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Remarks on Axiomatic Rejection in Aristotle's Syllogistic*

Abstract. In the paper we examine the method of axiomatic rejection used to describe the set of non-valid formulae of Aristotle's syllogistic. First we show that the condition which the system of syllogistic has to fulfil to be completely axiomatised, is identical to the condition for any first order theory to be used as a logic program. Than we study the connection between models used for refutation in a first order theory and rejected axioms for that theory. We show that any formula of syllogistic enriched with classical connectives is decidable using models in the domain with two members.

Key words: Aristotle's syllogistic, axiomatic rejection, Horn clause

1. The starting point for our consideration is the presentation of Aristotle's syllogistic given by J. Łukasiewicz[1]. The language of the systems consists of individual variables *S*, *M*, *P*, ..., two binary predicate symbols *a* and *i* with infix notation, standing for positive categorical predicates of syllogistic (e.g. *SaM* and *SiM* are read resp. *every S is M* and *some S are M*) and classical propositional connectives: $\neg$ (negation), $\wedge$ (conjunction) $\vee$ (dysjunction), $\rightarrow$ (implication).

The set of theorems of syllogistic is given by the usual rules Modus Ponens and Substitution for individual variables and the following axioms:

(Ax 0) any substitution of a theorem of the classical propositional calculus is an axiom

(Ax 1) SaS

(Ax 2) SiS

(Ax 3) $MaP \wedge SaM \rightarrow SaP$

(Ax 4) $MaP \wedge MiS \rightarrow SiP$

Moreover, Łukasiewicz, following suggestions given by Aristotle, formalises the set of non-theorems of the system. Initially he used for that purpose two rejected axioms and the rules of Reversed Modus Ponens and Reversed Substitution for individual variables of schemata :

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[1] All the concepts and results of this section comes from Łukasiewicz and Słupecki. They were first published in in [4] and in details in [6]. The same results are reported in an easier to access book [5].

(MP^{-1})

$$\frac{\vdash \alpha \rightarrow \beta,\ \dashv \beta}{\dashv \alpha}$$

(Subst^{-1})

$$\frac{\dashv s(\alpha)}{\dashv \alpha}$$

where $\dashv$ marks a rejected formula, $\vdash$ stands for theoremhood, α, β are formulae and s is a substitution for individual variables. We will call MP^{-1} and Subst^{-1} basic rules of rejection. Because MP and Subst are the only rules of the accepted part of the system the basic rules form together the following, intuitively very clear, metalogical transposition rule:

(Transp)

$$\frac{\alpha \vdash \beta,\ \dashv \beta}{\dashv \alpha}$$

where $\vdash$ stands for derivability, and α, β are formulae. Furthermore we have

PROPOSITION 1 Let T be a theory and Φ a set of rejected formulae of T given by the basic rules of rejection and set of rejected axioms Φ_0. Then for any α

$\alpha \in \Phi$ iff there exists a formula $\beta \in \Phi_0$ such that β is derivable from α in T.

Such a formalisation of non-theorems was sufficient to reject all non-valid syllogisms considered by Aristotle. However in the context of classical propositional calculus there are infinitely many non-valid formulae which are not rejected by the basic rules of rejection and any finite set of rejected axioms. The problem is solved by adding a new recursive rule schema (Rule of Słupecki) to the system:

(RS)

$$\frac{\dashv \neg\alpha \rightarrow (\gamma_1 \wedge \ldots \wedge \gamma_k \rightarrow \gamma),\ \dashv \neg\beta \rightarrow (\gamma_1 \wedge \ldots \wedge \gamma_k \rightarrow \gamma)}{\dashv (\neg\alpha \wedge \neg\beta) \rightarrow (\gamma_1 \wedge \ldots \wedge \gamma_k \rightarrow \gamma)}$$

where γ, $\gamma_1, \ldots, \gamma_k$ are atomic formulae or negations of such formulae, and α, β are atomic. In the presence of RS it is enough to use a single rejected axiom:

(Ax^{-1}) $\dashv\ SaM \wedge PaM \supset SiP$

to obtain the complete and decidable system of Aristotle's syllogistic.

2. The Rule of Słupecki, RS, was treated purely technically and restricted its application to the system of syllogistic (see [5,6]). The rule has, however, a more general meaning. It is easier to see it in the language of clauses.

The expression of a form:

$$X \leftarrow Y$$

where X,Y are finite sets of atoms represents a clause, i.e. implication, in which the condition is the conjunction of formulae from Y and the conclusion is the disjunction of formulae from X. If $|X| \leq 1$, then a clause is a Horn clause. For the sake of simplicity in clauses we will omit the usual set-theoretical brackets *{* and *}* and instead of a symbol $\cup$ of the sum we will put a comma.

We will call a theory T a Horn theory if there is an adequate axiomatisation of T in the language of Horn clauses, i.e. there exists a set of Horn clauses, from which all valid clauses are derivable and nothing more is. The significance of Horn theories lies in the fact that they can be used as logic programs (see for example [3]).

The following proposition is well known in the theory of Horn clauses (see for instance [1]).

PROPOSITION 2. A theory T is a Horn theory if and only if it possesses the following disjunction property:

(DP) $$X,Y \leftarrow Z \in T \;\; iff \;\; X \leftarrow Z \in T \; or \; Y \leftarrow Z \in T$$

where X, Y, Z are finite sets of atoms.

Now we come back to the Rule of Słupecki. In the language of clauses it takes an equivalent form:

$$\frac{\dashv X, \alpha \leftarrow Y \; ; \;\; \dashv X, \beta \leftarrow Y}{\dashv X, \alpha, \beta \leftarrow Y}$$

where X, Y and α, β are resp. finite sets of atoms and atoms.

It is obvious that it is equivalent to the rule:

$$\frac{\dashv X \leftarrow Z\, ;\ \dashv Y \leftarrow Z}{\dashv X, Y \leftarrow Z}$$

where X, Y, Z are finite sets of atoms. By MP^{-1} it can also be strengthened to the following property:

(DP*) $$\not\dashv X, Y \leftarrow Z \ \textit{ iff } \ \dashv X \leftarrow Z \ \textit{ and } \ \dashv Y \leftarrow Z$$

The correspondence with the condition DP is self-evident.

The presence of the rule DP* allows us to restrict the problem of rejection in syllogistic to Horn clauses. Moreover the rule RS is not used for the rejection of Horn clauses due to the following theorem.

THEOREM 1 Let T be a Horn theory, Φ a set of rejected formulae of T defined by the rules MP^{-1}, $Subst^{-1}$, RS and set of rejected axioms Φ_0. Let $\alpha \subseteq \Phi$ be a Horn clause. Then α can be rejected without the use of RS.

PROOF Let us consider the last use of RS in the rejection of α illustrated by a schema:

$$\frac{\dashv X \leftarrow Z\, ;\ \dashv Y \leftarrow Z}{\dashv X, Y \leftarrow Z}$$

where X,Y,Z are sets of atoms. Since it is the last use of RS α can be rejected using the basic rules of rejection from $\dashv X, Y \leftarrow Z$. Thus by Proposition 1 the clause $X, Y \leftarrow Z$ is derivable from α in T . Since T is a Horn theory and α a Horn clause, by DP $X \leftarrow Z$ or $Y \leftarrow Z$ is also derivable from α in T. Thus in the rejection of α we can use $\dashv X \leftarrow Z$ or $\dashv Y \leftarrow Z$ instead of $\dashv X, Y \leftarrow Z$ and the use of RS is superfluous. In the same way any other use of RS can be eliminated. □

It is surprising that the postulate of decomposition of a formula to Horn clauses, known from logic programming, expressed by the rule RS, together with the transposition rule and the single rejected axiom is sufficient for a complete characterisation of non-valid formulae of Aristotle's syllogistic. Since axioms Ax1 - Ax4 are Horn clauses syllogistic is a Horn theory and DP also holds. Thus, both acceptance and refutation of an arbitrary formula of the theory can be reduced to acceptance and rejection of Horn clauses.

3. The relation between models of the theory and its axiomatic system rejection is described by the following theorem:

THEOREM 2. Let T be a theory and Φ a set of rejected formulae of T given by the basic rules of rejection and set of rejected axioms Φ_0. Let M be a model for T in which all rejected axioms from Φ_0 are false. Then any formula from Φ can be rejected in model of a size less or equal than M.

PROOF It is enough to notice that both basic rules of rejection preserve the size of a rejecting model.

(i) If $\vdash \alpha \to \beta$ holds in T than $\alpha \to \beta$ is true in any model of T. Thus α can be rejected in any model in which β is rejected.
(ii) If $s(\alpha)$ is rejected in a model M, than α is can be rejected in the model M′ obtained from M by changing the interpretation of terms according to the substitution s. The size of M′ equals the size of M. □

We can apply Theorem 2 to syllogistic. Syllogistic has its well known model system in which terms are represented by non-empty sets and predicates *a* and *i* - respectively by set inclusion and having a non-empty intersection. The only rejected axiom of the theory, Ax^{-1}, can be rejected in a model in a domain of two members. Thus any Horn clause of syllogistic can be rejected in such a model. The models can be generated from the following matrices:

a	01	10	11
01	V	F	V
10	F	V	V
11	F	F	V

i	01	10	11
01	V	F	V
10	F	V	V
11	V	V	V

where arguments of the characterised functions *a* and *i* correspond to non-empty sets in a two-membered domain, and their values are truth values of respectively atomic formulae (V - verum, F - falsum).

Since in classical logic any formula can be reduced to a conjunction of clauses (a conjunctive normal form) and, as we have shown, decision problem for clauses can be reduced to Horn clauses we have the following corollary.

COROLLARY Any formula of Aristotle's syllogistic enriched by classical connectives is decidable using models in a domain with two members.

Similar problem is considered by F. Johnson in [2]. It is shown that for specific type of formulae, called Aristotelian chains, the domain of a model can be restricted to three members. To adopt matrices analogous to ours in the presentation of that result we should use three elements generating arguments. In this way we obtain eight arguments, from which two extreme are eliminated. Thus in Johnson's approach we have six arguments instead of three in ours.

References

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