The Topos of Emergence

or: putting the tower of turtles on a more *profound* basis

or: more about objects which are not even intended to be entities

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Abstract. The aim of this paper is to provide a sketch of a bit more formal underpinning concerning the conceptual intricacy of the notion of '(primordial) emergence' by applying category- and topos-theoretical means.[1] This relates most directly to what had been called 'the mêontology of primordial emergence' in the corresponding paper "The Tower of Turtles" (see the "The Philosophical Aspects of ANPA 24"), namely the 'objects which are not intended entities'.

A particularly interesting instance of such 'objects which are no entities' is the proposed mathematical structure from which in my understanding the most elementary (or fundamental) physical structure(s) have been emergent.

1 Introduction

In this paper[2] I will try to give an outline of a mathematical model of 'primordial emergence'. By 'primordial emergence' is meant an emergence of primordial physicality up from an unphysical purely intelligible mathematical state. Intimately intertwined with that I will also present a sketch of a mathematical structure apt to solve a seeming paradox of the reductionist program in science which I've discussed in the directly related paper 'The Tower of Turtles'[3] (see the volume containing 'The Philosophical Aspects of ANPA 24') where I stressed the point that the reductionist program cannot come to its very end, i.e. it cannot terminate as long as it is pursued as a project with the aim of discovering a

[1] The idea of applying features of topos-theory as means for analyzing the concept of emergence and subsequently formulating a theory of emergence had - to my knowledge - for the first time been proposed in P.Eisenhardt, D.Kurth, Emergenz und Dynamik, Cuxhaven 1993

[2] This paper is based on the talk I gave at the ANPA 24 meeting in Cambridge (UK), August 2002

[3] D.Kurth, The Tower of Turtles, in: The Philosophical Aspects of ANPA 24, Proceedings of ANPA 24, to appear

primordial and fundamental level of physical reality which doesn't stem from any other (non-physical) preceding structure.
I.e. I did state in this paper that it is a hidden (yet perhaps mostly unnoticed) implication of the reductionist project that physical existence itself has to be seen as being emergent from an underlying pre-physical level, which I would call a 'mêontological' level, by this indicating that this level has to be seen as being made of objects which are no entities and which cannot even be intended to be entities, i.e. a level of *not in any sense physically existing objects*. Thus these objects should - since I'm not in favour of spiritual entities (at least not in this context) - be *mathematical objects*.
As an inspiration for such kind of objects I had a look into somewhat related considerations of Leibniz where I found a sketch of something I've called 'dynamical Leibniz-point objects'.[4]
The means I now will use to give the Leibniz-point objects a modern spin are mainly the ones you might expect. Namely

a) n-categories or higher dimensional categorical algebra[5] (yet in our case it will turn out to be rather lower dimensional, i.e. negative ascending categorical algebra) and intricately intertwined with that

b) Topos theory

But before we will try to make a suggestion of how to bring such reconstructed Leibniz-point objects (despite the fact that they shall not be intended to be entities) into appearance, we must at first say a word about our interpretation of the *conatus* which by Leibniz was meant to be copresent with his special kind of points (i.e. points which have parts (or a structure) yet no extension). Leibniz obviously took the conatus as an infimum motus, i.e. as an infinitesimal motion and as such of course as an infinitesimal physical entity.
Such an infinitesimal physical entity could obviously not serve for our purpose of overcoming the difficulties of reductionism and therefore we will give this dynamical aspect attached to that unconventional Leibniz point object a different meaning. Like the Leibniz point object itself we also will understand the conatus as an entirely mathematical object, i.e. as something *dynamical in a purely mathematical sense*. There is already a tradition in doing so and taking in particular morphisms as formal analogies of dynamical action.[6] Thus we will suggest to see the conatus as an automorphism of a Leibniz point object onto itself. And motivated by this new aspect we now will also rename our objects in question and instead of 'Leibniz point objects' we will just call them 'automorphic objects' ($\mathbf{ob_{AM}}$).
Now two other preliminary terminological remarks (yet this time not about a terminology of our own making). The mathematical concept of an *n*-category had been developed in two different mathematical contexts and there are also two different types of *n*-categories, namely strict *n*-categories and weak *n*-categories. Since we will later make use of both these types let me give now a short characterisation.
Strict *n*-categories are the much older and in a sense less fruitful bunch. Yet they played an important role in metamathematical considerations e.g. in the mid sixties when they had

[4] For more details cf. D.Kurth, The Tower of Turtles, loc. cit.
[5] The idea of applying *n*-categorical concepts and means for analyzing emergence has already been introduced (and extensively discussed) in: P.Eisenhardt, D.Kurth, Complexity Categorified; in: Implications, Scientific Aspects of ANPA 22, (ed. Keith Bowden), London 2001
[6] Cf. as an example F.W.Lawvere, Categorical Dynamics, in: Topos theoretic methods in geometry (ed. A.Kock et. al.), pp. 1-28. Aarhus University Matematik Institut various publication series no. 30, Aarhus 1979

been used by F.W.Lawvere[7] in the business of foundational efforts. Lawvere's *n*-category (the category of categories **CAT**) then served a metamathematical purpose. That kind of *n*-categories later had been named strict *n*-categories (i.e. a kind of *n*-category in which the specific content or structure of the *n*-1 category is not so much of a particular relevance) in difference to the so called weak n-categories which are no metamathematical products of foundational endeavours yet categorical representations of increasingly enriched layers of hierarchical structures. I.e. in a *strict* *n*-category the object of the strict *n*-category is the respective *n*-1 category itself whereas in a *weak* *n*-category the object(s) of the weak *n*-category is (are) the *morphism(s)* of the respective *n*-1 category. By J.Baez, an important contributor to the development of higher dimensional categorical algebra (read as '*n*-category theory'), the definition and the significance of weak *n*-categories has been put into the following words

> "An *n*-category is an algebraic structure consisting of a collection of 'objects', a collection of 'morphisms' between objects, a collection of '2- morphisms' between morphisms, and so on up to *n*-morphisms, with various reasonable ways of composing these *j*-morphisms. A 0-category is just a set, while a 1-category is just a category. Recently *n*-categories for arbitrarily large *n* have begun to play an increasingly important role in many subjects including homotopy theory and topological quantum field theory. The reason is that they let us *avoid mistaking isomorphism for equality*."[8]

The mathematical concept or object of a *topos* had been originally designed by William F. Lawvere and Michael Tierney to be a dynamical version of a set.[9] By dynamical here is meant that there are (non-trivial) morphisms between any of the objects of the topos , where these objects are roughly speaking the categorical version of the elements of set theory. Further characteristics of a topos are amongst others that any topos has to have an initial object as well as a terminal object. The mentioned dynamical aspect then turns out to be equivalent to the various sequences of morphisms which can be found in the universe between the initial and the terminal object.
In the following we will make use of all three concepts we've just touched. And by making use of them we can maybe also give an additional philosophical spin to the difference of 'impact' related to strict respectively weak *n*-categories.

2 The topos of pre-physical emergence (PrePhys): automorphic objects with a (negative ascending) n-categorical enrichment

The following figure FIG.1 shows the mathematical structure I propose as being fit to carry the burden of the tower of turtles or being apt to stand as a profound basis for it. FIG.1 shows the topos **PrePhys** which is meant to represent an assumed pre-physical as well as pre-natural mêontological 'process' and that process itself is characterised by a strict *n*-categorical unfolding of automorphic objects $\mathbf{ob}_{\mathbf{AM}}$.

[7] Cf. F.W.Lawvere, The category of categories as a foundation for mathematics, in: S.Eilenberg et. al. (eds.), Proceedings of the Conference on Categorical Algebra in La Jolla, 1965
[8] Baez, J., An Introduction to *n*-Categories, in. 7th conference on Category Theory and Computer Science, ed. E.Moggi and G.Rosolini, Springer Lecture Notes in Computer Science vol. 1290 (1997) p 1
[9] Cf. F.W.Lawvere, An elementary theory of the category of sets, Proceedings of the National Academy of Sciences, U.S.A., 52, 1506-11; F.W.Lawvere, Continuously variable sets: algebraic geometry = geometric logic, in: Proceedings of the ASL Logic Colloquium, Bristol 1973 (ed. H.Rose, J.C.Shepherdson), pp. 135-56

So we mix toposic and *n*-categorical features in **PrePhys**. And by speaking of a 'mêontological process' we do even something more strange than that. The notion of a 'mêontological process' implies a process of (or in) the non-being.
Thus it must not come as a surprise that the automorphic objects $\mathbf{ob_{AM}}$, which are the particular sort of subobjects of **PrePhys**, are exactly the candidates for the objects which are not and cannot even be intended to be entities but which are intended to be the matrix from which such objects can arise (or emerge) which then could justifiably be intended to be entities.[10] To shed some light on such peculiar objects has been the motivation and purpose of this paper as had been explained in the introduction.

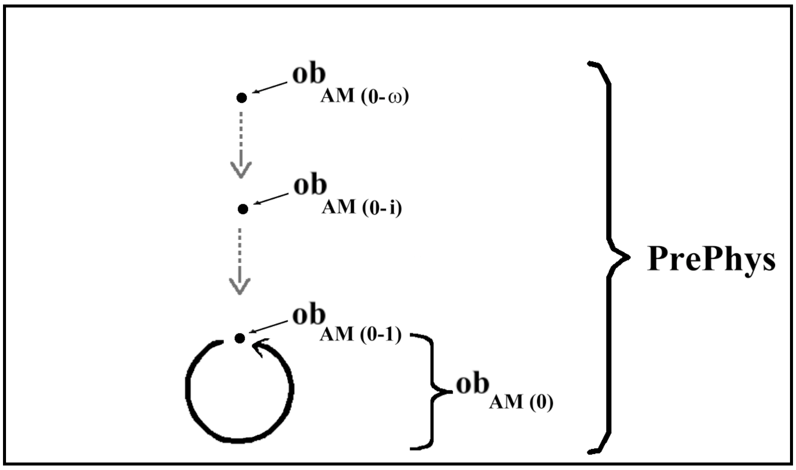

FIG.1: The topos of pre-physical emergence **PrePhys**: automorphic objects with a (negative ascending) strict *n*-categorical enrichment

For **PrePhys** the following definitions hold:

PrePhys: $\{\mathbf{ob_{AM(0-□)}}, \mathbf{ob_{AM(0-i)}}, \mathbf{ob_{AM(0)}}\}$

with $\mathbf{ob_{AM(0-□)}}$ (or a respective $\mathbf{ob_{AM(0-i)}}$) as initial object and $\mathbf{ob_{AM(0)}}$ as terminal object
for 1 □ i □ □
and with □ >> n, □ >> n+1, □ >> n+2, ..., □ >> n+j, □ >> n+k, ...;
with k >> j (for n, j, k □ $\mathbb{N}$)
□ then is an infinitely large Natural Number, and thus it holds □ ≠ .

Any of the automorphic objects $\mathbf{ob_{AM}}$ which are the subobjects of **PrePhys** then is itself also a category, consisting of an object $\mathbf{ob_{AM}}$ and an (auto)morphism $\mathbf{am_{obAM}}$. Yet it must be noted that the very structure of **PrePhys**, i.e. the relations between its subobjects $\{\mathbf{ob_{AM(0-□)}}, \mathbf{ob_{AM(0-i)}}, \mathbf{ob_{AM(0)}}\}$ is a strict *n*-categorical unfolding. Therefore it holds for arbitrary subobjects $\mathbf{ob_{AM}}$ of **PrePhys**:

a)Any subobject $\mathbf{ob_{AM}}$ of **PrePhys** has an *n*-categorical double $\mathbf{ob_{AM(-i)}}$.
b)An *n*-category $\mathbf{ob_{AM(-i)}}$ with the rank (or dimension) -i consists of an object $\mathbf{ob_{AM(-i-1)}}$ and the automorphism $\mathbf{am_{obAM(-i-1)}}$ onto that object $\mathbf{ob_{AM(-i-1)}}$ itself.

[10] For all the anti-epistemological, ontological and mêontological intricacies which had been implied in these remarks cf. D.Kurth, Actual Existence and Factual Objectivation, in: Movements, Philosophical Aspects of ANPA 23 (Proceedings of ANPA 23), Arleta D. Ford (ed.), London 2002 and D.Kurth, The Tower of Turtles, in: The Philosophical Aspects of ANPA 24, Proceedings of ANPA 24, to appear

(For $0 \leq i \leq \omega$; $i \in \mathbb{N}$ and ω being an infinitely large Natural Number)

As a further illustration of that *n*-categorical aspect of the internal making of **PrePhys** might serve FIG.2, which shows $\mathbf{ob}_{\mathbf{AM(0)}}$ and its internal structure consisting of $\mathbf{ob}_{\mathbf{AM(0\text{-}1)}}$ and the automorphism $\mathbf{am}_{\mathbf{ob(0\text{-}1)}}$ of $\mathbf{ob}_{\mathbf{AM(0\text{-}1)}}$ onto itself in some magnification.

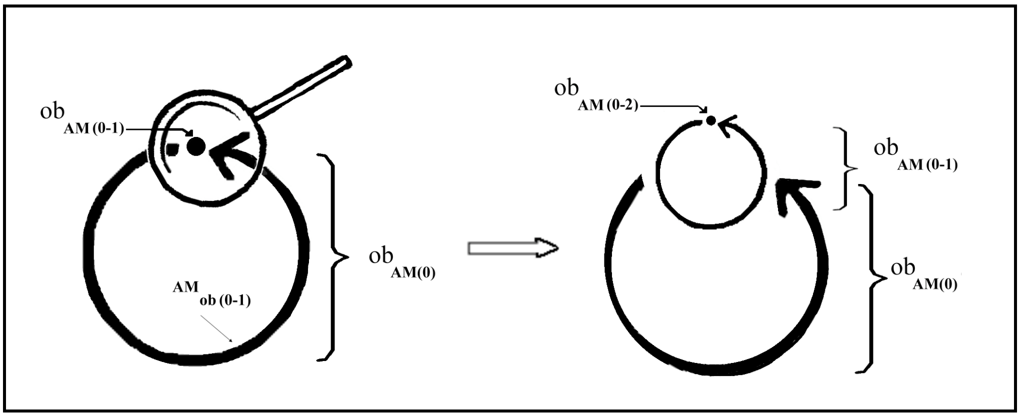

FIG.2: $\mathbf{ob}_{\mathbf{AM(0)}}$,$\mathbf{ob}_{\mathbf{AM(0\text{-}1)}}$ and $\mathbf{am}_{\mathbf{ob(0\text{-}1)}}$ as seen through a magnifying glass

The following figure FIG.3 then might give the reader a broader perspective on that mentioned mêontological process, which can be characterised as a potentially infinite regress into the entrails of the pre-physical mêon, the non-being, itself. And it might be a question of terminological taste whether to call that process an ‘unfolding’ or an ‘enfolding’.

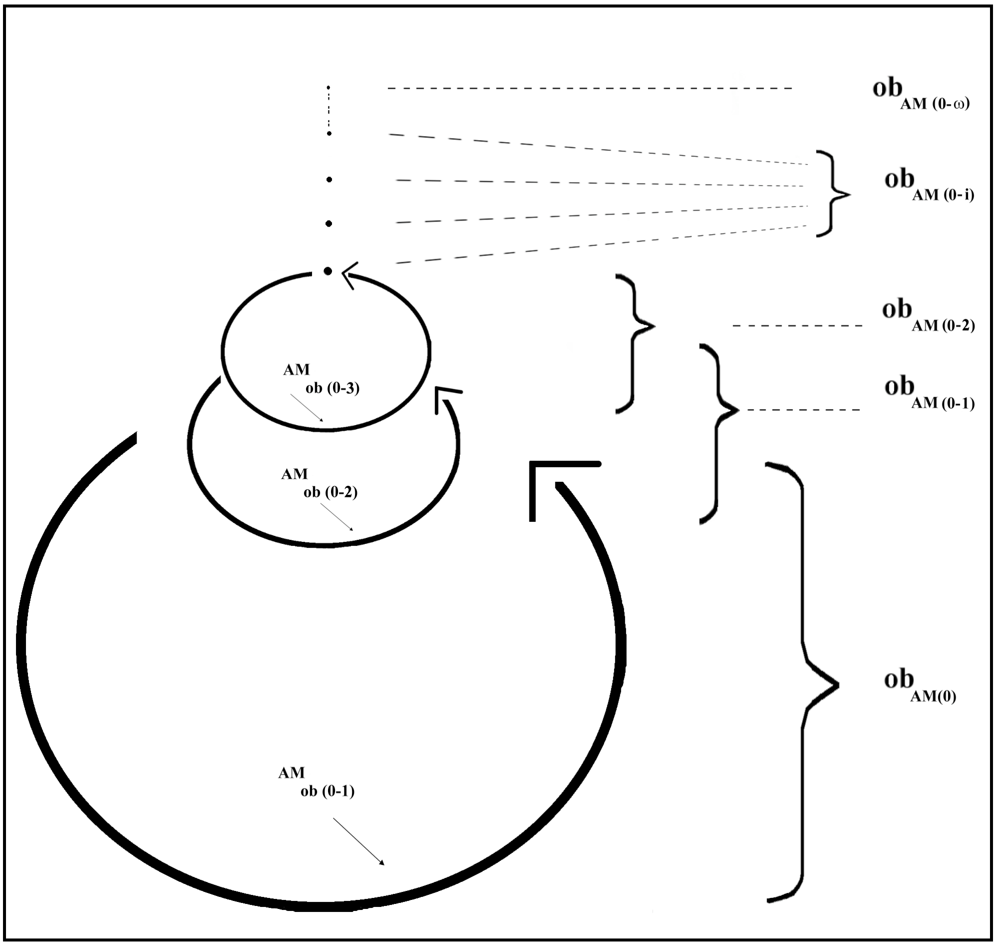

FIG.3: The potentially infinite involutive regress in **PrePhys**: a negative ascending inverted

hierarchy

PrePhys has the automorphic object $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$ or a respective $\mathbf{ob}_{\mathbf{AM(0-i)}}$ as its initial object and the automorphic object $\mathbf{ob}_{\mathbf{AM(0)}}$ as its terminal object, the functors between the various $\mathbf{ob}_{\mathbf{AM(0-i)}}$ (possibly *up from* $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$) terminating at $\mathbf{ob}_{\mathbf{AM(0)}}$ all being isomorphisms.
The automorphic object $\mathbf{ob}_{\mathbf{AM(0)}}$ and the various automorphic objects $\mathbf{ob}_{\mathbf{AM(0-i)}}$ (possibly including $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$) then are the subobjects of **PrePhys**.
In addition - and even somewhat in contrast - to these topos-theoretical characterisations there are also the respective *n*-categorical features namely that the objects $\mathbf{ob}_{\mathbf{AM(0-i)}}$ signify a (negatively ascending) hierarchy of respectively lower (or rather 'negatively higher') dimensional strict *n*-categories (possibly *up to* $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$) related to the one from which this 'descending' (i.e. negatively ascending) cascade starts namely $\mathbf{ob}_{\mathbf{AM(0)}}$. Thus it must be noted that even though $\mathbf{ob}_{\mathbf{AM(0)}}$ is - so to speak - the starting point of a *construction* of negatively increasing categorical dimension it very well also is - perhaps at the first look somewhat puzzling - the object with the *highest* dimension in the particular hierarchy produced by an assumed underlying *process* - simply due to the fact that $0 > 0\text{-}i$ (for $i \in \mathbb{N}$).
The mentioned contrast of the topos-theoretical characterisation of **PrePhys** and the characterisation of the *n*-categorical construction of the $\mathbf{ob}_{\mathbf{AM(0-i)}}$ (possibly *up to* $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$) (which also happen to be the subobjects of **PrePhys**) has - in my view - to do with two problems. Namely a) with the difference between a process and a construction, and then subsequently b) with the problem of how to envision a pre-physical or pre-natural process.
Concerning a) it seems obvious to me that the *n*-categorical approach has to be a construction. From that then follows that this construction has to start with $\mathbf{ob}_{\mathbf{AM(0)}}$ since the dimensions of the subsequently constructed $\mathbf{ob}_{\mathbf{AM(0-i)}}$ (possibly *up to* $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$) will *increase*.[11] Yet this will be a rather uncommon sort of increase, namely an increase - so to say - in negative direction.
But that directedness of the respective *n*-categorical construction doesn't matter at all for the assumed underlying *process* as such. For, if one could imagine such a process as being 'real', then of course such a 'real' process would have to start with an object of a respectively lowest dimension like $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$ or an appropriate $\mathbf{ob}_{\mathbf{AM(0-i)}}$. This relates obviously to the problem b) mentioned above. But even if it might be hard to envision or imagine such a pre-natural mêontological process one has - at least in the context of this argument - to assume such a process. And therefore questions of attainability of objects like $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$ do not matter in the topos-theoretical perspective. The intrinsic 'dynamics' of the topos **PrePhys** had certainly to start with $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$ (or an appropriate $\mathbf{ob}_{\mathbf{AM(0-i)}}$) and to terminate in $\mathbf{ob}_{\mathbf{AM(0)}}$.[12]

I've introduced the topos of pre-physical emergence **PrePhys** for mainly two reasons. At first to give a 'consistent' model of the most primordial emergence of structure, namely an even pre-physical emergence. By 'consistent' here I mean 'consistent' with a mode of emergence, which has also applications beyond just that pre-physical emergence itself. That 'mode of how emergence works' can in my view be described by an unfolding of new levels

[11] Such a *construction* obviously starts with the object of an at least minimal attainability (which is also the object with the respectively highest dimension), namely $\mathbf{ob}_{\mathbf{AM(0)}}$, i.e. it starts where a pre-natural mêontological 'process' should end. In contrast to $\mathbf{ob}_{\mathbf{AM(0)}}$ the $\mathbf{ob}_{\mathbf{AM(0-i)}}$ are significantly less attainable and $\mathbf{ob}_{\mathbf{AM(0-\infty)}}$ is definitely unattainable

[12] Again a bit puzzling is that - to my understanding - such intrinsic dynamics (or the generative structure) of **PrePhys** would by itself show no stratified structure. Thus it seems to me that emergentic features hide in pre-natural mêontological contexts not less than they do in natural ontological ones.

which again in some cases can be modelled by categorification[13], i.e. described by the means of dimensional increase as seen in n-category theory. Also foreshadowed in **PrePhys** is the - in my view - very important aspect of a heterotopic mapping[14], i.e. that emergent transitions are essentially characterised by a (stratified) increase of the topological genus, i.e. an emergent higher level has a kind of 'higher dimensional' topological genus, an example for that is 'Antoine's necklace' (a chain the links of which are chains the links of which again are chains etc.). In the case of **PrePhys** attaching the feature of topological genus might be rather metaphorical. Yet it will turn out that together with increasing ontological hardening by the same mechanism which works in **PrePhys** also the feature of topological genus will emerge - soon.

3 The topos of primordial physical emergence (PrimTor)

The following figure FIG.3 shows again a mathematical structure which bears toposic as well as n-categorical features, namely the topos of primordial physical emergence which I've called **PrimTor**.

PrimTor is meant to be a topos - so to speak - just on top of **PrePhys**. The objects of **PrimTor** are again n-categories but this time there is an important change to note in contrast to the mix up of n-categorical and toposic aspects which we have seen in the case of **PrePhys**. In **PrePhys** the virtually infinitary cascade of n-categories $\mathbf{ob}_{\mathbf{AM(-i)}}$ all had been *strict* n-categories, which are n-categories the *objects* of which are respective (n-1)-categories themselves. A classical example for such a *strict* n-category is the category of categories **Cat**.

Yet in the case of **PrimTor**, which has just two subobjects (namely $\mathbf{ob}_{\mathbf{AM(0)}}$ and $\mathbf{tor}_{\mathbf{AMob(0)}}$), the category $\mathbf{tor}_{\mathbf{AMob(0)}}$ (which stands for the terminal object) is a *weak* n-categorical extension of the other subobject $\mathbf{ob}_{\mathbf{AM(0)}}$ which stands in **PrimTor** for the initial object (but is already known as the terminal object of **PrePhys**). I.e. $\mathbf{tor}_{\mathbf{AMob(0)}}$ is a *weak* n-categorical dimensional increase of $\mathbf{ob}_{\mathbf{AM(0)}}$ with $\mathbf{am}_{\mathbf{ob(0)}}$ (i.e. the automorphism of $\mathbf{ob}_{\mathbf{AM(0)}}$ onto itself) as its object and $\mathbf{am}_{\mathbf{amob(0)}}$ (i.e. the automorphism of $\mathbf{am}_{\mathbf{ob(0)}}$ onto itself) as its n-morphism.

This then generates a sort of rolled-up or toroidal plane which would - embedded in a three-dimensional space - look like a cable torus.

Therefore I called this n-categorical extension of $\mathbf{ob}_{\mathbf{AM(0)}}$ '$\mathbf{tor}_{\mathbf{AMob(0)}}$'.

Yet this *weak* n-categorical dimensional increase (resp. 'enlargement', 'enrichment' or 'unfolding') is also of a particular philosophical relevance. Since in contrast to the *strict* n-categorical dimensional increase in **PrePhys** the *weak* n-categorical enrichment in **PrimTor** shows the most defining feature of a real emergent transition, namely a change in the making-up of the intended entities in question. I.e. the succeeding entities in an emergent transition have essentially to differ in their very structure from the preceding ones they stem from. *Weak* n-categorical dimensional enrichment or *categorification* is particularly apt to serve as model for emergent transitions or *complexification* since the $n+1$-category is not an enrichment of the objects of the n-category but takes the 'dynamical' aspect of the

[13] I use this term - which had originally be invented by L.Crane - following J.Baez and J.Dolan. Cf. J.Baez, J.Dolan, Categorification, math.QA/9802029 5 Feb. 1998

[14] The term 'heterotopic mapping' had been suggested to me by Hans van den Berg to describe a 'genus increasing' transformation.

respective n-category as its base.[15] This implies an important 'step aside' which then allows to give a model of the chain of beings not just as a quite linear and continuous sequence of variations but rather of a discontinuous unfolding of an emergent hierarchy of levels. **PrimTor** is meant to represent the most elementary of such levels.

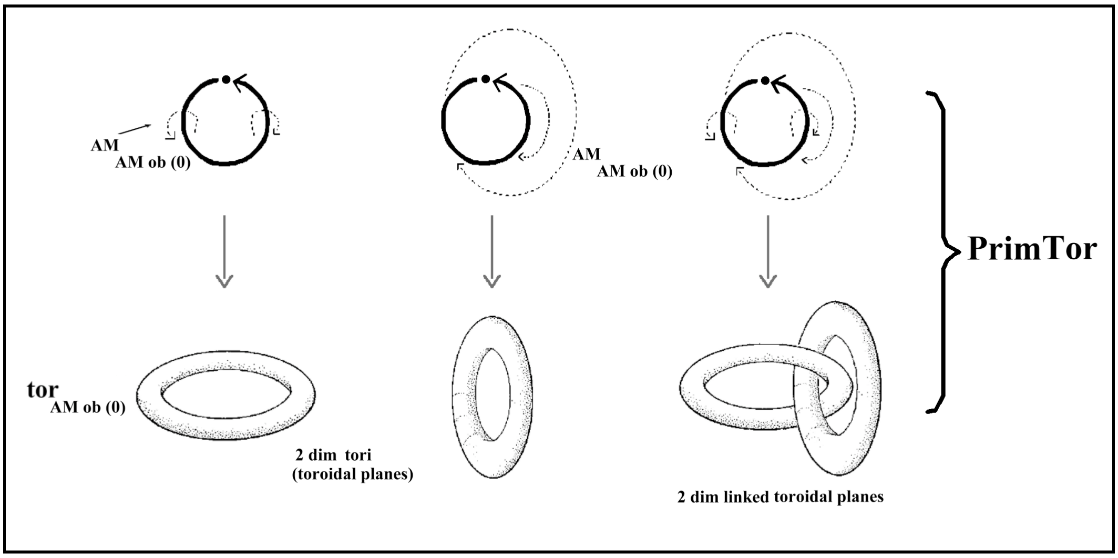

FIG.4: The topos of primordial physical emergence **PrimTor**: a *weak* n-categorical enrichment

For **PrimTor** the following definition holds:

PrimTor: $\{\mathbf{tor}_{\mathbf{AMob(0)}}, \mathbf{ob}_{\mathbf{AM(0)}}\}$

with $\mathbf{ob}_{\mathbf{AM(0)}}$ as initial object and $\mathbf{tor}_{\mathbf{AMob(0)}}$ as terminal object

In the light of the (anti)epistemological distinctions I've mentioned in the introduction to this paper[16] between objects which cannot be intended to be entities (i.e. purely intelligible, formal or mathematical objects) and objects which can be intended to be entities (i.e. potentially physical objects) the tori (or rolled-up toroidal planes) $\mathbf{tor}_{\mathbf{AMob(0)}}$ then are intended to represent such later potentially physical objects as being emerged from an underlying level of purely intelligible or mathematical objects ($\mathbf{ob}_{\mathbf{AM(0)}}$). And such an emergence then would also have to be seen as the emergence of potential physical existence from a underlying state of no physical existence and this again would provide the conceptual means of overcoming the seeming paradox of reductionism - mentioned in the introduction of this paper - without abandoning the reductionist project as such.

3.1 Linked toroidal objects

Since there is no limitation attached for the higher dimensional toroidal objects $\mathbf{tor}_{\mathbf{AMob(0)}}$ concerning either anything like 'orientation' or the number of the n-morphisms $\mathbf{am}_{\mathbf{amob(0)}}$,

[15] Cf. P.Eisenhardt, D.Kurth, Complexity Categorified; in: Implications, Scientific Aspects of ANPA 22, (ed. Keith Bowden), London 2001

[16] Cf. also D.Kurth, Actual Existence and Factual Objectivation, in: Movements, Philosophical Aspects of ANPA 23 (Proceedings of ANPA 23), Arleta D. Ford (ed.), London 2002 and D.Kurth, The Tower of Turtles, in: The Philosophical Aspects of ANPA 24, Proceedings of ANPA 24, to appear

which essentially constitute **tor**$_{\mathbf{AMob(0)}}$, there could very well be various ***linked*** **toroidal objects** come as a result of the respective n-(auto)morphisms **am**$_{\mathbf{amob(0)}}$.
Such linked toroidal objects then could be understood as something like elementary chain-elements, which again could be rated as first instances of what I regard to be the quintessential property of - in this case yet very elementary and consequently very ‘simple’ - ‘complex systems’, namely topological connectivity.

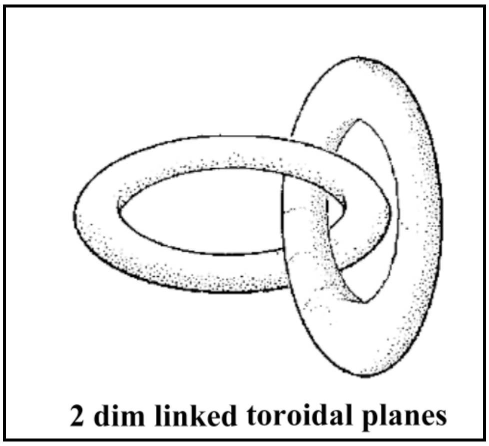

FIG.5: 2 dim linked rolled-up toroidal planes (embedded in $\mathbb{R}^3$): a first instance of potential physical complexity

The philosophical significance of the **linked toroidal objects** shown in FIG.5 lies in my view in the assumption that physical existence is linked to an at least minimal degree of complexity and that - as laid out above - the very feature of complexity is topological connectivity. That leads to the consequence that I doubt it being meaningful to speak of ‘elementary physical objects’, if by ‘elementary’ is meant ‘simple’ or ‘of no degree of complexity’. The reason for this point of view is of course that it is consistent with my view of reductionism and the required action to overcome its seeming paradox.[17] Just to make this point as clear as possible: In my view there can’t exist an absolutely elementary or simple ‘physical entity’, i.e. a physical entity of an absolutely least complexity. Quite on the contrary I hold that even the most elementary physical objects stem from mathematical ones and by this they already inherited a respective degree of complexity and since the physical objects emerged out of the mathematical ones their proper rank of complexity is also higher than that of their respective mathematical predecessors. Thus in my view the only objects of a truly least complexity could only be mathematical objects, e.g. of the kind of **ob**$_{\mathbf{AM(0-i)}}$.
In the light of these premises I want to highlight the transitorial as well as hermaphrodite nature of **PrimTor**. I.e. **PrimTor** represents just this transitorial stage in the emergence of physical existence, in which its terminal objects **tor**$_{\mathbf{AMob(0)}}$ are no more purely mathematical and not yet proper physical ones.[18] And the ***linked*** **2-dim rolled-up toroidal planes** then would just be a step further in the direction of emerging physical existence.

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[17] Cf. D.Kurth, The Tower of Turtles, in: The Philosophical Aspects of ANPA 24, Proceedings of ANPA 24, to appear
[18] And this provides a good excuse for a surely surprising and perhaps overly precautionary note, namely that an essential part of my argument in this papers relies exclusively on this mentioned double-nature of **PrimTor**. This relates to the emergence of physics out of mathematics. The proclaimed emergence of an elementary (almost) physical state up from a purely mathematical one would precisely take place in the transition from the **ob**$_{\mathbf{AM(0)}}$ to the **tor**$_{\mathbf{AMob(0)}}$ in **PrimTor**. The infinitary negative ascending hierarchy of **ob**$_{\mathbf{AM(-i)}}$ in **PrePhys** is not required for that emergent transition but just for providing a non-paradoxical infinite regress for the reductionist program.

4 Interpretation

4.1 Emergence categorified: *weak 2*-categorical localic/quantal spintori

After coming so near to the advent of physical existence a question carefully avoided till now might eventually come across and can no longer be evaded. This is the question of what particular mathematical objects the **ob**$_{\mathbf{AM}}$ actually are?
My answer to this question might appear as predominantly clear, but then a bit undecided as well: The ‘natural’ candidates for being the real **ob**$_{\mathbf{AM}}$ are either *locales* or *quantales*.
The reasons for this sort of choice (or choices) should be rather obvious. At first the entire context in which the **ob**$_{\mathbf{AM}}$ undergo their emergent endeavours - namely **PrePhys** and **PrimTor** - is thoroughly toposic. Locales as well as quantales again are of a toposic nature, thus having them as the subobjects of **PrePhys** and **PrimTor** would at least satisfy principles of simplicity as well as selfconsistency.
Yet for the argument in this paper another reason is at least as relevant. Locales (as well as quantales) are just by their very nature ideal candidates for giving the ‘Leibniz-point objects’[19] (once envisioned and postulated by Leibniz[20] in his *Theoria motus abstracti*), which had been the starting point of the considerations in this paper, a most rigorous mathematical setting, i.e. a reformulation in the terms of topos theory.

Locales are topos-theoretical doubles of the open sets of set theory and bear also a strong resemblance of the monads of non-standard analysis. All these concepts have in common that they somehow are apt to substitute the ordinary point by a kind of rather infinitesimal element which has - even if not proper parts in the sense of Leibniz then at least - an internal structure.[21] An elementary explanation of the notion of a locale in a somewhat broader context (which has yet the advantage to put this explanation in just that context which has been the starting point of this paper, namely that of topos theory) had been given by Saunders Mac Lane and Ieke Moerdijk:

> “A topos is, in a suitable sense, a generalized space, so should have (generalized!) points. Indeed, at a given point x of an ordinary topological space X, one can erect each set A as a sort of “skyscraper” sheaf A_x concentrated around the point x. The resulting mapping from the category of sets into that of sheaves on X is, in fact, the direct image of a geometric morphism **Sets** $\rightarrow$ Sh (X). But an arbitrary topos $\mathcal{E}$ may not have enough “points” **Sets** $\rightarrow$ $\mathcal{E}$ in this sense. In order to develop an adequate comparison between topoi and spaces, it is useful to alter the definition of a space by describing a space not in terms of its points, but in terms of its open sets. The objects so defined by a lattice of

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[19] Cf. the introduction to this paper. Cf. also D.Kurth, The Tower of Turtles, in: The Philosophical Aspects of ANPA 24, Proceedings of ANPA 24, to appear

[20] Specifically as subobjects of a topos **PrePhys**$_{\mathbf{loc}}$ locales **loc**$_{\mathbf{AM(-i)}}$ might also provide a conceptual basis for reconciling the hitherto seemingly contradictory relationalist and dynamist tendencies in Leibniz’ natural philosophy. The relata which are supposed to constitute a certain space then could be seen as respectively lower dimensional (abstract) spaces themselves being constituted by relata which again are respectively lower dimensional abstract spaces related to the one they constitute and themselves again being constituted by another set of such a kind of relata etc.
That possible reconciliation then would be just due to the double nature of **PrePhys**$_{\mathbf{loc}}$ as a topos and a structure of n-categorical un- or enfolding (see above). The relationist aspect then would be carried by its toposic nature and the dynamist aspect by its n-categorical dowry. I.e. in the toposic perspective **PrePhys**$_{\mathbf{loc}}$ has all the various locales **loc**$_{\mathbf{AM(-i)}}$ as ordinary subobjects (and in that sense as proper relata(!)), yet seen as elements of the respective *strict* n-categorical structure (which again happens to be **PrePhys**$_{\mathbf{loc}}$) any strict n-category **loc**$_{\mathbf{AM(-i)}}$ has also a strict *n-1* category **loc**$_{\mathbf{AM(-i-1)}}$ as its object. And since **loc**$_{\mathbf{AM(-i)}}$ would just be produced as the automorphism of **loc**$_{\mathbf{AM(-i-1)}}$ I would rate this as a genuinely dynamical generation in contrast to a merely static relation.

[21] The mathematical definition and an extensive explanation of locales can be found in: S.Mac Lane, I.Moerdijk, Sheaves in Geometry and Logic: A first Introduction to Topos Theory, New York Berlin 1992

> open sets are called locales. Since sheaves can be described in terms of coverings by open sets, one can construct a topos Sh(X) consisting of all the sheaves of sets on such a locale X. Moreover, any "continuous" map $Y \to X$ between locales gives rise to a geometric morphism Sh(Y) $\to$ Sh(X) between such sheaf topoi."[22]

In this paper locales $\mathbf{loc}_{\mathbf{AM}}$, which would stand for the originally introduced $\mathbf{ob}_{\mathbf{AM}}$, then would have to be seen as - in an emergence-theoretical sense preceding - constituents of the physical space and not as (in whatever ways derived) parts of it and therefore they would utterly lack the very feature of a pre-particle or of a spatial entity. This is in distinct contrast primarily to loops, which had been designed just as elementary *parts* of physical space, i.e. derived as discrete elements of space based on canonical quantization of the relativistic space-time manifold.[23]

This makes a significant difference because even if one may claim that such an eventually uncovered quantized and thereby discrete deep-structure of physical space would be the underlying and even - in that sense - 'true' structure of physical space one cannot claim that physical space would be emergent from that structure but just the opposite, namely that space always and ever was made up by that elementary structure (and thus *did not emerge* out of a radically non-spatial pre-structure).

Yet to claim that space itself is radically emergent from a pre-spatial structure requires a completely different approach, since - if by the epithet 'spatial' here a *physical space* is suggested - such a pre-spatial structure would have to be of an essentially non-spatial making.

Locales as well as their non-commutative siblings, quantales[24], now are just such mathematical objects, which neither had been derived from any concepts of physical space nor designed for the purpose to provide elements for a physical space to be built up by (or on) them. I.e. locales (and quantales) must *not* be seen as - in what ever way diluted or diminihed - *parts* of physical space, not at least because they cannot be taken as physical entities at all.

But if the assumed most imperceptibly emerging physicality does not stem from the *objects* in the topos(es) $\mathbf{SpinTor}_{\mathbf{loc}}$ (resp. $\mathbf{SpinTor}_{\mathbf{qtl}}$) - (both) shown (in a sort of superposition) in FIG.6 (which are just the respectively interpreted versions of **PrimTor,** see FIG.4) then - since nothing else is left - any possibly emerging physicality must stem from the respective weak *n*-morphisms in $\mathbf{SpinTor}_{\mathbf{loc}}$ (resp. $\mathbf{SpinTor}_{\mathbf{qtl}}$), i.e. the *n+1* (auto)morphisms $\mathbf{sp}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sp}_{\mathbf{AMqtl(0)}}$) of the *n*-(auto)morphism $\mathbf{AM}_{\mathbf{loc(0)}}$ (resp. $\mathbf{AM}_{\mathbf{qtl(0)}}$) onto themselves.

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[22] S.Mac Lane, I.Moerdijk, Sheaves in Geometry and Logic: A first Introduction to Topos Theory, p 8

[23] In such a sense (as well in many others aspects) the role attached in this paper to locales (or quantales) differs decisively from e.g. the role of Wilson Loops in the loop-theoretical approach to Quantum Gravity.

[24] Most of what has been said about locales above also relates to quantales, which are in fact a sort of quantized locales, i.e. locales which satisfy an additional noncommutative rule. Quantales had been originally introduced by C.J..Mulvey in 1986. For a detailed introduction into the subject cf. C.J.Mulvey, M.Nawaz, Quantales: Quantal sets, in: Non-classical Logics and their Application to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Set Theory, Kluwer 1995, pp 159-217; somewhat more related to the topic of this paper is: C.J.Mulvey, J.W.Pelletier; On the Quantisation of Points, J.Pure Appl. Algebra 159 (2001) 231-295

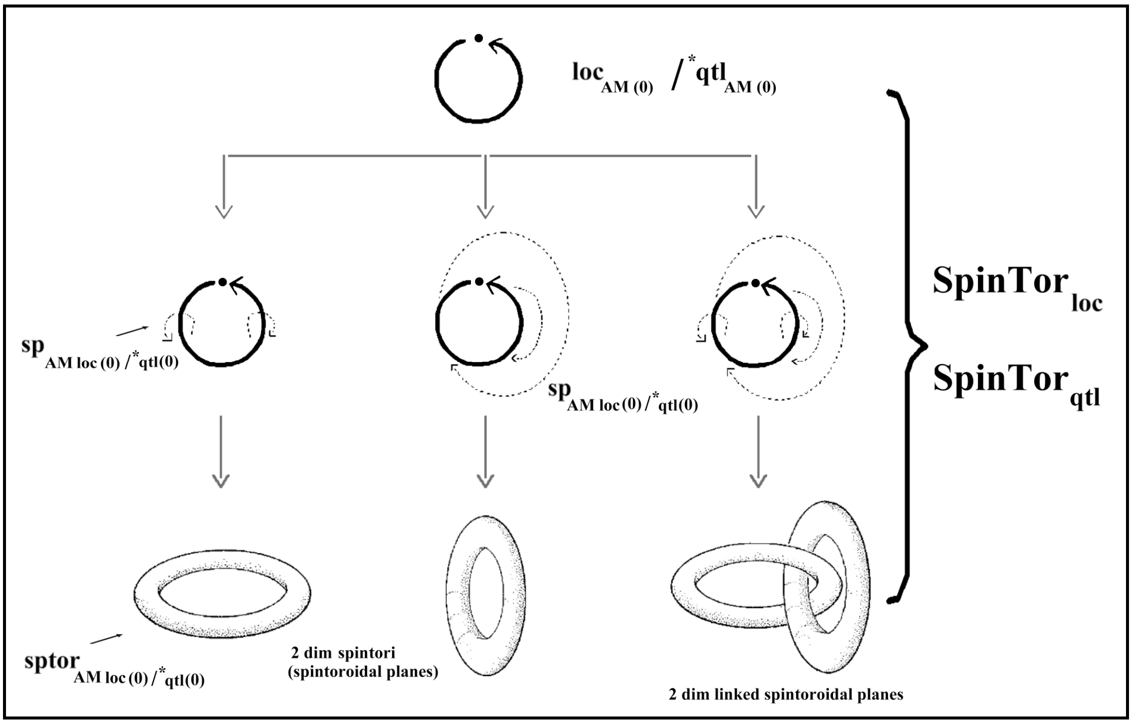

FIG.6: The topos(es) of emergence: SpinTor loc /*qtl
* The dash (/) in FIG.6 signifies that the objects in question are always supposed either to be read as proper *locales* or as their non commutative siblings, namely *quantales*.

For $\mathbf{SpinTor_{loc}}$ the following definition holds:

$\mathbf{SpinTor_{loc}}$: $\{\mathbf{loc_{AM(0)}}, \mathbf{sptor_{AMloc(0)}}\}$

with $\mathbf{loc_{AM(0)}}$ as initial object and $\mathbf{sptor_{AMloc(0)}}$ as terminal object

Respectively the following definition holds for $\mathbf{SpinTor_{qtl}}$:

$\mathbf{SpinTor_{qtl}}$: $\{\mathbf{qtl_{AM(0)}}, \mathbf{sptor_{AMqtl(0)}}\}$

with $\mathbf{qtl_{AM(0)}}$ as initial object and $\mathbf{sptor_{AMqtl(0)}}$ as terminal object

Thus the emergence of potential physicality is - in my view - utterly based on the dynamical - which here are in fact *the same as the emergentic* - aspects of the mathematical structure(s) ($\mathbf{SpinTor_{loc}}$ resp. $\mathbf{SpinTor_{qtl}}$) in question, namely the aspects, which are related to the *weak n*-categorical entrails of the topos(es) $\mathbf{SpinTor_{loc}}$ (resp. $\mathbf{SpinTor_{qtl}}$, i.e. the *weak 2*-categorical (auto)morphisms $\mathbf{sp_{AMloc(0)}}$ (resp. $\mathbf{sp_{AMqtl(0)}}$).[25] And this is perfectly in line with the philosophical premises which led me to all these conclusions[26] and which originated in Leibniz' radical dynamism as displayed in his *Theoria motus abstracti* (as well as in many others of his writings).

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[25] But in sharp contrast to the similar situation in the case of **PrePhys** here the mentioned "*weak n*-categorical entrails" are no more and not at all completely independent of the toposic structure of $\mathbf{SpinTor_{loc}}$ (resp. $\mathbf{SpinTor_{qtl}}$), but - just to the contrary - in the case of $\mathbf{SpinTor_{loc}}$ (resp. $\mathbf{SpinTor_{qtl}}$) the (toposic) morphisms, which lead from the initial object(s) $\mathbf{loc_{AM(0)}}$ (resp. $\mathbf{qtl_{AM(0)}}$) to the terminal object(s) $\mathbf{sptor_{AMloc(0)}}$ (resp. $\mathbf{sptor_{AMqtl\ (0)}}$) are just *identical* with the *weak n*-categorical (auto)morphism(s) $\mathbf{sp_{AMloc(0)}}$ (resp. $\mathbf{sp_{AMqtl(0)}}$).

[26] Cf. D.Kurth, The Tower of Turtles, in: The Philosophical Aspects of ANPA 24, Proceedings of ANPA 24, to appear; and D.Kurth, Actual Existence and Factual Objectivation, in: Movements, Philosophical Aspects of ANPA 23 (Proceedings of ANPA 23), Arleta D. Ford (ed.), London 2002

But then what could that insinuated physical feature of $\mathbf{sp}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sp}_{\mathbf{AMqtl(0)}}$) ever be? Since we haven't been parsimonious with exacting suggestions to the imaginative benevolence and faculty of the readers, we feel entitled to stress at least that benevolence (of the remaining ones) a little further.
I take or interpret the $\mathbf{sp}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sp}_{\mathbf{AMqtl(0)}}$) as a sort of (physical) spin. And I would even extend this interpretation to the preceding automorphism(s) $\mathbf{AM}_{\mathbf{loc(0)}}$ (resp. $\mathbf{Am}_{\mathbf{qtl(0)}}$), i.e. I would take them as being spins of either locales or quantales. With this surely exacting suggestion we now have reached the decisive point of the whole argument, which I put forward in this paper. The only excuse for all these exactions is the fact that the notion of 'emergence' inherently has a tinge of unreasonableness, since it implies to find a way (a sort of continuity) across an abyss (of a radic al discontiuous nature).
As always in the theory of emergence also in the case of the relation of the concepts of 'automorphism' on the one hand and of 'spin' on the other the reductionist view works fine and the proper emergentist one much less fine, if at all. This becomes obvious when one agrees that the physical spin can - as an abstract symmetric rotation - nicely be modelled as an instance of an automorphism, yet that an automorphism can obviously not be seen as an instance of a spin, simply because it lacks the physical features of a spin.
Yet as always in the theory of emergence one has to overcome the undeniable reasonableness of this objection. Since by not doing so one has to pay a much higher price in terms of reasonableness, i.e. one has either to admit the existence of miracles, e.g. in the case of the emergence of life up from prebiotic conditions, or of potential paradoxes as in the case of the reductionist project with respect to the question of the ontological status of a first physical entity.
So I hold that $\mathbf{SpinTor}_{\mathbf{loc}}$ (resp. $\mathbf{SpinTor}_{\mathbf{qtl}}$) provides a model of how automorphisms of the sort of $\mathbf{sp}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sp}_{\mathbf{AMqtl(0)}}$) and their 'predecessors' $\mathbf{AM}_{\mathbf{loc(0)}}$ (resp. $\mathbf{Am}_{\mathbf{qtl(0)}}$) once could have find a way across the abyss between the mêontology of mathematical objects and the ontology of intended physical entities by metamorphosing into spins.

4.2 From spinchains to contiguity: emergent topological connectivity = complexity

Yet of course we have still left out some further exacting details of $\mathbf{SpinTor}_{\mathbf{loc}}$ (resp. $\mathbf{SpinTor}_{\mathbf{qtl}}$), namely the status of their terminal objects $\mathbf{sptor}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sptor}_{\mathbf{AMqtl(0)}}$).
Now, these are apparently static images of the automorphisms $\mathbf{sp}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sp}_{\mathbf{AMqtl(0)}}$) of which they are 'made of'. I.e. insofar as they are statified automorphisms they are ontified as to be a sort of elements. And they are supposed to be elements - quite literally. This becomes evident in the bottom line of FIG.6 in the case of the object called '2-dim linked spintoroidal planes'.
I.e. the real potential physical role of $\mathbf{sptor}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sptor}_{\mathbf{AMqtl(0)}}$) as elements lies exclusively in their potential role as elements of a spinchain, i.e. a spinchain just of such elements as $\mathbf{sptor}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sptor}_{\mathbf{AMqtl(0)}}$).
Such spinchains then might - appropriately increased - grow into more complex physical entities. A possible example could be entities like 'strings', the proposed elements of such entities, namely $\mathbf{sptor}_{\mathbf{AMloc(0)}}$ (resp. $\mathbf{sptor}_{\mathbf{AMqtl(0)}}$) then being a sort of 'parton' of such spinchains or 'strings'. Yet serious physical stuff was never the topic of this paper so we stick to our philosophical suggestions. As a somewhat enlarged illustration of these philosophical suggestions see FIG.7.

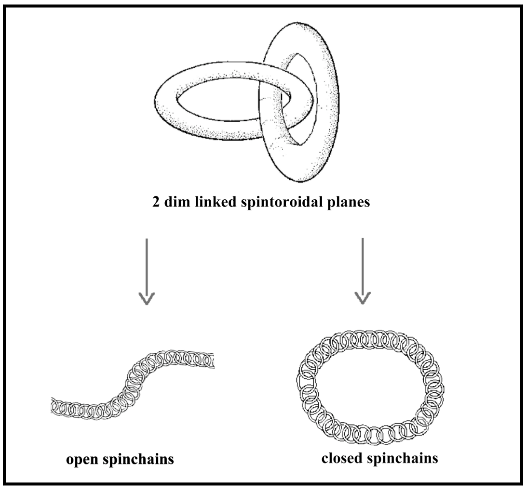

FIG.7: spinchains: emergent topological connectivity built-up of linked spintori

In this respect what matters most with the spinchains shown in FIG.7 is that the emerging physicality - claimed to be illustrated there - is essentially due to the growing topological connectivity which is as well characterised by the contiguous structure (of a chain).

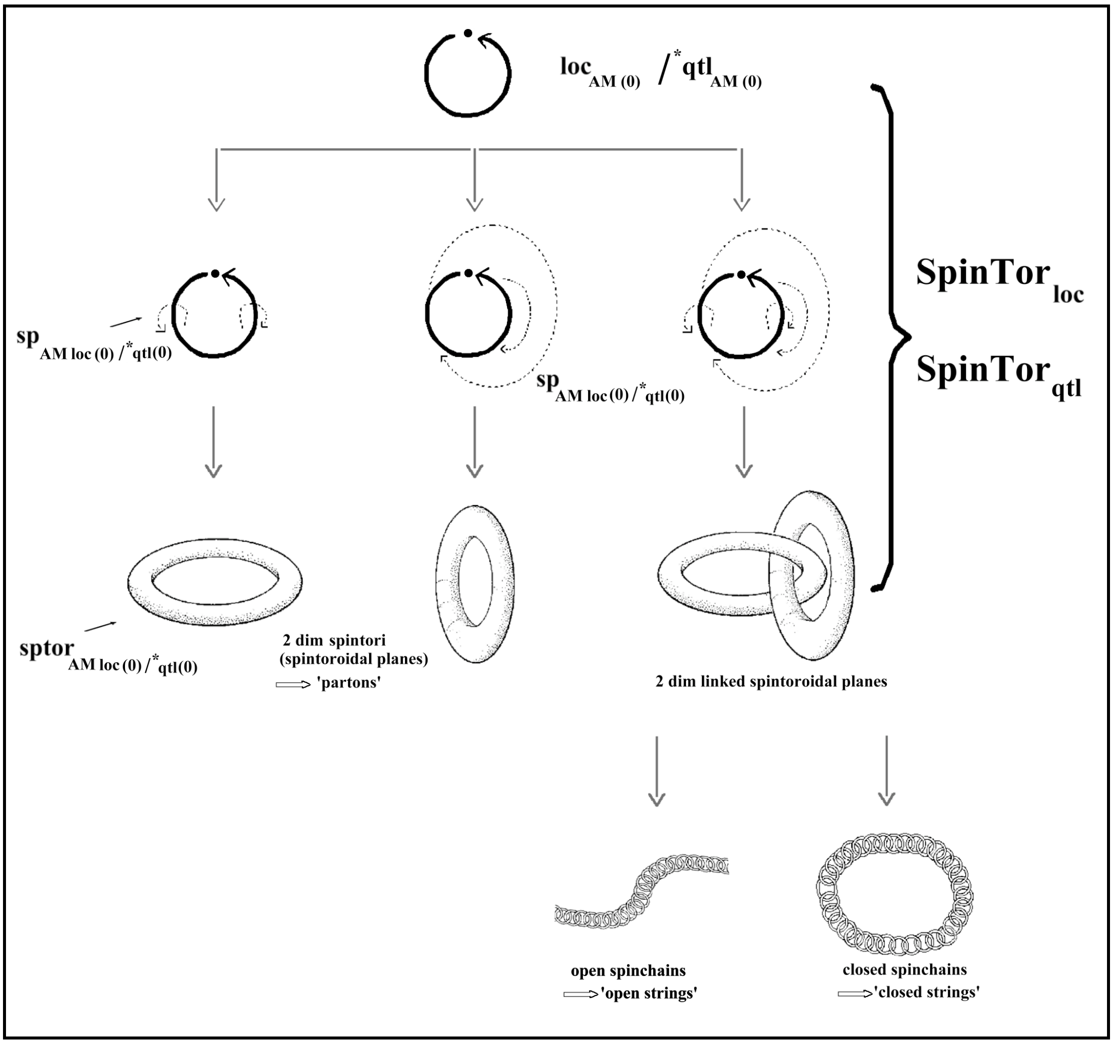

FIG.8: The topos(es) of emergence. And elementary spinchains (made of linked spintori) as first instances of emergent topological connectivity.

As the mentioned growing topological connectivity is producing more of that contiguous structure(s) by its own further growth it also produces more and more of potential actual physicality. Thus - I hold - that exactly such topological connectivity or contiguity is the underlying structure (and measure) of physical complexity and even the most elementary physical entities already are characterised by an increased complexity related to that of the mathematical objects from which they stem.[27]

Since this paper was - for good reason - not about physics at all but just about to think of it as emerging, it now already overstretched its scope. Therefore it may now come to an end after having presented you instead of a summary an overview of the proposed topos(es) of emergence in FIG.8.

[27] This is again in line with my claim that any assumed most elementary physical entity already has to have a certain degree of complexity which directly stems from (the increase of complexity having taken place in the pre-physical emergence of) its mathematical predecessors.