Contradictions, objects, and belief*

Srećko Kovač, Institute of Philosophy (Zagreb, Croatia)

Abstract

We show how some model-theoretical devices (local reasoning, modes of presentation, an additional accessibility relation) can be combined in first-order modal logic to formalize the consequence relation that includes *de dicto* and *de re* contradictory beliefs. Instead of special "sense objects", appearances of objects in an agent's belief are introduced and presented as ordered pairs consisting of an object and an individual constant. A non-classical identity relation is applied. A relation S on the set of possible worlds is introduced, which models possible distortions in an agent's picture of a (real) world. The application of such models in deontic logic is illustrated by a characteristic example.

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As is well known, a rational agent can have beliefs that contain contradictions, including disturbances of the identity of objects. Contradictions can arise not only in the *de re* sense of a belief (cf. the Hesperus—Phosphorus puzzle), but also, as Kripke has shown, in the *de dicto* sense. The aim of the logic of belief is, among other things, to formalize such "non-classical" states of affairs.

This paper attempts to show how some model-theoretical devices (local reasoning, modes of presentation, an additional accessibility relation) can be combined in first-order modal logic to formalize *de dicto* contradictory beliefs, as well as *de dicto* non-contradictory beliefs that have *de re* contradictory consequences. An agent's *de dicto* and *de re* contradictions, if presented to the agent, are an important motive for the agent's change of belief (we are not dealing here with the

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belief change itself). After some introductory remarks on local reasoning, intension functions, and mode of presentation in the literature of logic, an ontologically reductive version of the logic of belief is presented, without introducing special "sense objects", and keeping worlds (states) classical at least regarding the valuation of predicates (except $=$) and terms. Besides modal accessibility (to define the satisfaction of belief formulas), a special accessibility relation (S) on worlds is introduced by which a strong connection between *de dicto* and *de re* beliefs is established and a non-classical concept of satisfaction of atomic formulas is defined. Subsequently, we add an example of a possible application of the presented semantics in deontic logic.

1 Some related results

1.1 Local reasoning

The aim of the "local reasoning" approach (Fagin and Halpern [1]) was to model contradictory beliefs (even contradictory knowledge), defining the satisfaction of a belief formula $B_i\phi$ in the following way:

$$M, w \models_v B_i\phi \textit{ iff there is } T \in \mathcal{C}_i(w) \textit{ such that for each } t \in T,\ M, t \models_v \phi,$$

where T is a cluster of worlds, and $\mathcal{C}_i$ a function that maps each world to a set of clusters (M is a model, w a world, v a variable assignment). In this way, it is possible for an agent i to believe that ϕ in relation to one cluster of possible accessible worlds, and to believe that $\neg\phi$ in relation to another cluster of possible accessible worlds, i.e. $B_i\phi \wedge B_i\neg\phi$ (both times either in the *de dicto* or in the *de re* sense).

An agent is in fact modeled as a "society of minds" with a pluralism of beliefs (each cluster representing one "mind"). Here, $B_i(\phi \wedge \neg\phi)$ does not follow from $B_i\phi \wedge B_i\neg\phi$, because both occurrences of B_i need not be determined by the same cluster of i-accessible worlds. Thus, contradiction of belief disappears, no "explosion" of belief results, and the self-identity of objects and the rigidity of names can be preserved.

For example, if Peter believes that Paderewski had musical talent, and if he also believes that Paderewski had no musical talent, both times in the *de dicto* sense (as Kripke insists in his famous puzzle[1]), or in the *de re* sense, then those

[1]In the well known puzzle, Peter does not recognize that Paderewski, a Polish politician of the first half of the 20th century, is the very same person as Paderewski, a famous pianist [9].

two contradictory beliefs can be dependent on two different clusters ("frames of mind"), so that in no cluster of the two does the contradictory belief arise that Paderewski had musical talent and had no musical talent, or, consequently, that Paderewski is not Paderewski.

Nevertheless, if asked whether Paderewski, the politician, and Paderewski, the pianist, are the same person, Peter would probably say that they are not identical. Logically, he applies two different names, say, 'Paderewski$_1$' and 'Paderewski$_2$'.[2] Peter's belief that Paderewski$_1$ is not identical to Paderewski$_2$ is *de dicto* non-contradictory, but in the *de re* sense it is a contradiction. We cannot avoid that contradiction by local reasoning, since 'Paderewski$_1$ is not identical to Paderewski$_2$' is a literal (negated atomic formula) which cannot be distributed over different clusters. Moreover, we feel that Peter's *de re* contradictory belief is somehow a consequence of his *de dicto* non-contradictory belief. Hence, the two different appearances of Paderewski (from Peter's viewpoint) are somehow logically connected to Paderewski himself. In general, to model the contradictory (*de re*) side of some beliefs as a consequence of their non-contradictory (*de dicto*) side, some means are needed to trace the appearances of objects in the agent's beliefs to the real objects to which these appearances belong.

A mode of presentation or a similar intension function seems to be appropriate to model the relation of appearances to objects, as well as to model the diversity and changeability of the appearances of objects with respect to possible worlds and agents. In the next two subsections, some recent first-order modal logic approaches are sketched where individual concepts and a mode of presentation function are used.

1.2 Individual concepts as objects

In the **FOIL** quantified modal logic by M. Fitting (cf. [4] and [5], also [3]), intension (concept) is a (partial) function $G \longrightarrow D_0$, where G is a non-empty set of worlds, and D_0 a domain of objects. A special domain of intensions, D_I, is introduced in a model, M, where the model is defined as an ordered quintuple $\langle G, R, D_0, D_I, I\rangle$ (R is an accessibility relation on worlds, and I is an interpretation). The identity of objects (and, possibly, of intensions) is preserved across the worlds (in contrast to the counterparts semantics). Besides, object variables and intension variables are distinguished, as are the object types and intension types of the relation symbols. The set of agents can easily be supplied to accommodate

[2] For indexing names in Kripke's puzzles about belief, see, e.g., [2] and [18, p. 346].

FOIL to the logic of belief. *De re* and *de dicto* readings are disambiguated by the λ-abstraction device, as, for example, in $\langle \lambda x.B_i \langle \lambda y.\neg x = y\rangle(h)\rangle(h)$, where the left occurrence of 'h' is in the *de dicto* position and the right occurrence in the *de re* position. In general, the designation of an intension variable f is relativized with respect to worlds by the λ operator:

$$M, \Gamma \in G \models_v \langle \lambda x.\phi\rangle(f) \text{ iff } M, \Gamma \models_{v[v(f,\Gamma)/x]} \phi \tag{1}$$

(M is a model, Γ a world, and v a mapping of each object variable to an object and of each intension variable to an intension).

Objects (members of D_0) are conceived in a liberal way, so that, for instance, for some Γ, the object that is the value of the intension Phosphorus at Γ is different from the object that is the value of the intension Hesperus at Γ, since there are agents (say, the ancient Babylonians) who believe that Phosphorus and Hesperus are two different objects. In the "real world", both intensions have one and the same object (Venus) as their value. Hence, what we in the real world designate by 'Venus' is identical, as taken in the *de re* sense, to at most one of the two, to Phosphorus or to Hesperus as perceived by the Babylonians. Otherwise, according to (1), a contradiction also in the Babylonians' *de dicto* belief would arise (for instance, the same object would be and would not be a morning star). Accordingly, there is no *de re* contradiction in the ancient Babylonians' beliefs about Phosphorus and Hesperus either.—Similarly, for Kripke's belief agent Peter, Paderewski, the politician, and Paderewski, the pianist, should also be distinguished as two different objects.

Below, we will propose a semantics where a *de re* contradiction is allowed as a consequence of a *de dicto* non-contradictory belief.

Remark 1. *Besides* **FOIL***, Fitting proposed an epistemic logic where the quantification over reasons (evidences) is introduced [6]. That could be an interesting approach to model the situations where, for different reasons t and s, contradictory beliefs are held about one and the same object, for example, $\exists x B_i(t : \phi(x) \wedge s : \neg\phi(x))$.*

Close to **FOIL** models are, for example, "coherence models" (by M. Kracht and O. Kutz [8, 10]), where instead of many intensions, there is a unique surjective intension function ("trace function" τ), which maps each object at a world w to a thing (which is the trace of that object at w):

$$\tau : U \times W \longrightarrow T,$$

(U is a set of objects, W a set of worlds, and T a set of things). We could say that "objects" in a pair with the function τ are, in fact, individual concepts (the authors think of the objects themselves as "modal individuals", "transcendental" objects). So, for example, Paderewski, the politician, and Paderewski, the pianist, would be two objects with the same trace in the real world, but with different traces in each of Peter's accessible worlds. Here, contradictory *de re* consequences are avoided by the reduction of the identity of objects to the world-relative identity of the traces of objects.

1.3 Modes of presentation in the FMP logic

Let us pause also on the **FMP** logic of belief by R. Ye [19], and Ye and M. Fitting [20], where the mode of presentation function m plays the role of an intension function for agents. In **FMP**, for each name a, belief agent i, and world w, $m(a, i, w) \subseteq D(w)$, where D is a domain function on the set G of possible worlds. A model is an ordered set $\langle G, I, R_1, \ldots, R_{|I|}, D, \sigma, m, \pi\rangle$, where I is the set of belief agents, R_i the accessibility relation for an agent i, and σ and π functions that assign values to names and predicates, respectively.

The *de re* and *de dicto* sense of names are disambiguated by the abstraction notation (similar to Fitting's λ-abstraction), which indicates that a name (an individual constant) a designates each object referred to by a mode of presentation m of a for an agent i at a world w (in a model M). The following is the satisfaction condition of a *de dicto* belief:

$$\begin{aligned} &M, w \models_v B_i \, \langle x.\psi\rangle(a) \\ &\qquad \textit{iff for each } o \in m(a, i, w),\ M, w \models_{v[o/x]} B_i \, \psi. \end{aligned} \tag{2}$$

If we take it that $m(a, i, w) = \varnothing$, we can model *de dicto* contradictory beliefs of the form $B_i\langle x.\phi \wedge \neg\phi\rangle(a)$ because of the vacuous satisfaction of the right side of (2). Outside the abstraction notation, names are "weakly" rigid (the value of a name at w remains the same at each world accessible to w).

This approach is especially appropriate for the case where an agent believes of several objects to be one and the same object (e.g., if an agent believes that there is only one author of *Principia Mathematica*). In the case when an agent splits one object, in his/her belief, into several objects, disjoint sets of objects (probably containing, new, "sense objects") are to be introduced. For example, if an agent i believes at w that Hesperus and Phosphorus are two distinct objects, formally $B_i\langle x_1x_2.x_1 \neq x_2\rangle(h, p)$, that should mean, according to condition (2) above, that under the mode of presentation of i at w 'Hesperus' and 'Phosphorus' refer to two disjoint sets of objects (possibly just to two distinct singletons).[3] Venus could be (although it need not be) one of these objects, but cannot be a member of both sets, in which case it would and it would not be denoted by the same predicate (e.g., 'to be a morning star') at the same worlds, contrary to the definitions of the valuation and satisfaction in [19, 20]. Hence, *de re* contradictions do not result in the **FMP** logic from consistent *de dicto* beliefs (similarly as in the logics considered above), since such contradictions would follow only if one and the same object would be included under two non-equivalent modes of presentation (cf. [19, pp. 60–62]).

Remark 2. *The mode of presentation concept originates, in modern philosophy, in Frege's* On Sense and Reference *(1892) [7]. It is used in the contemporary philosophy of language, for example, on the basis of a Kripkean semantics as a "mode of acquaintance with propositions", a "proposition guise" [13, pp. 117] [14, pp. 255–256], or as an "extra descriptive information evident to the conversational participants" [17, p. 214] (see a discussion, for example, in [15] or [16]). Finally, the mode of presentation concept is introduced in modal logic (cf. [20, pp. 389, 406]). See also E. Zalta [21], where the author takes modes of presentation to be abstract objects of his previously developed intensional logic.*

2 The QBL logic of belief

One idea of the **QBL** logic now to be proposed is to allow an agent's *de re* contradictions as consequences of the agent's *de dicto* beliefs. The intuition is that an agent's *de re* contradictions should be sufficient reason for rejecting or revising the agent's corresponding *de dicto* beliefs. Pure *de dicto* contradictions (with all terms taken *de dicto*) will also be possible, but only in the sense of local reasoning (i.e. relativized by agent's different frames of mind). Both, *de re* and *de dicto* contradictions are a ground for a dynamics that is a topic of a possible dynamic logic of belief.

Related to the strong connection of *de dicto* beliefs with their *de re* counterparts is another idea, namely to propose a reductive ontology that does not presuppose a distinct object (a distinct set of objects) for each sense of a term. In **QBL** we need not presuppose such different things (or objects) like "Phosphorus", Hesperus", etc. Instead, we merely have appearances (aspects) of real objects, and represent appearances (in a simplified way) by the association of the

[3] See also [19, p. 57 Remark].

objects with their logical names (imagine these appearances, for example, like shadows in Plato's cavern from the seventh book of the *Republic*).

Technically, *de re* contradictions of beliefs will be modeled in **QBL** by a special accessibility relation (S), by which the satisfaction of atomic formulas will be defined, and which will make contradictions possibly true at a world without making the valuation of predicates (except $=$) at the world contradictory. The identity will be non-classical in order to account for the relationship between appearances and objects.

Remark 3. *For the concept of appearance, see in [13, p. 106] how, for Salmon, "a change in appearance" (either objective or subjective appearance) is responsible for the subject's failure to recognize an object. Let us also note that, for Salmon, "the mode of acquaintance by which one is familiar with a particular object", i.e. the appearance of an object, "is part of the mode of acquaintance by which one grasps a singular proposition involving that object" [13, p. 108].*

2.1 Syntax

For **QBL** we build a language $\mathscr{L}_{\mathbf{QBL}}$, with, in general, familiar first-order modal syntax including λ-abstraction formulas.

So, the vocabulary of $\mathscr{L}_{\mathbf{QBL}}$ consists of the set $\mathscr{C}$ of individual constants (a, b, c, $a_1, \ldots$), set $\mathscr{V}$ of individual variables ($x, y, z, x_1, \ldots$), set $\mathscr{P}$ of n-place predicates (P^n, Q^n, R^n, $P_1^n, \ldots$, and '$=$'), connectives '$\neg$' and '$\rightarrow$' (other connectives being defined), the quantifier symbol '$\forall$' ('$\exists$' is defined), 'λ' (abstraction operator), belief operators ($B_1, \ldots, B_n$), and parentheses.

The formulas are of the form $\Phi^n t_1 \ldots t_n$ (where Φ^n is a predicate, and t_i a term), $\neg\phi$, $(\phi \rightarrow \psi)$, $B_i\,\phi$, $\forall\alpha\,\phi$, and λ-abstraction formulas of the form $(\lambda\alpha.\phi)(\kappa)$ (where ϕ and ψ are formulas, κ an individual constant, and α a variable).

2.2 Semantics

In a *frame* we distinguish @ as a "real" world, which behaves in a classical way. Other worlds could behave, to some extent, in a non-classical way and serve to model agents' beliefs. There is a cluster function $\mathcal{C}_i$, which maps each world to a set of clusters of worlds. Further, there is an S-function, which has the role to model a possibly "broken" picture of a world, in the sense that the world can split, in an agent's view, into a set of mutually different worlds. For example, if an agent does not always recognize an object d to be one and the same object, and both ascribes and denies of it a property Φ, we model this situation by two S-accessible worlds, in one of which d has the property Φ, and in the other of which d does not have Φ. The possibility of an agent's correct picture of the world is retained. There is a domain D of "real objects", which exist in $@$, but which do not necessarily exist in each world accessible to some agent i (since i has not to be aware of the existence of each real object).

Definition 1 (Frame). *A frame is an ordered set $\mathcal{F} = \{W, @, \mathcal{C}_1, \ldots, \mathcal{C}_n, S, D, \mathcal{Q}\}$ where*

1. *W is a non-empty set of worlds (w will be a member of W),*
2. *$@ \in W$,*
3. *$\mathcal{C}_i(w) \in \wp\wp W$ (i is a belief agent),*
4. *$S \subseteq W \times W$, $@$ is S-related only to itself, S is reflexive and transitive,*
5. *D is a non-empty set of objects (d will be a member of D),*
6. *$\mathcal{Q} : W \longrightarrow \wp D - \{\varnothing\}$, $\mathcal{Q}(@) = D$ (we abbreviate '$\mathcal{Q}(w)$' as '$\mathcal{Q}_w$').*

For a cluster function $\mathcal{C}_i$, we introduce some further conditions, corresponding to a plausible concept of belief:

1. *each $T \in \mathcal{C}_i(w)$ is non-empty,*
2. *if $T \in \mathcal{C}_i(w)$, then for some $T' \in \mathcal{C}_i(w)$ and for each $w' \in T'$, $T \in \mathcal{C}_i(w')$,*
3. *for each w there is $T \in \mathcal{C}_i(w)$ such that for each $w' \in T$, $\mathcal{C}_i(w') \subseteq \mathcal{C}_i(w)$.*

The first of the conditions above models seriality. It can be easily shown that the second one models positive introspection, and the third one negative introspection.

In the definition of a *model* below, we have a twofold valuation of individual constants. The first one is rigid (condition 1), and the second one is non-rigid and implicitly includes "modes of presentation" (condition 2). Corresponding to the mentioned distinction between (real) objects and their appearances, we have, besides D (real objects), a set A of pairs $\langle object, name\rangle$, i.e. of appearances, which result from the non-rigid valuation of constants. A pair $\langle object, name\rangle$ should present an object as it appears (to an agent) in association with a logical name, so that the pair may be called a mode of presentation of the object. The (apparently "baroque") valuation of predicates (condition 3) has some restrictions in order to ensure the *de dicto* consistency of an agent's beliefs (condition 3a) and the correspondence of each *de dicto* belief to some *de re* belief (condition 3b). The valuation of predicates has also to ensure the classical behavior of $@$, as well as to account for the identity relation. Identity is interpreted in some respects like any other predicate and thus behaves in a non-classical way, except that the self-identity of appearances is ensured in each world, and the self-identity of (real) objects is ensured, for each world, in some S-accessible world (condition 3d). The idea is that the self-identity always holds in *de dicto* beliefs, but not necessarily always in *de re* beliefs. For example, the belief that Hesperus is an evening star, and that Phosphorus is not an evening star, is consistent if taken in the *de dicto* sense, but not if taken in the *de re* sense, since in the *de re* sense both 'Hesperus' and 'Phosphorus' refer to one and the same object, Venus. Finally, since $@$ is a real world, modes of presentation at $@$ entirely correspond to the rigid valuation of logical names (see condition 3c below).

Definition 2 (Model)**.** *A model is an ordered set* $\mathfrak{M} = \langle \mathcal{F}, V \rangle$ *where*

1. $V(\kappa) \in D$,
2. $V(\kappa, w) \in \wp D - \{\varnothing\}$, *in particular,* $V(\kappa, @) = \{V(\kappa)\}$;

   we use the following abbreviations:
   $A = \{\langle d, \kappa\rangle \mid d \in V(\kappa, w)$ *for some* $w\}$ *and*
   $U = D \cup A$ *(*u *will be a member of* U*),*
3. $V(\Phi^n, w) \in \wp U^n$, *where*
   - (a) *for each* w', w'' *with* wSw' *and* wSw'', $\langle\langle d_1, \kappa_1\rangle, \ldots, \langle d_n, \kappa_n\rangle\rangle \in V(\Phi^n, w')$ *iff* $\langle\langle d_1, \kappa_1\rangle, \ldots, \langle d_n, \kappa_n\rangle\rangle \in V(\Phi^n, w'')$,
   - (b) $\langle u_1, \ldots, u_n\rangle \in V(\Phi^n, w)$ *iff for each* n*-tuple* $e \in \{u_1, d_1\} \times \ldots \times \{u_n, d_n\}$ *there is* w' *with* wSw' *such that* $e \in V(\Phi, w')$, *where* $d_i \in u_i$ *if* $u_i \in A$, *otherwise* $u_i = d_i$,
   - (c) *there are following restrictions regarding* $@$*:*
     - i. $\langle u_1, \ldots, d, \ldots, u_n\rangle \in V(\Phi^n, @)$ *iff* $\langle u_1, \ldots, \langle d, \kappa\rangle, \ldots, u_n\rangle \in V(\Phi^n, @)$, *for each* d *and* κ *such that* $d \in V(\kappa, @)$,
     - ii. *for each* d, $\langle d, d\rangle \in V(=, @)$,
   - (d) *there are following general restrictions for the identity predicate:*

i. for each w *and* $\langle d, \kappa \rangle \in A$, $\langle \langle d, \kappa \rangle, \langle d, \kappa \rangle \rangle \in V(=, w)$,

ii. for each d *and* w, *there is* w' *with* wSw' *such that* $\langle d, d \rangle \in V(=, w')$,

(e) if $\langle u, u' \rangle \in V(=, w)$, *then* $\langle u_1, \ldots, u, \ldots, u_n \rangle \in V(\Phi^n, w)$ *iff* $\langle u_1, \ldots, u', \ldots, u_n \rangle \in V(\Phi^n, w)$.

Let us pause, first, on the non-rigid valuation of individual constants (condition 2). The non-rigid valuation has a non-empty set of objects as value, in order to account for a possible fusion of objects in an agent's perception. That valuation could be regarded as a simplified m of [19, 20] in that in the non-rigid valuation of **QBL** there is no argument for agents, and agents differ one from another only with respect to their accessible worlds. Hence another difference, namely, that non-rigid valuation in **QBL** is relativized to the agent's accessible worlds (not to a world at which the agent has a belief). Besides, for reasons already mentioned, **FMP** allows empty set as a value of a mode of presentation.—Let us remark that individual constants (names in a logical sense) need not always be conceived as names of ordinary language. For example, 'this' or 'that', too, could serve as logical names.[4] Hence, logical modes of presentation are not confined to the names of ordinary language.

Condition 3a says that all worlds that are S-accessible to the same world w agree on the properties and relations of appearances (but not necessarily also on the properties and relations of real objects). This feature will be used in the definition of satisfaction below (Definition 6) to model-theoretically ensure the consistency of *de dicto* beliefs in one cluster.

Condition 3b essentially says, informally, that at the ground of each property of an appearance there is the same property of the corresponding real object. The corresponding real object behind the appearance $\langle d, \kappa \rangle$ is d. More technically, 3b says that for each ordered n-tuple of entities (objects or appearances) with some property Φ at w, each corresponding n-tuple that could be obtained by replacing, in the original n-tuple, some or all (or none) appearances with the corresponding objects, has the property Φ at some S-accessible world w'. This feature of models, together with the transitivity of S-accessibility (see condition 4 of Definition 1), will serve in Definition 6 to ensure that for each *de dicto* belief an agent will also have all the corresponding *de re* beliefs. Note that at @ appearances and corre-

[4] As is known, Russell even states that 'this' and 'that' are 'the only words one does use as names in the logical sense' [12, p. 201].

sponding objects are equivalent regarding the extension of predicates (condition 3c).

Note also that for each w and d there is w' with wSw' such that $\langle d, d\rangle \in V(=, w')$ (see condition 3d), which will serve in Definition 6 to ensure the satisfaction of self-identity of objects in each world. The identity between entities (members of U) means only that they share the same properties (condition 3e), not that they are one classically (logically) identical entity. Hence, for example, $\langle d, \kappa\rangle$ and $\langle d', \kappa'\rangle$ could share all their properties, although they are two classically different entities.

In what follows, the definitions of a variable assignment and of a variant of a variable assignment are partially dependent on modes of presentation, since $A \subseteq U$.

Definition 3 (Variable assignment)**.** *A variable assignment is a mapping* $\mathfrak{v} : \mathscr{V} \longrightarrow U$.

Definition 4 (Variant of a variable assignment)**.** *A variant of a variable assignment* $\mathfrak{v}$ *is a variable assignment* $\mathfrak{v}[u/\alpha]$ *that differs from* $\mathfrak{v}$ *at most in assigning* u *to* α.

Definition 5 (Designation)**.** *A designation* $[\![\kappa]\!]^{\mathfrak{M},w}_{\mathfrak{v}}$ *of an individual constant and a designation* $[\![\alpha]\!]^{\mathfrak{M},w}_{\mathfrak{v}}$ *of an individual variable are defined in the following way:*

1. $[\![\kappa]\!]^{\mathfrak{M},w}_{\mathfrak{v}} = V(\kappa)$,
2. $[\![\alpha]\!]^{\mathfrak{M},w}_{\mathfrak{v}} = \mathfrak{v}(\alpha)$.

In Definition 6 below, we distinguish $\mathbf{t}$-satisfaction ("verification") and $\mathbf{f}$-satisfaction ("falsification"). The satisfaction of an atomic formula at w depends on the valuation of the predicate of the atomic formula at an S-accessible world (see case 1). Because of that dependency, an atomic formula can be both $\mathbf{t}$-satisfied and $\mathbf{f}$-satisfied at the same w, except at @ (i.e. both an atomic formula and its negation can be $\mathbf{t}$-satisfied). Consequently, in general, formula ϕ can also be both $\mathbf{t}$-satisfied and $\mathbf{f}$-satisfied at the same w, except at @. So @ is a possible (and real) world, while the other worlds could be impossible worlds—not as they are in themselves, but due to their different S-accessible worlds. We note that the world where $\langle d, d\rangle \notin V(=, w)$ is not in a strong sense impossible, since '$=$' is not, in fact, a logical predicate.

Case 6 of Definition 6 shows that, in general, the satisfaction of a λ-formula depends on the mode of presentation of an object d in association with the individual constant κ.

First, we introduce two new abbreviations:

$$A_w = \{\langle d, \kappa\rangle \mid d \in V(\kappa, w)\},$$
$$U_w = D_w \cup A_w.$$

Definition 6 ($\mathbf{t}$-satisfaction, $\mathbf{f}$-satisfaction)**.**

1. (a) $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \Phi t_1 \ldots t_n$ *iff for some* w' *with* wSw', $\langle [\![t_1]\!]^{\mathfrak{M},w}_{\mathfrak{v}}, \ldots, [\![t_n]\!]^{\mathfrak{M},w}_{\mathfrak{v}}\rangle \in V(\Phi, w')$,

   (b) $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} \Phi t_1 \ldots t_n$ *iff for some* w' *with* wSw', $\langle [\![t_1]\!]^{\mathfrak{M},w}_{\mathfrak{v}}, \ldots, [\![t_n]\!]^{\mathfrak{M},w}_{\mathfrak{v}}\rangle \notin V(\Phi, w')$,

2. (a) $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \neg\phi$ *iff* $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} \phi$,

   (b) $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} \neg\phi$ *iff* $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \phi$,

3. (a) $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} (\phi \to \psi)$ *iff* $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} \phi$ *or* $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \psi$,

   (b) $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} (\phi \to \psi)$ *iff* $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \phi$ *and* $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} \psi$,

4. (a) $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} B_i\,\phi$ *iff there is* $T \in \mathcal{C}_i(w)$, *such that for each* $w' \in T$, $\mathfrak{M}, w' \models^{\mathbf{t}}_{\mathfrak{v}} \phi$,

   (b) $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} B_i\,\phi$ *iff for each* $T \in \mathcal{C}_i(w)$ *there is* $w' \in T$ *such that* $\mathfrak{M}, w' \models^{\mathbf{f}}_{\mathfrak{v}} \phi$,

5. (a) $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \forall\alpha\,\phi$ *iff for each* $u \in U_w$, $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}[u/\alpha]} \phi$,

   (b) $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} \forall\alpha\,\phi$ *iff for some* $u \in U_w$, $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}[u/\alpha]} \phi$,

6. (a) $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} (\lambda\alpha.\phi)(\kappa)$ *iff for each* $d \in V(\kappa, w)$, $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}[\langle d,\kappa\rangle/\alpha]} \phi$,

   (b) $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}} (\lambda\alpha.\phi)(\kappa)$ *iff for some* $d \in V(\kappa, w)$, $\mathfrak{M}, w \models^{\mathbf{f}}_{\mathfrak{v}[\langle d,\kappa\rangle/\alpha]} \phi$,

As already mentioned, and according to condition 3b of Definition 2, for each $\mathbf{t}$-satisfied *de dicto* atomic formula, corresponding *de re* formulas are also $\mathbf{t}$-satisfied. Besides, note that $t = t$ is always $\mathbf{t}$-satisfied (cf. case 3d of Definition 2), but possibly also $\neg t = t$, except at @ (because @ has only itself as an S-accessible world).

Let us now define three concepts regarding the $\mathbf{t}$-satisfaction of formulas through worlds and models.

Definition 7 (Satisfiability)**.** *A set Γ of formulas is satisfiable iff there is a model $\mathfrak{M}$, a world w, and a variable assignment $\mathfrak{v}$ such that for each $\phi \in \Gamma$, $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \phi$.*

Definition 8 (Consequence)**.** *$\Gamma \models \phi$ iff $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \phi$ whenever for each $\psi \in \Gamma$, $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} \psi$.*

Definition 9 (Validity)**.** *A formula ϕ is valid iff it is satisfied at each world in each model, for each variable assignment.*

It can be shown that in the proposed $\mathbf{QBL}$ logic, formulas of the form ***K*** are not valid (due to the locality of belief). ***4*** and ***5*** are valid (due to the corresponding properties of $\mathcal{C}_i$). ***D*** (i.e. $B_i \neg \bot$) is valid if $\bot$ does not contain a mode independent (rigid) individual constant. Not only $B_i\phi \wedge B_i\neg\phi$, but also $B_i(\phi \wedge \neg\phi)$ and $B_i(\lambda\alpha.\phi \wedge \neg\phi)(\kappa)$ are satisfiable in $\mathbf{QBL}$ if ϕ contains a mode independent individual constant. In addition, the locality of belief enables the satisfiability of a set of formulas like $\{B_i\phi \wedge B_i\psi, \neg B_i(\phi \wedge \psi)\}$.

Further, for instance, the formulas of the form $B_i(\lambda x.\exists y\, y = x)(\kappa)$ are valid, i.e. each agent i believes that what is an appearance with respect to i is an existing thing (cf. analogously for **FMP** in [19, p. 62]). Namely, according to Definition 6, $\mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} B_i(\lambda x.\exists y\, y = x)(\kappa)$ iff $(\exists T \in \mathcal{C}_i(w))\, (\forall w' \in T)\, \mathfrak{M}, w' \models^{\mathbf{t}}_{\mathfrak{v}} (\lambda x.\exists y\, y = x)(\kappa)$. And further,

$$
\begin{aligned}
&\mathfrak{M}, w' \models^{\mathbf{t}}_{\mathfrak{v}} (\lambda x.\exists y\, y = x)(\kappa)\\
&\quad \textit{iff } (\forall d \in V(\kappa, w'))\, \mathfrak{M}, w' \models^{\mathbf{t}}_{\mathfrak{v}[\langle d,\kappa\rangle/x]} \exists y\, y = x\\
&\quad \textit{iff } (\forall d \in V(\kappa, w'))\, (\exists u \in U_{w'})\, \mathfrak{M}, w' \models^{\mathbf{t}}_{\mathfrak{v}[\langle d,\kappa\rangle/x, u/y]} y = x\\
&\quad \textit{iff } (\forall d \in V(\kappa, w'))(\exists u \in U_{w'})(\exists w'' w' S w'')\, \langle\langle d, \kappa\rangle, u\rangle \in V(=, w'').
\end{aligned}
$$

Because of the reflexivity of S the last line always holds, since if $d \in V(\kappa, w')$ then $\langle d, \kappa\rangle \in U_{w'}$. However, it can be shown that $B_i\, \exists x\, B_j(\lambda y.y = x)(\kappa)$ is not valid, i.e. an agent i does not need to believe that what is an appearance with respect to some (other) agent j is an existing thing for i.

2.3 Some examples

Let us define a model $\mathfrak{M}$ in the following way (we informally use individual constants h, p and v, and the predicate H^2):

1. $W = \{@, w_1, w_2, w_3, w_4, w_5\}$,

2. $T_1 = \{@\}, T_2 = \{w_1, w_2\}, T_3 = \{w_3, w_4, w_5\}$,
   $\mathcal{C}_i(@) = \mathcal{C}_i(w_1) = \mathcal{C}_i(w_2) = \{T_1, T_2, T_3\}$,
   $\mathcal{C}_j(@) = \mathcal{C}_j(w_1) = \mathcal{C}_j(w_2) = \mathcal{C}_j(w_3) = \mathcal{C}_j(w_4) = \mathcal{C}_j(w_5) = \{T_1, T_2, T_3\}$,

3. $S = \{\langle @, @\rangle, \langle w_1, @\rangle \langle w_1, w_1\rangle, \langle w_1, w_2\rangle, \langle w_2, @\rangle, \langle w_2, w_2\rangle, \langle w_3, w_3\rangle, \langle w_3, w_4\rangle, \langle w_3, w_5\rangle, \langle w_4, w_4\rangle, \langle w_4, w_5\rangle, \langle w_5, w_4\rangle, \langle w_5, w_5\rangle\}$,

4. *for each w, D_w includes the planet Venus,*

5. $V(v) = V(p) = V(h) = Venus\ (Phosphorus, Hesperus)$,

6. *for each w, $V(h, w) = V(p, w) = \{Venus\}$ (hence, set A_w includes $\langle Venus, h\rangle$ and $\langle Venus, p\rangle$),*

7. *for each w, $V(H^2, w) = \{\langle u, u'\rangle \mid u$ is hotter (on the surface) than $u'\}$:*
   $\langle\langle Venus, h\rangle, Venus\rangle \notin V(H^2, @)$,
   $\langle\langle Venus, h\rangle, \langle Venus, p\rangle\rangle, \langle\langle Venus, h\rangle, Venus\rangle, \langle Venus, Venus\rangle \in V(H^2, w_2)$,
   $\langle\langle Venus, p\rangle, \langle Venus, h\rangle\rangle \notin V(H^2, w_2)$
   $\langle\langle Venus, p\rangle, \langle Venus, h\rangle\rangle \in V(H^2, w_3)$,
   $\langle\langle Venus, p\rangle, \langle Venus, h\rangle\rangle \in V(H^2, w_4)$,
   $\langle\langle Venus, p\rangle, \langle Venus, h\rangle\rangle \in V(H^2, w_5)$,

8. $\langle Venus, Venus\rangle \in V(=, @)$,
   $\langle\langle Venus, h\rangle, Venus\rangle \in V(=, @)$,
   $\langle Venus, Venus\rangle \notin V(=, w_2)$,
   $\langle\langle Venus, h\rangle, Venus\rangle \notin V(=, w_2)$.

We can illustrate the model with Figure 1, where S-accessibility is indicated by dashed lines, the values of predicates at a world are indicated by (pseudo)literals, and individual constants that serve for a mode of presentation are put in brackets:

Example 1. $\mathfrak{M}, @ \models^{\mathbf{t}}_{\mathfrak{v}} B_i Hvv$.
Proof.
For each $w \in T_2$, there is w' with wSw' such that $\langle Venus, Venus\rangle \in V(H^2, w')$, since $\langle Venus, Venus\rangle \in V(H^2, w_2)$ and $w_1 S w_2, w_2 S w_2$,
hence, for each $w \in T_2, \mathfrak{M}, w \models^{\mathbf{t}}_{\mathfrak{v}} Hvv$,
therefore, $\mathfrak{M}, @ \models^{\mathbf{t}}_{\mathfrak{v}} B_i Hvv$, since $T_2 \in \mathcal{C}_i(@)$.

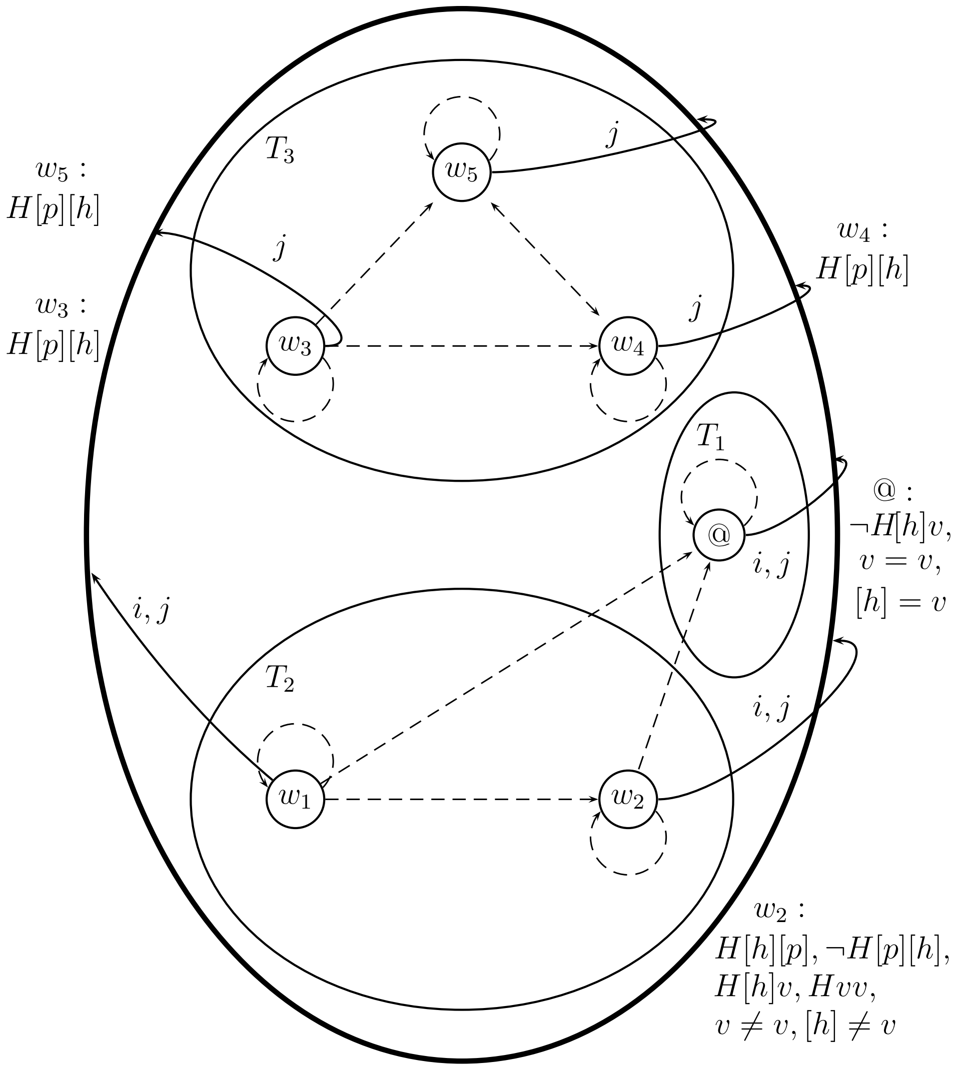

Figure 1: Model $\mathfrak{M}$

Example 2. $\mathfrak{M}, @ \models^{\mathbf{t}}_{\mathfrak{v}} B_i(\lambda x.Hxv \wedge \neg Hxv)(h)$
Proof.
$\mathfrak{M}, w_1 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,h\rangle/x]} Hxv$, *since* $\langle\langle Venus, h\rangle, Venus\rangle \in V(H^2, w_2)$ *and* $w_1 S w_2$,
also, $\mathfrak{M}, w_1 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,h\rangle/x]} \neg Hxv$, *since* $\langle\langle Venus, h\rangle, Venus\rangle \notin V(H^2, @)$ *and* $w_1 S @$,
therefore, $\mathfrak{M}, w_1 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,h\rangle/x]} Hxv \wedge \neg Hxv$,
thus, $\mathfrak{M}, w_1 \models^{\mathbf{t}}_{\mathfrak{v}} (\lambda x.Hxv \wedge \neg Hxv)(h)$, *since* $\{Venus\} = V(h, w_1)$.
Also, $\mathfrak{M}, w_2 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,h\rangle/x]} Hxv$, *since* $w_2 S w_2$,
and $\mathfrak{M}, w_2 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,h\rangle/x]} \neg Hxv$, *since* $w_2 S @$,
therefore, $\mathfrak{M}, w_2 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,h\rangle/x]} Hxv \wedge \neg Hxv$,
thus, $\mathfrak{M}, w_2 \models^{\mathbf{t}}_{\mathfrak{v}} (\lambda x.Hxv \wedge \neg Hxv)(h)$, *since* $\{Venus\} = V(h, w_2)$.
Therefore, $\mathfrak{M}, @ \models^{\mathbf{t}}_{\mathfrak{v}} B_i(\lambda x.Hxv \wedge \neg Hxv)(h)$, *since* $\{w_1, w_2\} = T_2$ *and* $T_2 \in \mathcal{C}_i(@)$.

Example 3. $\mathfrak{M}, @ \models_{\mathfrak{v}} B_i(\lambda x.B_j(\lambda y.Hxy)(h))(p)$
Proof.
$\mathfrak{M}, w_3 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,p\rangle/x,\langle Venus,h\rangle/y]} Hxy$, *since* $\langle\langle Venus, p\rangle, \langle Venus, h\rangle\rangle \in V(H^2, w_3)$ *and* $w_3 S w_3$
thus $\mathfrak{M}, w_3 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,p\rangle/x]} (\lambda y.Hxy)(h)$, *since* $\{Venus\} = V(h, w_3)$,
similarly, $\mathfrak{M}, w_4 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,p\rangle/x]} (\lambda y.Hxy)(h)$ *and* $\mathfrak{M}, w_5 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,p\rangle/x]} (\lambda y.Hxy)(h)$,
therefore, $\mathfrak{M}, w_1 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,p\rangle/x]} B_j(\lambda y.Hxy)(h)$, *since* $\{w_3, w_4, w_5\} = T_3$ *and* $T_3 \in \mathcal{C}_j(w_1)$.
Similarly, $\mathfrak{M}, w_2 \models^{\mathbf{t}}_{\mathfrak{v}[\langle Venus,p\rangle/x]} B_j(\lambda y.Hxy)(h)$, *since* $T_3 \in \mathcal{C}_j(w_2)$.
Further, $\mathfrak{M}, w_1 \models^{\mathbf{t}}_{\mathfrak{v}} (\lambda x.B_j(\lambda y.Hxy)(h))(p)$, *since* $\{Venus\} = V(p, w_1)$,
also, $\mathfrak{M}, w_2 \models^{\mathbf{t}}_{\mathfrak{v}} (\lambda x.B_j(\lambda y.Hxy)(h))(p)$, *since* $\{Venus\} = V(p, w_2)$.
Therefore, $\mathfrak{M}, @ \models^{\mathbf{t}}_{\mathfrak{v}} B_i(\lambda x.B_j(\lambda y.Hxy)(h))(p)$, *since* $\{w_1, w_2\} = T_2$ *and* $T_2 \in \mathcal{C}_i(@)$.

Example 4. $\mathfrak{M}, @ \models^{\mathbf{t}}_{\mathfrak{v}} B_i(v = v \wedge \neg v = v)$.
A sketch of the proof. Note that $\langle Venus, Venus\rangle \in V(=, @)$, *but* $\langle Venus, Venus\rangle \notin V(=, w_2)$. *Since both* $w_1 S @, w_1 S w_2$ *and* $w_2 S @, w_2 S w_2$, *we obtain that* $\mathfrak{M}, w_1 \models^{\mathbf{t}} v = v, \neg v = v$ *and* $\mathfrak{M}, w_2 \models^{\mathbf{t}} v = v, \neg v = v$. *Hence the proposition easily follows.*

Example 5. $\mathfrak{M}, @ \models^{\mathbf{t}}_{\mathfrak{v}} B_i(\lambda x.x = v \wedge \neg x = v)(h)$.
For the proof, note that $\langle\langle Venus, h\rangle, Venus\rangle \in V(=, @)$ *and* $\langle\langle Venus, h\rangle, Venus\rangle \notin V(=, w_2)$*, and that both* $w_1 S @, w_1 S w_2$ *and* $w_2 S @, w_2 S w_2$*. Also,* $\{Venus\} = V(h, w_1) = V(h, w_2)$*. With the help of this, the proposition follows.*

Let us note that examples with an agent who mistakes many objects for one and the same object can be modeled similarly as in [20], using the fact that the non-rigid valuation of individual constants has a set of objects (not a single object) as a value.

2.4 A note on a deductive system

A natural deduction system QBL can be proposed for the above semantically described **QBL**. The usual first-order modal logic rules are used with some exceptions. We need a restriction on the indirect subproof within a B_i-subproof, i.e. in a B_i subproof, the introduction and elimination rules for $\neg$ are valid only by means of *de dicto* formulas ϕ and $\neg\phi$. Also, for each formula $B_i\,\phi$, a new B_i-subproof has to be opened, where $B_i\,\phi$ should be reiterated in an appropriate way (local ***4*** reiteration, local ***5*** reiteration). In the introduction and elimination rules for $\forall\alpha\,\phi$, $Et \to \phi$ is used as the substitution instance, where Et abbreviates $\exists\alpha\,\alpha = t$. In the introduction and elimination rules for λ-abstraction we can use individual constants with one or more asterisks as instantiating mode dependent (non-rigid) terms, e.g., $(\lambda\alpha.\phi\alpha)(\kappa) \vdash \phi(\kappa^*/\alpha)$. We can then, within a B_i subproof, replace a mode dependent term with a mode independent (rigid) constant, but not vice versa: if $\Gamma^{B_i} \vdash \kappa_i^* = \kappa_j,\ \phi(\kappa_i^*)$, then $\Gamma^{B_i} \vdash \phi(\kappa_j//\kappa_i^*)$.

Soundness could be proved by mathematical induction on the number of lines of a proof, where the modal degree of a line should be taken into account. For a possible completeness proof, a Gallin style of proof could be proposed, with the construction of a system of saturated sets of sentences, and with a canonical model, where, for example, the cluster function $\mathcal{C}_i$ is defined as follows: $T \in \mathcal{C}_i(w)$ iff there is a non-empty set X such that $X \subseteq \sqrt[B_i]{w}$ and $(\forall v \in T)\ X \subseteq v$ (w, like v, is here a saturated set of sentences, and $\sqrt[B_i]{w}$ is set $\{\phi \mid B_i\phi \in w\}$).

3 An example in deontic logic

Local reasoning has been employed by L. Royakkers [11] to formalize the enactment of conflicting norms. In deontic language, the modal operator NA_i and the following kind of formulas are included:

$NA_i : \theta$ ('an authority A_i enacted a norm θ'),

where θ is a deontic formula (a norm), and there are no nested enactments.

In a way similar to **QBL**, S-accessibility and non-rigid valuation of constants (modes of presentation) can make it possible to model contradictory *de re* obligations being consequences of non-contradictory *de dicto* obligations. This is briefly illustrated in the following example, where we combine deontic logic with the logic of belief:

Example 6. *An authority i could simultaneously enact an obligation to arrest the person b, and to release the person c, without being aware that b and c are one and the same person. Thus, i in fact believes of one and the same person (taken in the* de re *sense) that he/she is b as well as c. The following enactment and beliefs are included in the situation:*

$NA_i : O(\lambda x.(\lambda y.Ax \wedge Ry)(c))(b),$
$B_{A_i}(\lambda x.x = c)(b),$
$B_{A_i}(\lambda x.x = c)(c),$

where 'A' and 'R' mean 'to be arrested' and 'to be released', respectively. Those enactment and beliefs should be expressed as being in the same frame of mind of the authority A_i, which can be accomplished by the following formula:

$$A_i.(NA_i : O(\lambda x.(\lambda y.Ax \wedge Ry)(c))(b) \wedge B_{A_i}((\lambda x.x = c)(b) \wedge (\lambda x.x = c)(c))), \quad (3)$$

where A_i simultaneously "bounds" the belief and the enactment operator. Now, from (3)

$$NA_i : O(\lambda y.Ac \wedge Ry)(c)$$

and

$$NA_i : O(Ac \wedge Rc)$$

logically follow as consequences. Note that, according to our semantics, the de dicto *identity of b and c, $B_{A_i}(\lambda x.(\lambda y.x = y)(c))(b)$, is not a consequence of the beliefs in (3). Thus, the following enactment:*

$$NA_i : O(\lambda x.(\lambda y.Ax \wedge Ry)(c))(c)$$

is not a consequence of (3) either.

4 Conclusion

Contradictory beliefs appear to be deeply rooted features of belief agents and are a strong motive for an agent's change of belief. The paper aims to show how an agent's contradictory beliefs can be modeled in a first-order modal setting, on the presuppositions of a reductive ontology without separate "sense objects".

Technically, we aim to show how local reasoning and modes of presentation can be combined and employed in modeling contradictory beliefs. Local reasoning distributes two contradictory beliefs over two different clusters of accessible worlds. Modes of presentation (non-rigid valuation of constants) and an additional S-accessibility relation help to model contradictions which occur in the scope of one and the same belief operator and which thus cannot be distributed over clusters.

The dynamics of belief is an interesting open problem for a future research. In dealing with that problem, it should be shown how two or more clusters conflate into one and how modes of presentation accommodate to *de re* references of terms in order to revise an agent's beliefs, once contradictions in the agent's beliefs have been discovered.

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