For the other direction, it follows by induction (or completeness over the class of all models) that if χ is free for p, and $\iota_1, \ldots, \iota_n$ are the free interpretation terms in ϕ, then $p = \chi, p =_{\iota_1} \chi, \ldots, p =_{\iota_n} \chi \Vdash \phi = \phi[\chi/p]$. Hence:

$\forall p\, \phi, p = \chi, \ldots, p =_{\iota_n} \chi \Vdash \phi[\chi/p]$	above, $\mathrm{Elim}_\forall$
$\downarrow i. \forall p\, \phi, \downarrow i.(p =_i \chi \,\&\, \cdots \,\&\, p =_{\iota_n} \chi) \Vdash \downarrow i. \phi[\chi/p]$	$\mathrm{Gen}_\downarrow$, $\mathrm{Idle}_\downarrow$, $\mathrm{Dist}_\downarrow$
$\forall p\, \phi, \downarrow i.(p =_i \chi \,\&\, \cdots \,\&\, p =_{\iota_n} \chi) \Vdash \phi[\chi/p]$	$\mathrm{Vac}_\downarrow$
$\forall p\, \phi, \exists p \downarrow i.(p =_i \chi \,\&\, \cdots \,\&\, p =_{\iota_n} \chi) \Vdash \phi[\chi/p]$	$\mathrm{RK}_\exists$, $\mathrm{VDist}_\exists$, $\mathrm{Vac}_\exists$
$\forall p\, \phi, \downarrow i. \exists p(p =_i \chi \,\&\, \cdots \,\&\, p =_{\iota_n} \chi) \Vdash \phi[\chi/p]$	$\mathrm{BF}_\downarrow$
$\forall p\, \phi \Vdash \phi[\chi/p]$	Ex_Φ, $\mathrm{Gen}_\downarrow$.

Df_Φ: I will only prove weak completeness here; it's easy to check that if Φ is finite, then strong completeness can be established via the same method. Suppose ϕ is $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-consistent. Enumerate the members of Φ as $\chi_1, \chi_2, \chi_3, \ldots$. Parallel to $\Gamma_1, \Gamma_2, \Gamma_3, \ldots$, we construct a new sequence of sets $\Delta_1, \Delta_2, \Delta_3, \ldots$. First, $\Gamma_1 = \{\phi\} \cup \{l_1^+, |l_1^+|_1\}$ and $\Delta_1 = \varnothing$. Next, define Γ_{k+1} as in the proof of Lemma A4.5. Finally, define Δ_{k+1} as follows:

$$\Delta_{k+1} = \begin{cases} \Delta_k \cup \{q^+ =_{l^+} \chi \mid (q^+ =_{l_1^+} \chi) \in \Delta_k\} & \text{if (*) holds} \\ \Delta_k \cup \{p^+ =_{l^+} \chi \mid l^+ \in \mathsf{lNom}^+ \text{ occurs in } \Gamma_{k+1}\} & \text{if (**) holds} \\ \Delta_k & \text{otherwise} \end{cases}$$

(*) $\phi_k = \neg @_\iota \psi$ and l^+ is the witness introduced in Γ_{k+1}

(**) $\phi_k = \exists p\, \psi$, where p^+ is the witness introduced in Γ_{k+1} and χ is the first of Φ such that $\Gamma_{k+1}, \Delta_k, \{p^+ =_{l^+} \chi \mid l^+ \in \mathsf{lNom}^+ \text{ occurs in } \Gamma_{k+1}\} \nVdash \bot$.

Finally, $\Gamma^+ = \bigcup_{k \geqslant 1} \Gamma_k$. We first show that for each k, $\Gamma_k \cup \Delta_k$ is $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-consistent. Clearly this holds for $k = 1$. And clearly if $\Gamma_k \cup \Delta_k$ is $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-consistent, then so is $\Gamma'_k \cup \Delta_k$ and $\Gamma_{k+1} \cup \Delta_k$. So we just need to show that if $\Gamma_{k+1} \cup \Delta_k$ is $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-consistent, then so is $\Gamma_{k+1} \cup \Delta_{k+1}$. If $\phi_k = \exists p\, \psi$, then $\Gamma_{k+1} \cup \Delta_{k+1}$ is $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-consistent by construction of Δ_{k+1}, assuming it's defined. Here's the proof that it is always defined, i.e., there always is such a χ in this case. Suppose otherwise. That means for all $\chi \in \Phi$, where $\gamma \coloneqq \bigwedge \Gamma_k$, $\delta \coloneqq \bigwedge \Delta_k$, and $l_1^+, \ldots, l_n^+$ are the nominals occurring in some formula of $\Gamma_{k+1}, \gamma, \delta \vdash \neg(p^+ =_{l_1^+} \chi \wedge \cdots \wedge p^+ =_{l_n^+} \chi)$. By $\mathrm{Gen}_{\forall\Phi}$, $\gamma, \delta \vdash \forall p \, \neg(p^+ =_{l_1^+} p \wedge \cdots \wedge p^+ =_{l_n^+} p)$. By $\mathrm{Elim}_\forall$, $\gamma, \delta \vdash \neg(p^+ =_{l_1^+} p^+ \wedge \cdots \wedge p^+ =_{l_n^+} p^+)$. Hence, by S5, $\gamma, \delta \vdash \bot$, $\lightning$.

Now suppose $\phi_k = \neg @_\iota \psi$ and suppose for reductio that $\Gamma_{k+1} \cup \Delta_{k+1}$ is $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-inconsistent. Then for some formula of the form $q_i^+ =_{l^+} \chi_i$, we have $\neg @_\iota \psi, l^+ \in \iota, \gamma, \delta, (q_1^+ =_{l^+} \chi_1), \ldots, (q_n^+ =_{l^+} \chi_n) \vdash @_{l^+} \psi$. Repeating the reasoning in Lemma A3.7, $\gamma, \delta, \downarrow i.(q_1^+ =_i \chi_1), \ldots, \downarrow i.(q_n^+ =_i \chi_n) \vdash @_\iota \psi$. Since $l_1^+, |l_1^+|_1 \in \Gamma_k$: $\gamma, \delta, (q_1^+ =_{l_1^+} \chi_1), \ldots, (q_n^+ =_{l_1^+} \chi_n) \vdash @_\iota \psi$. But $(q_1^+ =_{l_1^+} \chi_1), \ldots, (q_n^+ =_{l_1^+} \chi_n) \in \Delta_k$. Thus, $\gamma, \delta \vdash @_\iota \psi$. So $\Gamma_k' \cup \Delta_k$ is already $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-inconsistent, $\lightning$. Hence, $\Gamma_{k+1} \cup \Delta_{k+1}$ is $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-consistent. Therefore, $\Gamma^+ \cup \bigcup_k \Delta_k$ is $(\mathbf{QH} + \mathrm{Gen}_{\forall\Phi})$-consistent, and so by maximality, $\Delta_k \subseteq \Gamma^+$ for all k.

By construction, for each $p^+ \in \mathsf{Prop}^+$, there is a $\chi \in \Phi$ such that $(p^+ =_{l^+} \chi) \in \Gamma^+$ for all $l^+ \in \mathsf{lNom}^+$. From here, the completeness proof proceeds as before. To complete the proof, we show $D_{\mathbb{P}\Gamma}$ is definable in Φ. Where $P = P_{p^+} \in D_{\mathbb{P}\Gamma}$, let $\chi \in \Phi$ be such that $p^+ =_{l^+} \chi \in \Gamma^+$ for all $l^+ \in \mathsf{lNom}^+$. Then by Lemma A4.12, $P_{p^+}(c_\kappa) = [\![p^+]\!]^{c_\kappa} = [\![\chi]\!]^{c_\kappa}$. Hence, $P_{p^+} = [\![\chi]\!]$.

$\mathsf{Cl}_\Phi\mathsf{Df}_\Phi$: We use the same construction as in Df_Φ. We need to show (i) that we can dispense with the $\mathrm{Gen}_{\forall\Phi}$ rule in the proof above, and (ii) $D_{\mathbb{P}\Gamma}$ is closed under Φ. (To establish that $\mathbf{QH} \cup \mathrm{Hom}_\Phi \cup \mathrm{Ex}_\Phi = \mathbf{QH} \cup \mathrm{Hom}_\Phi \cup \mathrm{Ex}_\Phi^-$, simply observe that $\{\forall p((p = \chi) \supset (p =_{\iota_i} \chi)) \mid i \leqslant n\}, \mathsf{E}\chi \Vdash \exists p \,\&_{i=1}^n (p =_{\iota_i} \chi)$.)

For (i), note that $\mathrm{Gen}_{\forall\Phi}$ was only used to establish that in the Henkin construction, if $\phi_k = \exists p \, \psi$ is added to Γ_k' and p^+ is the witness introduced into Γ_{k+1}, then there is a $\chi \in \Phi$ such that $\Gamma_{k+1}, \Delta_k, \{p^+ =_{l^+} \chi \mid l^+ \in \mathsf{lNom}^+ \text{ occurs in } \Gamma_{k+1}\} \nvdash \bot$. For all $\chi \in \Phi$, there is an l^+ such that $(p^+ =_{l^+} \chi) \notin \Gamma_{k+1}$. Then for all $\chi \in \Phi$, there exist some $\gamma_1, \ldots, \gamma_n \in \Gamma_{k+1}$, some $\delta_1, \ldots, \delta_m \in \Delta_k$, some $\alpha_1, \ldots, \alpha_k \in \mathrm{Hom}_\Phi$, and some $\beta_1, \ldots, \beta_j \in \mathrm{Ex}_\Phi^-$ such that $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, p^+ =_{l_1^+} \chi, \ldots, p^+ =_{l_n^+} \chi \vdash \bot$. Since $\forall p(p = \chi \supset p =_{l_i^+} \chi) \in \mathrm{Hom}_\Phi$ for each l_i^+, we can assume these are included in $\widehat{\alpha}$. Hence, by $\mathrm{Elim}_\forall$, $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, p^+ = \chi \vdash \bot$. So by Lemma A4.3, where r is fresh, $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, r = \chi \vdash \bot$. By $\mathrm{Intro}_\exists$, $\mathrm{VDist}_\exists$ and $\mathrm{Vac}_\exists$, $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \widehat{\delta}, \mathsf{E}\chi \vdash \bot$. So, $\widehat{\gamma}, \widehat{\delta} \vdash_{\mathrm{Hom}_\Phi \cup \mathrm{Ex}_\Phi^-} \bot$, $\lightning$. (Notice we did not rely on Γ_{k+1} being finite, so the same strategy establishes strong completeness.) For (ii), let $\chi \in \Phi$. By Ex_Φ, $(p^+ = \chi) \in \Gamma$ for some $p^+ \in \mathsf{Prop}^+$. By Hom_Φ and $\mathrm{Elim}_\forall$, $(p^+ =_\iota \chi) \in \Gamma$ for all $\iota \in \mathsf{lTerm}^+$. Hence, $p^+ \in [[\![\chi]\!]^{c_\kappa}]_\kappa$ for all c_κ, i.e., $[\![\chi]\!] \in D_{\mathbb{P}\Gamma}$.

$\mathsf{Cl}^+_\Phi\mathsf{Df}_\Phi$: Similar to $\mathsf{Cl}_\Phi\mathsf{Df}_\Phi$, except using Ex_Φ as an axiom to show that $D_{\mathbb{P}\Gamma}$ is strongly closed (as in the proof of completeness over Cl^+_Φ).

Cb: Let $X_1 \in \pi_{c_{\kappa_1}}, \ldots, X_n \in \pi_{c_{\kappa_n}}$ where $c_{\kappa_1}, \ldots, c_{\kappa_n}$ are distinct. Let $p_1^+, \ldots, p_n^+ \in \mathsf{Prop}^+$ be such that $p_i^+ \in [X_i]_{\kappa_i}$. Since $c_{\kappa_1}, \ldots, c_{\kappa_n}$ are distinct, by the same reasoning as in Di, $(\kappa_i \neq \kappa_j) \in \Gamma$ for $i \neq j$. By Split and Bool, $\exists p \bigwedge_{i=1}^n (p =_{\kappa_i} p_i^+) \in \Gamma$. By witnessing $\exists$s, there is a $p^+ \in \mathsf{Prop}^+$ such that $p^+ =_{\kappa_i} p_i^+ \in \Gamma$ for $1 \leqslant i \leqslant n$. Hence, $P_{p^+}(c_{\kappa_i}) = X_i$.

To establish that $\mathbf{QH} + \mathrm{PII}^+ + \mathrm{Split} = \mathbf{QH} + \mathrm{PII}_1^+ + \mathrm{Split}$, we just need to show that PII^+ is $(\mathbf{QH} + \mathrm{PII}_1^+ + \mathrm{Split})$-derivable. By PII_1^+, it suffices to show that $\forall p(p^\iota = p^\kappa), |\iota|_1, |\kappa|_1, (\iota \neq \kappa) \Vdash |\pi_\iota|_1$ is derivable using Split:

$$
\begin{array}{rl}
\forall p(p^\iota = p^\kappa) \Vdash p^\iota = p^\kappa & \mathrm{Elim}_\forall \\
\forall p(p^\iota = p^\kappa) \Vdash r^\iota = r^\kappa & \mathrm{Elim}_\forall \\
\forall p(p^\iota = p^\kappa), p^\iota = q^\iota, p^\kappa = r^\kappa \Vdash q^\iota = r^\iota & \mathrm{S5} \\
\forall p(p^\iota = p^\kappa), \exists p(p^\iota = q^\iota \,\&\, p^\kappa = r^\kappa) \Vdash q^\iota = r^\iota & \mathrm{RK}_\exists, \mathrm{VDist}_\exists, \mathrm{Vac}_\exists \\
\forall p(p^\iota = p^\kappa), |\iota|_1, |\kappa|_1, (\iota \neq \kappa) \Vdash q^\iota = r^\iota & \mathrm{Split} \\
\forall p(p^\iota = p^\kappa), |\iota|_1, |\kappa|_1, (\iota \neq \kappa) \Vdash |\pi_\iota|_1 & \mathrm{RK}_\forall, \mathrm{Vac}_\forall, \mathrm{Dist}_@.
\end{array}
$$

CpSi: We must revise the definition of the canonical model so that $D_{\mathbb{P}\Gamma} = \mathbb{P}_{D_{\mathbb{H}\Gamma}}$. The only thing that needs to be redone is the $\forall$ inductive step of Lemma A4.12. The argument that if $\mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \forall p\, \phi$, then $@_\kappa \forall p\, \phi \in \Delta$ is the same. For the other direction, we first need the following intermediate result:

> **Claim**: For all ϕ and all $\lambda \in \mathsf{ITerm}^+$ such that $|\lambda|_1 \in \Gamma$, there is a formula $\phi^\uparrow$ such that where $\overline{p}$ are the free propositional variables in ϕ:
> (i) $\phi^\uparrow$ contains no interpretation binders $\downarrow i$
> (ii) if ι and κ occur in $\phi^\uparrow$ and ι isn't κ, then $(\iota \neq \kappa) \in \Gamma$
> (iii) for all $\Delta \in W_\Gamma$, $\mathcal{M}_\Gamma, \Delta, c_\lambda \Vdash \forall \overline{p}(\phi = \phi^\uparrow)$
> (iv) for all $\Delta \in W_\Gamma$, $\forall \overline{p}(\phi = \phi^\uparrow) \in \Delta$.

> *Proof*: First, since Γ witnesses $\neg @$s, for each free ι, there is an $l_\iota^+ \in \mathsf{INom}^+$ such that $(l_\iota^+ \in \iota) \in \Gamma$. By Sing and $\mathrm{Intro}_=$, $(l_\iota^+ = \iota) \in \Gamma$. Let l_ι^+ be the first in INom^+ with this property. By SubId, we can replace each ι that occurs free in ϕ with l_ι^+. Call the result ϕ'. Now proceed

as follows:

(a) If $\downarrow i$ does not occur in the scope of any $@_\kappa$ or any $\downarrow j$, replace each free i in its scope with l^+_λ. Then delete this $\downarrow i$.

(b) Repeat (a) on the result until there are no more $\downarrow i$s that do not occur in the scope of any $@_\kappa$ or any $\downarrow j$.

(c) For each subformula of the form $@_{l^+}\,\psi$ that does not occur in the scope of any $@_\kappa$ operator, repeat (a) and (b) on ψ except with l^+ in place of l^+_λ. Continue until there are no more binders $\downarrow i$ left. Call the result $\phi^\uparrow$.

It is now easy to verify that $\phi^\uparrow$ satisfies (i)–(iv). ■

So suppose $\mathcal{M}_\Gamma, \Delta, c_\kappa \nVdash \forall p\,\phi$. By the claim above, $\mathcal{M}_\Gamma, \Delta, c_\kappa \nVdash \forall p\,\phi^\uparrow$. Thus, there is a $P \in \mathbb{P}_{D_{\mathbb{H}\Gamma}}$ such that $(\mathcal{M}_\Gamma)^p_P, \Delta, c_\kappa \nVdash \phi^\uparrow$. Let $l^+_1, \ldots, l^+_n$ be the interpretation terms in $\phi^\uparrow$. By ClEx and $\Vdash \mathsf{E}p$ (by Intro$_\exists$), for each l^+_i, there is a p^+_i such that $p^+_i \in [P(c_{l^+_i})]_{l^+_i}$, i.e., $\Delta' \in P(c_{l^+_i})$ iff $@_i\,p^+_i \in \Delta'$. Since $(l^+_i \neq l^+_j) \in \Gamma$ when $i \neq j$, it follows by Split that $\exists p\,\&^n_{i=1}(p =_{l^+_i} p^+_i) \in \Gamma$. By witnessing $\exists$s, there is a p^+ such that $\&^n_{i=1}(p^+ =_{l^+_i} p^+_i) \in \Gamma$. Thus, for each i and Δ': $@_{l^+_i}\,p^+ \in \Delta'$ iff $@_{l^+_i}\,p^+_i \in \Delta'$. Hence, $\Delta' \in P(c_{l^+_i})$ iff $@_i\,p^+ \in \Delta'$. By Lemma A4.1, $(\mathcal{M}_\Gamma)^p_P, \Delta, c_\kappa \Vdash \phi^\uparrow$ iff $\mathcal{M}_\Gamma, \Delta, c_\kappa \Vdash \phi^\uparrow[p^+/p]$. Hence, $\mathcal{M}_\Gamma, \Delta, c_\kappa \nVdash \phi^\uparrow[p^+/p]$. By IH, $@_\kappa\,\phi^\uparrow[p^+/p] \notin \Delta$. By Elim$_\forall$, $\forall p\,@_\kappa\,\phi^\uparrow \notin \Delta$. By the claim above, $\forall p\,@_\kappa\,\phi \notin \Delta$. By CBF$_@$, $@_\kappa\,\forall p\,\phi \notin \Delta$.

B5.4 Theorem B3.10

In each case, it suffices to show that f_Γ satisfies the corresponding constraint given the axiom. Moreover, f_Γ is already defined to satisfy the relevant constraint when $[A] = \varnothing$. So assume throughout that $[A] \neq \varnothing$.

Suc: Suppose $\langle \Delta', c_\lambda \rangle \in f_\Gamma(A, \Delta, c_\kappa)$. Let $\phi \in [A]$. By Lemma B3.6 and Definition B3.7, if $(\phi \mathrel{\Box\!\!\rightarrow}_\kappa (\lambda \supset \psi)) \in \Delta$ where $\psi \in \mathcal{L}^{\mathsf{H}+}_{\Box\!\rightarrow}$, then $@_\lambda\,\psi \in \Delta'$. By Id$_{\Box\!\rightarrow}$ and RK$_{\Box\!\rightarrow_\kappa}$, $(\phi \mathrel{\Box\!\!\rightarrow}_\kappa (\lambda \supset \phi)) \in \Delta$. So $@_\lambda\,\phi \in \Delta'$. By Definition B3.5, $\langle \Delta', c_\lambda \rangle \in A$.

W: Let $\langle \Delta, c_\kappa \rangle \in A$ and $\phi \in [A]$ (so $@_\kappa\,\phi \in \Delta$). Suppose $\phi \mathrel{\Box\!\!\rightarrow}_\kappa (\kappa \supset \psi) \in \Delta$. By MP$_{\Box\!\rightarrow}$ and Ded, $\phi, \phi \mathrel{\Box\!\!\rightarrow} (\kappa \supset \psi), \kappa \Vdash \psi$. By Gen$_@$ and Ref, $@_\kappa\,\phi, \phi \mathrel{\Box\!\!\rightarrow}_\kappa (\kappa \supset \psi) \Vdash @_\kappa\,\psi$. Hence, $@_\kappa\,\psi \in \Delta$. By Definition B3.7, $\langle \Delta, c_\kappa \rangle \in f_\Gamma(A, \Delta, c_\kappa)$.

C: Suppose $\langle \Delta, c_\kappa \rangle \in A$. Let $\phi \in [A]$. Thus, $@_\kappa \phi \in \Delta$. Reasoning as above, we have $@_\kappa \phi \Vdash (\phi \,\Box\!\!\to_\kappa (\kappa \supset \psi)) \equiv @_\kappa \psi$. So if $(\phi \,\Box\!\!\to_\kappa (\kappa \supset \psi)) \in \Delta$, then $@_\kappa \psi \in \Delta$, meaning $\langle \Delta, c_\kappa \rangle \in f_\Gamma(A, \Delta, c_\kappa)$. Moreover, let $\langle \Delta', c_\lambda \rangle \in f_\Gamma(A, \Delta, c_\kappa)$. So for all $\psi \in \mathcal{L}^{\mathsf{H}+}_{\Box\!\to}$, if $(\phi \,\Box\!\!\to_\kappa (\lambda \supset \psi)) \in \Delta$, then $@_\lambda \psi \in \Delta'$. Now, by Cen, $i, \phi \Vdash (\phi \,\Box\!\!\to i)$. By $\text{Gen}_\downarrow$ and $\text{Vac}_\downarrow$, $\phi \Vdash \downarrow i.(\phi \,\Box\!\!\to i)$. By $\text{Gen}_@$ and $\text{DA}_@$, $|\kappa|_1, @_\kappa \phi \Vdash @_\kappa(\phi \,\Box\!\!\to \kappa)$. Since $|\kappa|_1 \in \Delta$, that means $(\phi \,\Box\!\!\to_\kappa \kappa) \in \Delta$. By $\text{RK}_{\Box\!\to_\kappa}$, $(\phi \,\Box\!\!\to_\kappa (\lambda \supset \kappa)) \in \Delta$. So $@_\lambda \kappa \in \Delta'$. By Rigid and Corollary A3.10, $|\kappa|_1 \in \Delta'$. By $\text{Intro}_=$, $(\kappa = \lambda) \in \Delta'$. By Lemma A3.14, $c_\kappa = c_\lambda$. We now show $\Delta' = \Delta$. We'll just show $\Delta \subseteq \Delta'$ to illustrate. Let $\psi \in \Delta$. By $\text{Intro}_@$, $\text{Elim}_@$, and Red, $@_\kappa @_{l_\Delta} \psi \in \Delta$. Thus, $\phi \,\Box\!\!\to_\kappa (\kappa \supset @_{l_\Delta} \psi) \in \Delta$. So $@_\kappa @_{l_\Delta} \psi \in \Delta'$ since $\langle \Delta', c_\kappa \rangle \in f_\Gamma(A, \Delta, c_\kappa)$. So by Red, Rigid, $\text{Intro}_@$, and $\text{Elim}_@$, $\psi \in \Delta'$.

Stal: Suppose $\langle \Delta_1, c_\lambda \rangle, \langle \Delta_2, c_\mu \rangle \in f(A, \Delta, c_\kappa)$. Let $\phi \in [A]$. Thus, for all $\psi \in \mathcal{L}^{\mathsf{H}+}_{\Box\!\to}$:

$$\begin{aligned} (\phi \,\Box\!\!\to_\kappa (\lambda \supset \psi)) \in \Delta \quad &\Rightarrow \quad @_\lambda \psi \in \Delta_1 \\ (\phi \,\Box\!\!\to_\kappa (\mu \supset \psi)) \in \Delta \quad &\Rightarrow \quad @_\mu \psi \in \Delta_2. \end{aligned}$$

Suppose $(\phi \,\Box\!\!\to_\kappa \sim\lambda) \in \Delta$. Thus, $(\phi \,\Box\!\!\to_\kappa (\lambda \supset \sim\lambda)) \in \Delta$ by $\text{RK}_{\Box\!\to_\kappa}$, and so, $@_\lambda \sim\lambda \in \Delta_1$, $\lightning$. Hence, $(\phi \,\Box\!\!\to_\kappa \sim\lambda) \notin \Delta$. By CEM, $(\phi \,\Box\!\!\to_\kappa \lambda) \in \Delta$. By $\text{RK}_{\Box\!\to_\kappa}$, $(\phi \,\Box\!\!\to_\kappa (\mu \supset \lambda)) \in \Delta$, and so, $@_\mu \lambda \in \Delta_2$. By Rigid and $\text{Intro}_=$, $(\lambda = \mu) \in \Gamma$ since $|\lambda|_1, |\mu|_1 \in \Gamma$. By Lemma A3.14 then, $c_\lambda = c_\mu$. Further, $(\lambda = \mu) \in \Delta_1 \cap \Delta \cap \Delta_2$ by Rigid.

We now show that $\Delta_1 \subseteq \Delta_2$ (the proof that $\Delta_2 \subseteq \Delta_1$ is symmetric). Suppose $\psi \in \Delta_1$. By $\text{Intro}_@$ and $\text{Elim}_@$, $@_{l_\Delta} \neg\psi \notin \Delta_1$. By Red, $@_\lambda @_{l_\Delta} \neg\psi \notin \Delta_1$. Since $\langle \Delta_1, c_\lambda \rangle \in f_\Gamma(A, \Delta, c_\kappa)$, $(\phi \,\Box\!\!\to_\kappa (\lambda \supset @_{l_\Delta} \neg\psi)) \notin \Delta$. By SubId, since $(\lambda = \mu) \in \Delta$, $(\phi \,\Box\!\!\to_\kappa (\mu \supset @_{l_\Delta} \neg\psi)) \notin \Delta$. By CEM, $(\phi \,\Box\!\!\to_\kappa \sim(\mu \supset @_{l_\Delta} \neg\psi)) \in \Delta$. Since $(\phi \,\Box\!\!\to_\kappa \mu) \in \Delta$, we have $(\phi \,\Box\!\!\to_\kappa (\mu \supset \sim @_{l_\Delta} \neg\psi)) \in \Delta$ by $\text{RK}_{\Box\!\to_\kappa}$. Since $\langle \Delta_2, c_\mu \rangle \in f_\Gamma(A, \Delta, c_\kappa)$, we have $\neg @_{l_\Delta} \neg\psi \in \Delta_2$ by Bool. So by $\text{Dist}_@$, $\text{Intro}_@$, and $\text{Elim}_@$, $\psi \in \Delta_2$.

Vac: Let $A(c_\kappa) = \varnothing$. Suppose for reductio $f_\Gamma(A, \Delta, c_\kappa) \neq \varnothing$. Let $\langle \Delta', c_\lambda \rangle \in f_\Gamma(A, \Delta, c_\kappa)$ and let $\phi \in [A]$. By Corollary A3.10 and $\text{Dist}_@$, $@_\kappa \sim \blacklozenge \phi \in \Delta$. By Vac and $\text{Gen}_@$, $\phi \,\Box\!\!\to_\kappa (\lambda \supset \bot) \in \Delta$. By Definition B3.7, $@_\lambda \bot \in \Delta'$, $\lightning$.

NC, NEC, SIC: We just do NC, since NEC and SIC are similar. It's left as an exercise to the reader to show that the two versions of the relevant axiom are coderivable. Let $\phi \in [A]$ and let $\langle \Delta', c_\lambda \rangle \in f_\Gamma(A, \Delta, c_\kappa)$. So for all ψ, if $(\phi \,\Box\!\!\to_\kappa (\lambda \supset \psi)) \in \Delta$, then $@_\lambda \psi \in \Delta'$. By Rigid and Bool, $@_\kappa \blacksquare \kappa \in \Delta$. By

NC and $RK_{\Box\!\!\rightarrow_\kappa}$, $\phi \Box\!\!\rightarrow_\kappa (\lambda \supset \kappa) \in \Delta$. Hence, $@_\lambda \kappa \in \Delta'$. By Rigid, $Intro_@$, and $Elim_@$, $(\kappa = \lambda) \in \Delta'$. Thus, $c_\lambda = c_\kappa$.

R_0: We revise the definition of a canonical hyperconvention as we did for Theorem B2.2 so that $\pi_{c_\kappa} = \{X \mid [X]_\kappa \neq \varnothing\}$. Given this, let $\langle \Delta', c_\lambda \rangle \in f(A, \Delta, c_\kappa)$ and let $\alpha \in [A]$. Thus, for all χ, if $\alpha \Box\!\!\rightarrow_\kappa \chi \in \Delta$, then $@_\lambda \chi \in \Delta'$. We will just show the ¬-case, i.e., that $\neg_{c_\kappa} = \neg_{c_\lambda}$, since the others are similar.

First, observe that $\pi_{c_\kappa} = \pi_{c_\lambda}$, since, e.g., if $\phi \in [X]_\lambda$, then $@_\lambda \phi \in [X]_\kappa$ by Red. So let $X \in \pi_{c_\lambda}$ and let $\phi \in [X]_\lambda$. By R_0 and $Gen_@$, $@_\kappa \downarrow i. \Box_\alpha \downarrow j.(\neg \phi = @_i \neg @_j \phi) \in \Delta$. By $DA_@$, $@_\kappa \Box_\alpha \downarrow j.(\neg \phi = @_\kappa \neg @_j \phi) \in \Delta$. Hence, $@_\lambda \downarrow j.(\neg \phi = @_\kappa \neg @_j \phi) \in \Delta'$. By $DA_@$, $@_\lambda(\neg \phi = @_\kappa \neg @_\lambda \phi) \in \Delta'$. By $Dist_@$ and Red, $(@_\lambda \neg \phi = @_\kappa \neg @_\lambda \phi) \in \Delta'$. By Lemma B5.1 (since $@_\lambda \phi \in [X]_\kappa$), $\neg_{c_\kappa} X = \neg_{c_\lambda} X$.

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