Accepted manuscript. Published in: Marcin Będkowski, Anna Brożek, Alicja Chybińska, Stepan Ivanyk & Dominik Traczykowski (eds.), *Formal and Informal Methods in Philosophy*. Leiden: Brill Rodopi, 2020, pp. 163-179.

Remarks on the Origin and Foundations of Formalisation

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1. The formalisation and historical origins of logic

According to modern standards of the certainty and exactness of knowledge, one cannot be fully satisfied with a given theory until it is formalised, that is, presented in the shape of a formal system. A formalised theory should precisely define its language (a set *S* of sentences) and make explicit its axioms, especially its logic: logical axioms and rules of inference (the relation $\vdash$ of the derivability of sentences from a set of sentences), and is thus definable as an ordered pair $\langle S, \vdash\rangle$.[1] Such standards were established by the founders of modern logic at the end of the 19th and beginning of the 20th centuries. In this context, Łukasiewicz remarked in 1922:

> We are no longer satisfied with ordinary mathematical deductions, which usually start somewhere "in the middle", reveal frequent gaps, and constantly appeal to intuition. ... We want to know the axioms on which each system is based, and the rules of inference of which it makes use. (Łukasiewicz 1967a, 20)

Regarding the philosophy of his time, Łukasiewicz was even more critical:

> Philosophy must be reconstructed from its very foundations; it should take its inspiration from scientific method and be based on new logic. (Łukasiewicz 1967a, 21)

He added: "This is a work for generations and for intellects much more powerful than those yet born" (p. 21). These standards have their origins in ancient times. Łukasiewicz and Bocheński showed that Aristotle developed a deductive system of "formal logic" and that he established a general theory of axiomatisation. Admittedly, they concluded that Aristotle did not arrive at a formalised theory of logic, implemented in rigorously defined language, deeming that this had been achieved by the Stoic logicians.[2] However, they pointed out that Aristotle's concept and construction of "formal logic" was primarily due to his employment of term letters (variables), which enabled him to abstract from the associated meaning of a sentence and pay exclusive attention to the form (of a sentence) relevant for deductive inference. As stressed, for instance, by

1 See, e.g., Tarski's early account in (1983). Cf. Béziau (2005).

2 Sometimes giving almost exclusive emphasis to the linguistic expression of logical forms, Łukasiewicz (1957) denies that Aristotle succeeded in establishing a logical formalism owing to what Łukasiewicz considers an incorrect way of substituting concrete terms into logical forms and to the synonymous use of logical terms (e.g., "belongs to" and "is predicated to", Łukasiewicz 1957, 17-18, see Bocheński 2002, 113, transl. in Bocheński 1961). He recognises that Aristotle developed an axiomatic system of syllogisms (e.g., Łukasiewicz 1957, pp. 44, 73; in particular, see Bocheński 2002, 74, 81, 84, 86), albeit without fulfilling modern "formalistic" standards.

Bocheński, Aristotle's formal logic is characterised by the use of variables and deals with "formulas", that is, with sentences (including laws) where descriptive ("constant") terms are replaced by variables.[3] Moreover, Bocheński continues, there is a "further groundbreaking contribution of Aristotle to logic": the axiomatisation ("notwithstanding its weaknesses") of syllogistic ("categorical", restricted to subject-predicate propositions) (Bocheński 2002, for example, pp. 3-5, 74, 84-87, cf. Bocheński 1961). According to Bocheński, Aristotle's general theory of "categorical" syllogism, as presented in *Prior Analytics*, is a historical paradigm of what we should understand as formal logic.

In spite of some shortcomings in Aristotle's presentation and wording,[4] it is evident that his establishment of formal logic and general theory of axiomatics aimed at rigorous norms (especially if we consider Aristotle's time period) and constituted a significant step towards the attainment of a fully formalised formal logic. Although Aristotle claimed that syllogism is a matter of "internal speech" (ἔσω λόγος, *An. Post.* A 10, 76b 27[5]), he intended to implement his theory of syllogism and axiomatic theories in general in an appropriate, artificial language, not only by using term letters ("variables"), but also by expressing the structure of sentences and inferences in an unambiguous way, even if this structure departs from ordinary ways of speaking.[6] Thus, in his *Analytics* he clearly distinguished logical words ('all', 'not', 'some', 'belongs') and expressed a logical structure of sentences. He rendered the ordinary way of saying 'All Bs are As' (which he had used earlier, e.g., in *De interpretatione*, still without term letters) as 'A belongs to each B' (or, metatheoretically, 'A is predicated to each B'), and similarly for other sentences of the so-called "logical square". He often expressed a syllogism in the form of a conditional statement. By these means, Aristotle could strictly express the necessary "following" of a conclusion from its premises (*An. Pr*. A 1, 24b 18-21) so that it was possible to explicate all implicit assumptions (for example, conversion in the second and third figures of a syllogism) in order to reduce reasoning to "perfect" syllogisms, where no implicit assumptions of this kind are present (see, for example, *An. Pr*. A4 26b 28-30, A5 27a 15-18). The foundational principles of a syllogism (the principles of contradiction and of the excluded middle), beyond the "working" formal logic (the theory of syllogism), are addressed by Aristotle in *Metaphysics* (Aristotle 1973), especially in book Γ. Thus, Aristotle was not very far from strictly defining an artificial, formalistic language, which together with logical axioms and rules would provide a formal system of logic (or some other special theory).[7] Despite using expressions of natural language along with term letters and artificial phrases, Aristotle's approach resembled the requirements for a formal system as formulated by Frege: (a) a formal language ("ideography", *Begriffsschrift*) should express only what is relevant for reasoning, excluding any tacit connotations that may derive from natural language and discourse context ("ideography" contains only special symbols, not "words" of a natural language), and (b) reasoning should be described without any gaps (*lückenlos*), with all necessary axioms, inference rules and definitions made explicit (Frege 1988, X, 3; Frege 1998, V-VII).

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3 "... the use of letters instead of constant words gave birth to formal logic" (Bocheński 2002, 80-81).

4 See, e.g., in Łukasiewicz (1957, 16-19).

5 For *Analytica Priora et Posteriora* and their translations, see (Aristotle 1964, 2002, 2010, 1994-2000).

6 Aristotle's "structural discrepancy between abstract and the concrete forms of the syllogisms", stressed by Łukasiewicz (1957, 17), may be considered evidence of the artificiality of Aristotle's idiom in order to more exactly express logical forms.

7 An intensional semantics of Aristotle's syllogistic system is proposed in (Kovač 2013).

2. Formalism between sensible perception and concepts

A formal system (formalism), as conceived by Frege, seems to be essentially concerned with abstract, non-sensible "entities" such as concepts, propositions, inferences and thoughts (cf. Frege's "pure thought" or "pure concepts"). Nevertheless, a formal system is established only via a "mapping" of these abstract entities onto sensibly perceivable, written signs ("formal language", "ideography"). For Frege, proofs of a formal system are presented "to the eye" as a sequence of formulas (Frege 1998, V).[8] This indicates that the foundations of a formal system include the requirement of sensible givenness, which is something beyond logic if logic is understood strictly as an intellectual activity of formal reasoning.[9]

It should be noted that inscriptions in themselves, merely as objects of a sensible experience, do not need to reveal anything about what they are inscribing. For only if we in advance know that a given inscription is associated with a certain expression, particularly with a certain occurrence of an expression (e.g., with the first occurrence of 'A' in 'A v non-A'), can we read and grammatically or proof-theoretically check, say, given inscriptions of a proof. Moreover, we only logically understand an expression such as 'A v ¬A' if we understand that this is just one way we can choose a formal language to express some logical law (besides 'p v ¬p', 'ApNp', etc.) with which this expression has to have some structural similarity. Therefore, although exactness, strictness and (grammatical and proof-) checkability in a formalism stems on the one hand from sensible evidence of written expressions, it presupposes on the other some abstract and "ideal" or "conceptual" pre-understanding of expression-types and expressed "forms" themselves (cf., for example, Frege's or Gödel's "concepts" and "thoughts", Aristotelian ἔσω λόγος).

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[8] The visual accessibility of a written language is a pre-condition of the "strictness of proving" (*Strenge der Beweisführung*) and "sharpness of distinguishing" (Frege 1998, VI; Frege 1988, XI). See Kant's reflection in footnote 11 below.

[9] The sensible intuitive givenness of the expressions in a proof written in a formal language was especially highlighted by Hilbert, irrespective of all of the differences from Frege's philosophical position:

> If a logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given *intuitively*, together with the objects, as something that neither can be reduced to anything else nor requires reduction.
>
> (Hilbert 1967, 376, our emphasis)

Kant plays here an essential philosophical role:

> ... we find ourselves in agreement with philosophers, especially with Kant. Kant already taught ... that mathematics has at its disposal a content secured independently of all logic and hence can never be provided with a foundation by means of logic alone. ... [S]omething must already be given to our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. (Hilbert 1967, 376)

Hilbert's considerations about Kant's role were further deepened by Gödel on the grounds of his incompleteness theorems (see the last two paragraphs of 1995b).

In connection with this “conceptual” component in the understanding of formalised inscriptions, there is also the question of the design and choice of a formal system (its axioms, definitions and rules). For the formalisation of empirical knowledge, it is clear that the design of the system should consider and be based on empirical results. On the other hand, a question may be posed regarding the criterion of our choice of *logical* principles that should be incorporated into a formal system and accepted without being formally proven. Regardless of how one may respond to this question, the decision should obviously be made at least partly by means of conceptual considerations and one’s (= logical agent’s) self-evidence beyond proof procedures. As illustration, an example can be found in Łukasiewicz’s investigation of the laws of excluded middle and bivalence in the context of Aristotle’s discussion in *De interpretatione* (see Aristotle 1974). As Łukasiewicz emphasises, in the foundations of logic we should rely on personal “self-evidence”:

> Because it [the principle of bivalence] lies at the very foundations of logic, the principle under discussion cannot be proved. One can only believe it, and he alone who considers it *self-evident* believes it. To me, *personally*, the principle of bivalence does not appear to be self-evident. Therefore, I am entitled not to recognize it, and to accept the view that besides truth and falsehood there exist other truth-values, including at least one more, the third truth-value.
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> (Łukasiewicz 1967a, 36-37, our emphasis)[10]

While on the one hand the sensible givenness of signs facilitates exactness, on the other hand, if taken literally, that is, without sufficient abstract consideration of the signs, it leads to antinomies involving not just apparent, non-actual “entities” (such as Aristotle’s “goatstag”), but logically self-denying “subjects” and sentences (cf. “this sentence” in the formulation of the Liar, and Curry’s paradox).

3. The machine as a symbol of a formalism

The aforementioned Tarskian set-theoretic general definition of a formal system takes the concepts of a sentence and derivability (“consequence” of early Tarski) as primitive and describes them axiomatically by using set-theoretic notions; that is, it defines a formal system by means of a specially designed meta-theoretical axiomatic system, which has, in turn, its own pre-formal presuppositions (see Gödel 1990). The concept of a formal system can be rendered precise in its “abstract” (“absolute”) sense independently of any formalism (i.e., not defined in some given formal system), and at the same time, fully exact and strict. As is known, according to Gödel, who follows Turing (1965), the universal concept of a formal system is given independently of a

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10 See in his earlier text: “Yet in all reasoning there is inherent *formal* creative reasoning: a logical *principle of reasoning. ...* Logic is an *a priori* science. Its theorems are true on the strength of definitions and axioms derived from reason and not from experience. This science is a sphere of pure mental activity.” (Łukasiewicz 1970a, 11). And later, in the 1930s: “We are concerned with that meaning, with the thought and ideas expressed by signs, even if we do not know what these meanings are, and not with the signs as such” (Łukasiewicz 1970b, 241). Nevertheless, Łukasiewicz laid much stress on the linguistic expressions of logical laws (the “nominalistic guise” of logic, Łukasiewicz 1970c, 222) and sometimes radically dispensed with “thinking” as the object of logic: “It is not true, however, that logic is the science of the laws of thought. It is not the object of logic to investigate how we are actually thinking or how we ought to think. The first task belongs to psychology, the second to a practical art of a similar kind to mnemonics.” (Łukasiewicz 1957, 12, cf. also Łukasiewicz 1970c).

particular formalism (but not informally, see Crocco 2017) by a clear and precise perception of the concept of a mechanical procedure, defined by a Turing machine which can write down all the theorems of a given formal system (Gödel 1995a, 308): a "formal system is a mechanical procedure for producing formulas". For Gödel, the "essence" of a formal system "is that reasoning is completely replaced by mechanical operations on formulas" (1986a, 195; 1986c, 370), i.e., it is equivalent to a Turing machine ("mechanical procedure", "algorithm") for writing theorems of the system (Gödel 1995a, 308, see Crocco 2017, cf. Kennedy 2014).[11]

In the definition of a formal system by means of a Turing machine, the concept of a formal system is reduced to mechanical (and thus causal) terms and rendered objective.[12] Such a definition should not come as a surprise. We claim that a mechanical (or at least generally causal) perspective was essentially involved in devising and defining central logical concepts (not merely in the sense of helpful techniques and tools for drawing conclusions) from the beginnings of the history of logic. It can be shown, for example, that Aristotle's understanding of syllogism was basically dependent on causal terms (e.g., premises as causes of a syllogism). A syllogistic inference (of a "perfect" syllogism), according to an Aristotelian approach, is nothing but a (mechanical) "computation" (συλλογίζειν) of the terms of a syllogism according to the quality of the major (universal) premise and the quantity of the minor (affirmative) premise on the basis of the transitivity of predication.[13] "Necessity" in Aristotle's definition of a syllogism amounts to a mechanical causation of the conclusion by the premises as its "causes". Although the conclusion might be understood as the end (τέλος) of a syllogism, the conclusion follows (συμβάινει) merely due to the premises, without any additional, external cause ("through the being of premises", διὰ τὸ ταῦτα εἶναι). In addition, what is in accordance with a causal understanding of a syllogism are the non–reflexivity, non-monotonicity, and transitivity of syllogistic reasoning as well as the non-validity of not-P ⊨ P (see Kovač 2013).

Furthermore, specifically contributing to a mechanistic "picture" of Aristotel's syllogistic is his rule of ecthesis ("exposition"), presupposed in the foundational layer of his syllogistic. As pointed out by Hintikka (1967),[14] this is similar to Euclid's ecthesis accompanied by construction (κατασκευή, "preparation", "machinery"). By means of Aristotelian ecthesis, a construction of an instantiating term (concept) X is initiated as a sort of artificial, mechanical device that instantiates the terms of the premises and then "automatically" shows whether it also has some property P in question.[15] Given that the ecthetic term is "lower" than all of the terms of a considered syllogism, it could be understood as a singular term, intended at representing a singular object. It seems that in a

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11 Kant's "fabric of syllogisms" is a sort of anticipation: "Von der fabric der Vernunftschlüsse. Man sucht jederzeit die Vernunft zuletzt technisch zu machen, damit, indem man sie der Behandlung der Sinne unterwirft, man wegen der Fehler gewiß sey" (Kant 1924, 742, refl. 3256). Kant outlined, to his standards, a provably *complete* "system" of formal logic (including non-Aristotelian hypothetical and disjunctive propositional forms) on which this "fabric" should have been founded (cf., for example, Kant 1968, B VIII-IX, 94-101, 131-142, 359-361; Kant 1924).

12 The objectivity of the inherent "mechanical" thought of logical formalism is stressed by J. Salamucha (2003). He also thinks that "logical 'mechanization'", cultivating clarity, precision and objectivity, is essential for the development of "sound individuality" (p. 68).

13 On a specific "arithmetical interpretation of syllogistic" by Leibniz, cf. in Łukasiewicz (1957, 126-129).

14 See Hintikka's discussion on ecthesis in connection with Euclidean construction as a preparation for a proof.

15 On Aristotelian ecthesis, see, for example, Żarnecka-Biały (1993).

possible reduction of syllogistic to its foundations – the principles of non-contradiction and the excluded middle (*Met*. Γ, in Aristotle 1973) – ecthesis should play an essential role (cf. the ecthetic style of Aristotle's definition of universal propositions in *An. Pr.* 24b 28-30), regardless of the fact that in his "working logic" Aristotle uses ecthesis only occasionally, such as in proving conversion and third figure syllogisms (Kovač 2013).[16]

A Turing machine or a constructed ecthetic entity have a very general, symbolic character. To better understand their essence, we refer to some of Wittgenstein's striking reflections on machines.

> We use a machine, or the drawing of a machine, to symbolize a particular action of the machine. For instance, we give someone such a drawing and assume that he will derive the movement of the parts from it. (Wittgenstein 1958, 78e)
>
> We might say that a machine, or the picture of it, is the first of the series of pictures which we have learnt to derive from this one. (Wittgenstein 1958, 78e)[17]

According to Wittgenstein, a machine (or picture of a machine) can be used as a *symbol for a certain way of operation* (causation) or motion (activity), in distinction to a real machine, which additionally includes accidental properties such as deformability. In the symbolic sense, a machine has its determinate way of operation, its possible motions (excluding its additional behaviour as a given real machine), "*in itself"*. Wittgenstein suggests that a symbolic (picture of a) machine, containing possible motions, does not just depict motions, but rather, as a symbol, has some closer, non-empirical relation to them[18] (in *Tractatus*, Wittgenstein 1976, the possibility of a motion, and thus a machine, would indeed be understood as a picture of motion). Wittgenstein's symbolic machine shares its abstractness with a Turing machine. We can conceive of a Turing machine (or its picture, presenting its scanning/writing head, its tape, and its flowchart) as a *symbol* of the specific provability (*possible* steps in proofs) of the corresponding formal system (not just as a device doing an assigned job). A Turing machine symbolises this formal system, containing in "itself" all provability "moves" of the system. Machines and pictures of machines in themselves constitute a universal language, independent of given formalisms, but nonetheless fully exact and strict and thus able to directly evince the rigouristic nature of a formalism.[19] Let us

16 The following Aristotelian causal analysis of a syllogism is proposed in (Kovač 2013): the premises as the material cause (*Met.* Δ2, 1013b 20-21), the figure of a syllogism (position of the middle term) as its formal cause, ecthesis as the moving cause, and the conclusion as the final cause.

17 "Wir gebrauchen eine Maschine, oder das Bild einer Maschine, als Symbol für eine bestimmte Wirkungsweise. Wir teilen z.B. Einem dieses Bild mit und setzen voraus, daß er die Erscheinungen der Bewegung der Teile aus ihm ableitet*."* (Wittgenstein 1958, 78). "Wir könnten sagen, die Maschine, oder ihr Bild, sei der Anfang einer Reihe von Bildern, die wir aus diesem Bild abzuleiten gelernt haben." (Wittgenstein 1958, 78).

18 Cf. "... 'so the possibility of the movement stands in a unique relation to the movement itself; closer than that of a picture to its subject'; for it can be doubted whether a picture is the picture of this thing or that. ... but we do not say "Experience will shew whether this gives the pin the possibility of this movement"..." (Wittgenstein 1958, 79e).

19 Wittgenstein's philosophical interest in machines can be naturally connected with his study of aeronautics and in particular with the books containing machine drawings that he possessed in his private library, including works by Leonardo da Vinci (Les manuscrits de Leonardo de la Bibliotheque de l' Institut de France, six vols., Paris, 1881-1891), Faust Vrančić (Faustus Verantius, Machinae novae, Venetiis, 1615/16, see Vrančić 1993) and Georg Andreas Böckler

note that in a Wittgensteinian sense, an Aristotelian ecthetic (singular) term X can be understood as a *symbol* of the interrelations of the terms as proposed by the premises of a syllogism, where we abstract from any other "real" properties X might otherwise possess, i.e., we can understand X as symbolising this particular syllogism as well as the conversions and other syllogisms it logically includes.

4. Formalism, determinism and Ł3

To better understand some aspects of the foundations of formal systems, we will briefly analyse Łukasiewicz's three-valued logic Ł3, which is well-known for having been inspired by the discussion on determinism, causality and contingency. Given that formal reasoning (proofs in a formal system) is a mechanical and thus causal affair, it should possess general features of determinacy. Determinism is for Łukasiewicz "the belief" that

> if *A* is *b* at instant *t* it is true at any instant earlier than *t* that *A* is *b* at instant *t*.
>
> (Łukasiewicz 1967a, p. 22)

Since a TM is an idealized device (independent of any circumstances in the physical world), we can take that, ideally, a TM for a formal system S is available at any instant *t*' (independently of the indeterminacies of the physical world) and thus, if a TM which starts working at *t′* is to produce a theorem T of S at the moment *t*, then it is and was always true that the TM for S, starting at *t′*, will produce T at *t*. Łukasiewicz did not give a formal definition of formal provability, but rather, the definition of necessity (L) for Ł3:

$$L\varphi = \neg(\varphi \rightarrow \neg\varphi),$$

which truth-functionally excludes indeterminacy and falsehood, and covers an essential feature of the work of a (deterministic) TM that produces theorems of S.[20] Possibility is defined dually, $M\varphi = \neg L\neg\varphi = \neg\varphi \rightarrow \varphi$ (Łukasiewicz 1967b, pp. 55, 57).

Ł3 and its reformulations make it possible to study in general how the indeterminacy of events interacts with deterministic structures (including formalisms) and how particular deterministic conditions are embedded in a deterministic, causal structure as a whole.

Deterministic justification and indeterminism in Ł3 can be made more explicit if Ł3 is reformulated by using explicit necessity and possibility, as in Minari's (2002) $W^{\Box}$, a "modal" equivalent of Ł3 (i.e., of its axiomatisation W by Wajsberg). For convenience, we repeat Minari's $W^{\Box}$ (2002, 172):

$W^{\Box}.1$	$\varphi \rightarrow (\psi \rightarrow \varphi)$
$W^{\Box}.2$	$(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
$W^{\Box}.3$	$(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$
$W^{\Box}.4$	$\varphi \rightarrow \neg\neg\varphi, \quad \neg\neg\varphi \rightarrow \varphi$
$W^{\Box}.5$ (=**T**)	$L\varphi \rightarrow \varphi$

(Theatrum machinarum novum, Nürnberg, 1661). See Spadoni (1985, 25-45) and http://digitalcollections.mcmaster.ca/russell-lib/media/machinae-novae-fausti-verantii-siceni. These books may have played a significant role in inspiring Wittgenstein in his early "picture philosophy" (sign and machine as pictures), as well as in his later philosophy of language games. Wittgenstein probably did not know that Vrančić was also the author of a small logic (Vrančić 2018), containing in its second edition an argumentation against the possibility of metaphysics.

[20] In Ł3, the negation of indeterminacy returns indeterminacy, and the valuation v of $\varphi \rightarrow \psi$ returns $\min(1, 1 - v(\varphi) + v(\psi))$ (a conditional between indeterminacies returns truth).

$W^{\Box}$.6 (=**B**) $\varphi \to LM\varphi$

$W^{\Box}$.7 $(L\varphi \to (L\varphi \to \psi)) \to (L\varphi \to \psi)$

$W^{\Box}$.8 $\varphi \to (\varphi \to L\varphi)$

Rules: Modus ponens, Necessitation,

where φ, ψ, and χ are metavariables for formulas of $W^{\Box}$.[21] A set Γ is inconsistent iff $\Gamma \vdash \bot$, where ⊥ is short for $\neg(\varphi \to \varphi)$.

The counterparts of modal formulas **K**, 4 and 5 are theorems. In distinction to modal system S5 (and S4, which is used in Gödel's 1933 modal translation of IPC), **we** recall that in Ł3 (and so in $W^{\Box}$) some classical tautologies are not valid (due to the possible indeterminacy of subformulas), for example, $\varphi \vee \neg\varphi$, $\neg(\varphi \wedge \neg \varphi)$, $(\varphi \wedge \neg\varphi) \to \psi$ (although $\varphi \to (\neg\varphi \to \psi)$ is valid). At the same time, some S5 (and S4) non-valid formulas become valid in Ł3. For example, $(M\varphi \wedge M\psi) \to M(\varphi \wedge \psi)$, which seems to be "counterintuitive" if we consider it from the viewpoint of possible world semantics. However, in one-world semantics with indeterminacies (like truth-functional semantics for Ł3 and $W^{\Box}$), it might be something quite natural: all possibilities (say, φ and ψ) are now "incorporated" in one and the same world as its indeterminacies or, as the case may be, truths. Technically, since in Ł3 $M\chi = \neg\chi \to \chi$ and the conditional between indeterminacies returns truth, then $M(\varphi \wedge \psi)$ never decreases the value of $M\varphi \wedge M\psi$. On the other hand, $\varphi \to L\varphi$ ("modal collapse") is classically valid under the aforementioned truth-functional definition of L in in Ł3, but is non-valid in Ł3 due to the potential indeterminacy of φ, which falsifies Lφ. To verify the "modal collapse", Ł3 requires φ itself as a pre-condition: $\varphi \to (\varphi \to L\varphi)$, i.e., *if* φ is assumed, then its "modal collapse" results; indeed, indeterminate φ (truly) implies indeterminate $\varphi \to L\varphi$. Let us also note that $L\neg L\bot$ is a theorem of Ł3 (as well as of S4 and S5, in their original sense of 'L'), indicating that 'L', if viewed from the standpoint of Ł3 ($W^{\Box}$), cannot substitute the formal provability concept (because of the second incompleteness theorem), but rather generally accounts for the causal impossibility of a deterministic (mechanical) system being in some self-contradictory state.

Łukasiewicz's necessity should be distinguished from a universal concept of proof (Gödel's "absolute proof", "abstract proof", see Gödel 1986b, 1995d), which is not reducible to formal provability or determinism, as well as from Gödel's ontotheological concept of necessity (Gödel 1995c), which, despite its S5 propositional base, leads to "modal collapse" as a natural consequence due to the specific higher-order ("abstract") concepts and perspective involved.

Besides necessity in general, a state *s* of a deterministic mechanical system (like a deterministic TM) is necessitated (caused, justified) by particular configurations that precede *s.* In order to be able to formally present *particular* deterministic necessitation, we sketch out logic W^J, where besides L: φ, where L is a term (justification by means of a whole causal structure or an unspecified part of it), we introduce the form t: φ to allow specific causal justifications ("t presently necessarily justifies φ", "t actually causally justifies φ"). In W^J, we partially apply tools of justification logic (e.g., Fitting 2005), without possible worlds semantics. Vocabulary consists of L, e_i, :, +, ×, p_i, ¬, →, (,) (*i* is a positive integer); justification term $t ::= L \mid e_i \mid (t_1+t_2) \mid (t_1 \times t_2)$; formula $\varphi ::= p \mid \neg\varphi \mid \varphi_1 \to \varphi_2 \mid t{:}\ \varphi$. W^J has axioms analogous to $W^{\Box}$.1-4, while $W^{\Box}$.5-8 are replaced by W^j.5-8, respectively:

W^j.5 (=W^j**T**) $t{:}\ \varphi \to \varphi$

[21] W consists of the axioms $W^{\Box}$.1, $W^{\Box}$.2, $(\neg\psi \to \neg\varphi) \to (\varphi \to \psi)$, $((\varphi \to \neg \varphi) \to \varphi) \to \varphi$, and modus ponens.

$W^j.6$ (=$W^j\mathbf{B}$) $\neg\varphi \rightarrow L{:}\ \neg t{:}\ \varphi$

$W^j.7$ $(t{:}\ \varphi \rightarrow (t{:}\ \varphi \rightarrow \psi)) \rightarrow (t{:}\ \varphi \rightarrow \psi)$

$W^j.8$ $\varphi \rightarrow (\varphi \rightarrow L{:}\ \varphi)$

where t is an arbitrary causal justification (possibly 'L'). 'L: φ', which could be read as 'there is a cause that φ', can be understood as referring to a *whole* actual causal structure (e.g., the causal structure of a Turing machine under consideration), within which we know that φ is the case (effectuated). $W^j.8$ expresses a weakened "modal collapse".

The additional propositional axioms (involving an interplay between distinct justification terms) of W^j are:

$W^j.9$ $t{:}\ \varphi \rightarrow (t{+}u){:}\ \varphi$, $u{:}\ \varphi \rightarrow (t{+}u){:}\ \varphi$

$W^j.10$ $t{:}\ (\varphi \rightarrow \psi) \rightarrow (u{:}\ \varphi \rightarrow (L{:}\ \psi \rightarrow (t{\times}u){:}\ \psi))$

$W^J.11$ $t{:}\ \varphi \rightarrow L{:}\ \varphi$

Term (t+u) is the sum of deterministic causal justifications t and u, whereas (t×u) is the application of t to u (corresponding to the evidence in justification logic). In W^j, for simplicity, the necessitation rule allows 'L:' to be prefixed in front of a proven formula. The following proposition shows in which way the W^J counterparts of **K**, **4** and **5** involve a deterministic combination of causes and their embedding in a whole causal structure (L).

Proposition.

$W^J\mathbf{K}$ $t{:}\ (\varphi \rightarrow \psi) \rightarrow (u{:}\ \varphi \rightarrow (t{\times}u){:}\ \psi)$

$W^J\mathbf{4}$ $t{:}\ \varphi \rightarrow ((L{\times}t){\times}t){:}\ L{:}\ \varphi$

$W^J\mathbf{5}$ $\neg L{:}\ \varphi \rightarrow (L{\times}L){:}\ \neg t{:}\ \varphi$

Proof.
(a) $W^J\mathbf{K}$ is proven by adapting the sketch of the proof for **K** in Minari (2002), starting from $t{:}\ (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \psi)$ ($W^j.\mathbf{T}$) and $(u{:}\ \varphi \rightarrow \varphi) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (u{:}\ \varphi \rightarrow \psi))$ ($W^j.2$) to get $t{:}\ (\varphi \rightarrow \psi) \rightarrow (u{:}\ \varphi \rightarrow \psi)$, and combining the use of $W^j.8$ with $W^j.10$ to obtain $W^J\mathbf{K}$.
(b) Proof of $W^J\mathbf{4}$ (cf. hints for **4** in Minari 2002): 1. $\varphi \rightarrow (\varphi \rightarrow L{:}\ \varphi)$, 2. $L{:}\ (\varphi \rightarrow (\varphi \rightarrow L{:}\ \varphi))$, 3. $L{:}\ (\varphi \rightarrow (\varphi \rightarrow L{:}\ \varphi)) \rightarrow (t{:}\ \varphi \rightarrow (L{\times}t){:}\ (\varphi \rightarrow L{:}\ \varphi))$, 4. $t{:}\ \varphi \rightarrow (L{\times}t){:}\ (\varphi \rightarrow L{:}\ \varphi)$, 5. $(L{\times}t){:}\ (\varphi \rightarrow L{:}\ \varphi) \rightarrow (t{:}\ \varphi \rightarrow ((L{\times}t){\times}t){:}\ L{:}\ \varphi)$, 6. $t{:}\ \varphi \rightarrow (t{:}\ \varphi \rightarrow ((L{\times}t){\times}t){:}\ L{:}\ \varphi)$, 7. $t{:}\ \varphi \rightarrow ((L{\times}t){\times}t){:}\ L{:}\ \varphi$.
(c) Proof of $W^J\mathbf{5}$ (cf. hints for **5** (=**E**) in Minari 2002): 1. $\neg L{:}\ \varphi \rightarrow L{:}\ \neg((L{\times}t){\times}t){:}\ L{:}\ \varphi$ ($W^J\mathbf{B}$), 2. $\neg((L{\times}t){\times}t){:}\ L{:}\ \varphi \rightarrow \neg t{:}\ \varphi$ ($W^J\mathbf{4}$), 3. $L{:}\ (\neg((L{\times}t){\times}t){:}\ L{:}\ \varphi \rightarrow \neg t{:}\ \varphi)$, 4. $L{:}\ \neg((L{\times}t){\times}t){:}\ L{:}\ \varphi \rightarrow (L{\times}L){:}\ \neg t{:}\ \varphi$, 5. $\neg L{:}\ \varphi \rightarrow (L{\times}L){:}\ \neg t{:}\ \varphi$ (1,4). □

The W^J counterpart of modal formula **4** may be informally conceived of as the positive introspection of deterministic evidence, leading to deeper foundations of a deterministic chain within a holistic causal structure. We note that $W^J\mathbf{5}$ avoids requiring one and the same particular justification of any φ not justified by t, and instead states necessity in general, (L×L), as being responsible for ¬t: φ (see the discussion on the problem with the axiom candidate $\neg t{:}\ \varphi \rightarrow ?t{:}\ \neg t{:}\ \varphi$ in Artemov and Fitting 2016, but cf. also Artemov et al. 1999).

Corollary. $W^{\square} \subset W^J$, $W \subset W^J$. Proof. Given that L is also a t, with $W^J.11$ the first part easily follows. The second part is obvious, because W and $W^{\square}$ are equivalent (see Minari 2002, Theorem 2.6 ii). □

Semantics. The usual truth-functional semantics for Ł3 is extended to define the semantic properties of justification terms and the formulas that include them. To this end, a special function of deterministic evidence, **E,** is added**.** Let 1, ½ and 0 be the values true, indeterminate and false, respectively. Then, a W^J model **M** = ⟨V, D, v, **E**⟩, where the set of values V = {1, ½, 0}, the set of designated values D = {1}, $v(\varphi) \in V$ such that $v(p) \in V$, $v(\neg\varphi) = 1 - v(\varphi)$, $v(\varphi \to \psi) = \min(1, 1 - v(\varphi) + v(\psi))$, with the addition that

$v(t{:}\ \varphi) = 1$ iff $v(\neg(\varphi \to \neg\varphi)) = 1$ and $t \in \mathbf{E}(\varphi)$, otherwise $v(t{:}\ \varphi) = 0$,

and where the following conditions for $t \in \mathbf{E}(\varphi)$ hold:

(1) if $t \in \mathbf{E}(\varphi)$, then $(t+u) \in \mathbf{E}(\varphi)$; if $\mathbf{u} \in \mathbf{E}(\varphi)$, then $(t+u) \in \mathbf{E}(\varphi)$,
(2) if $v(\neg\varphi) = 1$ or $v(\neg\varphi) = ½$, then $L \in \mathbf{E}(\neg t{:}\ \varphi)$ for any t,
(3) if $t \in \mathbf{E}(\varphi \to \psi)$ and $u \in \mathbf{E}(\varphi)$, then $(t \times u) \in \mathbf{E}(\psi)$,
(4) if $v(\neg(\varphi \to \neg\varphi)) = 1$, then $L \in \mathbf{E}(\varphi)$,
(5) if $t \in \mathbf{E}(\varphi)$, then $L \in \mathbf{E}(\varphi)$.

Minari (2002) has proven that $W^{\Box}$ is equivalent to W (as mentioned above), and thus sound and complete with respect to the same models as W. For the soundness and completeness of W^j, we focus on the formulas that include subformulas of the shape t: φ.

(a) *Soundness*. Let us take some examples. (a) Axiom W^j.5 is semantically obvious from the definition of the satisfaction of t: φ. (b) For W^j.6, suppose that ¬φ is true in **M** and thus φ is false in **M**. It follows that for any t, t: φ is false, and thus ¬t: φ and $\neg(\neg t{:}\ \varphi \to t{:}\ \varphi)$ are true in **M**. With the condition (2) for **E**, we obtain the truth of L: ¬t: φ, and thus the truth of W^j.6. Suppose, alternatively, that ¬φ, and so φ, are indeterminate in **M**. Hence ¬t: φ is true and, by condition (2) for **E,** L: ¬t: φ is true as well, which gives the truth of W^j.6. (c) For W^j.10, assume that t: (φ → ψ), u: φ and L: ψ of W^j.10 = $t{:}\ (\varphi \to \psi) \to (u{:}\ \varphi \to (L{:}\ \psi \to (t \times u){:}\ \psi))$ are true in **M**. It follows that ψ, and thus ¬(ψ → ¬ ψ), are true in **M** as well. By the condition (3) for **E**, the truth of (t×u): ψ results.
(b) *Completeness*. The proof can be adapted from Minari (2002, Section 4), with L generalised to any justification term t. Let us sketch out such an adaptation. We define that a set Γ is W^j-maximal consistent iff $\Gamma \nvdash \bot$ and for each formula φ and each justification t, either (a) $\Gamma \vdash t{:}\ \varphi$, or (b) $\Gamma \vdash \neg t{:}\ \neg\varphi$ and $\Gamma \vdash \neg t{:}\ \varphi$, or (c) $\Gamma \vdash t{:}\ \neg\varphi$ (MAX3J, cf. Minari's Definition 4.1). A W^j consistent set Γ can be consistently extended, for each φ and t, with only one of the subsets {t: φ}, {¬t: ¬φ, ¬t: φ} and {t: ¬φ} (cf. Minari's Lemma 4.2). In addition, it can be shown (cf. Minari, Lemma 4.3) that for a W^j consistent set Γ, Minari's following rewritten conditions hold: (i) if $\Gamma \vdash t{:}\ (\varphi \to \psi)$ for some t, then exclusively either (a) $\Gamma \vdash v{:}\ \psi$ for some v, or (b) $\Gamma \vdash u{:}\ \neg\varphi$ for some u, or (c) $\Gamma \vdash \neg u'{:}\ \neg\varphi$, $\Gamma \vdash \neg u'{:}\ \varphi$, $\Gamma \vdash \neg v'{:}\ \neg\psi$, and $\Gamma \vdash \neg v'{:}\ \psi$ for any u′ and v′; (ii) if $\Gamma \vdash t{:}\ \neg(\varphi \to \psi)$ for some t, then $\Gamma \vdash u{:}\ \varphi$ for some u and $\Gamma \vdash v{:}\ \neg\psi$ for some v; (iii) if $\Gamma \vdash \neg t{:}\ \neg(\varphi \to \psi)$ and $\Gamma \vdash \neg t{:}\ (\varphi \to \psi)$ for any t, then either (a) $\Gamma \vdash u{:}\ \varphi$ for some u, and $\Gamma \vdash \neg v{:}\ \psi$ and $\Gamma \vdash \neg v{:}\ \neg\psi$ for any v, or (b) $\Gamma \vdash \neg u{:}\ \varphi$ and $\Gamma \vdash \neg u{:}\ \neg\varphi$ for any u, and $\Gamma \vdash v{:}\ \neg\psi$ for some v. – For the proof, for instance, of (i), assume (1) $\Gamma \vdash t{:}\ (\varphi \to \psi)$, but (2) $\Gamma \nvdash v{:}\ \psi$ for any v, and (3) for each u, $\Gamma \nvdash u{:}\ \neg\varphi$. From (2) and MAX3J it follows: $\Gamma \vdash v{:}\ \neg\psi$ or both $\Gamma \vdash \neg v{:}\ \neg\psi$ and $\Gamma \vdash \neg v{:}\ \psi$. Thus, $\Gamma \nvdash v{:}\ \neg\psi$, from (3), since $\{t{:}\ (\varphi \to \psi), v{:}\ \neg\psi\} \vdash u{:}\ \neg\varphi$ for some u. Therefore, $\Gamma \vdash \neg v{:}\ \neg\psi$ and $\Gamma \vdash \neg v{:}\ \psi$ for each v. Also, from (3) and MAX3J, $\Gamma \vdash u{:}\ \varphi$ or both $\Gamma \vdash \neg u{:}\ \neg\varphi$ and $\Gamma \vdash \neg u{:}\ \varphi$. But $\Gamma \nvdash u{:}\ \varphi$, from (1), (2) and $W^j\mathbf{K}$. Therefore, $\Gamma \vdash \neg u{:}\ \neg\varphi$ and $\Gamma \vdash \neg u{:}\ \varphi$ for each u. The remaining instances of (i) and cases (ii) and (iii) can be proven by similar reasoning. – Furthermore, in analogy to Lindenbaum's lemma (cf. Lemma 4.4 in Minari 2002), it holds that the extension of Γ in some of the three mutually exclusive ways, i.e., by {t: φ}, {¬t: ¬φ, ¬t: φ} or {t: ¬φ}, is consistent. Finally (cf. Theorem 4.5 by Minari), a canonical valuation v^c can be defined by associating values 1, ½, and 0 to an atomic formula p in correspondence with alternatives (a)-(c) of MAX3J, respectively, and by the condition that in a canonical model $\mathbf{M^c}$, $t \in \mathbf{E}^c(\varphi)$ iff $\Delta^{max} \vdash t{:}\ \varphi$, where Δ^{max} is a maximal extension of Δ. Using (i)-(iii) above, the correspondence of the canonical valuation (values 1, ½, and 0 of φ) and the derivability (of t: φ, of both ¬t: ¬φ and ¬t: φ, or of t: ¬φ) from the related W^j-maximal consistent set ("truth lemma") can by proven by induction, with strong completeness following as a result.

Concluding note

Since ancient times, formal logic (including ancient formalisms or ancient incipient formalisms, as well as modern logical systems) has possessed a causal, mechanical sense, as only precisely stated by Turing and Gödel in the 20th century. Wittgenstein's reflections reveal the symbolic character of a machine. Łukasiewicz's philosophical views and formal analyses can be used to better understand the embedding of deterministic justifications in general in a broader causal context which includes indeterminacy and contingencies. Łukasiewicz's viewpoint was strongly influenced by the considerations on logical and physical concepts of necessity in the context of the causality of human decisions. This concept of necessity should be distinguished from Gödel's wider concepts of "abstract" provability and ontotheological necessity.

Acknowledgments

I am grateful to the reviewers, to Kordula Świętorzecka, and to the audience of the conference *Formal and Informal Methods in Philosophy*, Warsaw, 25-27 June 2018, for their helpful remarks and comments.

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