



A Solution of Zeno's Paradox of Motion —based on Leibniz' Concept of a Contiguuum

Author(s): DAN KURTH

Source: *Studia Leibnitiana*, Bd. 29, H. 2 (1997), pp. 146-166

Published by: [Franz Steiner Verlag](#)

Stable URL: <http://www.jstor.org/stable/40694322>

Accessed: 24/05/2013 20:14

Your use of the JSTOR archive indicates your acceptance of the Terms & Conditions of Use, available at
<http://www.jstor.org/page/info/about/policies/terms.jsp>

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.



Franz Steiner Verlag is collaborating with JSTOR to digitize, preserve and extend access to *Studia Leibnitiana*.

<http://www.jstor.org>

A Solution of Zeno's Paradox of Motion – based on Leibniz' Concept of a Contiguum*

By

DAN KURTH (FRANKFURT AM MAIN)

Zusammenfassung

In der vorliegenden Arbeit soll eine Lösung der zenonischen Paradoxie des ruhenden Pfeils vorgestellt werden, die auf möglichen Implikationen des Kontiguumbegriffs beruht, wie ihn Leibniz in mehreren Arbeiten zu den Grundlagen der Dynamik entwickelt hat. Wesentlich sind dabei wechselseitige thematische Bezüge seiner *Theoria Motus Abstracti* und seines Dialogs *Pacidius Philalethi*. Aus der von Leibniz durchgeführten Analyse des Kontiguum als einer Voraussetzung der Möglichkeit von Bewegung ergibt sich, daß das (scheinbar zwischen Kontinuum und Diskretheit angesiedelte) Kontiguum – in heutiger Terminologie – nicht durch solche Merkmale wie Mächtigkeit oder Dichte bestimmt werden kann, sondern vielmehr eine besondere (topologische) Zusammenhangsstruktur aufweisen muß. In der Arbeit wird gezeigt, daß die dynamisch begründeten Anforderungen an eine solche Zusammenhangsstruktur von geeigneten topologischen Modellen einer Kette erfüllt werden.

I. The Philosophical Matrix: Zeno's Paradox of Motion

The central argument of Zeno against the possibility of motion is known as his – referring to Aristotle's listing of Zeno's arguments – 'third argument against (the possibility of) motion' or as his 'arrow paradox' or as 'the paradox of the resting arrow'. Its preparation will be found in Aristotle's *Physics* (Phys. Z 239b 5-7) and it then finally is stated in Phys. Z 239b 30.

The argument runs as follows:

- (1) Anything which occupies a space of exactly the same size as itself is at rest.
- (2) At any particular instant (i. e. the Aristotelian 'vuv') anything (including anything in motion) occupies a space of exactly the same size as itself.
- (3) From (1) and (2) follows: Anything (including anything in motion) is at any particular instant at rest.

* I wish to thank Prof. W. G. Saltzer and the members of his Colloquium at the Institut für Geschichte der Naturwissenschaften at the Johann Wolfgang Goethe-Universität (Frankfurt am Main) for interesting discussions of topics related to the argument of this paper. Special thanks goes to Dipl.-Phys. Frank Linhard and Dipl.-Ing. Dieter Brendel for helpful discussions of a previous version of this paper. I also wish to thank Dr. Vera Hohlstein for her perseverance and help by improving my original sketches to presentable Figures. By drafting the central parts of this paper I eventually found out how much I am indebted to Klaus Ripke who – more than seven years ago – introduced the concept of Antoine's necklace to me – in the context of the reconstruction of stoic philosophy, which may tacitly lie at the bottom of the more ostensible arguments.

(4) The flying arrow (the arrow in motion) is at rest.

This argument may be summarized as follows:

If an object at any instant has to be (at rest) in a particular commensurate part or point of space, then it impossibly can move (on).

From the above argument (1), (2), (3), (4) – as yet explicated – it obviously does not follow the impossibility of motion. Any opponent would be entitled to argue that even if the premises (1) and (2) and the conclusion (3) hold, the flying arrow might be at any instant at rest, but then would fly nicely nevertheless.

That opponent simply had to maintain that the arrow would rest at the first instant in a first part or point of space, at the second instant in a second part or point of space and so on, till it finally would arrive. Thus the opponent would hold that motion is essentially built up from a succession of rests.

But then such an opponent would dangerously underestimate the not so obvious strength of Zeno's argument.

This strength becomes more clearly manifest when the underlying paradox of division or of the unwarrantability of discreteness and continuity comes to light. I. e. Zeno would ask the respective opponent how the arrow could move from a state S_1 (i. e. when it is at rest at the first instant in a first part or point of space) to the state S_2 (when it is at rest at the second instant in a second part or point of space). By this question the focus of the paradox instantaneously shifts from the problem of *m o t i o n* to the problem of *i n f i n i t e s i m a l m o t i o n*, and to a hell of a bunch of therewith attached problems as well. And it is just here where Zeno's argument really puts its grip on.

Clearly the opponent now can no longer assume motion to be constituted by a succession of rests. He already got involved in the paradox of discreteness and continuity. If he chooses the succession $S_1, S_2, \dots, S_n, S_{n+1}, \dots$; to be of a continuous nature he never could reach the state S_2 by starting on S_1 because 'at first' he had to traverse an infinite number of intervening states S_i . And this holds for any two states or points in such a continuum.

If he then chooses the mentioned succession to be discrete, he gets a problem to explain what the arrow does in between two of such discrete states S_n and S_{n+1} . This is a real problem because by presupposition he has to admit that there actually is no part or point of space and therefore no arrow 'in between', not at last because there is nothing than the *h i a t u s*, i. e. an abyss of *n o t h i n g n e s s* 'in between'. This again holds true no matter how 'large' or – what might be of even greater significance – how 'small' the supposed 'distance' ever might be 'in reality'. Thus the advocate of discreteness gets – just as the arrow – devoured by the *h i a t u s* which separates S_n and S_{n+1} .

For our reason to expose the *r e a l* paradox of motion, i. e. to show that – by turning out to be a *p a r a d o x o f i n f i n i t e s i m a l m o t i o n* – it boils down to the paradox of discreteness and continuity, this may be enough – at least for now.

Before we now come to the attempt(s) of solving this real paradox of motion, some considerations about the 'philosophical' or – more generally – the 'theoretical' classification or evaluation of Zeno's paradox might be spent.

In his fine study of *Zenons Paradoxien der Bewegung und die Struktur von Raum und Zeit*¹ Rafael Ferber says "[wir] sehen [...] nämlich in den Paradoxien nicht mathematische bzw. physikalisch-mathematische, sondern physikalisch-empirische Probleme"².

In contrast to Ferber I would regard Zeno's paradoxes not at all as empirical problems (or challenges) of physics, and I would rather doubt that Zeno himself would have regarded it as such, even if one leaves out of account the question if any presocratic thinker would have been able to grasp the rather modern (one might say at best 17th century) concept of empiricity.

But I would agree that the paradoxes would not have been accounted for by Zeno as just logical or formal mathematical mental exercises or even pastimes. They had rather been intended to be mathematical-ontological arguments against the atomistic – or rather against any naturalistic – challenge of the eleatic ontocosmological convictions, quite in the spirit of Plato's later ontocosmo-theological efforts (e. g. in his *Timaios*) to ultimately beat off that kind of challenge. In both of those attempts the degree of success might lack somewhat compared with the degree of the fervor with which they then had been put forward.

II. The Construction of the Contiguum: From Cantor's Discontinuum by Antoine's Necklace to Leibniz' Chain Armor

The solution of Zeno's paradox as proposed in this paper is essentially based on two assumptions.

The first of these assumptions is that a zenoproof space has to be a kind of discontinuum.

The second refers to what kind of discontinuum it ought to be, i. e. to the topological (connectivity) structure of the considered discontinuum.

The guiding hypothesis is that this topological connectivity structure should be of such a kind that it

a) provides a sound model of the structure described by Leibniz as a 'contiguum', and

b) thus provides the basis for a comparably sound solution of Zeno's paradox.

1 R. Ferber: *Zenons Paradoxien der Bewegung und die Struktur von Raum und Zeit*, München 1981.

2 Ibid, p. 2.

1. The Definition of the Contiguum

For the definition of the *c o n t i g u u m*, i. e. the topological structure which is assumed to provide the basis for a zenoproof space, I follow Leibniz' reflections on this subject as pointed out in his dialogue *Pacidius Philalethi*³ and his *Theoria Motus Abstracti*⁴. Leibniz' concept of the *c o n t i g u u m* essentially stems from Aristotle's notion of *h a p t o m e n o n*⁵, but even though Aristotle discussed this concept also in the context of his considerations of Zeno's paradox he didn't even come near to any conclusion of using it effectively to solve these paradoxes. Instead he again and again fervently propagated – as the 'only concept' which could be regarded as apt to represent the fundamental structure of space – the *c o n t i n u u m* (συνεχής) as the 'true and only commensurate of space'.

Even though Leibniz is as well famous as one of the greatest advocates of the continuist view he in striking contrast to Aristotle came – one is attempted to say: 'infinitesimally' – near to the proper use of the *c o n t i g u u m* as a solution of the paradox, as will be shown in this paper.

The most explicit references to the concept of the *c o n t i g u u m* Leibniz makes in his *Pacidius Philalethi*. In the first of these he refers to the related notion of Aristotle's *h a p t o m e n o n*.

Reference I:

"The (ophilus): [Memini] Aristotelemquoque contiguum à continuo ita discernere, ut *c o n t i n u a* sint, quorum extrema unum sunt, *c o n t i g u a* quorum extrema simul sunt" (C, 601).

In the following second reference Leibniz (again in his *Pacidius Philalethi*) brings the now established concept of the *c o n t i g u u m* in the context of the explanation of motion, and thus seemingly prepares the ground for a concept of a contiguous (or – to better expose the timelike character of this motion – contingent) *m o t i o n*. Here he even refers to the *c o n t i g u u m* as generated by nature (seemingly for the reason to entail motion).

Reference II:

"Pa(cidius): Sed quo jure id negas, cum nulla sit in linea uniformi < continua > praerogativa unius puncti prae altero?

Ch(a)rinus): At nobis hic sermo non est de linea aliqua uniformi < continua > in qua duo ejusmodi puncta sibi immediata B et D ne sumi quidem potuissent, sed de linea AC jam actu in partes secta à natura, quia ponimus mutationem ita factam, ut uno momento existeret mobile in unius ejus partis AB extremo B, et altero in alterius partis DC extremo D. Estque discrimen inter has lineas duas actu < a se > divisas < contiguas >, et unam indivisam seu continuam

3 *Pacidius Philalethi*; C, 594-627.

4 *Theoria Motus Abstracti*; GM VI, 61-80; also in: GP IV.

5 See Aristotle *Phys.* E 226b 18ff; *Phys.* E 227a 20ff; *Phys.* Z 231a 21ff; *Phys.* Z 232a 8ff; *De Gen. et Corr.* 322b – 323b; *Metaph.* K 1069 a.

manifestum, quod, ut jam Aristoteles notavit, extrema B, D in duobus contiguis lineis differunt, in una continua coincidunt [...]" (C, 620/621).

In the third reference given here Leibniz comes to some conclusion and as well starts the preparation for his own 'solution' of the problem of a thorough explanation of motion, namely the so called *transcreation*, which unfortunately is no contiguous motion at all. Thus finally he didn't escape Zeno's trap when he admits a 'problem' of mapping a seemingly contiguous structure of motion to the seemingly continuous structure of the underlying space, and then inevitably fails in this trial.

Reference III:

"Ch(arinus): Itaque jam divisionis ac difformitatis causam habemus, et quomodo hoc potius quàm illo modo instituaturs divisio punctaque assignentur explicare possumus. [...] Tota res ergo eò redit: quolibet momento quod actu assignatur dicemus mobile in novo puncto esse. Et momenta quidem atque puncta assignari infinita, sed nunquam in eadem linea immediata sibi plura duobus, necque enim indivisibilia aliud quam terminos esse" (C, 622).

Thus unfortunately Leibniz in his *Pacidius Philalethi* finally didn't take the last step to set up a 'physical' concept of contiguous motion, but instead rather tried to substitute the physical concept of motion by the rather obscure notion of *transcreation*⁶, which strongly resembles a concept of Islamic philosophy called *kalam*⁷.

Another reference to the *contiguum* – this time of a slightly more technical kind can be found in the theorem 17 of Leibniz' *Theoria Motus Abstracti*.

Reference IV:

"Duae aliquae contiguae corporis partes *cohaerent* tum demum sibi, si se premunt, seu si is est corporis motus, ut una alterum impellat, id est in alterius locum sit successura" (GM VI, 73; also in: GP IV, 234).

But then it is not one of these four explicit references (I – IV) to the *contiguum* which the solution of Zeno's paradox as proposed in this paper resumes, but rather two considerations in both of which Leibniz not even directly mentions the *contiguum* at all. These are

Reference V:

"[...] ac proinde divisio continui non considerata ut arenae in grana, sed ut chartae vel tunicae in plicas, itaque licet plicae numero infinito, aliae aliis minores fiant, non ideò corpus unquam in puncta seu minima dissolvitur. [...] atque ita non fit dissolutio in puncta usque, licet quodlibet punctum à quolibet motu differat. Quemadmodum si tunicam, plicis in infinitum multiplicatis, ita signari ponamus ut nulla sit plica tam parva, quin nova plica subdividatur: atque ita nullum punctum in tunica assignabile erit, quin diverso à vicinis motu cieatur, non tamen ab iis divellatur, necque dici poterit tunicam in puncta usque resolutam esse, sed plicae

⁶ *Pacidius Philalethi*; C, 624/625.

⁷ See K. Stiegler: *Das Problem der Bewegung im kalam und im Pacidius Philalethi des jungen G.W.Leibniz*, in: K. Figalla, E. H. Berninger (eds.): *Arithmos-Arrhythmos, Festschrift für Joachim Otto Fleckenstein zum 65. Geburtstag*, München 1979.

licet aliae aliis in infinitum minores, semper extensa sunt corpora, et puncta nunquam partes fiunt, sed semper extrema tantum manent" (C, 615)

and, in my view, one of the most significant of all his considerations concerning the required prerequisites for a zenoproof theory of motion, namely the following evidence from his *Theoria Motus Abstracti*:

Reference VI:

"Cohaesionis, qualitatis tam obviae, rationem reddidit nemo: quid prodest ramos, hamos, uncos, annulos, aliaque corporum implicamenta comminisci, cum opus futurum sit hamis hamorum in infinitum?" (GM VI, 78; also in: GP IV, 239)

In these two references the very characteristics of the solution of Zeno's paradox of motion as proposed in this paper are insinuated as intimate as nowhere else in the history of the debate about this puzzling topic.

2. The Discontinuous Structure of – nearly – the Contiguum: A first Attempt of its Construction

The leading idea to finally solve Zeno's paradox of motion which had been proposed by me as well as others⁸ several times before is that motion should be represented or explained neither continuously nor discretely, but rather discontinuously.

The concept or rather a model of the discontinuum had been originally introduced by Georg Cantor in 1883⁹. It is well known as *Cantor discontinuum* or *Cantor dust* and defined as a "kompakte, perfekte, nirgends dichte Menge mit der Mächtigkeit c und dem Maß Null"¹⁰.

The *Cantor discontinuum* can be easily constructed by starting with an arbitrary bar of some finite length, and then in a next step taking out an actual part in its middle, e. g. of a third of the original length. This partition or the act of eliminating a part of that relative length then has to be infinitely

8 See Ferber (see note 1); P. Eisenhardt/D. Kurth/H. Stiehl: *Du steigst nie zweimal in denselben Fluß*, Reinbeck 1988; I. Tóth: *Le problème de la mesure dans la perspective de l'Être et du non-Être – Zénon et Platon, Eudoxe et Dedekind: une généalogie philosophico-mathématique*, in: R. Rashed (ed.): *Mathématiques et Philosophie de l'Antiquité à l'Âge classique*, Paris 1991; P. Eisenhardt/D. Kurth: *Emergenz und Dynamik*, Cuxhaven 1993.

9 See G. Cantor: *Über unendliche lineare Punktmannichfaltigkeiten, Grundlagen einer allgemeinen Mannichfaltigkeitslehre*, in: *Math. Ann.* 21 (1883), pp. 545-591. Also in: G. Cantor: *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts, mit erläuternden Anmerkungen sowie mit Ergänzungen aus dem Briefwechsel Cantor-Dedekind, nebst einem Lebenslauf Cantors von A. Fraenkel*, ed. by E. Zermelo, Berlin 1932 (repr. Hildesheim 1962).

10 W. Purkert/H. J. Ilgauds: *Georg Cantor*, Basel – Boston – Stuttgart 1987, p. 69.

iterated, i. e. stepwise to be applied to all remaining parts of the original bar. The construction can be ‘seen’ in the following Figure 1.

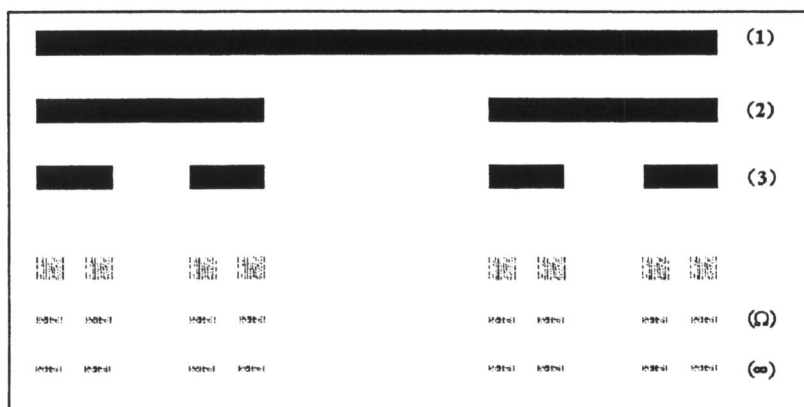


Figure 1: Cantor Discontinuum

There are three possible results of this construction, or rather two results and one mistake.

In a case a) when the iteration is finite, i. e. terminates after a countable finite number of steps the remaining parts then turn out to be bars of a respective length, and such this result is obviously none – and above all it is no discontinuum – but rather a mistake of the construction. Clearly such mistake also was not a part of Cantors original considerations about the discontinuum.

This doesn't hold for the next case b) where we assume an intersection of infinite partitions. The ‘remaining elements’ of the discontinuum then are isomorphic to points of the continuum, i. e. they are points themselves. This is the result of Cantors original construction of the discontinuum.

Then there is a third case c) which will be of major significance for the discussion of the contiguum. It may be called a Ω -construction (its details will be given in the following). Here the ‘remaining elements’ of the discontinuum turn out to be infinitesimals, i. e. infinitely small bars.

But regardless of the question if we get the result of case b) or that of case c) the generated Cantor dust anyway looks rather as the opposite of Aristotle's touching boundaries (haptomena) or Leibniz' ‘contigua quorum extrema sunt simul’.

Yet despite this disenchanting fact we stick to our assumption that the contiguum is to be of a discontinuous nature.

In such obstinacy our conviction that the feature of discontinuity is essential not just for the contiguum but as well for the construction – and thus also for any possible solution – of Zeno's paradoxes is matched by that of Imre Tóth. In his paper *Le problème de la mesure dans la perspective de l'Être et du non-Être* he especially highlights the strong structural resemblance of (the underlying

operation of generating) Zeno's paradox of dichotomy and the Cantor discontinuum¹¹. He even goes a decisive step further by implying that there might be more than just a strong resemblance but something like a similar approach to a common problem, namely the analysis of infinity or of the power of the continuum. Here Tóth brings Zeno's paradoxes of motion – especially that of dichotomy as well as the Achilles¹² – in close relation to the – for ancient Greek mathematics – notorious problem of incommensurability¹³, by stressing the undoubtedly not just coincidental relationship¹⁴ of Zeno's method of cutting the continuum and the method of Antanahairesis¹⁵.

But despite this seeming operational similarity the objective of Zeno's argument against the possibility of motion is essentially different from the problems concerning the discovery of incommensurability¹⁶ or the resulting attempts to find a sufficient mathematical representation of the irrational numbers¹⁷. This later quest leads to a geometrical representation of arithmetical entities, i. e. irrational numbers, whereas Zeno's argument asks for physico-geometrical – or rather ontogeometrical – entities as a proper base for motion. Such entities obviously could have been real infinitesimals or indivisibles¹⁸ (which

11 See Tóth (see note 8), pp. 40-48.

12 See *ibid.*, pp. 68-87.

13 Obviously there is an even closer relation between Zeno's paradoxes of (the possibility of) plurality or multitude – especially the entailed paradox of measure – and the problem of incommensurability.

14 See – for the historical context – also: H. Hasse/ H. Scholz: *Die Grundlagenkrise der Griechischen Mathematik*, Berlin 1928.

15 See Tóth (see note 8), pp. 49-67. Antanahairesis was introduced by Hippasus of Metapontum as a method of deriving a contingent common measure of two geometrical entities e. g. two lines by a kind of iterated division (comparable to a geometrical equivalent of a continued fraction), i. e. by the marking off of an (smaller) one a_2 on an (larger) one a_1 . If then a_2 doesn't fit in a_1 without a remainder, such a remainder a_3 is to be marked off on a_2 , and this kind of division has to be repeated until either a rational proportion $a_1 : a_{i+1}$ (i. e. without a remainder) is reached or – in case the iterated divisions do not terminate – $a_1 : a_2$ finally turns out to be irrational.

16 For a comprehensive presentation of the context and the relevance of this discovery see K. v. Fritz: *The Discovery of Incommensurability by Hippasus of Metapontum*, in: *Annals of Mathematics* 46 (1954), pp. 242-264; (German translation: *Die Entdeckung der Inkommensurabilität durch Hippasos von Metapont*, in: O. Becker (ed.): *Zur Geschichte der griechischen Mathematik*, Darmstadt 1965, pp. 271-307; also in: K. v. Fritz: *Grundprobleme der Geschichte der antiken Wissenschaft*, Berlin – New York 1971, pp. 545-575).

17 These attempts later lead – in the confinements to geometrical methods of ancient Greek mathematics – to the Eudoxian theory of *lógoi* as a geometrical representation of irrational numbers as proportions of incommensurable line segments. This Eudoxian theory somewhat foreshadowed – in a geometrical way – the modern extension of the field of rational numbers to that of the real numbers by the Dedekind cut.

18 Yet – to my knowledge – there is only one source in ancient Greek literature which unequivocally mentions indivisibles, namely *De Lineis Insecabilibus*, see Aristotle: *De Lineis Insecabilibus (peri atomôn grammôn, On Indivisible Lines)*, in: W. D. Ross: *The Works of Aristotle*, vol. VI, Oxford 1952, 968a –972b. Here the discussed concept of

should not be confused with material atoms). Yet such indivisibles are hardly found in ancient Greek physico-mathematics¹⁹.

The Cantor discontinuum in the Ω -construction however generates – in an even somehow zenonian manner – plenty and more of real infinitesimals, but unfortunately it also generates the same amount of gaps between them. So as a first attempt to regain enough of the lost density of the continuum to get a proper base for motion we then might try to somewhat ‘refill’ the unwanted gaps in the Cantor discontinuum.

To gain the required structure of the contiguum, i. e. a structure of infinitesimal near neighborhood of the elements of the discontinuum one has to decisively modify the original construction of the Cantor discontinuum. For reasons of terminological clarity I will call the resulting entity a discontiguum. Yet as decisive the needed modifications of the original construction ever may be, as simple they also are.

At any step i of the original construction of the Cantor discontinuum one not just has to take out or eliminate a part of the (remaining) bar(s), but also to refill a part of the such generated blank(s). Such a part then may be of arbitrary length, provided it is actually smaller than the respective take-out. This construction of the discontiguum can be ‘seen’ in Figure 2.

This modified construction comes obviously very close to the concepts of a contiguum as proposed by Aristotle and Leibniz, one might even say it comes infinitesimally near to its proposed structure, but then regrettably it doesn’t terminate.

In the (∞) -case of the intersection of infinite partitions and – here – as well refillings the discussed discontiguum again like the original discontinuum is of the power of the continuum, (i. e. of the set of the Real Numbers) \mathbb{R} . Thus the elements of the *d i s c o n t i g u u m* in this case again are isomorphic to the

indivisibles is even brought in some relation to the problem of incommensurability (968b 4-20), but just for finally coming to the conclusion that they do not exist. This pseudoaristotelian text is casually cited by I. Tóth (see Tóth, see note 8, p. 45), but doesn’t play an important role for the ends of his argument. See also P. Eisenhardt/D. Kurth: *Nichtstandard Topologie und Prägeometrie*, in: A. v. Gotstedter (ed.): *Ad Radices: Festband zum fünfzigjährigen Bestehen des Instituts für Geschichte der Naturwissenschaften der Johann Wolfgang Goethe-Universität Frankfurt am Main*, Stuttgart 1994.

- 19 Except in *De Lineis Insecabilibus*, see the preceding footnote. In contrast to I. Tóth, who claims, that “Platon désigne cette relation hétérodoxe – ou non-standard – comme un *phantasme d’égalité* (... *Parm.* 165A) et parle de sa participation à l’idée de l’*Égalité et de la similitude* (... *Parm.* 140E) [...] En effet la relation définit un lieu diachronique un *Hen*, un et unique, un instant indivisible [...]” (Tóth, see note 8, p. 86) to my understanding neither time-like (‘instants’) nor space-like nor any kind of mathematical indivisibles (or non-standard entities) are the topic of Plato’s *Parmenides*. Plato was – of course – very well aware of the difference between an *u n i t*, which by definition cannot be an indivisible in the genuine sense, and a *q u a n t i t y*, which may or may not be an indivisible. His *Parmenides* is essentially concerned with the ontological status of the ‘Hen’ and related eleatic confusions of these concepts.

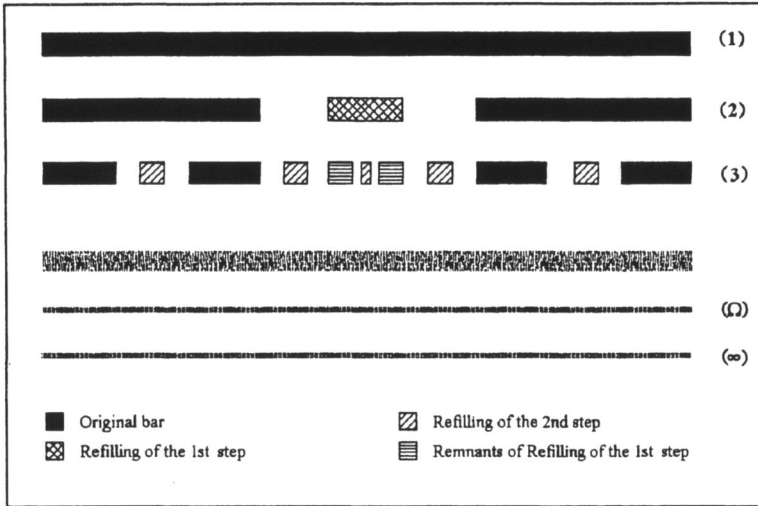


Figure 2: Discontiguum

points of the continuum, and then they are points. But now the annoying gaps or blanks of the discontinuum are completely gone. The neighborhood of any of the points of the discontinuum is infinitely dense, i. e. between any two points of the discontinuum lie infinitely many further points. So, is the discontinuum in the (∞) -case actually the continuum? As far as I can see this is not the case. The discontinuum in the (∞) -case might be of the same order as the continuum, but then there is still an ephemeral difference in the neighborhood relation. Thus the discontinuum in the (∞) -case seems for me to be some kind of a dynamic double of the rather static continuum.

In the – for our purpose of defeating Zeno – much more significant (Ω) -case (or rather the Ω -construction) the elements of that construction display an infinitesimal dense neighborhood whilst having the power of the set of the Natural Numbers \mathbb{N} , i. e. between any two of these infinitesimal elements lies no further element. These elements of the discontiguum in the (Ω) -case obviously are not isomorphic to the points of the continuum, and simply are no points but infinitesimals instead.

Yet however regrettably one cannot defeat Zeno by the means of the discontiguum. The reasons are simply the same as they always have been.

In the (∞) -case (or continuous case) too many – to be precise: infinitely many – points come in between the point the arrow (or rather the arrow-head) actually rests in and the next to which it aspires. Case closed.

In the (Ω) -case then we still do not overcome Zeno's argument against the defender of discreteness. In fact the discontiguum in the (Ω) -case is not discrete, because it is a misplaced notion of discreteness, if one cannot discern – or 'pick' – the elements of a so called 'discrete' structure. And undoubtedly no one

ever will be able to discern – or ‘pick’ – the elements of the discontiguum in the (Ω) -case. The reason for this is – as I will show in the next paragraph – the unattainability or uncountability of Ω .

Yet, as I have shown above, Zeno’s argument in the (arrow) paradox of motion is – against all appearances (at least as far as Aristotle is concerned) – essentially about infinitesimal motion. And then there is no doubt that the discontiguum in the (Ω) -case is an infinitesimal structure. And then stubborn Zeno quite unimpressed says that he is not interested in the question how large or small the hiatus is the arrow or its head may plunge in, as long as they plunge in or stay at rest. Stubborn, but incontestable. Case closed.

One may wonder why I deliberated at quite a length about the discontiguum, if in the end it doesn’t lead us to the promised solution of Zeno’s paradox of motion. There had been two reasons for this considerations about the discontiguum.

The first reason is that the discontiguum – in particular, as some already might have expected, its Ω -construction – actually leads us very near to the later presented solution. That after all will become evident – with hindsight.

The other reason is my conviction that the discontiguum, again in the (Ω) -case, might bear some potential insights into the structure of space which reach far beyond the challenge of defeating Zeno, and which might also not have been completely entailed in the prerequisites to the solution I will present in the following.

3. Ω – Construction

In the preceding paragraph I several times mentioned ‘ Ω -construction’ and ‘the (Ω) -case of the discontiguum’, so I now will give the already announced details.

‘ Ω ’ refers to the number of iterations, i. e. the iterated partitions resp. partitions and refillings, required for the construction of a particular case of the Cantor discontinuum. resp. of the discontiguum. In the subsequent explanation of how Ω can be constructed I closely follow Laugwitz²⁰.

Ω is supposed to be an *i n f i n i t e l y* large number. Ω will be constructed as an extension of the number field of e. g. the Real Numbers, i. e. as an adjoint to the $K(\mathbb{R})$. To gain such new ‘ideal elements’ adjoint to $K(\mathbb{R})$ the following sequence of inequalities is sufficient²¹.

$$(1) \quad \Omega \gg 1, \Omega \gg 2, \Omega \gg 3, \dots, \Omega \gg 100, \dots;$$

a more generalized version of (1) then is

$$(1.1) \quad \Omega \gg n, \Omega \gg n+1, \Omega \gg n+2, \dots, \Omega \gg n+i, \dots; \\ \text{(for } n, i \in \mathbb{N})$$

20 D. Laugwitz: *Zahlen und Kontinuum*, Darmstadt 1986.

21 For more details and a technically advanced definition see *ibid.*, pp. 83-90.

Ω itself then is an infinitely large Natural Number, and thus it holds $\Omega \neq \infty$.

4. The Contiguum Revealed: Antoine's Necklace or Linked Discontinuity

Now that we got through the professionally required epicycles – and by the way prepared our arms – we might a little more straightforwardly attack our proper goal of solving Zeno's paradox of motion and – by this way – defeating him.

From our fruitless endeavors to achieve this goal with means of the contiguum we should have learned one thing: Zeno's paradox is not – or at least not in the first place – about the paradoxes of infinity. And then it is also not – or at least not in the first place – about the neighborhood relation of the respective parts or points of space (and motion). But about what it then may be?

Here we suddenly remember Leibniz' almost facetious remarks in his *Theoria motus abstracti* (quoted as reference No. VI in the first paragraph), where Leibniz came as near to the effective solution as possible, but then unfortunately refused to further 'comminisci'. Therefore it shall be displayed once more.

"Cohaesionis, qualitatis tam obviae, rationem reddidit nemo: quid prodest ramos, hamos, uncos, annulos, aliaque corporum implicamenta comminisci, cum opus futurum sit hamis hamorum in infinitum?" (GM VI, 78; also in: GP IV, 239)

If Leibniz would have gone further into the entanglements and intertwinings of hooks and barbs and – first of all – the tiny rings or rather links, and if he just had gone on a little further with 'comminisci', then he probably would have found that the solution of the queries of motion, which he emphasized in his *Theoria Motus Abstracti* – and therefore the puzzles of Zeno's paradox of motion as well – are essentially due to the kind of topological connectivity of the underlying space.

Still we stick to our premise that a structure apt to bear the solution of Zeno's paradox has to be discontinuous.

Now joining these two premises:

a) the solution of Zeno's paradox of motion has to do with the kind of topological connectivity of the underlying space, and

b) a structure apt to bear the solution of Zeno's paradox has to be discontinuous, the details of the intended solution slowly might become evident.

One can dissolve Zeno's paradox of motion and therefore defeat him, if one assumes the underlying space to be of a discontinuous connectivity structure.

But before I will go in the details of such a required discontinuous connectivity space, I will already attack Zeno barely with the means of topological connectivity. A topological connectivity which is needed and sufficient to defeat Zeno in a case very similar to the original 'discrete'

case is provided by any ordinary chain, the links of which may even be made of 'annuli'²².

If we assume, as Zeno did, that at any instant an object has to be (at rest) in a particular commensurate part or point of space and if we then assume that such a commensurate part or point of space is to be a link of a chain then motion is possible under these assumptions, provided that commensurateness is strictly warranted. The reason is the following:

If the object is at any commensurate part or point of space, i.e. – by assumption – on a ('first') link of the respective chain, then it is as well already on the next. This is an unavoidable consequence of the topological connectivity of a chain. In the next step the object will mainly be positioned on the 'second' link but as well already on the 'third', and so on. This is simply alike the pseudo-solution of the original discrete case, mentioned in the first paragraph, but now something very important has changed: the hiatus disappeared. To fully grasp the simplicity of that solution one must remember that that hiatus originally was Zeno's only weapon against motion in the original discrete case. But now it looks, as if time and space may be 'discrete' as long as space is close-knit, or rather: well connected. In fact space in this case would not be discrete but rather 'macro-contiguous'. And although motion here would look somehow 'stroboscopic'²³, it actually would not be discrete in the proper meaning of that term, simply for the reason, that there won't be a hiatus. Instead motion in this case always and with no exception will be characterized by an overlapping of the positions the arrow seemingly 'rests in'. For this strictly follows from the topological connectivity structure of a chain (with no negative tension in any of its parts)! This overlapping of those positions therefore is an inevitable consequence of the presupposed contiguity of the underlying space, or of its

- 22 Only the case (of a chain K), in which an element or link $L(K)_{n+2}$ is shifted backwards (in the succession direction towards $L(K)_n$) in such a way that it partially lies or starts before its preceding element or link $L(K)_{n+1}$ is excluded. It is at first excluded by the reasonable condition that any such chain must not have a negative tension in any of its parts. Furthermore it is excluded by the presupposition that the arrow is strictly commensurate to the part or point of space it lies in. For the reason that the arrow impossibly alters its size by moving, the respective links of the chain also are strictly all of the same size. We then add to the presupposition of commensurateness the as well reasonable condition that the length of the arrow is always commensurate to the longest possible diameter of these respective elements or links of the chain. Then at last it is required that these elements or links are exclusively non degenerate closed conic sections, i.e. ellipses or circles with no negative curvature in any part of their perimeters. By these conditions the anyway rather 'exotic' case mentioned in the first line of this footnote is effectively excluded.
- 23 This characterization for the kind of a contiguous motion had been originally proposed by Peter Eisenhardt – independently of the central topic of this paper, namely a solution of Zeno's paradox of motion.

linked connectivity. Thus the overlapping of the positions is precisely the expression of the contiguity of the motion itself. And such the arrow flies nicely – along the links of that chain, (as long as they are commensurate). Case closed.

How the motion of that arrow would look like on such a ‘macro-contiguous’ path can be seen in Figures 3a, 3b and 3c.

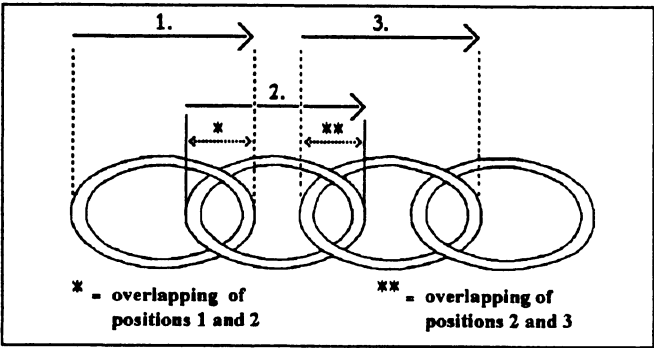


Figure 3a: Contiguous motion on a chain with arbitrary tension

The difference between Figure 3a and Figure 3b is, that in 3a we see a chain with arbitrary tension, but in 3b a chain with maximal tension. The case with maximal tension is of some significance because one here can see perhaps most convincingly that from the contiguity of the chain inevitably follows an overlapping in the arrows (positions in its) motion, i. e. its contiguous motion.

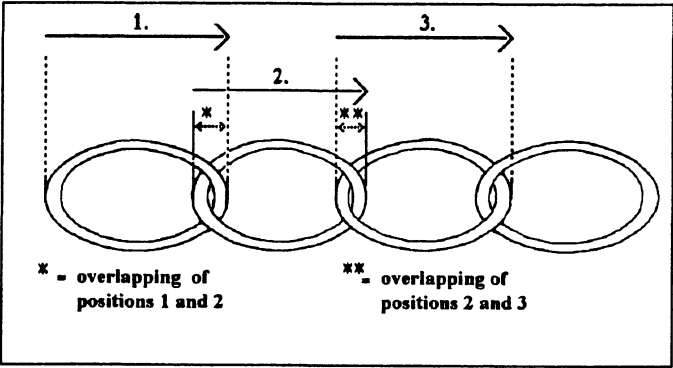


Figure 3b: Contiguous motion on a chain with maximal tension

Figure 3c then displays the case of a chain with an admissible minimal tension.

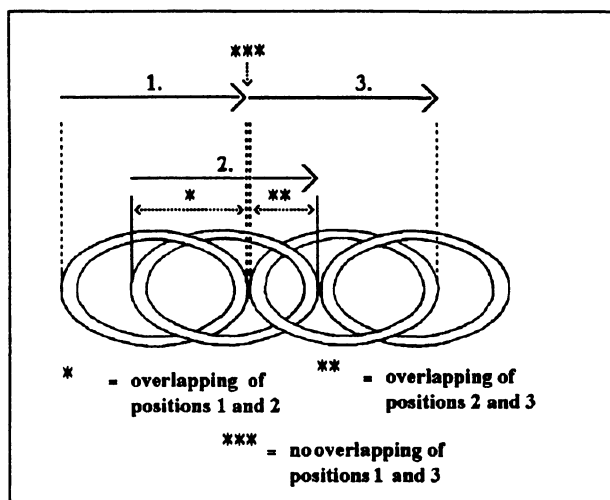


Figure 3c: Contiguous motion on a chain with admissible minimal tension

For this solution the focus of the paradox also does not instantaneously shift from the problem of motion to the problem of infinitesimal motion. Zeno is defeated at – nearly – any scale (the infinitely small left aside for the moment), the larger the better. And this advantage of this solution is its problem.

There is mainly one reason not to be content with this solution, although it obviously is a solution of Zeno's paradox.

That reason is that space simply doesn't look like being made of links (of chains) of arbitrary magnitude.

If it is made of links (of chains), one rather would suggest, that it is made of very, very small 'annuli', i. e. of infinitely small or infinitesimal links. Thus, if the focus now does not instantaneously shift from the problem of motion to the problem of infinitesimal motion by itself, we will make it shift, i. e. we will look for the already announced discontinuous (rather than 'discrete' or better: macroscopic) connectivity structure.

An established example of such a discontinuous connectivity structure, which then is required, is known as Antoine's necklace²⁴. Antoine's necklace is supposed to be a chain (of an arbitrary scale c) of links, which links again are chains (of a scale $c - 1$), and the links of which again are chains (of a scale $c - 2$) and so on, ad – the abhorred – infinitum. Antoine's necklace not only looks suspiciously familiar, it simply is isomorphic to the Cantor discontinuum, and then conclusively the discontinuum. But then it is different as well, in an other aspect, namely its topological connectivity. A two-scale fragment of Antoine's necklace is shown in Figure 4.

24 See for details of Antoine's necklace: D. Rolfsen: *Knots and Links*, Berkeley 1976.

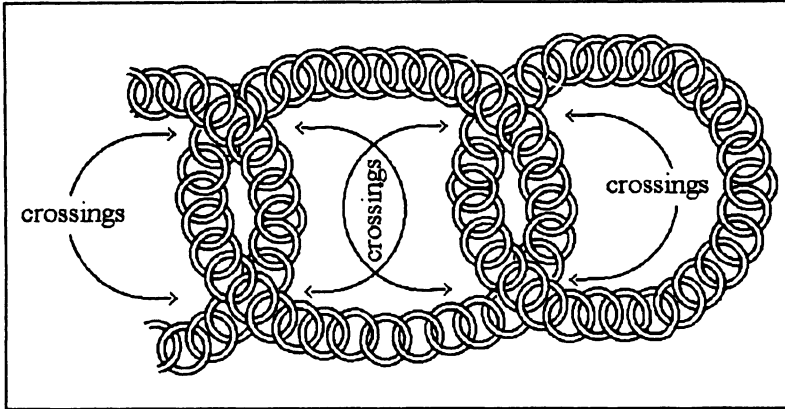


Figure 4: A two-scale fragment of Antoine's necklace

Now, how does motion work on Antoine's necklace. As well as on every fine chain, one might assume. And that is – sort of – true. Yet regrettably motion then is a bit cyclic, and in the 'real' Antoine's necklace, i. e. the one with infinite dissections (formerly: partitions) these cycles are even actual-infinitely small ones. The reason is that one can move in the original Antoine's necklace just on the elements, i. e. links (of scale $c - 1$) of only that chain (of scale c) on which one has started, not regarding that this chain (of scale c) might be itself a link of a chain (of scale $c + 1$) and so on, or that these links (of scale $c - 1$) are themselves chains composed of links (of scale $c - 2$) and so on. In the preferred (Ω)-case, which applies of course as well to Antoine's necklace as to the Cantor discontinuum or the discontinuum, things aren't much better, just about the size of the difference between the infinite and the infinitesimal.

But this very difference matters. It means that in the (∞)-case one doesn't even have any link to start from. And just this provides an insight into the reason why Zeno cannot be defeated at the 'continuous horn' of his paradox. Different from the usual representation of the continuum, where it seems as if actual-infinitely many points would come to lie in between any two points of the continuum, in the (∞)-case of Antoine's necklace the true reason why there can be no motion on a continuous structure turns out to be that such a structure implies an intrinsic infinite downscaling (something one might call an 'absolute dissolution' or breakdown of that – or any – structure itself).

In the (Ω)-case however, there is a link to start from. But then we have just an infinitesimal cycle to move on, and we rather dislike that.

To overcome this confinement to infinitesimal cyclicity we do claim that there is a zenoproof contiguous path through the entire Antoine's necklace (on whatever scale) if just one additional condition to the original construction of Antoine's necklace is satisfied. This condition says, that at any arbitrary crossing of two elements K^1 , K^2 of a relatively higher scale c of the chain the

elements $L(K^1)$ of the respective directly lower scale $c - 1$ are linked not only among themselves as they do in the original construction, but are linked also to the elements $L(K^2)$ of the same scale $c - 1$, which belong to (or are embedded in) that element K^2 of scale c , which is just crossed by the element K^1 to which the elements $L(K^1)$ actually belong (or are embedded in). I call such a linking a 'crossing linking'. The difference of the normal kind of crossing (in the original Antoine's necklace) and the respective crossing linking is shown in Figures 5 resp. 6 (but one has at least to look twice to see it).

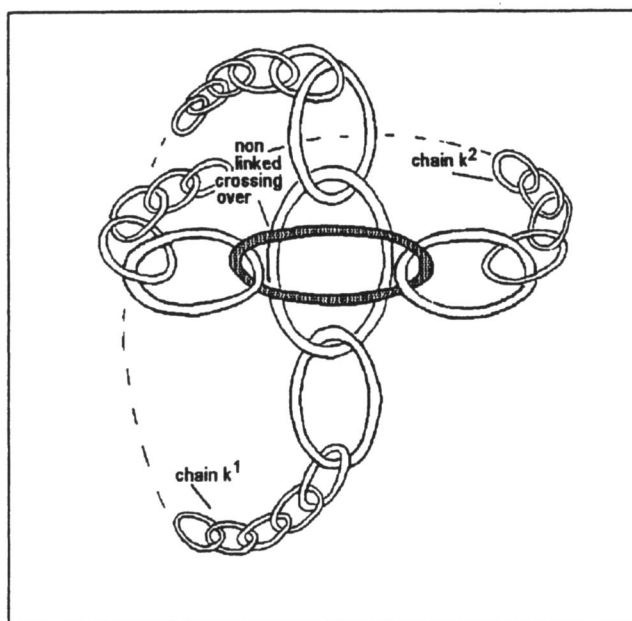


Figure 5: An ordinary fragment of Antoine's necklace (with no crossing linking)

Now we finally reached the zenoproof contiguous path through the entire 'Antoine's necklace with crossing linking' (on an arbitrary scale, but effectively of course only on the same scale on which the motion started).

Yet it should well be understood, that such a respective crossing linking does not follow from the original algorithmic construction rule of Antoine's necklace, and that it also cannot simply be added to such an algorithmic rule of generation, but rather has to be added – so to speak – by hand at any particular scale, when or where required.

And at last it might have become evident that the (Ω) -case of 'Antoine's necklace with crossing linking' is a model of the *contiguum* Aristotle, Leibniz and we had been after. Motion on a zenoproof contiguous path through the entire 'Antoine's necklace with crossing linking' (on whatever scale) over arbitrary distance is possible, even if it is a bit curvy. How such a crossing

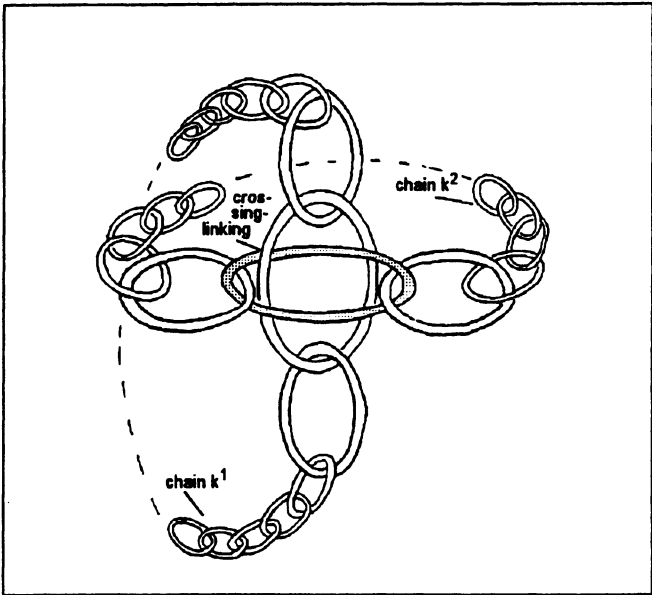


Figure 6: A fragment of Antoine's necklace (w i t h crossing linking)

linking looks from the perspective of a respective object in motion – for the end to be a bit mean, I will call it a ‘zenomobil’- is shown in Figure 7. The zenomobil starts at α , then is on β , and then can choose between γ_1 or γ_2 as its next position. From the perspective of the zenomobil that looks just like a bifurcation.

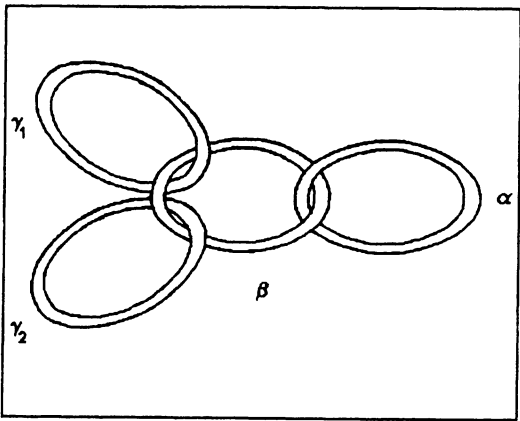


Figure 7: A crossing linking from the perspective of a zenomobil (while moving on a fragment of ‘Antoine’s necklace with crossing linking’)

Finally we also want to get rid of that curvyness of our contiguous path through the entire ‘Antoine’s necklace with crossing linking’. We will no longer submit to these links and chains. We want to move freely in all dimensions (at least three of them).

So we leave Antoine’s necklace (with or without crossing linkings) and prepare ourselves with *Leibniz’ chain armor* for our struggle for the freedom of motion.

III. Leibniz’ Chain Armor Zeno’s Arrow won’t pierce: the Contiguous Link-Space

The idea behind the design of *Leibniz’ chain armor* is to extend the contiguous structure of the (Ω) -case of ‘Antoine’s necklace with crossing linking’ to a three-dimensional space (e. g. with a Riemannian curvature at large scale) just alike the one which we believe to live – and move – in. Such a space then will be a *contiguous space*, i. e. an infinitely dense connected link-space. (By our presupposition this contiguous link structure then is confined to an infinitesimal scale!)

And not in the least for virtue of his ‘rami, hami, unci’ and – above all – his ‘annuli’ this space shall be called ‘*Leibniz’ chain armor*’. In Figure 8 one might get – in quite a magnification – a glimpse of how Leibniz’ chain armor might be imagined on its infinitesimal scale.

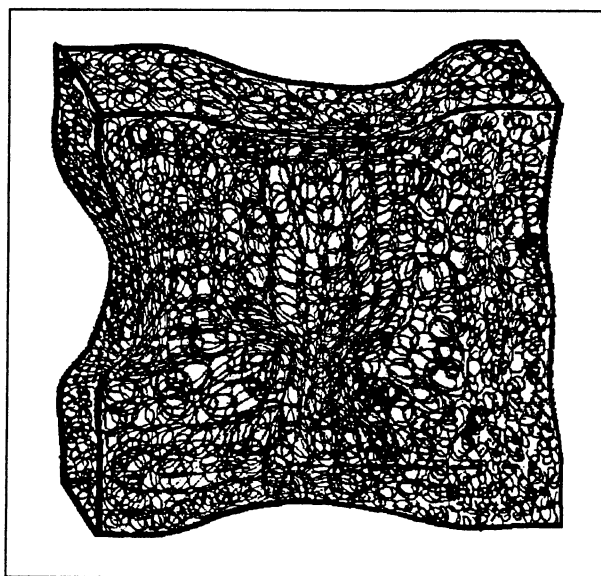


Figure 8: A fragment of Leibniz’ chain armor

Concerning the mentioned motion it evidently has to be conceived as a motion on arbitrary two dimensional surfaces of intersection through this space, one of which – with some imagination – can be 'seen' in Figure 9.

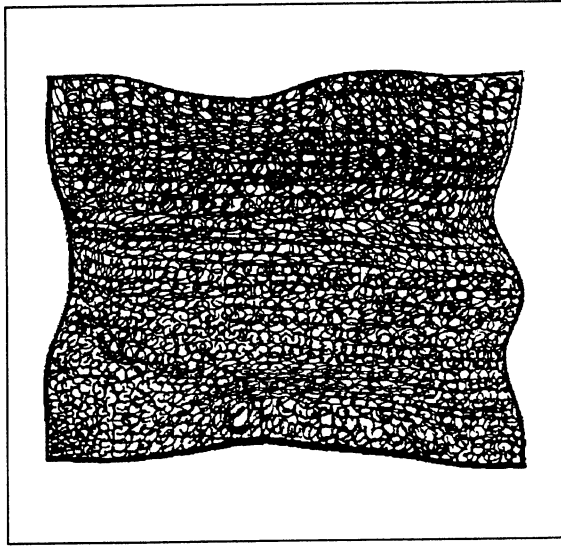


Figure 9: Leibniz' tunica: a surface of intersection through Leibniz' chain armor

For reasons, which soon will become evident, we will call such a surface of intersection through Leibniz' chain armor 'Leibniz' tunica'. Leibniz' tunica is in my view not just a compelling step on the way from the respective rather aprioristic considerations of Leibniz concerning the ontotopology of the contiguum to a more physical interpretation in an even rather modern sense, but first of all it is a strikingly fitting model for the central idea or metaphor of reference V, the first of these two exceptional allusions in Leibniz writings to the very structure of the contiguum mentioned in the above paragraph concerning 'The Definition of the Contiguum'. I then characterised these two references as the ones most intimately resuming the argument of this paper, a characterisation already proven for 'the barbs and hooks and tiny rings, and hooks of hooks'. So now it shall be proven for the infinitely enfolded cloth or tunica as well.

Therefore the respective evidence shall be displayed once more – as well for reasons of supporting the recollection as for emphasizing that striking resemblance to our surface of intersection through Leibniz' chain armor, which from now on rightfully shall be called: Leibniz' tunica.

"[...] ac proinde divisio continui non considerata ut arenae in grana, sed ut chartae vel tunicae in plicas, itaque licet plicae numero infinito, aliae aliis minores fiant, non ideò corpus unquam in puncta seu minima dissolvetur. [...] atque ita non fit dissolutio in puncta usque, licet quodlibet punctum à quolibet motu differat. Quemadmodum si tunicam, plicis in infinitum multiplicatis, ita signari ponamus ut nulla sit plica tam parva, quin nova plica subdividatur: atque ita nullum punctum in tunica assignabile erit, quin diverso à vicinis motu cieatur, non tamen ab iis divelletur, necque dici poterit tunicam in puncta usque resolutam esse, sed plicae licet aliae aliis in infinitum minores, semper extensa sunt corpora, et puncta nunquam partes fiunt, sed semper extrema tantum manent" (C, 615).

Thus our attack on Zeno's paradox of motion led us – prepared with Leibniz' armings and cloths – far beyond our original object. So now, at last, only a few remarks might be spent regarding the entanglement of the (Ω) -case discontinuum – already left so far behind – and Leibniz' chain armor.

At the infinitesimal scale of space, which both these structures refer to, there might be something like an actual intertwining of these structures – or at least of structures related to these. Modern theories like superstring theory²⁵ and/or quantum gravity are about the structure of space or space-time at such scale.

Furtheron one might feel a slight allusion to superstrings imagining the discontinuum, and Leibniz' chain armor – or the contiguous link-space – might not less be a suggestion of the Ashtekar Rovelli Smolin 'loop (space) representation' in quantum gravity²⁶. And what has made Zeno's paradox so puzzling once, might have had to do with the deeply concealed ways of how – sit venia verbo – to move from the one to the other.

But certainly that's going too far and so – at least for now – we'll lay such speculation to rest – with Zeno.

Dan Kurth, M. A., Institut für Geschichte der Naturwissenschaften, Johann Wolfgang Goethe-Universität, Robert Mayer-Straße 1, D-60054 Frankfurt am Main

25 See as an earlier survey: M. B. Green/J. H. Schwarz/E. Witten: *Superstring Theory*, vols. I, II, Cambridge 1987.

26 See A. Ashtekar: *Old Problems in the Light of new Variables*; C. Rovelli: *Loop Representation in Quantum Gravity*; L. Smolin: *Nonperturbative Quantum Gravity via the Loop Representation*, all in: A. Ashtekar/J. Stachel (eds.): *Conceptual Problems of Quantum Gravity*, Boston 1991.