Two faces of obligation

Piotr Kulicki and Robert Trypuz*
John Paul II Catholic University of Lublin
Faculty of Philosophy
Al. Racławickie 14, 20-950 Lublin, Poland
{kulicki,trypuz}@kul.pl

February 11, 2013

Abstract

In the paper we discuss different intuitions about the properties of obligatory actions in the framework of deontic action logic based on boolean algebra. Two notions of obligation are distinguished – abstract and processed obligation. We introduce them formally into the system of deontic logic of actions and investigate their properties and mutual relations.

Introduction

The attempts to provide an adequate logical system of deontic notions such as obligation, permission and prohibition have been made systematically at least since the early fifties of the 20th century when the papers of G. H. von Wright and J. Kalinowski [17, 7] were published. None of the papers is commonly regarded as successful. One of the reasons is that there exists no single meaning of deontic notions. They occur in law, morality, technical regulations, rules of a game, etc. and they vary in many aspects. Different intuitions concerning the meaning of the deontic notions coexist in different contexts. In the present paper we study this phenomenon for the case of obligation.

We are interested in a philosophical justification of the notion of obligation based on its intuitive meaning aiming at a more philosophically then purely formally oriented reader. Most of the formal results we use in this paper were already presented in our earlier works (see [14, 15, 9]). The basic formalism introduced there to describe a deontic characterization of the possible behaviour of an agent in a single particular situation is flexible and can be extended to cover multiple situations (cf. [16]) and a sequential composition of actions (cf. [8]). Thus the problems discussed within that framework can be transferred to more complex settings.

*This research was supported by the National Science Center of Poland (DEC-2011/01/D/HS1/04445).

1 Atomic boolean algebra as a model of the action space of an agent

In the present paper, following [12, 3, 13], we use boolean algebra (BA) to model the variety of the possible actions of a rational agent. It is a formal tool that is well known[1] and simple but still powerful enough to express the aspects of an action we want to study.

We shall interpret the elements of BA as descriptions of actions in contrast to execution of a particular action[2]. Thus some of the elements of BA correspond to many possible behaviours of an agent. For the sake of simplicity we shall, however, call any element of BA just 'action'. In the symbolic language we shall use the Greek letters: α, β, γ, ... for actions. The special elements of BA $\mathbf{0}$ and $\mathbf{1}$ represent respectively an impossible action (a description that cannot be fulfilled) and a universal action (a description covering every possible behaviour).

There are three basic operators in BA, a unary one – a complement of an element (represented in this paper symbolically by an overline) – and two binary ones: a sum and a product (represented respectively by symbols "$\sqcup$" and "$\sqcap$"). In the case of BA of actions we shall understand "$\overline{\alpha}$" as the complement of an action α, i.e. as an action describing doing anything that does not fall under α, the sum $\alpha \sqcup \beta$ of actions α and β – as a free choice between the behaviours defined by α and β, and the product $\alpha \sqcap \beta$ of actions α and β – as an action falling under α and β at the same time (or, in other words, a parallel execution of α and β).

To form an algebra of actions it is enough to choose the elements of agent's behaviour to be included in the formalisation. Formally such elements are *generators* of the algebra – every element of the algebra can be obtained from the generators by the consecutive application of the operators. Generators are basic actions (in fact they are description of actions) that form a particular ontology or the space of a discourse abstracted from the possibly infinitely rich space of real actions.

We assume that the number of actions is finite. That makes our BA atomic – there exists a finite set of actions which we shall call *atoms*, that are pairwise disjunctive (for any two atomic actions α and β such that $\alpha \neq \beta$ we have $\alpha \sqcap \beta = \mathbf{0}$) and any action other then $\mathbf{0}$ can be defined as a sum of a certain set of atoms (in particular $\mathbf{1}$ is a sum of all atoms). Atoms can be seen as a complete description of a behaviour of an agent. The completeness of atoms is relative to the choice of ontology expressed in the choice of basic actions.

Example. *For an illustration let us consider an example of an agent witnessing an accident in which a person is hurt happening inside an office building. The most important actions to consider are calling the emergency service and providing first aid. In the example we do not take into account the order in which actions are performed we just consider the actions undertaken by the agent in a short period after the accident.*

The situation can be formalised by a BA with two generators. The structure of the algebra is presented in figure 1. The actions of calling the emergency

[1] For introduction to the theory and application of BA see for example [10].

[2] A more detailed discussion about the intuitive meaning of an algebra of actions can be found in [5] and [9].

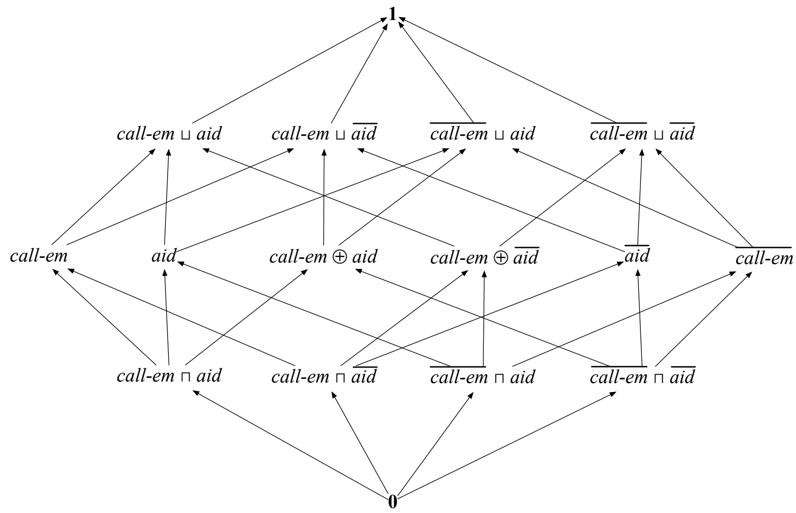

Figure 1: The structure of action algebra with two basic actions: *call-em* and *aid*. "⊕" is defined as follows: $call\text{-}em \oplus aid =_{df} (call\text{-}em \sqcap aid) \sqcup (\overline{call\text{-}em} \sqcap \overline{aid})$.

service and providing first aid are abbreviated in the figure by 'call-em' *and* 'aid' *respectively.*

By adding more basic actions to our action ontology we may receive a more complex structure of actions.

Let us consider two more basic actions: smoking a cigarette and calling the victims family. The structure that occurs now is too complicated to depict so we just list all atoms of the algebra ('call-fam' *stands for calling the victims family and* 'smoke' *– for smoking a cigarette):*

(atom 1) $call\text{-}em \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 2) $call\text{-}em \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 3) $call\text{-}em \sqcap \overline{aid} \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 4) $call\text{-}em \sqcap \overline{aid} \sqcap call\text{-}fam \sqcap smoke$

(atom 5) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 6) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 7) $call\text{-}em \sqcap aid \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 8) $call\text{-}em \sqcap aid \sqcap call\text{-}fam \sqcap smoke$

(atom 9) $\overline{call\text{-}em} \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 10) $\overline{call\text{-}em} \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 11) $\overline{call\text{-}em} \sqcap \overline{aid} \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 12) $\overline{call\text{-}em} \sqcap \overline{aid} \sqcap call\text{-}fam \sqcap smoke$

(atom 13) $\overline{call\text{-}em} \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 14) $\overline{call\text{-}em} \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 15) $\overline{call\text{-}em} \sqcap aid \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 16) $\overline{call\text{-}em} \sqcap aid \sqcap call\text{-}fam \sqcap smoke$

Intuitively, for example, atom 1 corresponds to the situation when an agent just calls the emergency services and does not perform any other of our basic actions (does not provide first aid, does not call victim's family and does not smoke a cigarette).

Example. *Note that the basic action of calling the victim's family is possible for an agent only if the agent knows the victim and knows his or her family. If it is not the case the set of atoms shrinks to the following eight ones:*

(atom 1) $call\text{-}em \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 2) $call\text{-}em \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 5) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 6) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 9) $\overline{call\text{-}em} \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 10) $\overline{call\text{-}em} \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 13) $\overline{call\text{-}em} \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 14) $\overline{call\text{-}em} \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap smoke$

Example. *To see how arbitrary actions can be constructed from atoms let us consider the basic action of calling emergency. In the case with 16 atoms this action can be defined as a sum of atoms 1 – 8 from the list. To construct the same action in the algebra reduced to 8 atoms it is enough to take the sum of atoms 1, 2, 5 and 6. That means that we can call emergency in 8 (or 4 in the second case) different ways – in the sense of combining that action with different other basic actions.*

Let us now introduce an axiomatic characterisation of the above theory of actions.

Axioms. *A standard list of axioms of BA consists of (see [10]):*

$$\alpha \sqcup \beta = \beta \sqcup \alpha,\ \alpha \sqcap \beta = \beta \sqcap \alpha \tag{1}$$

$$(\alpha \sqcup \beta) \sqcup \gamma = \alpha \sqcup (\beta \sqcup \gamma),\ (\alpha \sqcap \beta) \sqcap \gamma = \alpha \sqcap (\beta \sqcap \gamma) \tag{2}$$

$$(\alpha \sqcap \beta) \sqcup \beta = \beta,\ \alpha \sqcap (\alpha \sqcup \beta) = \alpha \tag{3}$$

$$\alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma) \tag{4}$$

$$\alpha \sqcup \overline{\alpha} = \mathbf{1},\ \alpha \sqcap \overline{\alpha} = \mathbf{0} \tag{5}$$

It is also reasonable to require that BA of actions is not degenerated, i.e. that there exists at least one possible action. Formally that requirement can be expressed by the following axiom:

$$\mathbf{0} \neq \mathbf{1} \tag{6}$$

We also define the order in BA in a standard way:

$$\alpha \sqsubseteq \beta =_{df} \alpha \sqcap \beta = \alpha \tag{7}$$

Intuitively, in such a situation α is a description of agent's behaviour that is more specific than β.

2 Permission and forbiddance

Boolean algebra of actions presented in the previous section forms our basic formal representation of actions. Now we introduce a deontic characteristics of actions into the discourse. We start from the notions of permission and prohibition (forbiddance). We find them useful for the further discussion of the notion of obligation being the main subject of the present paper and, at the same time, simpler and less controversial then obligation.

In our approach, for an action to be permitted (forbidden) means permitted (forbidden) in any possible circumstances, i.e. in combination with any other action. Such an understanding of prohibition is quite natural and straightforward. For permission it is not so obvious. In many contexts an action is regarded as permitted if there exist some ways of performing it that are permitted or, in other words, if it is not forbidden. However, our understanding of the notion is also in use and it is present in the literature as *strong permission* (cf. [18, 6]). The alternative understanding of permission is called *weak permission.* We shall introduce such a permission as a defined notion.

As theory of permission and prohibition we shall use the atomic closed system from [14][3], which we find more intuitive than the several similar systems discussed there. To express the notions of permission and prohibition in the symbolic language we shall use unary proposition forming operators $\mathtt{P}$ and $\mathtt{F}$ taking action names as arguments.

Axioms. *The axiomatization of the system is founded on the classical propositional calculus with Modus Ponens and Substitution for action variables. The set of axioms includes those of BA listed in the previous section, identity axiom:*

$$\alpha = \beta \to (\varphi \to \varphi(\alpha/\beta)), \tag{8}$$

where $\varphi(\alpha/\beta)$ is any sentence obtained from φ by replacing some or all the occurrences of α with β, and finally specific axioms for deontic operators:

$$\mathtt{P}(\alpha \sqcup \beta) \equiv \mathtt{P}(\alpha) \land \mathtt{P}(\beta) \tag{9}$$

$$\mathtt{F}(\alpha \sqcup \beta) \equiv \mathtt{F}(\alpha) \land \mathtt{F}(\beta) \tag{10}$$

$$\alpha = \mathbf{0} \equiv \mathtt{F}(\alpha) \land \mathtt{P}(\alpha) \tag{11}$$

$$\mathtt{F}(\gamma) \lor \mathtt{P}(\gamma), \ \textit{for } \gamma \textit{ being an atom of algebra.} \tag{12}$$

Axiom (9) says that a free choice between two actions is permitted if and only if each of them is permitted. Axiom (10) has a similar meaning for forbidden actions. Axiom (11) expresses the fact that only the impossible action is at the same time permitted and forbidden. Finally, axiom (12) states that every

[3] In the paper the system is referred to as $\mathcal{DAL}^4$.

atomic action is either permitted or forbidden. In other words any concrete behaviour that is not forbidden is permitted.

The following theorems can be proven:

$$\mathtt{P}(\alpha) \wedge \beta \sqsubseteq \alpha \rightarrow \mathtt{P}(\beta) \tag{13}$$

$$\mathtt{F}(\alpha) \wedge \beta \sqsubseteq \alpha \rightarrow \mathtt{F}(\beta) \tag{14}$$

$$\mathtt{P}(\alpha) \wedge \mathtt{F}(\beta) \rightarrow \alpha \sqcap \beta = \mathbf{0} \tag{15}$$

In our example let us assume that since the scenario take place in an office smoking is forbidden there. The situation of the accident does not change that norm[4].

The above mentioned weak permission (symbolically represented by the operator $\mathtt{P_W}$) can be defined in the following way[5]:

$$\mathtt{P_W}(\alpha) =_{df} \neg\mathtt{F}(\alpha) \tag{16}$$

3 Attempts to define the notion of obligation

The easiest way to introduce the notion of obligation to the deontic logic from the previous section would be by defining it within the system by means of the notions of permission and prohibition. In [15] we discussed and criticized three such definitions coming from [12] and [3]. Let us vey briefly recall this discussion.

Segerberg in [12] gives two definitions of obligation:

$$\mathtt{O_P}(\alpha) =_{df} \neg\mathtt{P}(\overline{\alpha}) \tag{17}$$

$$\mathtt{O_F}(\alpha) =_{df} \mathtt{F}(\overline{\alpha}) \tag{18}$$

With the first definition we can prove the following theorem:

$$\mathtt{F}(\beta) \wedge \alpha \sqsubseteq \beta \wedge \alpha \neq \beta \rightarrow \mathtt{F}(\alpha) \wedge \mathtt{O_P}(\alpha). \tag{19}$$

It says that any action that is properly included in a forbidden action is at the same time forbidden and obligatory. Such a property is obviously unacceptable. It would perhaps be more intuitive if we replaced strong permission with weak permission in the first of the above definition. Then, however, we obtain a definition which is equivalent to the second one – definition (18).

The second definition has other consequences usually regarded in deontic logic as paradoxical. One of them:

$$\mathtt{O_F}(\alpha) \rightarrow \mathtt{O_F}(\alpha \sqcup \beta) \tag{20}$$

is a well known *Ross paradox*. The other consequence of the introduction of definition (18) into the system is that the universal action is obligatory, formally:

$$\mathtt{O_F}(\mathbf{1}) \tag{21}$$

[4] Moreover, it is the case that cigarette smoke may have bad effect on the victim

[5] Let us note that weak permission is free from "deontic collapse" expressed by thesis (15) (see more about it in [15]).

We believe that it should not be a theorem of deontic logic. In our opinion any obligation needs a positive justification and having no alternative does not create an obligation. An action which is necessary is not necessarily obligatory.

Another definition of obligation comes from [3] and is as follows:

$$\mathtt{O}^{\mathtt{F}}_{\mathtt{P}}(\alpha) =_{df} \mathtt{P}(\alpha) \wedge \mathtt{F}(\overline{\alpha}) \tag{22}$$

This one is free from the problems of the definitions presented by Segerberg and entails many intuitive properties of obligation. However, somehow unexpectedly, it can be proved that at most one action can be obligatory, formally:

$$\mathtt{O}^{\mathtt{F}}_{\mathtt{P}}(\alpha) \wedge \mathtt{O}^{\mathtt{F}}_{\mathtt{P}}(\beta) \rightarrow \alpha = \beta \tag{23}$$

Our first reaction to that fact was an opinion that it is clearly against intuitions and renders the definition useless like the ones mentioned earlier. Let us return to our example. Obviously, calling emergency service and providing first aid are different actions. Moreover, quite naturally, we tend to agree that both of them are obligatory for any agent that faces the situation. Thus we have two different obligatory actions[6]. However the study of the paper of J. Czelakowski [4] has changed our views. Intuitions similar to Czelakowski's are also presented in the paper of O. Roy, A.J.J. Anglberger and N. Gratzl [11] considering deontic logic in the context of game theory. In both papers the authors explicitly refer to the unique action that is obligatory in certain situations. Such an action precisely defines the space of acceptable behaviour of an agent that obeys the norms governing its actual situation.

Thus, we have an obvious conflict of intuitions about the notion of obligatory actions. How to resolve this conflict of intuitions? We believe that we have to do here with two different notions of obligation. In the following sections we will try to define them, express in our logic and relate to each other and to other deontic notions.

4 Abstract versus processed obligation

The two notions of obligation identified in the previous section are connected with two different perspectives on norms, especially obligations, which can be regarded as external and internal.

In the first one we look at the problem situation from the outside and identify norms, especially prohibitions and obligations, that hold in it. We are interested in general norms, orders, promises and even agent's own personal resolutions[7] that may influence the deontic status of agent's actions in a certain situation. We can call them sources of deontic status. Obviously there can be many of them. In our example the role of the sources of deontic status can be played by a directive forbidding smoking in an office building and a general norm that makes it obligatory to safe the life of people in danger, that can be specified

[6] Taking into account this intuition in [15] we proposed an axiomatic characteristics of obligation avoiding the uniqueness of obligation

[7] We do not want to discuss the subject of sources of deontic situation of an agent, which are widely discussed in the literature. For example, let us just mention classical works of J.M. Bocheński's on the power of establishing norms and giving orders (deontic authority) [2] and P. S. Atiyah's book on promises [1]. A unified logical account of promises and orders is presented in [19]

by obligations to call the emergency service and to provide first aid. In the present considerations we assume that the union of sources' requirements is consistent. In general it may not be the case. If they are inconsistent, we have a conflict of norms that has to be resolved. We leave the job of expressing this very interesting and widely discussed problem in our framework for further investigations.

In the internal perspective we take the position of an agent facing the situation and we are interested in what the agent should actually do. We have to take into account all norms holding in the situation in question, process them and find a unique normative interpretation of the situation on the basis of which decision about a particular action can be made. K. Świrydowicz pointed out in a private conversation[8] that activity of a judge in court is similar. For example, in a criminal process, a judge, on the basis of legal norms, looks for the unique deontic characteristics of a situation and then confronts it with the actual behaviour of a defendant.

These two perspectives lead to two different notions of obligation, which we shall call *abstract obligation* and *processed obligation* respectively. We shall try to characterize the abstract obligation introducing an axiomatic system for it. The system is similar to, but not identical with, the one from [15]. For the processed obligation we use the definition of $\mathtt{O}_\mathtt{P}^\mathtt{F}$ from the previous section.

Let us employ the symbol $\mathtt{O}_\mathtt{A}$ for abstract obligation. We can characterize that obligation by showing the basic laws concerning it and its relation to forbiddance.

Axioms. *The fundamental principle is deontic consistency ensuring that there is no actual norm conflict (i.e. that no action is at the same time obligatory and forbidden) and potential executability of obligation (i.e. that it is possible to perform an obligatory action, or in other words an action that is impossible cannot be obligatory[9]) can be expressed by the following two axioms:*

$$\neg(\mathtt{O}_\mathtt{A}(\alpha) \wedge \mathtt{F}(\alpha)) \tag{24}$$

$$\neg\mathtt{O}_\mathtt{A}(\mathbf{0}) \tag{25}$$

Other laws enable us to generate new obligations and prohibitions from the ones already stated. They include the law of obligation combination:

$$\mathtt{O}_\mathtt{A}(\alpha) \wedge \mathtt{O}_\mathtt{A}(\beta) \rightarrow \mathtt{O}_\mathtt{A}(\alpha \sqcap \beta) \tag{26}$$

obligation trimming:

$$\mathtt{O}_\mathtt{A}(\alpha) \wedge \mathtt{F}(\beta) \rightarrow \mathtt{O}_\mathtt{A}(\alpha \sqcap \overline{\beta}) \tag{27}$$

and obligation economy:

$$\mathtt{O}_\mathtt{A}(\alpha) \rightarrow \mathtt{F}(\overline{\alpha}) \tag{28}$$

The first of them, namely (26), states that the parallel execution of two obligatory actions is also obligatory.

[8] The conversation took place during one of the coffee brakes at the conference "Applications of Logic in Philosophy and Foundations of Mathematics", held in Szklarska Poreba in 2012.

[9] That principle originates from the Roman laws (*ultra posse nemo obligatur* or *ad inpossibilia nemo tenetur*). It is also related to Kant's law known in moral philosophy stating that obligation entails possibility.

Example. *To illustrate it let us come back again to the accident example. As we have already mentioned we regard actions of calling emergency service and providing first aid as obligatory. In the symbolic language we have* $\mathsf{O}(call\text{-}em)$ *and* $\mathsf{O}(aid)$*. Thus, by (26) we can obtain* $\mathsf{O}(call\text{-}em \sqcap aid)$*. In our algebra action* $call\text{-}em$ *is equal to the sum of the listed below atoms:*

(atom 1) $call\text{-}em \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 2) $call\text{-}em \sqcap \overline{aid} \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 3) $call\text{-}em \sqcap \overline{aid} \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 4) $call\text{-}em \sqcap \overline{aid} \sqcap call\text{-}fam \sqcap smoke$

(atom 5) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 6) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 7) $call\text{-}em \sqcap aid \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 8) $call\text{-}em \sqcap aid \sqcap call\text{-}fam \sqcap smoke$

and action aid *is equal to the sum of the following ones:*

(atom 5) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 6) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 7) $call\text{-}em \sqcap aid \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 8) $call\text{-}em \sqcap aid \sqcap call\text{-}fam \sqcap smoke$

(atom 13) $\overline{call\text{-}em} \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 14) $\overline{call\text{-}em} \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 15) $\overline{call\text{-}em} \sqcap aid \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 16) $\overline{call\text{-}em} \sqcap aid \sqcap call\text{-}fam \sqcap smoke$

Consequently $call\text{-}em \sqcap aid$ *is equal to the sum of atoms which both actions have in common, i.e.:*

(atom 5) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap \overline{smoke}$

(atom 6) $call\text{-}em \sqcap aid \sqcap \overline{call\text{-}fam} \sqcap smoke$

(atom 7) $call\text{-}em \sqcap aid \sqcap call\text{-}fam \sqcap \overline{smoke}$

(atom 8) $call\text{-}em \sqcap aid \sqcap call\text{-}fam \sqcap smoke$

Thus, we can see that the law of obligation combining (26) can be used to make obligation more precise. The law of obligation trimming (27) has a similar character. It says that if we have an obligatory action and forbidden action that overlap, then the part of the obligatory action that is outside the forbidden action is also obligatory. Again, this law enables us to make obligation more precise.

Example. *Let us use the law in our example. We already have* $\mathtt{O}(\textit{call-em} \sqcap \textit{aid})$*. The fact that smoking is forbidden takes the symbolic form:* $\mathtt{F}(\textit{smoke})$*. By (27) we can obtain* $\mathtt{O}(\textit{call-em} \sqcap \textit{aid} \sqcap \overline{\textit{smoke}})$*. The action that occurs in the last formula is equal to the sum of the following atoms:*

(atom 5) $\textit{call-em} \sqcap \textit{aid} \sqcap \overline{\textit{call-fam}} \sqcap \overline{\textit{smoke}}$

(atom 7) $\textit{call-em} \sqcap \textit{aid} \sqcap \textit{call-fam} \sqcap \overline{\textit{smoke}}$

The last of the three laws – the law of obligation economy (28) reflects the fact that every obligation induces a prohibition of not obeying the obligation.

Note that we do not introduce the opposite implication:

$$\mathtt{F}(\overline{\alpha}) \rightarrow \mathtt{O_A}(\alpha) \tag{29}$$

That is because it would lead us to the equivalence corresponding to definition (18) which we have already rejected in section 2.

Let us now consider the relation between the two notions of obligation: abstract obligation $\mathtt{O_A}$, satisfying laws (24)–(28), and processed obligation $\mathtt{O_P^F}$ introduced by definition (22). The following auxiliary thesis will be useful:

$$(\mathtt{O_A}(\alpha) \wedge \mathtt{P}(\beta)) \rightarrow \beta \sqsubseteq \alpha \tag{30}$$

For the proof assume that $\mathtt{O_A}(\alpha)$ and $\mathtt{P}(\beta)$. By (28) $\mathtt{F}(\overline{\alpha})$. Since $(\beta \sqcap \overline{\alpha}) \sqsubseteq \overline{\alpha}$ by (14) we have $\mathtt{F}(\beta \sqcap \overline{\alpha})$. By (13) we also have $\mathtt{P}(\beta \sqcap \overline{\alpha})$. Thus by (11) $\beta \sqcap \overline{\alpha} = \mathbf{0}$ which means that $\beta \sqsubseteq \alpha$.

Intuitively law (30) states that *any obligatory action has to contain all permitted actions.*

Now let us state that if any action is obligatory in the abstract sense then processed obligation entails abstract obligation (providing there is some obligatory action in abstract sense):

$$\mathtt{O_A}(\beta) \rightarrow (\mathtt{O_P^F}(\alpha) \rightarrow \mathtt{O_A}(\alpha)) \tag{31}$$

For the proof assume that $\mathtt{O_A}(\beta)$, $\mathtt{P}(\alpha)$ and $\mathtt{F}(\overline{\alpha})$. By (30) $\alpha \sqsubseteq \beta$. By (27) $\mathtt{O_A}(\beta \sqcap \overline{\overline{\alpha}})$, which is equivalent to $\mathtt{O_A}(\beta \sqcap \alpha)$. Since $\alpha \sqsubseteq \beta \rightarrow \alpha \sqcap \beta = \alpha$ we obtain that $\mathtt{O_A}(\alpha)$.

In [4] and [11] obligation is described as the weakest (the most general) permission. This is the case for our processed obligation. Moreover, processed obligation is also the strongest of abstract obligations (if any action is obligatory in the abstract sense).

Due to the limited expressive power of our object language those properties cannot be expressed as theorems in it. Thus, we have to express them in a metalanguage as the properties of our logic in the following theorem:

Theorem 1. *The two following conditions are equivalent:*
(i) $\mathtt{O_P^F}(\alpha)$
(ii) α *is the weakest permitted action, i.e.* $\mathtt{P}(\alpha)$ *and for any action* β *such that* $\mathtt{P}(\beta)$*,* $\beta \sqsubseteq \alpha$*.*

Proof. The implication from (i) to (ii) can be expressed in the object language by the following equivalent formulas:

$$\mathtt{O_P^F}(\alpha) \rightarrow \mathtt{P}(\alpha) \wedge (\mathtt{P}(\beta) \rightarrow \beta \sqsubseteq \alpha) \tag{32}$$

$$\mathtt{O}^{\mathtt{F}}_{\mathtt{P}}(\alpha) \wedge \mathtt{P}(\beta) \rightarrow \beta \sqsubseteq \alpha \tag{33}$$

By definition (22) of $\mathtt{O}^{\mathtt{F}}_{\mathtt{P}}$ it is equivalent to

$$\mathtt{P}(\alpha) \wedge \mathtt{F}(\overline{\alpha}) \wedge \mathtt{P}(\beta) \rightarrow \beta \sqsubseteq \alpha \tag{34}$$

This, however, is an immediate consequence of theorem (15).

To prove the implication from (ii) to (i) let us assume that α is the weakest permitted action. Thus obviously α is permitted. Since α is the weakest permitted action, any atom not contained in α is not permitted. Thus all such atoms are forbidden. By (10) $\overline{\alpha}$ is also forbidden. Thus by definition (22) of $\mathtt{O}^{\mathtt{F}}_{\mathtt{P}}$ we receive $\mathtt{O}^{\mathtt{F}}_{\mathtt{P}}(\alpha)$ □ □

Theorem 2. *If any action is obligatory in the abstract sense, then the two following conditions are equivalent:*
(i) $\mathtt{O}^{\mathtt{F}}_{\mathtt{P}}(\alpha)$
(ii) α *is the strongest obligatory action in the abstract sense, i.e.* $\mathtt{O}_{\mathtt{A}}(\alpha)$ *and for any action* β *such that* $\mathtt{O}_{\mathtt{A}}(\beta)$, $\alpha \sqsubseteq \beta$.

Proof. The implication from (i) to (ii) can be expressed by formula (31) that is already proven.

For the implication from (ii) to (i) let us assume that α is the strongest obligatory action in the abstract sense. Thus none of the atoms contained in α is forbidden (otherwise in the presence of axiom (27) α would not be the strongest obligatory action) and consequently all atoms contained in α are permitted. By (9) α itself is also permitted. On the other hand, by (28) $\overline{\alpha}$ is forbidden. Thus, by definition (22) we have $\mathtt{O}^{\mathtt{F}}_{\mathtt{P}}(\alpha)$. □ □

Example. *Let us again return to our example. We have already stated that the action*

$$\textit{call-em} \sqcap \textit{aid} \sqcap \overline{\textit{smoke}}$$

being equal to the sum of the following atoms:

(atom 5) $\textit{call-em} \sqcap \textit{aid} \sqcap \overline{\textit{call-fam}} \sqcap \overline{\textit{smoke}}$

(atom 7) $\textit{call-em} \sqcap \textit{aid} \sqcap \textit{call-fam} \sqcap \overline{\textit{smoke}}$

is obligatory (in the abstract sense). By the law of atomic closure (12) both atoms are permitted (none of them is a part of explicitly forbidden action smoke and none of them is outside any obligatory action). By (9) their sum that is equal to $\textit{call-em}\sqcap\textit{aid}\sqcap\overline{\textit{smoke}}$ *is also permitted. All atoms other then atom 5 and atom 7 are forbidden and consequently not permitted. Thus* $\textit{call-em}\sqcap\textit{aid}\sqcap\overline{\textit{smoke}}$ *is the weakest permitted action. Thus, it is the action that is obligatory in the processed sense. As such it gives a space of choice for an agent that wants to obey the given norms. Since the agent has to call the emergency service, has to provide first aid to the victim and cannot smoke the only choice is between calling the victim's family or not doing it. This choice, as we noticed earlier, is not available for all agents, and for those, who do not have the possibility of calling the family there is no choice since there is an obligatory atomic action (namely atom 5) to be performed.*

Conclusions and further work

We have pointed out two different interpretations of obligation based on two different points of view on a normative situation. We have introduced both of them into the system of deontic action logic based on boolean algebra as abstract and processed obligation. We presented basic laws concerning both obligations and the relation between them.

The possibility of expressing the two notions of obligation proves that the formalism of deontic logic based on boolean algebra of actions is strong enough to conduct interesting philosophical investigations.

Since the present paper is intended to focus on a philosophical exposition of the introduced problems we leave the metalogical issues concerning the sketched logic of abstract obligations and transition between abstract and processed obligation such as completeness of the logic and its independent axiomatisation for further work.

Acknowledgements. We would like to thank professor Berislav Žarnić for his very constructive and detailed comments.

References

[1] P.S. Atiyah. *Promises, Morals, and Law.* Clarendon paperbacks. Clarendon Press, 1983.

[2] J.M. Bocheński. *Was ist Autorität?: Einf. in d. Logik d. Autorität.* Herder, 1974.

[3] Pablo F. Castro and T.S.E. Maibaum. Deontic action logic, atomic boolean algebra and fault-tolerance. *Journal of Applied Logic*, 7(4):441–466, 2009.

[4] Janusz Czelakowski. Action and deontology. In E. Ejerhed and S. Lindström, editors, *Logic, action, and cognition: essays in philosophical logic*, Trends in logic, pages 47–88. Kluwer Academic, 1997.

[5] J. Michael Dunn and Gary M. Hardegree. *Algebraic Methods in Philosophical Logic*, volume 41 of *Oxford Logic Guides.* Clarendon Press, Oxford, August 2001.

[6] D. Føllesdal and R. Hilpinen. Deontic logic: an introduction. In R. Hilpinen, editor, *Deontic Logic: Introductory and Systematic Reading*, pages 1–35. D. Reidel, Dordrecht, 1971.

[7] J. Kalinowski. Theorie des propositions normativess. *Studia Logica*, 1:147–182, 1953.

[8] Piotr Kulicki and Robert Trypuz. A deontic action logic with sequential composition of actions. In Thomas Ågotnes, Jan Broersen, and Dag Elgesem, editors, *Deontic Logic in Computer Science, Lecture Notes in Computer Science*, volume Volume 7393/2012, pages 184–198. Springer, 2012.

[9] Piotr Kulicki and Robert Trypuz. How to build a deontic action logic. In *The Logica Yearbook 2011*, pages 107–120. College Publications, 2012.

[10] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*. Monografie matematyczne. Państwowe Wydawn. Naukowe, 1963.

[11] Olivier Roy, Albert J.J. Anglberger, and Norbert Gratzl. The Logic of Obligation as Weakest Permission (Short Version). In Thomas Ågotnes, Jan Broersen, and Dag Elgesem, editors, *Deontic Logic in Computer Science, Lecture Notes in Computer Science*, volume Volume 7393/2012, pages 139–150. Springer Berlin / Heidelberg, 2012.

[12] Krister Segerberg. A deontic logic of action. *Studia Logica*, 41:269–282, 1982.

[13] Robert Trypuz. Simple theory of norm and action. In Anna Brożek, Jacek Jadacki, and Berislav Žarnić, editors, *Theory of Imperatives from Different Points of View*, Logic, Methodology and Philosophy of Science at Warsaw University 6, pages 120 – 136. Wydawnictwo Naukowe Semper, 2011.

[14] Robert Trypuz and Piotr Kulicki. A systematics of deontic action logics based on boolean algebra. *Logic and Logical Philosophy*, 18:263–279, 2009.

[15] Robert Trypuz and Piotr Kulicki. Towards metalogical systematisation of deontic action logics based on boolean algebra. In *Proc. 10th International Conference Deontic Logic in Computer Science*, volume 6181 of *Lecture Notes in Computer Science*, pages 132–147. Springer, 2010.

[16] Robert Trypuz and Piotr Kulicki. A norm-giver meets deontic action logic. *Logic and Logical Philosophy*, 20:59–72, 2011.

[17] G. H. von Wright. Deontic logic. *Mind*, LX(237):1–15, 1951.

[18] G. H. von Wright. *An Essay in Deontic Logic and the General Theory of Action*. North Holland, 1968.

[19] Tomoyuki Yamada. Acts of promising in dynamified deontic logic. In *JSAI 2007*, volume 4914 of *Lecture Notes in Computer Science*, pages 95–108, 2008.