Modal Collapse in Gödel's Ontological Proof *

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1 Introduction

As is well known, the modal second- (or higher-) order system in which Gödel 1970 sketched his ontological proof of God's existence [19, pp. 403–404] has proposition $\phi \to \Box\phi$ as a theorem. This was proven by J. H. Sobel in [34] (see [35, 36]). Gödel's system has the S5 propositional base. Since $\Box\phi \to \phi$ is a theorem of S5 (by $\Box$ Elimination), modal collapse, $\Box\phi \leftrightarrow \phi$, is provable. Modal collapse is also provable with the KB propositional base (see Theorem 1.8).

Let us briefly describe Gödel's ontological system in a variation that we shall call GO.[1] The language of GO is a second-order modal language with first-order self-identity ($t = t$), λ-abstracts, and a third-order term of positivity ($\mathcal{P}$). In the linear natural deduction format (originating from Jaśkowski), the rules of GO are the following: Assumption, Reiteration (derivable by $\forall$2I and $\forall$2E, see below), S5 modal rules, free first-order quantification rules ($\forall$1I, $\forall$1E, $\exists$1I, $\exists$1E), $\exists xEx$ (axiom of actual existence, see [15]), classical second-order quantification rules ($\forall$2I, $\forall$2E, $\exists$2I, $\exists$2E), =I (axiom scheme $c = c$) and λ-abstraction rules (λI, λE). Here are the rules for first-order quantification and λ-abstraction ($c \notin \mathcal{C}(\Delta)$ is a constant not occurring in the members of Δ):

$$\frac{\Gamma \vdash Ec \to \phi}{\Gamma \vdash \forall x\phi(x/c)}\ {\scriptstyle c\notin\mathcal{C}(\Gamma)} \qquad \frac{\Gamma \vdash \forall x\phi(x)}{\Gamma \vdash Ec \to \phi(c)} \qquad \frac{\Gamma \vdash Ec \wedge \phi(c)}{\Gamma \vdash \exists x\phi(x)} \qquad \frac{\Gamma \vdash \exists x\phi(x) \quad \Gamma, Ec \wedge \phi(c) \vdash q}{\Gamma \vdash q}\ {\scriptstyle c\notin\mathcal{C}(\Gamma,\exists x\phi(x),q)}$$

$$\frac{\phi(c_1/x_1,\ldots,c_n/x_n)}{(\lambda x_1\ldots x_n.\phi)(c_1\ldots c_n)} \qquad \frac{(\lambda x_1\ldots x_n.\phi)(c_1\ldots c_n)}{\phi(c_1/x_1,\ldots,c_n/x_n)}$$

* *Ontological Proofs Today*, Miroslaw Szatkowski (ed.), Frankfurt: Ontos (now Berlin: de Gruyter), 2012, 323–343.

[1] In analogy with PA for 'Peano arithmetic', and following Hájek's nomenclature [21, 23].

For simplicity, we do not include second-order identity ($X = Y$), nor first-order identity except for self-identity. They do not occur in Gödel's ontological proof from 1970, although they are useful for proving some further theorems.

We use Gödel's following abbreviations:

God, God-like $Gx =_{def} \forall X(\mathcal{P}X \to Xx)$

Essence $\mathcal{E}\!\int\!\!\int(X,x) =_{def} Xx \wedge \forall Y(Yx \to \Box\forall y(Xy \to Yy))$[2]

Necessary existence $Nx =_{def} \forall Y(\mathcal{E}\!\int\!\!\int(Y,x) \to \Box\exists xYx)$

The following axioms describe the concept of positivity:

GA1 $\forall X(\mathcal{P}\neg X \leftrightarrow \neg\mathcal{P}X)$

GA2 $\forall X\forall Y((\mathcal{P}X \wedge \Box\forall x(Xx \to Yx)) \to \mathcal{P}Y)$

GA3 $\mathcal{P}G$[3]

GA4 $\forall X(\mathcal{P}X \to \Box\mathcal{P}X)$

GA5 $\mathcal{P}N$

We list the propositions proved within the ontological argument:[4]

Proposition 1.1 $\mathcal{P}(\lambda x.x = x)$

Theorem 1.1 $\forall X(\mathcal{P}X \to \Diamond\exists xXx)$[5]

Corollary 1.1 $\Diamond\exists xGx$

Proposition 1.2 $\forall x(Gx \to \forall X(Xx \to \mathcal{P}X))$

Theorem 1.2 $\forall x(Gx \to \mathcal{E}\!\int\!\!\int(G,x))$

Theorem 1.3 $\exists xGx \to \Box\exists xGx$

Theorem 1.4 $\Box\exists xGx$

[2] In 1970 *definiens* lacks the left conjunct, which is present in an earlier note by Gödel [19, p. 431] and is required by D. Scott (see [35, p. 146]).

[3] It is Scott's version of Gödel's axiom [35, p. 145]. Fitting's version (see [13, p. 148]) is expressed in higher-order logic, and formalizes Gödel's formulation [19, p. 403] that, if ϕ and ψ are positive, so is their conjunction, "and for any number of summands" (infinitely many of them, too). Fitting then has $\mathcal{P}G$ as a provable proposition.

[4] The soundness and completeness proofs from [29] and `http://filist.ifzg.hr/~skovac/WeakenedCorrections.pdf` can be adapted to apply to GO.

[5] Gödel proves Theorem 1.1 from Proposition 1.1. There is a shorter way, independent of any non-empty property being provably positive [35, p. 120] [23].

For proofs, see in Gödel [19], Sobel [34, 35], Fitting [13], Chermak [11], Hájek [23]. Sobel proved the following theorem, too:

Theorem 1.5 (Modal collapse) $\forall X \forall x(Xx \leftrightarrow \Box Xx)$

Proof See [35] or [13]. (a) From left to right. Roughly, from the assumption Pc (and Ec) derive $(\lambda x.Pc)(d)$ and from the further assumption Gd and $\mathcal{E}\int\int(G, d)$ derive $\Box\forall y(Gy \rightarrow (\lambda x.Pc)(y))$. From there and from Theorem 1.4 derive $\Box Pc$. Hence, $Pc \rightarrow \Box Pc$ and $\forall X \forall x(Xx \rightarrow \Box Xx)$ follow (by $\rightarrow$I and $\forall$I). (b) From right to left. Apply $\Box$ Elimination.

Let us add some propositions that are closely related to modal collapse. The first collapses positivity to being, or, equivalently, "raises" being to positivity.

Theorem 1.6 (Positivity as being) $\forall X \forall x(Xx \leftrightarrow \mathcal{P}(\lambda y.Xx))$

Proof (a) From left to right (positivity of being). Assume Pc and Ec and from there and from Theorem 1.4, assuming Gd, derive $\Box\forall x(Gx \rightarrow (\lambda x.Pc)(x))$, as in the proof of the modal collapse above. Then, from GA2 derive $(\mathcal{P}G \wedge \Box\forall x(Gx \rightarrow (\lambda x.Pc)(x))) \rightarrow \mathcal{P}(\lambda x.Pc)$. Since $\mathcal{P}G$ is an axiom, $\mathcal{P}(\lambda x.Pc)$ follows by $\rightarrow$E. (b) From right to left (being of positivity). This follows from Theorem 1.4 and Theorem 1.2.

Corollary 1.2 (Positivity as necessity) $\forall X \forall x(\Box Xx \leftrightarrow \mathcal{P}(\lambda y.Xx))$

Proof From theorems 1.5 and 1.6.

The following theorem proved by Hájek [23, p. 311] makes explicit the equivalencies between being God-like, positivness, and necessity:

Theorem 1.7 $\forall x(Gx \leftrightarrow \forall Y(\mathcal{P}Y \leftrightarrow \Box Yx))$

If we replace $\mathsf{S5}$ propositional base in GO with KB, we obtain GOKB.

Theorem 1.8 (Modal Collapse in GOKB) $\mathsf{GOKB} \vdash \forall X \forall x(Xx \leftrightarrow \Box Xx)$.

Proof (a) From left to right. The same as for Theorem 1.5 above, except that, instead of by Theorem 1.4, the justification is by $\exists x Gx$, which is a theorem of GOKB (see [29]). (b) From right to left. Assume $\Box Pc$ (and Ec); then, from $\neg Pc$ and Corollary 1.1 a contradiction is derivable in the following way: in a $\Box$ subproof from assumption $\exists x Gx$ we derive $\Diamond\neg Pc$ (by B reiteration). Then derive, in the same $\Box$ subproof, $\Box Pc$. $\Box Pc$ is derivable in the same way as in the proof from left to right: by means of Theorem 1.2 (which holds in GOKB, too) and $\Box\forall x(Gx \rightarrow (\lambda x.Pc)(x))$.

Remark 1.1 *We note that, as a consequence of Proposition 1.1, there is no being with only negative properties, since each being is self-identical.*

Since $\Box\exists xGx$ is provable in GO, quantification is never vacuous (semantically: each world has a non-empty domain) and the rule of actual existence E is derivable. Moreover, since modal collapse is provable in GO, the Barcan formula and the converse Barcan formula are also provable (semantically: we have a constant domain across worlds).

Had we extended the system to a full logic with identity, e.g., with the interchangeability of identicals in atomic formulas, modal collapse would make the full substitution rule derivable and all first-order terms rigid:

$$\frac{t_1 = t_2, \phi(t_2)}{\phi(t_1/t_2)}$$

It would also make the necessity of identity and non-identity derivable:

$$\frac{t_1 = t_2}{\Box t_1 = t_2} \qquad\qquad \frac{\neg t_1 = t_2}{\Box\neg t_1 = t_2}$$

(The necessity of identity is also derivable from the full substitutivity of identicals, and from there, using the $\mathsf{S5}$ or B propositional base, so is the necessity of non-identicals (see [25, pp. 312–314].) As we can see, Gödel's positivity axioms are sufficient to transform free second-order modal logic with non-rigid terms to a classical variant of second-order logic.

Gödel gave two interpretations to his system – one is moral-aesthetic (from the standpoint of moral-aesthetic ontology) and the other is attributive (from the standpoint of ontology proper). From both standpoints, modal collapse seems to deny freedom (moral, aesthetic, ontic), to imply determinism, and as such seems to be hardly acceptable. Therefore, several ways have been proposed to emend the system in order to exclude modal collapse. Probably the best known is Andersons's given in [2] and modified in [3]. Andersonian systems were further explored and critically discussed and modified, for instance by Hájek [21, 22, 23] and Szatkowski (e.g., [38, 39]). Sobel's proposal is to exclude modal collapse by deleting Axiom 5 ([34, 35], preventing the provability of Theorem 1.4, too). Another approach was proposed by Hájek [21], consisting in the weakening of the ontological system to a belief system with the $\mathsf{KD45}$ propositional base, where Theorem 1.4 is not provable. Fitting proposed a change from intensional to extensional types of variables, preserving the validity of Theorem 1.4. There are still other proposals, such as to modify Axiom 5 (Koons [28]), or simply to restrict the comprehension/λ-conversion schema (Koons [28], Kovač [29];[6] see Sobel in [37, 36]). Good bibliographies on Gödel's ontological proof can be found in [10] and [12].

In the next part of the paper, we show that modal collapse is what Gödel, most probably, intended to have as a result, and we put modal collapse in the broader context of Gödel's philosophy. Thereafter, we propose a redefinition of Gödel's system from the standpoint of reinterpreted justification logic in a way that does not exclude modal collapse, but can give it an explicit, causal sense.

2 Is modal collapse in Gödel's ontology incidental?

2.1 Confirmations of modal collapse in Gödel's text

We aim to show that Gödel's texts and reflections confirm that modal collapse was intended and is part of Gödel's general philosophical view.[7] As a confirmation of this intention of Gödel, we have referred in 2003 [29, p. 582] and in [30] to page 435 of [19], where Gödel says that

$$\phi(x) \to \Box\phi(x) \tag{1}$$

should only be derived from the existence of God, and not vice versa, the existence of a thing "for every compatible system of properties" (including God) from (1).[8] According to Gödel, the proof from assumption (1) is "the bad way" ("der schlechte Weg", translated in [19] somewhat misleadingly as "the inferior way"). Gödel's approach is obviously, first, to prove the existence of God from ontological axioms, and only thereafter to prove modal collapse.

In the cited place (p. 435, "Ontological proof", nr. 4), Gödel compares two assumption candidates from which the existence of God is derivable. The first was at that time adopted by him as a crucial axiom:

$$\mathcal{P}X \to \mathcal{P}(\lambda x.\Box Xx) \tag{2}$$

[6] In [29] we proposed a restriction on λ-abstraction to block the provability of modal collapse, but, at the same time, stated that modal collapse was Gödel's intention (and put it in analogy with Gödel's cosmological collapse of time).

[7] On Adams' discussion on this point see [1] (and below).

[8] We have referred to this place also in the correspondence with Sobel on 6 February 2004. In the reply (7 February), Sobel allowed that Gödel, at least in 1956, "was easy" with the idea of modal collapse. In 2006 [36] and referring to [1], Sobel, too, expressed the view that there is a strong evidence that, for Gödel, modal collapse was a "welcomed feature" of his metaphysics.

that is, “the necessity of a positive property is positive”. Gödel mentions (in a footnote to a text several lines above) the dual form, too, for this assumption

if $M\phi$ is a perfective, then ϕ is too

i.e., $\mathcal{P}(\lambda x.\Diamond Xx) \rightarrow \mathcal{P}X$. Here, “perfective” is only a special way (corresponding to the later “moral aesthetic” interpretation) of how to interpret the “positive” (another acceptable interpretation of “positive” being “assertion”).

An alternative, unsatisfactory way is to assume (1), as provable from the essence of x. In fact, the provability of (1) from the essence of x would not be welcomed by Gödel, since this would have as a result the “bad way” of proving the existence of God simply from modal collapse. The concept of essence is not mentioned in the axiomatic outline on p. 435, but has been defined in other places: in an earlier outline of the ontological proof (p. 431) as well as in the last sketch (from 1970, with a slight difference). It is this definition of essence (p. 431, Scott) with respect to which Sobel's refutation of the provability of (1) from the essence of x holds (Sobel [36] and the reply in the correspondence with me on 7 February 2004). We may note that the inconclusiveness proved by Sobel need not be an error on the side of Gödel, but something that blocks the “bad” ontological proof. Why does Gödel name such an ontological proof a “bad way”? Gödel obviously wanted necessitarianism (and individual essences) eventually to be founded on God and on the “whole”. Hence, he had to exclude the provability of modal collapse directly from separate individual essences, and to exclude the provability of the existence of God from such (naturalistic or fatalistic) modal collapse.

It appears R. M. Adams [1, p. 400] was the first to quote (in 1995) from most probably the same place in Gödel's notes, and this directly from the manuscript (see [36, p. 402]). He quotes what he calls Gödel's thesis that “for every compatible system of properties there is a thing”. However, taking into account a suggestion of Ch. Parsons, he thinks that the interpretation of the thesis is “not obvious”, although it “looks strongly necessitarian” (Gödel's necessitarianism as a “somewhat speculative” suggestion, p. 401).[9]

[9]Similarly, later, A. P. Hazen in [24, p. 369]. In F. Orilia in 1994 [32, p. 130] we find the following general idea about modal collapse: “...there are some philosophical and theological doctrines according to which every truth is necessary. A theologian who embraces one such doctrine could see Sobel's result as a confirmation of his/her view, rather than as a refutation of Gödel's system” (although Orilia does not “tend to side with” such a view).

There is still another place confirming Gödel's adoption of modal collapse. Gödel says:

> The positive and the true sentences are the same, for different reasons ... [19, p. 433]

Closed sentences are zero-place properties. If God is defined as a being having all positive properties, and God necessarily exists, then whatever is true is necessarily true. It is hard to imagine that Gödel would not be aware of this consequence.[10] The above quotation expresses, in fact, our Theorem 1.6, with 'true' instead of 'being'.

2.2 Modal collapse and modal rise in Gödel's philosophy

Gödel's adoption of modal collapse suits perfectly well his general philosophical views as documented in his manuscripts, in reports by Hao Wang [41, 42], and in a particular way corresponds to Gödel's philosophy of time (see interpretations of his philosophy of time by P. Yourgrau, e.g., [43, 44]).

As Wang reports, according to Gödel, the separation of force (wish) and fact (wish is force "as applied to thinking beings, to realize something"), and the overcoming of this separation (in the "union of fact and wish"), are the "meaning of the world" [42, pp. 311, 309]. This separation and overcoming stand under the "maximum principle for the fulfilling of wishes", in the direction of building up the (Leibnizian) best possible world [42, p. 312].[11] Hence, the overcoming of the separation of force (wish) and fact should be perfect, in the sense of all possibilities being realized: "... as many beings as possible will be produced - and this is the ultimate ground of diversity ..." [19, p. 433]. In these views it is not hard to recognize modal collapse in terms of the possibility: $\Diamond\phi \to \phi$.

Far from being disappointed with the perspective of modal collapse, since it means the realization of all possibilities in the perfect world, Gödel states: "our total reality and total existence are beautiful and meaningful" (where "the short period of misery may even be necessary for the whole")[42, p.

[10] Adams [1, pp. 400–401] quotes another place from the same Gödel manuscript to the same effect, namely as a confirmation that God has, for Gödel, the property of coexisting (or knowing) each truth, and consequently, that each truth is a "perfective" (positive) since God's coexisting with the truth (as one of God's properties) is "perfective". As already said, Adams had some reserves about Gödel's necessitarianism.

[11] "... [since there are so many unrealized possibilities in this world, it must be a] preparation for another world" [42, p. 312]. Also, according to Gödel, our "very imperfect life ... may be necessary for and adequately compensated for by[,] the perfect life afterwards" [42, p. 317].

317]. The collapse of modalities is for Gödel, in fact, the *rise* of modalities to the perfect being.

Besides, despite the (final) modal collapse/rise, there is in Gödel another, clearly distinguishable, although inferior, view on modalities: strong "separation" of force and fact as characteristic of some fragments of the whole ("this world"), along with the "superior" view from the standpoint of "the whole" (where modalities unite).

Such a reading of Gödel's philosophical remarks is strengthened by Gödel's much more precisely elaborated cosmology and philosophy of time.[12] Gödel contributed to cosmology through his discovery of the possibility (in the sense of natural laws) of rotating universes without definable "absolute time". Further, in non-expanding rotating universes "time travel" would be physically possible.[13] On the ground of the mere physical possibility of rotating universes, and on the ground of the reflection that it is highly philosophically unsatifactory to think of time as dependent on the (accidental) arrangement of matter and motion in the universe, Gödel endorsed the view that there is no objective lapse of time (or change), and that time is only subjective ("an illusion or appearance due to our special mode of perception"), without any objective lapse.[14] Hence, there is objectively no future realization of presently non realized possibilities, but each possibility is "already" realized in cosmological spacetime. Yourgrau especially elaborated this side of Gödel's philosophy, but stressing the resulting dual perspective of the world: "from within and also *sub specie aeternitatis*".[15]

Let us add two remarks. In Gödel's non-expanding rotating universes, time behaves, from the same basic logical viewpoint, like modalities in his ontological proof – it satisfies the conditions of the S5 propositional base (reflexive euclidean models). This can be clearly seen from Gödel's following

[12] On the correspondence between modal collapse in Gödel's ontological proof and the temporal collapse in Gödel's cosmology, see briefly in Kovač [29, p. 582] and in [30, pp. 160–161]. This is strongly connected with some of the Yourgrau's reflections (see footnote 14).

[13] For a recent physicist's account of Gödel's cosmology, see W. Rindler [33].

[14] Yourgrau points out one characteristic general feature of Gödel's reasoning, which consists in the transition from the possible to the actual. It is manifest in mathematics (possible mathematical objects are as such mathematically actual), in cosmology (from the possible to the actual non-existence of time), as well as in the ontological argument (from the possible to the actual existence of God) [43, p. 44] [44, p. 130].

[15] "Yet, somehow, we, the individual selves, must be able to support *both* perspectives. Then what *are* we? 'We do no know *what* we are (namely, in essence and seen eternally)' (Gödel, in Wang 1987, p. 215)" [43, p. 191]. A. Ule has emphasized that, beside our experience of the lapsing time, we also have temporary non-lapsing time consciousness [40].

statement on his cosmological results about rotating universes:

> . . . it is possible in these worlds to travel into any regions of the past, present, and future, and back again, exactly as it is possible in other worlds to travel to distant parts of space [17, p. 205].

Second, there is another interesting analogy between the ontological and the cosmological proof of modal collapse which we can reconstruct. The "bad way" in cosmology would be first to assume that there is no lapse of time, and then from there to prove the existence of relativistic spacetime. Gödel proceeded the other way around, first presupposing natural laws, and coming from there to relativistic spacetime, he constructs his rotated universes, by means of which he concluded that time collapses.

3 Modal collapse and causality

Since time and modalities collapse, the question about what we are then left with in ontology remains. Gödel's answer is: causality. In his timeless ontology ("time is no specific character of being" [42, p. 320]), time structure is replaced by causal structure:

> The real idea behind time is causation; the time structure of the world is just its causal structure [42, p. 320].

Of course, the causal structure is itself unchanging:

> Causation is unchanging in time and does not imply change. It is an empirical—but not a priori—fact that causation is always accompanied by change. [42, p. 320]

What is the ontological status of collapsing modalities in general? According to Gödel, they should be ("perhaps") derived from causality, too:

> The fundamental philosophical concept is cause. It involves: will, force, enjoyment, God, time, space. . . .
>
> . . . Perhaps the other Kantian categories (that is, the logical ⟦ categories ⟧, including necessity) can be derived in terms of causality, and the logical (set-theoretical) axioms can be derived from the axioms for causality. [Property = cause of the difference of the things] [19, pp. 433–434].

Force ("connected to objects"), as well as concepts ("being general"), should be explained by means of causation (see [42, p. 312, nr. 9.4.5]).[16] Let us add that, while (the lapse of) time disappears in the objective causal structure of the world and remains in ontology only subjectively, as the "frame of reference" for the mind [42, p. 319], space is for Gödel the "possibility of influence" [19, pp. 434–435] – where possibility is conceived as "a weakened form of being", "synthesis of being and nonbeing" [42, p. 313].

Gödel's reflections give some hints as to how to understand ontological concepts in the causal, fundamental perspective. As we see from the quotation above, properties cause the difference of things. Further, according to Gödel, a mathematical theorem causes its consequences [42, p. 320]. It can be understood that positive properties, in "atributive" interpretation, are the properties that affirm being,[17] and that the "affirmation of being" is the *causal* meaning of positiveness:

> The affirmation of being is the cause of the world" [42, p. 433].

Finally, God should be the last cause of the world, a "necessity in itself", which does not require any further cause.[18]

This much could be sufficient as a corroboration of the thesis that modal collapse is a constituent part of Gödel's ontology and philosophy in general, and, secondly, that causation is for Gödel the real and fundamental concept behind modalities.

4 Causality and justification logic

Let us now see in more detail how modality in Gödel's ontological proof could be replaced by causality.

The formulation "if ϕ, then necessarily ϕ" is very general. It is not precise about the *kind* of necessity, except that it is understood that this is an *ontological* necessity, defined by the propositional $\mathsf{S5}$ base and by Gödel's specific axioms of positivity. As remarked in [7] (though from the epistemic perspective), there are two ways of reading $\Box$: as a universal quantifier over possible worlds (truth in all possible worlds), and as an existential quantifier over reasons (truth for a reason). The existential reading of $\Box$ in modal

[16] In [42, p. 297] Gödel says: "*Force* should be a primitive term in philosophy".

[17] See Gödel's attributive interpretation of positiveness, "as opposed to 'privation' " — the disjunctive normal form of a positive property contains a member without negation [19, p. 404].

[18] "…according to the Principle of Sufficient Reason the world must have a cause. This must be "necessity in itself" (otherwise it would require a further cause)" [19, p. 431].

collapse theorem of GO formally expresses Leibniz's principle of sufficient reason: each truth has its sufficient reason. But which reason in each particular case is it? Justification logic defines "realization" algorithms for the replacement of each occurrence of $\Box$ by a particular reason ("proof term", evidence) in such a way that each theorem of, e.g., propositional S4 or S5, and of first-order S4 (FOS4), after realization, remains (in a modified form) a theorem of the corresponding justification logic LP, LPS5 and FOLP, respectively (see Artemov [5], Fitting [14], Artemov and Yavorskaya [8]).

Actually, "reason" indicates a special modality beside possibility and necessity. From Leibniz on, and especially from Kant, we can trace a distinction between the principles of non-contradiction (possibility, "problematic judgments"), of sufficient reason (existence, reality, truth, "assertions"), and of excluded middle (necessity, "apodeictic judgments") (see [27, B 99–101, 106][19]). The existential reading of $\Box$ is in fact "assertoric" ("existential") modality, which can be obtained through a sort of reduction of necessity each time to some existing reason. "It is necessary that ϕ" is reduced (in a propositional setting) to "t is a reason for ϕ", $t\colon \phi$, for some existing reason t. Since the principle of excluded middle is intuitionistically not valid, it is quite natural that Gödel came to the idea to replace S4 modal propositional logic, as a modal formalization of intuitionism, with a corresponding justification logic.[20] Gödel saw that in S4 with $\Box$ conceived simply as "provability", the contradiction with his second incompleteness theorem follows.[21]

"Reason" is a term general enough to cover the epistemological as well as ontological sense of assertoric modality. It is conceived in these two senses, for example, by Leibniz as well as by Kant (who makes a sharper distinction between them). An appropriate and established ontological sense of "reason" is causality.[22] And, as already mentioned, causality is Gödel's explicitly intended primary ontological concept, from which all other "categories" should be derived. Therefore, causality is the best candidate to be conceived as reason in the "justificationally" transformed Gödel's ontological system. Accordingly, each occurrence of $\Box$ in Gödel's ontological

[19] Kant gives a nice explanation in his letter to Reinhold from 12. 5. 1789 [26, vol. 9]. Also, see [31].

[20] This proposal of Gödel was published posthumously in his "Zilsel lecture" in 1995 [20]. At that time, Artemov had already independently worked out the same idea, and published it the same year in [4].

[21] See Gödel [20] and Artemov [5] for further details.

[22] For example, for Kant, to the three logical principles of non-contradiction, sufficient reason and excluded middle, there correspond three ontological categories ("relations"): substance – accident, causality, and community (mutual influence), respectively [27, B 106].

proof should be read as an (ontological) cause, and $\Box\exists xGx$ should mean not merely that God necessarily exists, but should explicitly name the cause for which God exists. Similarly, modal collapse in Gödel's ontology should not simply mean that every fact is necessarily true, but should explicitly name the cause ("sufficient reason") for which the truth obtains. Let us add that the modal collapse theorem discriminates ontological justification logic from the epistemological one in that in the epistemological case not every truth should be evident (and therefore epistemological justification logic can be intuitionistically appropriate), whereas in the ontological case, at least from some traditional viewpoint, we naturally expect each truth to have its cause.

Besides, taking into account Gödel's distinction between the moral-aesthetic and the attributive (assertive) interpretations of positivity, we can in a natural way distinguish between moral-aesthetic causes, related to the affirmation or negation of what is "purely good" [19, p. 433], of an objective moral or aesthetic value [18, p. 375], and, on the other side, attributive causes, related simply and generally to the affirmation or negation of being.

To illustrate how a causal ontological proof might look, we describe the system CGO, a "justificationally" transformed GO in the causal sense. To that end, we extend the first-order justification logic FOLP by Artemov and Yavorskaya, and give it (informally) a causal interpretation.[23] CGO is an extension to a second-order logic with first-order self-identity, with the justificational $\mathsf{S5}$ propositional base (see Fitting [14]) and with modified Gödel's ontological axioms. In vocabulary, there are cause variables and cause constants, and otherwise no constant terms except first-order $=$ and $\mathcal{P}$. Connectives other than $\neg$ and $\rightarrow$, as well as existential quantifiers, are defined in the classical way. Justification terms receive causal meaning in the following way: $t+s$ means cause t or s, $t\cdot s$ means the affirmation (activation) of the cause t by means of cause s, $!t$ is the affirmation of a cause t having some specific effect, $?t$ is the affirmation of a cause t not having some specific effect, and $\mathsf{gen}(t)$ is the general application of a cause. We add $\mathsf{lam}(t)$ as a property maker, and introduce two further ontological cause constants:

g – cause of ("moral-aesthetic") positivity,

$exs(t)$ – cause of existence, "affirmation of being".

Justification formulas are built as in FOLP, with the addition of second-order variables in $\mathcal{X}$ of $t{:}_{\mathcal{X}}\,\phi$.

[23] The possiblity that t in $t{:}\,\phi$ could mean "something like set of causes or counterfactuals" is mentioned by Artemov in [6, 478]

The abbreviations of GO, except for the predicate G, are slightly transformed, so that essence and necessary existence are defined in the following way:

$\mathcal{E}\!\int\!\!\int_t(X,x) =_{def} Xx \wedge \forall Y(Yx \to t\!:_{\{X,Y\}} \forall y(Xy \to Yy))$ (essence)

$N_t x =_{def} \forall Y(\mathcal{E}\!\int\!\!\int_t(Y,x) \to exs(t)\!:_{\{Y\}} \exists x Yx)$ (necessary existence)

Axioms are the axioms of classical first-order logic, self-identity $x = x$, and the following ones (axiom schemes **CVCons**–**C4** and **C**$\forall$ are second-order generalizations of the corresponding schemes of FOLP [8]):

$\forall$**2a** $\forall X(\phi) \to \phi(T/X)$ (T is substitutable for X in ϕ)

$\forall$**2b** $\forall X(\phi \to \psi) \to (\phi \to \forall X\psi)$, X does not occur free in ϕ

λ**Conv** $(\lambda x.\phi)(y) \leftrightarrow \phi(y/x)$, y is substitutable for x in ϕ

CVCons $t\!:_{\mathcal{X},y}\, \phi \to t\!:_{\mathcal{X}}\, \phi$, y does not occur free in ϕ
$t\!:_{\mathcal{X},Y}\, \phi \to t\!:_{\mathcal{X}}\, \phi$, Y does not occur free in ϕ

CVMon $t\!:_{\mathcal{X}}\, \phi \to t\!:_{\mathcal{X},y}\, \phi \qquad t\!:_{\mathcal{X}}\, \phi \to t\!:_{\mathcal{X},Y}\, \phi$

CMon $s\!:_{\mathcal{X}}\, \phi \to (s+t)\!:_{\mathcal{X}}\, \phi, \qquad t\!:_{\mathcal{X}}\, \phi \to (s+t)\!:_{\mathcal{X}}\, \phi$

CK $s\!:_{\mathcal{X}}\, (\phi \to \psi) \to (t\!:_{\mathcal{X}}\, \phi \to (s \cdot t)\!:_{\mathcal{X}}\, \psi)$

CT $t\!:_{\mathcal{X}}\, \phi \to \phi$

C4 $t\!:_{\mathcal{X}}\, \phi \to !t\!:_{\mathcal{X}}\, t\!:_{\mathcal{X}}\, \phi$

C5 $\neg t\!:_{\mathcal{X}}\, \phi \to ?t\!:_{\mathcal{X}}\, \neg t\!:_{\mathcal{X}}\, \phi$

C$\forall$ $t\!:_{\mathcal{X}}\, \phi \to \mathsf{gen}_x(t)\!:_{\mathcal{X}}\, \forall x\phi$, $x \notin \mathcal{X} \qquad t\!:_{\mathcal{X}}\, \phi \to \mathsf{gen}_X(t)\!:_{\mathcal{X}}\, \forall X\phi$, $X \notin \mathcal{X}$

Cλ $t\!:_{\mathcal{X}}\, \phi(x/y) \to \mathsf{lam}_x(t)\!:_{\mathcal{X}}\, (\lambda y.\phi)(x)$

Gödel's ontological axioms **CGA1**–**CGA5** are the same as **GA1**–**GA5**, respectively, with the exception of the second (which is, in fact, an axiom scheme) and the fourth: **CGA2**: $\forall X \forall Y((\mathcal{P}X \wedge t\!:_{\{X,Y\}} \forall x(Xx \to Yx)) \to \mathcal{P}Y)$, **CGA4**: $\forall X(\mathcal{P}X \to g\!:_{\{X\}} \mathcal{P}X)$.

Rules are modus ponens (MP), first- and second-order generalization (gen1, gen2), and axiom necessitation (ANec): if $\vdash \phi$, then $\vdash c\!: \phi$, where ϕ is an axiom, and c a cause constant ("axiom causation").

Technical metatheoretical results about CGO will be presented in a separate paper. Here, we prove in CGO the theorems of Gödel's ontological argument. Corresponding to Gödel's argument, we start from proving the positivity of self-identity. In the annotations and indices, PL is propositional logic, and FOL first-order logic with self-identity and λ-abstraction.

Proposition 4.1 $\mathcal{P}(\lambda x.x = x)$ *(the same as 1.1).*

Proof Start from the axiom $x = x$, apply axiom necessitation, for example, with the cause constant c, to get $c : x = x$; by Cλ and MP, we obtain $\mathsf{lam}_x(c) : (\lambda x.x = x)(x)$; enter the axiom $(\lambda x.x = x)(x) \rightarrow ((\lambda x.\neg x = x)(x) \rightarrow (\lambda x.x = x)(x))$, and derive, by axiom necessitation, $d : ((\lambda x.x = x)(x) \rightarrow ((\lambda x.\neg x = x)(x) \rightarrow (\lambda x.x = x)(x)))$; therefore, by CK,

$$d : ((\lambda x.x = x)(x) \rightarrow ((\lambda x.\neg x = x)(x) \rightarrow (\lambda x.x = x)(x))) \rightarrow$$
$$(\mathsf{lam}_x(c) : (\lambda x.x = x)(x) \rightarrow (d \cdot \mathsf{lam}_x(c)) : ((\lambda x.\neg x = x)(x) \rightarrow (\lambda x.x = x)(x)))$$

By MP, $(d \cdot \mathsf{lam}_x(c)) : ((\lambda x.\neg x = x)(x) \rightarrow (\lambda x.x = x)(x))$ and, by C$\forall$ and MP, $\mathsf{gen}_x(d \cdot \mathsf{lam}_x(c)) : \forall x((\lambda x.\neg x = x)(x) \rightarrow (\lambda x.x = x)(x))$. Finally, by CGA2, CGA1, we derive $\mathcal{P}(\lambda x.x = x)$.

Theorem 4.1 $\forall X(\mathcal{P}X \rightarrow \neg u{:_{\{X\}}}\ \forall x \neg Xx)$

Proof

1 $c_{FOL}{:_{\{X\}}}\ (\forall x \neg Xx \rightarrow \forall x(Xx \rightarrow (\lambda x.\neg x = x)(x)))$
2 $u{:_{\{X\}}}\ \forall x \neg Xx \rightarrow (c_{FOL} \cdot u){:_{\{X\}}}\ \forall x(Xx \rightarrow (\lambda x.\neg x = x)(x)))$
  1, CK, MP
3 $(\mathcal{P}X \wedge (c_{FOL} \cdot u){:_{\{X\}}}\ \forall x(Xx \rightarrow (\lambda x.\neg x = x)(x))) \rightarrow \mathcal{P}(\lambda x.\neg x = x)$
  CGA2, JVCons
4 $(\mathcal{P}X \wedge u{:_{\{X\}}}\ \forall x \neg Xx) \rightarrow \mathcal{P}(\lambda x.\neg x = x)$ 2,3 PL
5 $\mathcal{P}X \rightarrow (u{:_{\{X\}}}\ \forall x \neg Xx \rightarrow \mathcal{P}(\lambda x.\neg x = x))$ 4 PL
6 $\neg \mathcal{P}(\lambda x.\neg x = x)$ CGA 1, Prop. 4.1
7 $\mathcal{P}X \rightarrow \neg u{:_{\{X\}}}\ \forall x \neg Xx$ 5,6 PL
8 $\forall X(\mathcal{P}X \rightarrow \neg u{:_{\{X\}}}\ \forall x \neg Xx)$ 7 gen2

Term u is a cause variable. Cause term c_{FOL} in line 1 is the result of the successive application of ANec to first-order logic axioms (with axioms for self-identity and λ-abstraction), and of the successive combination of the obtained cause terms into comlex causal terms ("polynomials").

Corollary 4.1 $\neg u : \forall x \neg Gx$

Proof From CGA3 ($\mathcal{P}G$) and from Theorem 4.1, $\neg u{:_{\{X\}}}\ \forall x \neg Gx$ and hence (by JVCons) $\neg u : \forall x \neg Gx$ follow.

Corollary 1a $\neg((c_{PL}\cdot a)\cdot ?exs(c_{SOL}\cdot g))\colon \forall x\neg Gx$

Proof In Theorem 4.1, we replace u by $((c_{PL}\cdot a)\cdot ?exs(c_{SOL}\cdot g))$ (this causal justification will be used in Theorem 4.4). Hence, $\neg((c_{PL}\cdot a)\cdot ?exs(c_{SOL}\cdot g))\colon \forall x\neg Gx$ follows.

Proposition 4.2 $\forall x(Gx \to \forall X(Xx \to \mathcal{P}X))$ *(the same as Proposition 1.2).*

Proof We leave it as an exercise.

Theorem 4.2 $\forall x(Gx \to \mathcal{E}\int\int_{c_{SOL}\cdot g}(G,x))$

Proof Left as an exercise. Let us note that in the first part of the proof, we use second-order logic axioms and their necessitation, to come to $c_{SOL}\colon_{\{X\}} (\mathcal{P}X \to \forall y(Gy \to Xy))$, where c_{SOL} is a combination of necessitations of second-order logic axioms (logical causality, preserving consistency of properties). We then derive $g\colon_{\{X\}} \mathcal{P}X \to (c_{SOL}\cdot g)\colon_{\{X\}} \forall y(Gy \to Xy)$, and, by Proposition 4.2, deduce the theorem.

Theorem 4.3 $\exists xGx \to exs(c_{SOL}\cdot g)\colon \exists yGy$

Proof

1	$Gx \to \forall X(\mathcal{P}X \to Xx)$	Def. G (Theorem $\phi \to \phi$)
2	$Gx \to (\mathcal{P}N \to Nx)$	
3	$\mathcal{P}N$	CGA5
4	$Gx \to Nx$	
5	$Gx \to \forall Y(\mathcal{E}\int\int_{c_{SOL}\cdot g}(Y,x) \to exs(c_{SOL}\cdot g)\colon_{\{Y\}} \exists yYy)$	Def. N
6	$Gx \to \mathcal{E}\int\int_{c_{SOL}\cdot g}(G,x)$	Th. 4.2
7	$Gx \to exs(c_{SOL}\cdot g)\colon_{\{Y\}} \exists yGy$	5,6 SOL
8	$Gx \to exs(c_{SOL}\cdot g)\colon \exists yGy$	7 CVCons
9	$\forall x(Gx \to exs(c_{SOL}\cdot g)\colon \exists yGy)$	8 gen1
10	$\exists xGx \to exs(c_{SOL}\cdot g)\colon \exists yGy$	9 FOL

Theorem 4.4 $exs(c_{SOL}\cdot g)\colon \exists yGy$

Proof

1 $a\colon (\exists x Gx \to exs(c_{SOL} \cdot g)\colon \exists y Gy)$ see note below
2 $c_{PL}\colon ((\exists x Gx \to exs(c_{SOL} \cdot g)\colon \exists y Gy) \to$
$(\neg exs(c_{SOL} \cdot g)\colon \exists y Gy \to \neg\exists x Gx))$ see note below
3 $(c_{PL} \cdot a)\colon (\neg exs(c_{SOL} \cdot g)\colon \exists y Gy \to \neg\exists x Gx)$ 1,2 CK, MP
4 $?exs(c_{SOL} \cdot g)\colon \neg exs(c_{SOL} \cdot g)\colon \exists y Gy \to$
$((c_{PL} \cdot a) \cdot ?exs(c_{SOL} \cdot g))\colon \neg\exists x Gx$ 3 CK, MP
5 $\neg((c_{PL} \cdot a) \cdot ?exs(c_{SOL} \cdot g))\colon \neg\exists x Gx \to$
$\neg ?exs(c_{SOL} \cdot g)\colon \neg exs(c_{SOL} \cdot g)\colon \exists y Gy$ 4 PL
6 $\neg((c_{PL} \cdot a) \cdot ?exs(c_{SOL} \cdot g))\colon \neg\exists x Gx$ Coroll. 4.1a
7 $\neg ?exs(c_{SOL} \cdot g)\colon \neg exs(c_{SOL} \cdot g)\colon \exists y Gy$ 5, 6 MP
8 $exs(c_{SOL} \cdot g)\colon \exists y Gy$ 7 C5

Abbreviation a in line 1 stands for the causal term that results from the transformation of the proof of Theorem 4.3 by the cumulative necessitation (based on ANec) of the axioms used. Similarly, c_{PL} is the abbreviation of a causal term obtained by the cumulative necessitation of propositional logic axioms used.

Informally, Theorem 4.4 says that the affirmation of logic and positivity is the cause of the existence of a God-like being. In some way, this should be understood as the explication of the "necessity in itself", without "further cause", and as somehow (in analogy with Th. 1.6, Def. of G) contained in the God-like being itself (see footnote 18).

Theorem 4.5 (Modal collapse)
$\forall z \forall X (Xz \to ((c_{SOL} \cdot g) \cdot exs(c_{SOL} \cdot g)) :_{\{X,z\}} Xz)$

Proof Similar to axiomatic non-justificational proof, see [13, 163–164].

1 $Gx \to (Gx \wedge \forall Y(Yx \to (c_{SOL} \cdot g){:}_{\{Y\}}\, \forall y(Gy \to Yy)))$ Th. 4.2
2 $Gx \to \forall Y(Yx \to (c_{SOL} \cdot g){:}_{\{Y\}}\, \forall y(Gy \to Yy))$ PL
3 $Gx \to \forall Y(Yx \to (c_{SOL} \cdot g){:}_{\{X,Y,z\}}\, \forall y(Gy \to Yy))$ CVMon
4 $Gx \to ((\lambda x.Xz)(x) \to (c_{SOL} \cdot g){:}_{\{X,Y,z\}}\, \forall y(Gy \to (\lambda x.Xz)(y)))$ $\forall$2a
5 $Gx \to ((\lambda x.Xz)(x) \to (c_{SOL} \cdot g){:}_{\{X,z\}}\, \forall y(Gy \to (\lambda x.Xz)(y)))$ CVCons
6 $Gx \to (Xz \to (c_{SOL} \cdot g){:}_{\{X,z\}}\, \forall y(Gy \to Xz))$ λConv
7 $\forall x(Gx \to (Xz \to (c_{SOL} \cdot g){:}_{\{X,z\}}\, \forall y(Gy \to Xz)))$ 6 gen1
8 $\exists x Gx \to (Xz \to (c_{SOL} \cdot g){:}_{\{X,z\}}\, (\exists y Gy \to Xz))$ 7 FOL
9 $exs(c_{SOL} \cdot g){:}\, \exists x Gx$ Th. 4.4
10 $\exists x Gx$ 9 CT
11 $Xz \to (c_{SOL} \cdot g){:}_{\{X,z\}}\, (\exists y Gy \to Xz)$ 8,10 MP
12 $Xz \to (exs(c_{SOL} \cdot g){:}_{\{X,z\}}\, \exists y Gy \to ((c_{SOL} \cdot g) \cdot exs(c_{SOL} \cdot g)){:}_{\{X,z\}}\, Xz)$ consequent of 11, CK, MP
13 $exs(c_{SOL} \cdot g){:}_{\{X,z\}}\, \exists x Gx$ 9 CVMon
14 $Xz \to ((c_{SOL} \cdot g) \cdot exs(c_{SOL} \cdot g)){:}_{\{X,z\}}\, Xz$ 12,13 PL
15 $\forall z \forall X(Xz \to ((c_{SOL} \cdot g) \cdot exs(c_{SOL} \cdot g)){:}_{\{X,z\}}\, Xz)$ gen1, gen2

The above-given axiomatic outline gives an explicit causal, although rather metaphysical (theological), answer to the meaning of modal collapse in Gödel's ontological proof.

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