The Logic of Sequence Frames

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Abstract

This paper investigates and develops generalizations of two-dimensional modal logics to any finite dimension. These logics are natural extensions of multidimensional systems known from the literature on logics for a priori knowledge. We prove a completeness theorem for propositional n-dimensional modal logics and show them to be decidable by means of a systematic tableau construction.

Keywords: Modal Logic · Multidimensional Modal Logic · Epistemic Logic · A Priori · Actuality · Necessity · Completeness · Decidability

Introduction

Two-dimensional modal logics have been the object of increasing philosophical interest given intuitive interpretations assigned to the modal operators. In particular, that such logics provide us with a logical analysis of metaphysical necessity, actuality, and a priori knowledge is now an important facet of prominent philosophical views in philosophy of language, epistemology, and metaphysics. Logics developed with the purpose of shedding light on a priori reasoning and its relation with the modal notions of necessity and actuality arguably originated with Davies and Humberstone (1980), and have recently been investigated by Restall (2012), Fritz (2013, 2014), Fusco (forthcoming), and others. Furthermore, the semantic treatment of epistemic and indexical terms in the works of Evans (1979), Kaplan (1989), and Chalmers (2004,

2014) make essential use of elements present in two-dimensional modal logics, while Weatherson (2001) and Wehmeier (2013) adopt similar two-dimensional semantics to provide formal treatments of both subjunctive and indicative conditionals.

The goal of this paper is to generalize two-dimensional modal logics with actuality operators to *any finite dimension*. To be more precise, the logics investigated here are generalizations of the logic for epistemic two-dimensional semantics studied by Fritz (2014) under the name **2Dg**, which in a certain sense is the same logic as those investigated proof-theoretically in Restall (2012) and Lampert (2018)—although the logics in Lampert (2018) contain first-order quantifiers. The formal language defined here contains several modal operators, one for each dimension, including indexed boxes, $\Box_i$, as well as actuality operators, $@_i$. As will be seen below, the resulting logics can be characterized as logics of generalized diagonal sequences, and therefore as natural generalizations of two-dimensional modal logics in which formulas in the scope of certain operators are evaluated at the diagonal points of *square models*, that is, models based on frames containing ordered pairs of worlds. Notwithstanding the formal character and scope of this paper, whose principal concern is with model- and proof-theoretic investigations of certain classes of frames and models for modal logics, the development of n-dimensional modal logics is in fact philosophically motivated and arises from considerations regarding the expressive power of modal languages for necessity, actuality, and apriority. In what follows we briefly describe some of these motivations as well as the plan for the rest of the paper.

Background

Crossley and Humberstone (1977) introduced the actuality operator, @, to remedy an expressive deficit in the first-order modal language,[1] i.e. the language of first-order logic augmented by the modal operators $\Box$ and $\Diamond$. What they observed was that there is no formula in that language that expresses the following truth condition:

(1) $\exists w \forall \mathsf{x}(R\mathsf{x}\text{-at-}z \to S\mathsf{x}\text{-at-}w)$,

where z is the "actual world" of the underlying frame.[2] According to (1),

[1] See also Hazen (1976).

[2] It is common to notate the actual world as w^* (see, for instance, Crossley and Humberstone (1977) and Davies and Humberstone (1980)), or even as @, if another symbol is used

there is a world w such that every object that is R in the actual world z is S in w. As Crossley and Humberstone (1977: 12) point out, there are readings of English sentences with truth conditions corresponding to (1), such as "It is possible for every red thing to be shiny". But the most obvious attempts to represent (1) in the object language all fail: $\Diamond\forall\mathsf{x}(R\mathsf{x} \rightarrow S\mathsf{x})$ expresses the truth condition $\exists w\forall\mathsf{x}(R\mathsf{x}\text{-at-}w \rightarrow S\mathsf{x}\text{-at-}w)$, $\forall\mathsf{x}\Diamond(R\mathsf{x} \rightarrow S\mathsf{x})$ the truth condition $\forall\mathsf{x}\exists w(R\mathsf{x}\text{-at-}w \rightarrow S\mathsf{x}\text{-at-}w)$ and, if we assume that truth in a model is defined as *real-world truth*,[3] i.e. truth at the actual world of the model—in which case formulas are initially evaluated at z—$\forall\mathsf{x}(R\mathsf{x} \rightarrow \Diamond S\mathsf{x})$ expresses the truth condition $\forall\mathsf{x}(R\mathsf{x}\text{-at-}z \rightarrow \exists w(S\mathsf{x}\text{-at-}w))$. None of these truth conditions are equivalent to (1). Yet, in the presence of the actuality operator (1) can be expressed by the formula $\Diamond\forall\mathsf{x}(@R\mathsf{x} \rightarrow S\mathsf{x})$. This is so because the semantic evaluation clause for the actuality operator invariably takes back the evaluation of a formula in its scope to the actual world of the frame. Let $\mathfrak{M} = (W, z, \mathcal{R}, \mathcal{D}, V)$ be a model for the first-order modal language containing @, where W is a set of possible worlds, $z \in W$, $\mathcal{R} \subseteq W \times W$, $\mathcal{D}$ is a domain of objects, and V is a valuation function defined as usual.[4] For any formula φ,

$$\mathfrak{M}, w \vDash @\varphi \Longleftrightarrow \mathfrak{M}, z \vDash \varphi.^{5}$$

So even if the formula $@R\mathsf{x}$ occurs within the scope of $\Diamond$, the semantic entry for @ makes us consider the objects in the domain of the world introduced by $\Diamond$ that are R in the actual world z, thereby, as it were, temporarily suspending the scope of the $\Diamond$ operator.

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for the actuality operator (see, for instance, Wehmeier (2014)).

[3]This terminology comes from Crossley and Humberstone (1977: 15), where *real-world validity* is defined as truth at the actual world of every model, and *general validity* as truth at every world of every model. The distinction is not without a difference: the formula $@\varphi \rightarrow \varphi$ for example, being real-world but not generally valid. Since, however, there can be worlds at which $@\varphi \rightarrow \varphi$ is false, real-world validity gives rise to contingent logical truths, a point explored by Zalta (1988) in his defense of real-world validity over general validity as the correct generalization of the Tarskian notion of logical truth for modal languages. A reply to Zalta can be found in Hanson (2006), but we also direct the reader to Nelson and Zalta (2012), French (2012), Hanson (2014), and Wehmeier (2014).

[4]We shall not be concerned with first-order models in this paper apart from this section. For simplicity, though, one can just assume that the constant symbols in the first-order modal languages mentioned here are interpreted rigidly, even though this undermines somewhat the purpose of extending two-dimensional modal languages with the first-order quantifiers, where it is possible to distinguish, say, the metaphysical rigidity of proper names, subsumed under the constant symbols of the language, from their epistemic non-rigidity.

[5]For simplicity, we leave the assignment function implicit here.

There is, however, a subtle difference in the semantic clause for @ when we move to a full-blooded two-dimensional modal logic.[6] In this case, formulas are evaluated against pairs of possible worlds, (w, v), where w is taken intuitively as a counterfactual world and v as a world *considered as actual* (or an *epistemic scenario*, such as in Fritz (2014)), and so the set of evaluation points W in the frames can now be defined as the Cartesian product of some non-empty underlying set of worlds, say, S. This means that rather than having a single fixed actual world as a part of the frame, any world can be actual as long as it occupies the second coordinate of the pair of worlds relative to which a formula is evaluated. That is, to use a metaphor from Humberstone (2004: 26), we first *dethrone* the single actual world from the identity of the frames or models, and formulate the semantic clause for @ as

$$\mathfrak{M}, (w, v) \vDash @\varphi \Longleftrightarrow \mathfrak{M}, (v, v) \vDash \varphi,^{7}$$

where v can be any element of S. Then the truth conditions for $\Box_1\varphi$, where $\Box_1$ is now used for the metaphysical necessity operator, can be given with the aid of its corresponding accessibility relation, $\mathcal{R}_{\Box_1}$, which relates a pair (w, v) to any pair (x, y) such that $y = v$. That is, the actual world is held fixed while the counterfactual world varies. Additionally, two-dimensional modal languages are equipped with a diagonal necessity operator, interpreted intuitively as an apriority operator, which is sometimes taken as a primitive operator in the language (Restall (2012), Fritz (2014), Lampert (2018)), or defined in terms of other operators (Davies and Humberstone (1980)). Here we notate it as $\Box_2$, and its corresponding accessibility relation will be $\mathcal{R}_{\Box_2}$, which in turn relates a pair (w, v) to any pair (x, y) such that $x = y$. Intuitively, the semantic entry for the apriority operator then says that a formula is true a priori if and only if, no matter which world turns out to be actual, that formula remains true in that world. The reason why we call this a diagonal operator is because when models are defined with a domain of pairs of possible worlds the operator $\Box_2$ projects precisely along the pairs consisting of identical coordinates. This can be illustrated by distributing the pairs constructed out of possible worlds w, v, u in a 2-D matrix, in which the X axis contains counterfactual worlds

[6] We distinguish the number of dimensions based on the evaluation points in the frames: if formulas are evaluated against a possible world w, the semantics is one-dimensional. By contrast, if formulas are evaluated against a pair (w, v) of possible worlds, we say that the semantics is two-dimensional; similarly for triples (w, v, u), and so on.

[7] This can be compared with Davies and Humberstone (1980: 4), although they use a different notation.

and the Y axis worlds taken as actual. $\Box_2\varphi$ is then true if φ holds along the diagonal points of the matrix, as seen in Figure 1 below.

$$\begin{pmatrix} & w & v & u \\ w & \varphi & - & - \\ v & - & \varphi & - \\ u & - & - & \varphi \end{pmatrix}$$

Figure 1: 2-D matrix

It is possible to subsume the semantic clauses for $\Box_1$ and $\Box_2$ under a single schema as follows. Let σ and τ be sequences (of length n) of possible worlds. Then:

(GB) $\mathfrak{M}, \sigma \vDash \Box_i\varphi \Longleftrightarrow$ for every $\tau \in W$, if $\sigma\mathcal{R}_{\Box_i}\tau$, then $\mathfrak{M}, \tau \vDash \varphi$,

where $\sigma\mathcal{R}_{\Box_i}\tau$ if and only if the first i coordinates in τ are all identical and σ and τ are identical beyond i, i.e. for all $j > i$, the jth coordinate of σ is the same as the jth coordinate of τ.[8] It is simple to check that (GB) delivers the correct semantic entries for both $\Box_1$- and $\Box_2$-formulas. And since there is no single distinguished point in the underlying frames anymore, we can add to them a set $D = \{(z, z) \mid z \in S\}$ of diagonal points, and define truth in a model relative to the elements of D, in which case D is the set of distinguished points of the underlying frame. This is by and large the approach used by Fritz (2014) to define a class of what he calls *matrix frames with distinguished elements* for the propositional two-dimensional language.

Of a first-order two-dimensional framework we may ask whether the following relative of the truth condition in (1) is expressible in it:

(1*) $\exists(w, z)\forall\mathsf{x}(R\mathsf{x}\text{-at-}(z, z) \rightarrow S\mathsf{x}\text{-at-}(w, z))$.

The answer is positive if @ is present in the language, for (1*) is expressed by the formula $\Diamond_1\forall\mathsf{x}(@R\mathsf{x} \rightarrow S\mathsf{x})$. In fact, it would seem that standard proofs for the analogous inexpressibility result in the one-dimensional setting would transfer, more or less directly, to the two-dimensional setting if the latter did not contain @ in the language. This can be done with Wehmeier's (2003) proof, for example, under the obvious generalization of the models for the

[8] Such sequences are defined more carefully in Definition 1.2.

two-dimensional case. Now, assuming the availability of @, and given that $\Diamond_2$ quantifies over diagonal pairs of worlds in W, it seems reasonable to expect that the truth condition in (2) be expressible, too:

(2) $\exists(w,w)\forall \mathsf{x}(R\mathsf{x}\text{-at-}(z,z) \to S\mathsf{x}\text{-at-}(w,w))$.

But it is not obvious whether (2) can be expressed by a formula in this language; the most obvious candidate, $\Diamond_2\forall \mathsf{x}(@R\mathsf{x} \to S\mathsf{x})$, expresses the truth condition $\exists(w,w)\forall \mathsf{x}(R\mathsf{x}\text{-at-}(w,w) \to S\mathsf{x}\text{-at-}(w,w))$. This is so because within the scope of $\Diamond_2$, @ is idle. Thus, the expressive deficit of the first-order (one-dimensional) modal language that motivated the introduction of @ in the first place seems to reoccur in the two-dimensional case: while @ is able to, as it were, temporarily suspend the scope of $\Diamond_1$, there is apparently no analogous device in the two-dimensional language that has the same effect on the scope of $\Diamond_2$. Furthermore, there are readings of English sentences such as "It is not a priori that not every red thing is shiny" whose truth conditions correspond to (2). That is, besides the apparent inability to express truth conditions that, on the face of it, should be expressible with the two-dimensional operators, it does not seem to be possible to formalize relevant portions of the kind of natural-language discourse targeted by two-dimensional semantics.

One way to fix this is to add a single distinguished point (pair of worlds) back in the frames and a new operator, say, A, that forces formulas in its scope to be evaluated at that distinguished point. In other words, A does just what @ did in the one-dimensional case (before it was dethroned). If we let this point be (z,z), the semantic clause for A will be:

$$\mathfrak{M},(w,v) \vDash \mathsf{A}\varphi \iff \mathfrak{M},(z,z) \vDash \varphi.$$

Now the truth condition (2) can be expressed by the formula $\Diamond_2\forall \mathsf{x}(\mathsf{A}R\mathsf{x} \to S\mathsf{x})$. Alternatively, we can take a point z from the underlying set S as distinguished, and instead of adding A to the language, add an operator D with the following semantic clause:

$$\mathfrak{M},(w,v) \vDash \mathsf{D}\varphi \iff \mathfrak{M},(w,z) \vDash \varphi.$$

The truth condition (2) can then be expressed by the formula $\Diamond_2\forall \mathsf{x}(\mathsf{D}@R\mathsf{x} \to S\mathsf{x})$, and A can be simulated in this language by the compound operator $\mathsf{D}@$.

Yet another option in a similar spirit is to dethrone the single distinguished element of S and thereby move to three-dimensional frames, just as the original actual world was dethroned in the introduction of two-dimensional frames.

In this setting, formulas are evaluated relative to triples of worlds, that is, $W = S^3$, and a set of distinguished elements is defined in the frames as $D = \{(z, z, z) \mid z \in S\}$. In this new language we can have three $\Box_i$ operators whose semantic entries can be fully specified by the (GB) clause above, this time for $i \in \{1, 2, 3\}$. Accordingly, $\Box_1\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{\Box_1}$-related triple (x, y, z), where $y = v$ and $z = u$; $\Box_2\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{\Box_2}$-related triple (x, y, z), where $x = y$ and $z = u$ (i.e. at any triple (x, x, u)); and, finally, $\Box_3\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{\Box_3}$-related triple (x, y, z), where $x = y = z$ (i.e. at any triple (x, x, x)). Now, just as Fritz (2014) adds an accessibility relation for the actuality operator in the two-dimensional frames,[9] we can relabel the actuality operator @ as $@_2$, add a new actuality operator, $@_3$, to the language, and accessibility relations to describe their semantic clauses schematically as follows:

(GA) $\mathfrak{M}, \sigma \vDash @_i\varphi \iff$ for every $\tau \in W$, if $\sigma\mathcal{R}_{@_i}\tau$, then $\mathfrak{M}, \tau \vDash \varphi$,

where $\sigma\mathcal{R}_{@_i}\tau$ if and only if the first i coordinates in τ are all identical and σ and τ are identical beyond $i - 1$. Thus, according to (GA), $@_2\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{@_2}$-related triple (x, y, z), where $x = y = v$ and $z = u$ (i.e. at the triple (v, v, u)); and $@_3\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{@_3}$-related triple (x, y, z), where $x = y = z = u$ (i.e. at the triple (u, u, u)). Additionally, (GA) gives us an $@_1$ operator which turns out to be redundant, for $@_1\varphi$ is true at a triple (w, v, u) if and only if φ is true at any $\mathcal{R}_{@_1}$-related triple (x, y, z), where $x = w$, $y = v$, and $z = u$ (i.e. at the triple (w, v, u) itself). If we now adapt the truth condition (2) to the three-dimensional environment we obtain

(2*) $\exists(w, w, z)\forall\mathsf{x}(R\mathsf{x}\text{-at-}(z, z, z) \to S\mathsf{x}\text{-at-}(w, w, z))$,

which is expressible by the formula $\Diamond_2\forall\mathsf{x}(@_3R\mathsf{x} \to S\mathsf{x})$.

What motivated the introduction of the original actuality operator, and the subsequent move to two dimensions, is really the same thing that motivates the introduction of the new actuality operator into the two-dimensional language, and the subsequent move to three dimensions. And even though it is not obvious what natural language correspondents for operators such as $\Box_3$ and $@_3$

[9] There are many advantages in doing this, also illustrated by the fact that the axioms involving actuality operators can then be defined as Sahlqvist formulas, and so the Sahlqvist completeness theorem can be applied (see §1.3).

might be, it is clear that the original expressive deficit in the first-order modal language, which reoccurred in the two-dimensional case, seems to appear once again in the three-dimensional setting since $@_3$ does not seem able to reset the point of evaluation within the scope of $\Diamond_3$. That is to say, the following truth condition is apparently not expressible by any formula in the three-dimensional language under consideration:

(3) $\exists(w, w, w)\forall \mathsf{x}(R\mathsf{x}\text{-at-}(z, z, z) \rightarrow S\mathsf{x}\text{-at-}(w, w, w))$.

The most obvious candidate, $\Diamond_3\forall \mathsf{x}(@_3 R\mathsf{x} \rightarrow S\mathsf{x})$, is unsuccessful for the same reason $\Diamond_2\forall \mathsf{x}(@R\mathsf{x} \rightarrow S\mathsf{x})$ fails to express the truth condition in (2).

It is now evident that analogous cases of expressive deficit will seem to occur for any finite dimension n. To be more specific, where $n \geq 2$, a first-order n-dimensional language contains operators $\Box_i$ and $@_i$ for each $i \in \{1, ..., n\}$, and a frame for this language is defined by a set $W = S^n$, where S is a non-empty set of worlds, relations $\mathcal{R}_{\Box_i}$ and $\mathcal{R}_{@_i}$ for each $i \in \{1, ..., n\}$, a set $D = \{\overline{s} \mid s \in S\}$ of distinguished points, where $\overline{s}$ is that sequence $\sigma \in W$ for which $\sigma_i = s \in S$ for all $1 \leq i \leq n$, besides a domain of objects $\mathcal{D}$. The truth conditions for $\Box_i$- and $@_i$-formulas are set by (GB) and (GA). Then, where $\sigma = \overline{s}$ and $\tau = \overline{t}$, the following truth condition would not seem to be expressible by any formula in the n-dimensional language:

(n) $\exists\sigma\forall \mathsf{x}(R\mathsf{x}\text{-at-}\tau \rightarrow S\mathsf{x}\text{-at-}\sigma)$.

In Lampert (2018), some of the philosophical ramifications of this general case for two-dimensional semantics were explored. In particular, the move from two to three-dimensions was motivated by the fact that some of the other solutions to the expressive deficit of the first-order two-dimensional language—namely, the ones involving the addition of the operators A or D—would result in the frames validating $\mathsf{A}\varphi \rightarrow \Box_2\mathsf{A}\varphi$ or $\mathsf{D}@\varphi \rightarrow \Box_2\mathsf{D}@\varphi$, respectively. But according to the intuitive interpretation we assign to these operators, such formulas should not be valid: if φ is true in the *real* actual world, it does not follow intuitively that it is a priori that it is true in that world. However, a similar issue ultimately arises in the three-dimensional language, as $@_3\varphi \rightarrow \Box_2@_3\varphi$, too, would be valid in its respective class of frames.

In this paper, however, we will not be concerned with philosophical implications of employing multidimensional frameworks to model a priori knowledge of finite or ideal agents. Rather, we shall focus on the logical generalization to higher-dimensions of logics for necessity, actuality, and a priori knowledge, which do raise a variety of questions from a purely logical point of view. The

n-dimensional frames with distinguished elements defined earlier are natural generalizations of the two-dimensional matrix frames with distinguished elements defined in Fritz (2014). Fritz proves that such frames, which he defines for a propositional two-dimensional language, admit of a finite axiomatization through a more general class of *matrix frames*, which do not contain distinguished elements. Fritz calls the logic of such frames **2Dg**. It is natural to ask, in light of the constructions above, whether **2Dg** is a special case of a more general framework, and whether similar results transfer to n-dimensional frames defined for n-dimensional propositional languages.[10] Are these frames finitely axiomatizable? Are the logics decidable? We settle these and more questions in the course of this paper. In §1 we work towards proving a completeness theorem for the logic of n-dimensions relative to n-dimensional frames without distinguished elements, and then we use this result to show that n-dimensional frames with distinguished elements are also complete. In §2 we prove that the resulting logics are decidable by means of n-dimensional tableaux and a systematic procedure guaranteeing termination for every tableau so constructed. At the end of the paper we suggest some additional questions to be investigated in future work.

1 The logic of sequence frames

1.1 Syntax and semantics

Definition 1.1 (Multidimensional language) Let PROP be a denumerable set of propositional variables $p, q, ...$ The language $\mathcal{L}_n^{@}$ is recursively generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box_i\varphi \mid @_i\varphi$$

for all $1 \leq i \leq n$, where $n \geq 2$. The other Boolean connectives are defined as usual, and $\Diamond_i\varphi := \neg\Box_i\neg\varphi$. Finally, let $\mathcal{O} \in \{\Box, @\}$.

Definition 1.2 (Sequences) Let S be a non-empty set and $W = S^n$, i.e. W is the n-fold Cartesian product of S, so that W contains sequences $\sigma = (s_1, ..., s_n)$ of elements $s_1, ..., s_n \in S$. When $\sigma = (s_1, ..., s_n)$, we write σ_i for s_i. For $s \in S$, $\overline{s}$ is that sequence σ for which $\sigma_i = s$ for all $1 \leq i \leq n$. Moreover, we say that

[10] It is reasonable to investigate the propositional languages first, especially because the whole technology consisting of finitely many $\Box_i$ and $@_i$ operators is already present at the propositional level.

a tuple σ is *i-diagonal* just in case $\sigma_1 = \sigma_2 = ... = \sigma_i$, and for any sequence $\sigma \in W$, let σ^i_z be that sequence τ for which $\tau_1 = \tau_2 = ... = \tau_i = z$ and for $j > i$, $\tau_j = \sigma_j$, so that σ^i_z is always i-diagonal. We say that σ and τ are *identical beyond* i if for all $j > i$, $\sigma_j = \tau_j$.[11]

Now we define *sequence frames* for n-dimensional modal logics. The number of dimensions dictates the length of the sequences in the frame, which can be thought of as sequences of possible worlds. Strictly speaking, all that is needed to fully specify the desired frames are pairs (S, n), where S is a non-empty set. Yet, it will be helpful to add the relations $\mathcal{R}_{\Box_i}$ and $\mathcal{R}_{@_i}$ to the official definition of the frames, as these play an essential role in the Sahlqvist completeness theorem proved in §1.3, besides making several definitions easier to articulate. Thus the official definition of n-dimensional sequence frames is the following:

Definition 1.3 (n-dimensional sequence frame) Let $n \geq 2$. An *n-dimensional sequence frame* for $\mathcal{L}^{@}_n$ is a triple, $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n})$, such that $W = S^n$ for some non-empty set S, $\mathcal{R}_{\Box_i} \subseteq W \times W$ and $\mathcal{R}_{@_i} \subseteq W \times W$ for all $1 \leq i \leq n$, such that for any sequences $\sigma, \tau \in W$,

- $\sigma \mathcal{R}_{\Box_i} \tau$ iff (i) τ is i-diagonal, and (ii) σ and τ are identical beyond i.
- $\sigma \mathcal{R}_{@_i} \tau$ iff (i) τ is i-diagonal, and (ii) σ and τ are identical beyond $i - 1$.

Additionally, let F be the class of n-dimensional sequence frames.

To make the behaviour of each accessibility relation more explicit, we register the main characteristic properties that hold in every n-dimensional sequence frame. We begin with the properties involving only the $\mathcal{R}_{\Box_i}$ relations, moving in turn to the properties involving only the $\mathcal{R}_{@_i}$ relations, and then to the properties involving both families of relations. The proofs of these facts are straightforward, and so we omit them.

Proposition 1.1 *Let $1 \leq j \leq i \leq n$ and $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n})$ be an* n-dimensional sequence frame. *Then:*

(1) $\mathcal{R}_{\Box_1}$ is an equivalence relation.

(2) $\mathcal{R}_{\Box_i}$ is serial, i.e. for all sequences σ there is a τ such that $\sigma \mathcal{R}_{\Box_i} \tau$.

[11] To illustrate some of the items defined above, in the case of basic modal logic σ is always a one-place sequence, while in a two-dimensional modal logic σ is an ordered pair. Also, if, for example, $\sigma = (x, u, z, w)$, then $\sigma^3_u = (u, u, u, w)$.

(3) (Upward transitivity) If $\sigma\mathcal{R}_{\Box_j}\tau$ *and* $\tau\mathcal{R}_{\Box_i}\upsilon$*, then* $\sigma\mathcal{R}_{\Box_i}\upsilon$*.*

(4) (Upward Euclideanity) If $\sigma\mathcal{R}_{\Box_j}\tau$ *and* $\sigma\mathcal{R}_{\Box_i}\upsilon$*, then* $\tau\mathcal{R}_{\Box_i}\upsilon$*.*

(5) (Downward weak density) If $\sigma\mathcal{R}_{\Box_i}\tau$*, there is a* υ *such that* $\sigma\mathcal{R}_{\Box_j}\upsilon$ *and* $\upsilon\mathcal{R}_{\Box_i}\tau$*.*

(6) (Downward shift reflexivity) If $\sigma\mathcal{R}_{\Box_i}\tau$*, then* $\tau\mathcal{R}_{\Box_j}\tau$*.*

(7) (Strictly decreasing weak density) When $1 < i \leq n$*, if* $\sigma\mathcal{R}_{\Box_{i-1}}\tau$*, there is a* υ *such that* $\sigma\mathcal{R}_{\Box_i}\upsilon$ *and* $\upsilon\mathcal{R}_{\Box_{i-1}}\tau$*.*

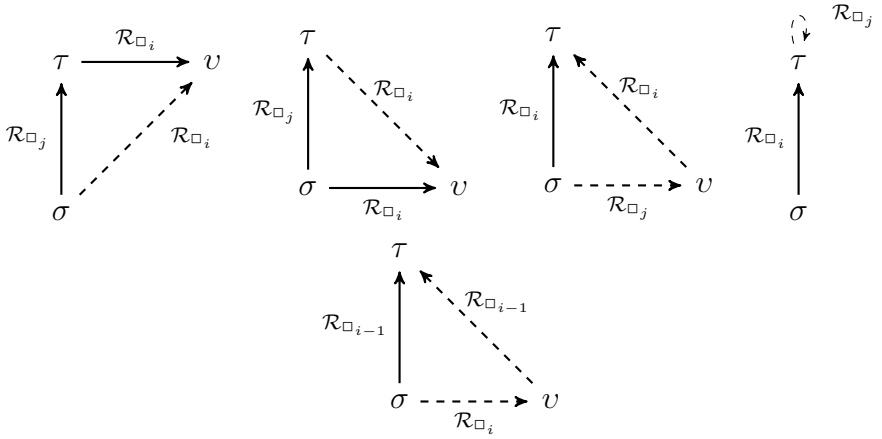

Figure 2: Upward transitivity, upward Euclideanity, downward weak density, downward shift reflexivity, and strictly decreasing weak density properties.

Proposition 1.2 *Let* $1 \leq i \leq n$ *and* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ *be an* n-dimensional sequence frame*. Then:*

(1) $\mathcal{R}_{@_1}$ *is reflexive.*

(2) $\mathcal{R}_{@_i}$ *is serial.*

(3) $\mathcal{R}_{@_i}$ *is functional, i.e. if* $\sigma\mathcal{R}_{@_i}\tau$ *and* $\sigma\mathcal{R}_{@_i}\upsilon$*, then* $\tau = \upsilon$*.*

(4) (Upward-downward transitivity) When $1 < i \leq n$*, if* $\sigma\mathcal{R}_{@_i}\tau$ *and* $\tau\mathcal{R}_{@_{i-1}}\upsilon$*, then* $\sigma\mathcal{R}_{@_i}\upsilon$*.*

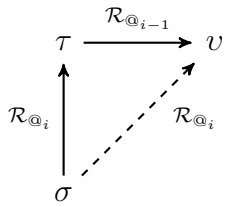

Figure 3: Upward-downward transitivity.

Remark 1.1 *By Proposition 1.2(3), it follows that each $\mathcal{R}_{@_i}$ is a function. Additionally, because $\mathcal{R}_{@_1}$ is reflexive, $\mathcal{R}_{@_1}$ is the identity function on the frame.*

Proposition 1.3 *Let $1 \leq j < i \leq n$ and $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n})$ be an n-*dimensional sequence frame*. Then:*

(1) (Strictly decreasing act-box) When $1 < i \leq n$, if $\sigma \mathcal{R}_{@_i} \tau$, then $\sigma \mathcal{R}_{\Box_{i-1}} \tau$.

(2) (Act-box) If $\sigma \mathcal{R}_{@_i} \tau$, then $\sigma \mathcal{R}_{\Box_i} \tau$.

(3) (Mixed upward transitivity) If $\sigma \mathcal{R}_{\Box_j} \tau$ and $\tau \mathcal{R}_{@_i} \upsilon$, then $\sigma \mathcal{R}_{@_i} \upsilon$.

(4) (Mixed shift reflexivity) If $\sigma \mathcal{R}_{\Box_i} \tau$, then $\tau \mathcal{R}_{@_i} \tau$.

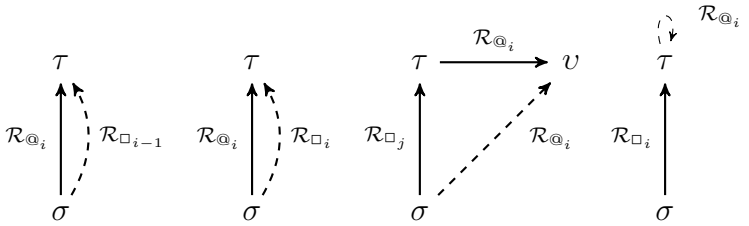

Figure 4: Strictly decreasing act-box, act-box, mixed upward transitivity, and mixed shift reflexivity properties.

Note that downward shift reflexivity now follows immediately from mixed shift reflexivity together with act-box and strictly decreasing act-box. Similarly, the properties (5) and (7) in Proposition 1.1 are provable from the other properties involving the $\mathcal{R}_{@_i}$ relations. With respect to (3) and (4), that is, upward transitivity and upward Euclideanity, they can also be derived from the other properties alongside (simple) transitivity and the Euclidean property. Thus, in order to characterize sequence frames syntactically we only need

to add the modal axioms for transitivity and the Euclidean property alongside the other axioms listed below. Moreover, from the properties listed in Proposition 1.1 one can show that sequence frames have the property of *left commutativity* as well as the *Church-Rosser* property, that is:

Corollary 1.1 *Let* $1 \leq j \leq i \leq n$, *and* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ *be an* n-dimensional sequence frame. *Then:*

(1) (Left commutativity) If $\sigma\mathcal{R}_{\Box_j}\tau$ *and* $\tau\mathcal{R}_{\Box_i}\upsilon$, *then there is a* σ' *such that* $\sigma\mathcal{R}_{\Box_i}\sigma'$ *and* $\sigma'\mathcal{R}_{\Box_j}\upsilon$.

(2) (Church-Rosser) If $\sigma\mathcal{R}_{\Box_j}\tau$ *and* $\sigma\mathcal{R}_{\Box_i}\upsilon$, *then there is a* σ' *such that* $\tau\mathcal{R}_{\Box_i}\sigma'$ *and* $\upsilon\mathcal{R}_{\Box_j}\sigma'$.

Proof. (1). If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\tau\mathcal{R}_{\Box_i}\upsilon$, it follows that $\sigma\mathcal{R}_{\Box_i}\upsilon$, by upward transitivity. But then $\upsilon\mathcal{R}_{\Box_j}\upsilon$, by downward shift reflexivity. Therefore, there is a sequence σ', namely, υ itself, such that $\sigma\mathcal{R}_{\Box_i}\sigma'$ and $\sigma'\mathcal{R}_{\Box_j}\upsilon$.

(2). If $\sigma\mathcal{R}_{\Box_j}\tau$ and $\sigma\mathcal{R}_{\Box_i}\upsilon$, then $\tau\mathcal{R}_{\Box_i}\upsilon$, by the upward Euclidean property. But then $\upsilon\mathcal{R}_{\Box_j}\upsilon$, by downward shift reflexivity. Therefore, there is a sequence σ', namely, υ itself, such that $\tau\mathcal{R}_{\Box_i}\sigma'$ and $\upsilon\mathcal{R}_{\Box_j}\sigma'$. □

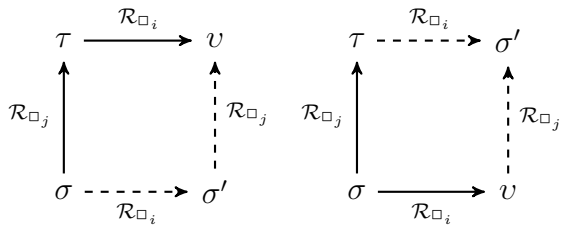

Figure 5: Left commutativity and Church-Rosser properties.

These properties are known for playing an important role in characterizing product frames for product logics (see Gabbay et al (2003: 222)). Note, however, that the property of *right commutativity*, which holds in product frames, does not hold in general for n-dimensional sequence frames.[12] Consider, for instance, the two-dimensional sequence frame $\mathfrak{F} = (W, \mathcal{R}_{\Box_1}, \mathcal{R}_{\Box_2}, \mathcal{R}_{@_1}, \mathcal{R}_{@_2})$ displayed in Figure 6, where $W = S^2$ and $S = \{w, v\}$. Then each of the properties listed in Proposition 1.1, 1.2, and 1.3 hold in that frame, but while $(v, w)\mathcal{R}_{\Box_2}(v, v)$ and $(v, v)\mathcal{R}_{\Box_1}(w, v)$, there is no pair σ' such that $(v, w)\mathcal{R}_{\Box_1}\sigma'$ and $\sigma'\mathcal{R}_{\Box_2}(w, v)$.

[12]Right commutativity in our framework corresponds to the property that for all σ, τ, υ, if $\sigma\mathcal{R}_{\Box_i}\tau$ and $\tau\mathcal{R}_{\Box_j}\upsilon$, then there is a σ' such that $\sigma\mathcal{R}_{\Box_j}\sigma'$ and $\sigma'\mathcal{R}_{\Box_i}\upsilon$, for $1 \leq j \leq i \leq n$.

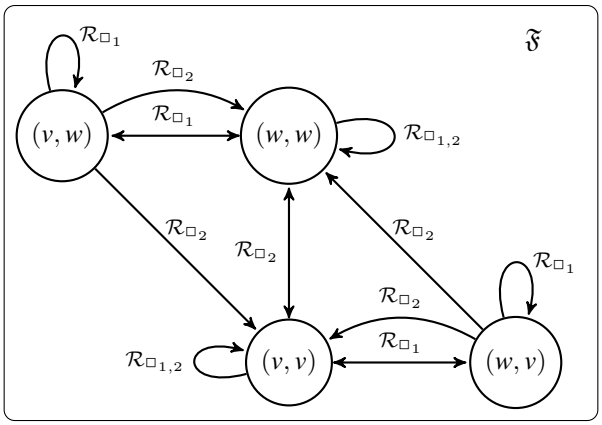

Figure 6: Two-dimensional sequence frame displaying only the $\mathcal{R}_{\Box_i}$ relations falsifying right commutativity.

Even though no accessibility relation except for $\mathcal{R}_{\Box_1}$ is in general reflexive or symmetric, there is at least a weak sense in which n-dimensional sequence frames can be seen as generalizations to n dimensions of **S5**-frames, a sense that emerges when we restrict attention to the relations between sequences that are i-diagonal for some i. Consider the following definition:

Definition 1.4 Let S be non-empty and $W = S^n$ as before. $\mathcal{R}_{\Box_i}$ is *pseudo-reflexive* if it is reflexive on W^i; it is *pseudo-symmetric* if for all $\sigma, \tau \in W^i$, $\sigma \mathcal{R}_{\Box_i} \tau$ implies $\tau \mathcal{R}_{\Box_i} \sigma$; it is *pseudo-transitive* if for all $\sigma, \tau, \upsilon \in W^i$, $\sigma \mathcal{R}_{\Box_i} \tau$ and $\tau \mathcal{R}_{\Box_i} \upsilon$ implies $\sigma \mathcal{R}_{\Box_i} \upsilon$; and *pseudo-Euclidean* if for all $\sigma, \tau, \upsilon \in W^i$, $\sigma \mathcal{R}_{\Box_i} \tau$ and $\sigma \mathcal{R}_{\Box_i} \upsilon$ implies $\tau \mathcal{R}_{\Box_i} \upsilon$. Finally, $\mathcal{R}_{\Box_i}$ is a *pseudo-equivalence relation* if it is pseudo-reflexive and pseudo-Euclidean.

Note that if $n = 1$ (and the language is one-dimensional), the pseudo notions coincide with the original notions, so they are legitimate generalizations of the latter. Now it can be shown that every n-dimensional sequence frame contains a subframe on which every relation $\mathcal{R}_{\Box_i}$ is an equivalence relation:

Proposition 1.4 *Let* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n})$ *be an* n*-dimensional sequence frame. For each* $1 \leq i \leq n$, $\mathcal{R}_{\Box_i}$ *is an equivalence relation on* W^i*. In other words, for each* $1 \leq i \leq n$, $\mathcal{R}_{\Box_i}$ *is a pseudo-equivalence relation on* W*.*

Proof. Obviously, $\mathcal{R}_{\Box_i}$ is pseudo-reflexive, since any member of W^i is i-diagonal and identical with itself beyond i. Moreover, it is easy to see that $\mathcal{R}_{\Box_i}$ is pseudo-Euclidean, for if $\sigma \mathcal{R}_{\Box_i} \tau$ and $\sigma \mathcal{R}_{\Box_i} \upsilon$, then σ, τ, and υ are all identical beyond i, so that $\tau \mathcal{R}_{\Box_i} \upsilon$ as long as $\upsilon \in W^i$. □

In addition, observe that any n-dimensional sequence frame has a certain kind of universality property governing its last accessibility relation, $\mathcal{R}_{\Box_n}$, in the sense that every point in W is $\mathcal{R}_{\Box_n}$-related to every n-diagonal point. This property, which we may call *shift universality*, plays an important role in the completeness theorem below, for we use the fact that if a point is n-diagonal, then it is $\mathcal{R}_{\Box_n}$-accessible to any point in the frame. Yet, shift universality is not modally definable. To verify this, let $\mathfrak{F}_1 = (W_1, \mathcal{R}_1)$ be a usual Kripke frame with points w, u, v such that $\mathcal{R}_1 = \{(w, v), (v, v), (u, v)\}$, and $\mathfrak{F}_2 = (W_2, \mathcal{R}_2)$ a frame with points a, b, c such that $\mathcal{R}_2 = \{(a, b), (b, b), (c, b)\}$. Then the first-order formula expressing shift universality, that is, $\forall x \exists y (y \mathcal{R} x \rightarrow \forall z \, z \mathcal{R} x)$, is valid in both $\mathfrak{F}_1$ and $\mathfrak{F}_2$, but not in their disjoint union, as illustrated in Figure 7. Therefore, by the Goldblatt-Thomason theorem—which states that a first-order definable class of frames is modally definable if and only if it is closed under taking disjoint unions, generated subframes, bounded morphic images, and reflects ultrafilter extensions—shift universality is not modally definable.[13]

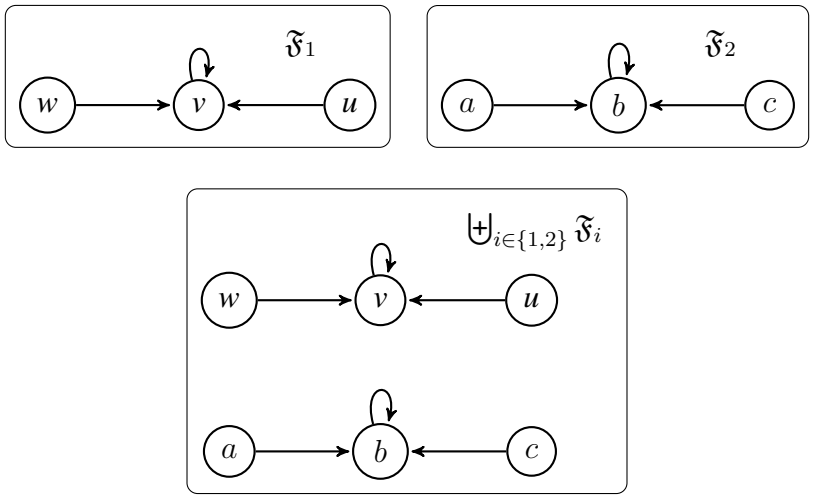

Figure 7: Non-definability of shift universality.

Next we define concepts such as models based on n-dimensional sequence frames, truth of a formula at a sequence in a model, as well as the core logical notions of validity and consequence.

[13]The Goldblatt-Thomason theorem is proved in Blackburn et al. (2001: 180-183), §3.8, Theorem 3.19. The reader can also find definitions for the notions of disjoint unions, generated subframes, bounded morphic images, and ultrafilter extensions in Blackburn et al. (2001), §3.3. The notions of generated subframes and bounded morphic images will be used later in the proof of Lemma 1.5.

Definition 1.5 (n-dimensional sequence models) An *n-dimensional sequence model* is a quadruple, $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n}, V)$, where V is a function assigning to each $p \in \mathsf{PROP}$ a subset $V(p) \subseteq W$. We say that $\mathfrak{M}$ is *based on* an n-dimensional sequence frame, $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n})$.

Definition 1.6 (Truth) We define 'φ is true at σ in a model $\mathfrak{M}$', written $\mathfrak{M}, \sigma \vDash \varphi$, by recursion on φ. For a sequence $\sigma \in W$, and a valuation V in $\mathfrak{M}$,

$$
\begin{array}{lcl}
\mathfrak{M}, \sigma \vDash p & \Longleftrightarrow & \sigma \in V(p) \\
\mathfrak{M}, \sigma \vDash \neg\varphi & \Longleftrightarrow & \mathfrak{M}, \sigma \nvDash \varphi \\
\mathfrak{M}, \sigma \vDash (\varphi \wedge \psi) & \Longleftrightarrow & \mathfrak{M}, \sigma \vDash \varphi \text{ and } \mathfrak{M}, \sigma \vDash \psi \\
\mathfrak{M}, \sigma \vDash \Box_i\varphi & \Longleftrightarrow & \text{for every } \tau \in W, \text{ if } \sigma \mathcal{R}_{\Box_i} \tau, \text{ then } \mathfrak{M}, \tau \vDash \varphi \\
\mathfrak{M}, \sigma \vDash @_i\varphi & \Longleftrightarrow & \text{for every } \tau \in W, \text{ if } \sigma \mathcal{R}_{@_i} \tau, \text{ then } \mathfrak{M}, \tau \vDash \varphi
\end{array}
$$

Definition 1.7 (Logical notions) A sentence φ is *satisfiable* in a model $\mathfrak{M}$ iff there is a sequence σ in $\mathfrak{M}$ such that $\mathfrak{M}, \sigma \vDash \varphi$, and *satisfiable* in a frame $\mathfrak{F}$ iff it is satisfiable in a model based on $\mathfrak{F}$. A sentence φ is *valid in an n-dimensional sequence model* $\mathfrak{M}$, written $\mathfrak{M} \vDash \varphi$, iff $\mathfrak{M}, \sigma \vDash \varphi$ for every $\sigma \in W$. A sentence φ is *valid at a point σ in a frame* $\mathfrak{F}$, written $\mathfrak{F}, \sigma \vDash \varphi$, iff φ is true at σ in every model $\mathfrak{M}$ based on $\mathfrak{F}$, and φ is *valid in a frame* $\mathfrak{F}$, written $\mathfrak{F} \vDash \varphi$, iff it is valid at every point in $\mathfrak{F}$. A sentence φ is *valid on a class of frames* C, written $\mathsf{C} \vDash \varphi$, iff φ is valid in every $\mathfrak{F} \in \mathsf{C}$. A sentence φ is a *logical consequence* of a set of sentences Γ over a class of frames C if and only if for every $\mathfrak{M}$ in C and sequence σ in $\mathfrak{M}$, if $\mathfrak{M}, \sigma \vDash \gamma$ for all $\gamma \in \Gamma$, then $\mathfrak{M}, \sigma \vDash \varphi$.

1.2 Axiomatization

In this section we axiomatize n-dimensional sequence frames. The logic of n-dimensional sequence frames is called $\mathbf{S@}_n$:

Definition 1.8 (The logic $\mathbf{S@}_n$) Let $1 \leq i \leq n$, where $n \geq 2$. $\mathbf{S@}_n$ is the

classical normal modal logic defined by the following axioms:

$$
\begin{array}{ll}
(\mathbf{T}_{\Box_1}) & \Box_1 p \rightarrow p. \\
(\mathbf{4}_{\Box_i}) & \Box_i p \rightarrow \Box_i \Box_i p. \\
(\mathbf{5}_{\Box_i}) & \Diamond_i p \rightarrow \Box_i \Diamond_i p. \\
(\mathbf{T}_{@_1}) & @_1 p \rightarrow p. \\
(\mathbf{D}_{@_i}) & @_i p \rightarrow \neg @_i \neg p. \\
(\mathbf{F}_{@_i}) & \neg @_i \neg p \rightarrow @_i p. \\
(\mathbf{A}_i) & \Box_{i-1} p \rightarrow @_i p, \text{ for } 1 < i \leq n. \\
(\mathbf{AT}_i) & \Box_i p \rightarrow @_i p. \\
(\mathbf{A4}_i) & @_i p \rightarrow \Box_j @_i p, \text{ for } 1 \leq j < i \leq n. \\
(\mathbf{AR}_i) & \Box_i (@_i p \rightarrow p).
\end{array}
$$

Where $\vdash_n$ denotes the provability relation in $\mathbf{S}@_n$, $\Gamma \vdash_n \varphi$ if and only if there are formulas $\psi_1, ..., \psi_n$ in Γ such that $\vdash_n \psi_1 \wedge ... \wedge \psi_n \rightarrow \varphi$.

As derived rules we can also add the *rule of regularity* and the *rule of congruentiality*, similarly to the basic modal case:

$$
\begin{array}{ll}
(\mathbf{RR}) & \text{If } \vdash_n p \rightarrow q, \text{ then } \vdash_n \Box_i p \rightarrow \Box_i q. \\
(\mathbf{RC}) & \text{If } \vdash_n p \leftrightarrow q, \text{ then } \vdash_n \Box_i p \leftrightarrow \Box_i q.
\end{array}
$$

Moreover, left commutativity $\Box_i \Box_j p \rightarrow \Box_j \Box_i p$ and the Church-Rosser axiom $\Diamond_j \Box_i p \rightarrow \Box_i \Diamond_j p$ are both derivable from the axioms above.

With the exception of $\mathbf{T}_{@_1}$, and by setting $n = 2$, the axioms above are just the ones found in Fritz (2014: 391), for the logic **2Dg**. Note that by setting $n = 2$ there are two box-like operators in the language, namely, $\Box_1$, corresponding to the (metaphysical) necessity operator $\Box$ in **2Dg**, and the final box-like operator $\Box_2$, corresponding to the a priori operator A in **2Dg**. The reason why $\mathbf{T}_{@_1}$ is not found in **2Dg** is because its language has a single actuality operator which, provided $n = 2$, corresponds to $@_2$ in $\mathcal{L}_2^@$. Having a single actuality operator in the language is, after all, expected since **2Dg** is designed to be a logic of necessity, actuality, and the a priori, and hence $@_2$ is just the traditional actuality operator found in other two-dimensional modal logics, such as Davies and Humberstone's (1980) logic for deep necessity, for example. As mentioned before, we have added $@_1$ to the language for the sake of generality, so that there are n boxes and actuality operators. Nevertheless, it is now easy to see that $p \rightarrow @_1 p$ is derivable from the axioms above, and hence

$p \leftrightarrow @_1 p$ is also derivable, by $\mathbf{T}_{@_1}$, which in turn means that $@_1$ is directly eliminable from the language $\mathcal{L}_2^@$ (in fact, $\mathcal{L}_n^@$) without loss of expressive power. Therefore, by discounting the $@_1$ operator, $\mathbf{S}@_n$ is a generalization of **2Dg** to n dimensions with many actuality operators.

1.3 Completeness

The axioms listed above are all Sahlqvist formulas, and hence we can compute their locally corresponding frame conditions by the general Sahlqvist algorithm for converting modal formulas into first-order formulas (see Blackburn et al. (2001, chapter 3.6) for a general description), from which it follows that $\mathbf{S}@_n$ is strongly complete with respect to the class of frames it defines.

Informally, we say that a class of frames C is *defined* by a set of formulas Γ (respectively, formula φ) if for every frame $\mathfrak{F}$, $\mathfrak{F} \in \mathsf{C}$ just in case $\mathfrak{F} \vDash \Gamma$ (respectively, $\mathfrak{F} \vDash \varphi$). Additionally, let $\psi(x)$ be a formula of first-order logic where x is the only free variable in ψ. Then $\psi(x)$ *locally corresponds* to a modal formula φ if $\psi(x)$ expresses a condition satisfiable on a point w in a first-order structure $\mathfrak{F}$ just in case φ is valid at w in $\mathfrak{F}$, where $\mathfrak{F}$ is taken as a Kripke frame. That is, for any frame $\mathfrak{F}$ and point w in $\mathfrak{F}$, $\mathfrak{F}, w \vDash \varphi$ just in case $\mathfrak{F}, \alpha[x/w] \vDash \psi$, where $\alpha[x/w]$ is the assignment function that sends x to w. Then the following first-order formulas locally correspond to the axioms of $\mathbf{S}@_n$:

$\mathbf{T}_{\Box_1}$	$\Box_1 p \rightarrow p$	$w\mathcal{R}_{\Box_1} w$
$\mathbf{4}_{\Box_i}$	$\Box_i p \rightarrow \Box_i \Box_i p$	$\forall v z((w\mathcal{R}_{\Box_i} v \wedge v\mathcal{R}_{\Box_i} z) \rightarrow w\mathcal{R}_{\Box_i} z)$
$\mathbf{5}_{\Box_i}$	$\Diamond_i p \rightarrow \Box_i \Diamond_i p$	$\forall v z((w\mathcal{R}_{\Box_i} v \wedge w\mathcal{R}_{\Box_i} z) \rightarrow v\mathcal{R}_{\Box_i} z)$
$\mathbf{T}_{@_1}$	$@_1 p \rightarrow p$	$w\mathcal{R}_{@_1} w$
$\mathbf{D}_{@_i}$	$@_i p \rightarrow \neg @_i \neg p$	$\exists v(w\mathcal{R}_{@_i} v)$
$\mathbf{F}_{@_i}$	$\neg @_i \neg p \rightarrow @_i p$	$\forall v z((w\mathcal{R}_{@_i} v \wedge w\mathcal{R}_{@_i} z) \rightarrow v = z)$
$\mathbf{A}_i$	$\Box_{i-1} p \rightarrow @_i p$	$\forall v(w\mathcal{R}_{@_i} v \rightarrow w\mathcal{R}_{\Box_{i-1}} v)$, for $1 < i \leq n$
$\mathbf{AT}_i$	$\Box_i p \rightarrow @_i p$	$\forall v(w\mathcal{R}_{@_i} v \rightarrow w\mathcal{R}_{\Box_i} v)$
$\mathbf{A4}_i$	$@_i p \rightarrow \Box_j @_i p$	$\forall v z((w\mathcal{R}_{\Box_j} v \wedge v\mathcal{R}_{@_i} z) \rightarrow w\mathcal{R}_{@_i} z)$, for $1 \leq j < i \leq n$
$\mathbf{AR}_i$	$\Box_i(@_i p \rightarrow p)$	$\forall v(w\mathcal{R}_{\Box_i} v \rightarrow v\mathcal{R}_{@_i} v)$

Theorem 1.1 (Sahlqvist completeness) $\mathbf{S}@_n$ *is sound and strongly complete with respect to the class of frames* $\mathsf{F}_{\mathbf{S}@_n}$ *(that is, the class of first-order frames defined by* $\mathbf{S}@_n$*).*

Proof. By Theorem 4.42 in Blackburn et al. (2001), the proof of which can be found in chapter 5. □

Let an $\mathbf{S}@_n$-frame be any frame in $\mathsf{F}_{\mathbf{S}@_n}$. It is clear that the axioms above are sound, too, with respect to the class of all n-dimensional sequence frames, given their correspondence to the first-order properties of these frames. An important question is whether the axioms above are also sufficient, that is, whether they settle the completeness problem for $\mathbf{S}@_n$ with respect to n-dimensional sequence frames. This question is settled here by generalizing the approach in Fritz (2014), in which a completeness theorem for **2Dg** is derived by first passing through an intermediate class of frames—in that case, the frames defined in Restall (2012). But rather than using an intermediate class of frames such as Restall's, we take the point-generated subframe of a frame in $\mathsf{F}_{\mathbf{S}@_n}$ and construct a bounded morphism from a sequence frame onto this point-generated subframe. Still, the proof strategy extends the main ideas employed in Fritz (2014), Lemmas 2.5-8.

First we prove a series of lemmas in order to establish the main completeness result for $\mathbf{S}@_n$. In particular, Lemmas 1.1 to 1.4 are instrumental in the proof of Lemma 1.5, which is in turn the principal lemma for the completeness theorem, namely, Theorem 1.3.

Lemma 1.1 *The following Sahlqvist formulas are derivable in* $\mathbf{S}@_n$*. Let* $1 \leq j < i \leq n$*:*

(1) (Seriality) $\vdash_n \Box_i p \rightarrow \Diamond_i p$*.*

(2) (Upward transitivity) $\vdash_n \Box_i p \rightarrow \Box_j \Box_i p$*. This formula corresponds to the frame condition* $\forall vz((w\mathcal{R}_{\Box_j} v \wedge v\mathcal{R}_{\Box_i} z) \rightarrow w\mathcal{R}_{\Box_i} z)$*.*

(3) (Upward Euclideanity) $\vdash_n \Diamond_i p \rightarrow \Box_j \Diamond_i p$*. This formula corresponds to the frame condition* $\forall vz((w\mathcal{R}_{\Box_j} v \wedge w\mathcal{R}_{\Box_i} z) \rightarrow v\mathcal{R}_{\Box_i} z)$*.*

Proof. (1). This is derivable from $\mathbf{AT}_i$ and its dual, $@_i p \rightarrow \Diamond_i p$.

(2). Consider the following derivation:

1.	$\vdash_n \Diamond_i p \rightarrow \Box_i \Diamond_i p$	$\mathbf{5}_{\Box_i}$
2.	$\vdash_n \Box_i \Diamond_i p \rightarrow @_i \Diamond_i p$	$\mathbf{AT}_i$
3.	$\vdash_n \Diamond_i p \rightarrow @_i \Diamond_i p$	$1, 2$
4.	$\vdash_n \neg @_i \neg \Box_i \neg p \rightarrow \Box_i \neg p$	3
5.	$\vdash_n @_i \Box_i p \rightarrow \Box_i p$	$\mathbf{D}_{@_i}, 4$
6.	$\vdash_n \Box_j @_i \Box_i p \rightarrow \Box_j \Box_i p$	$\mathbf{Nec_j}, \mathbf{K}_{\Box_j}, 5$
7.	$\vdash_n @_i \Box_i p \rightarrow \Box_j @_i \Box_i p$	$\mathbf{A4}_i$
8.	$\vdash_n @_i \Box_i p \rightarrow \Box_j \Box_i p$	$6, 7$
9.	$\vdash_n \Box_i p \rightarrow \Box_i \Box_i p$	$\mathbf{4}_{\Box_i}$
10.	$\vdash_n \Box_i \Box_i p \rightarrow @_i \Box_i p$	$\mathbf{AT}_i$
11.	$\vdash_n \Box_i p \rightarrow @_i \Box_i p$	$9, 10$
12.	$\vdash_n \Box_i p \rightarrow \Box_j \Box_i p$	$8, 11$

(3). Consider the following derivation:

1.	$\vdash_n \Box_i \Diamond_i p \rightarrow \Diamond_i \Diamond_i p$	(1)
2.	$\vdash_n \Diamond_i \Diamond_i p \rightarrow \Diamond_i p$	Dual of $\mathbf{4}_{\Box_i}$
3.	$\vdash_n \Box_i \Diamond_i p \rightarrow \Diamond_i p$	$1, 2$
4.	$\vdash_n \Box_j \Box_i \Diamond_i p \rightarrow \Box_j \Diamond_i p$	$\mathbf{Nec}_j, 3, \mathbf{K}_{\Box_j}$
5.	$\vdash_n \Box_i \Diamond_i p \rightarrow \Box_j \Box_i \Diamond_i p$	(2)
6.	$\vdash_n \Diamond_i p \rightarrow \Box_i \Diamond_i p$	$\mathbf{5}_{\Box_i}$
7.	$\vdash_n \Diamond_i p \rightarrow \Box_j \Diamond_i p$	$4, 5, 6$

□

For the next lemmas, $\mathcal{R}_{\mathcal{O}_i}[X]$ is the image of the set X under the relation $\mathcal{R}_{\mathcal{O}_i}$, and for each function $\mathcal{R}_{@_i}$, $\mathcal{R}_{@_i}(w)$ is the unique v such that $w\mathcal{R}_{@_i}v$, and $im(\mathcal{R}_{@_i})$ is the image of the function $\mathcal{R}_{@_i}$.

Lemma 1.2 *Let* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n})$ *be an* $\mathbf{S}@_n$*-frame,* $w \in W$*, and* $\mathfrak{F}^w = (W^w, (\mathcal{R}^w_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}^w_{@_i})_{1 \leq i \leq n})$ *the subframe of* $\mathfrak{F}$ *generated by* w*. Then:*

(1) $W^w = \mathcal{R}^w_{\Box_1}[\mathcal{R}^w_{\Box_2}[...[\mathcal{R}^w_{\Box_i}[...[\mathcal{R}^w_{\Box_n}[\{w\}]]]]]]$.

(2) $v\mathcal{R}^w_{\Box_n}u$ *if and only if* $u \in im(\mathcal{R}^w_{@_n})$.

Proof. (1). One inclusion is clear. For the other direction, let

$$Z = \mathcal{R}^w_{\Box_1}[\mathcal{R}^w_{\Box_2}[...[\mathcal{R}^w_{\Box_i}[...[\mathcal{R}^w_{\Box_n}[\{w\}]]]]]].$$

We first prove that (1a) $w \in Z$, and then that (1b) Z is closed under every relation $\mathcal{R}^w_{\mathcal{O}_i}$ in the frame $\mathfrak{F}^w$.

(1a). Let z be the element in W^w such that $\mathcal{R}^w_{@_n}(w) = z$, and for each i such that $1 \leq i < n$, let $\mathcal{R}^w_{@_i}(w) = z_i$ and $\mathcal{R}^w_{@_i}(z_i) = v_i$. Then $w\mathcal{R}^w_{\Box_n}z$, by $\mathbf{AT}_n$, and both $w\mathcal{R}^w_{\Box_{n-1}}z_{n-1}$ and $z_{n-1}\mathcal{R}^w_{\Box_{n-1}}v_{n-1}$, by $\mathbf{AT}_{n-1}$. So $w\mathcal{R}^w_{\Box_{n-1}}v_{n-1}$, by transitivity of $\mathcal{R}^w_{\Box_{n-1}}$. But $w\mathcal{R}^w_{\Box_{n-1}}z$ follows by $\mathbf{A}_n$, and so $z\mathcal{R}^w_{\Box_{n-1}}v_{n-1}$, since $\mathcal{R}^w_{\Box_{n-1}}$ is Euclidean. Then

$$w\mathcal{R}^w_{\Box_n}z\mathcal{R}^w_{\Box_{n-1}}v_{n-1}.$$

Moreover, both $w\mathcal{R}^w_{\Box_{n-2}}z_{n-2}$ and $z_{n-2}\mathcal{R}^w_{\Box_{n-2}}v_{n-2}$ follow by $\mathbf{AT}_{n-2}$. So $w\mathcal{R}^w_{\Box_{n-2}}v_{n-2}$, by transitivity of $\mathcal{R}^w_{\Box_{n-2}}$. Also, both $w\mathcal{R}^w_{\Box_{n-2}}z_{n-1}$ and $z_{n-1}\mathcal{R}^w_{\Box_{n-2}}v_{n-1}$, by $\mathbf{A}_{n-1}$. So $w\mathcal{R}^w_{\Box_{n-2}}v_{n-1}$, again by transitivity of $\mathcal{R}^w_{\Box_{n-2}}$. Hence $v_{n-1}\mathcal{R}^w_{\Box_{n-2}}v_{n-2}$, as $\mathcal{R}^w_{\Box_{n-2}}$ is Euclidean. Then

$$w\mathcal{R}^w_{\Box_n}z\mathcal{R}^w_{\Box_{n-1}}v_{n-1}\mathcal{R}^w_{\Box_{n-2}}v_{n-2}.$$

By repeating this argument multiple times, it follows that there is a chain

$$w\mathcal{R}^w_{\Box_n}z\mathcal{R}^w_{\Box_{n-1}}v_{n-1}\mathcal{R}^w_{\Box_{n-2}}v_{n-2}...\mathcal{R}^w_{\Box_3}v_3\mathcal{R}^w_{\Box_2}v_2.$$

Now, by $\mathbf{A}_2$, both $w\mathcal{R}^w_{\Box_1}z_2$ and $z_2\mathcal{R}^w_{\Box_1}v_2$, whence $w\mathcal{R}^w_{\Box_1}v_2$ and so $v_2\mathcal{R}^w_{\Box_1}w$, by transitivity and symmetry of $\mathcal{R}^w_{\Box_1}$, respectively, thereby extending the chain above to

$$(*)\ w\mathcal{R}^w_{\Box_n}z\mathcal{R}^w_{\Box_{n-1}}v_{n-1}\mathcal{R}^w_{\Box_{n-2}}v_{n-2}...\mathcal{R}^w_{\Box_3}v_3\mathcal{R}^w_{\Box_2}v_2\mathcal{R}^w_{\Box_1}w.$$

Therefore, $w \in Z$.

(1b). For $\mathcal{R}^w_{\Box_1}$, suppose that $v \in \mathcal{R}^w_{\Box_1}[Z]$. Then there is a $u \in Z$ such that $u\mathcal{R}^w_{\Box_1}v$. But since $u \in Z$, it follows by an argument similar to that which established $(*)$ in (1a) that there are $z_n, ..., z_2 \in W^w$ such that

$$w\mathcal{R}^w_{\Box_n}z_n\mathcal{R}^w_{\Box_{n-1}}z_{n-1}\mathcal{R}^w_{\Box_{n-2}}z_{n-2}...\mathcal{R}^w_{\Box_3}z_3\mathcal{R}^w_{\Box_2}z_2\mathcal{R}^w_{\Box_1}u.$$

By transitivity of $\mathcal{R}^w_{\Box_1}$, it then follows that $z_2\mathcal{R}^w_{\Box_1}v$, and so $v \in Z$.

For $\mathcal{R}^w_{\Box_i}$, where $1 < i \leq n$, suppose that $v \in \mathcal{R}^w_{\Box_i}[Z]$. Then there is a $u \in Z$ such that $u\mathcal{R}^w_{\Box_i}v$. But since $u \in Z$, it follows by an argument similar to the one establishing $(*)$ in (1a), that there are $z_n, ..., z_2 \in W^w$ such that

$$w\mathcal{R}^w_{\Box_n}z_n\mathcal{R}^w_{\Box_{n-1}}z_{n-1}\mathcal{R}^w_{\Box_{n-2}}z_{n-2}...\mathcal{R}^w_{\Box_3}z_3\mathcal{R}^w_{\Box_2}z_2\mathcal{R}^w_{\Box_1}u.$$

By upward transitivity, it follows that $z_2\mathcal{R}^w_{\Box_i}v$. Now, if $i = 2$, then $z_3\mathcal{R}^w_{\Box_2}v$, by transitivity of $\mathcal{R}^w_{\Box_2}$, from which it follows that $v \in Z$, as $\mathcal{R}^w_{\Box_1}$ is reflexive. If, on the other hand, $2 < i < n$, since $z_3\mathcal{R}^w_{\Box_2}z_2$, it follows that $z_3\mathcal{R}^w_{\Box_i}v$, by upward transitivity; since $z_4\mathcal{R}^w_{\Box_3}z_3$ and $z_3\mathcal{R}^w_{\Box_i}v$, it follows that $z_4\mathcal{R}^w_{\Box_i}v$ by upward transitivity; after multiple repetitions of this argument, since $z_i\mathcal{R}^w_{\Box_{i-1}}z_{i-1}$ and $z_{i-1}\mathcal{R}^w_{\Box_i}v$, it follows that $z_i\mathcal{R}^w_{\Box_i}v$ by upward transitivity. Then $v\mathcal{R}^w_{@_i}v$, by $\mathbf{AR}_i$, whence $v\mathcal{R}^w_{\Box_{i-1}}v$, by $\mathbf{A}_i$, and so $v\mathcal{R}^w_{@_{i-1}}v$, by $\mathbf{AR}_{i-1}$, whence $v\mathcal{R}^w_{\Box_{i-2}}v$, by $\mathbf{A}_{i-1}$, and so on. After multiple repetitions of this argument, $v\mathcal{R}^w_{@_3}v$, by $\mathbf{AR}_3$, whence $v\mathcal{R}^w_{\Box_2}v$, by $\mathbf{A}_3$. Also $v\mathcal{R}^w_{\Box_1}v$, since $\mathcal{R}^w_{\Box_1}$ is reflexive, and so $v \in Z$. In case $i = n$, since $w\mathcal{R}^w_{\Box_n}z_n$, and $z_n\mathcal{R}^w_{\Box_n}v$ follows as in the argument above, it then follows that $w\mathcal{R}^w_{\Box_n}v$ by transitivity of $\mathcal{R}^w_{\Box_n}$. The rest of the argument for the descending chain is exactly as the case $2 < i < n$ above.

For $\mathcal{R}^w_{@_1}$, suppose that $v \in \mathcal{R}^w_{@_1}[Z] = Z$, since $\mathcal{R}^w_{@_1}$ is the identity function. So $v \in Z$.

For $\mathcal{R}^w_{@_i}$, where $1 < i \leq n$, suppose that $v \in \mathcal{R}^w_{@_i}[Z]$. Then there is a $u \in Z$ such that $u\mathcal{R}^w_{@_i}v$. So $u\mathcal{R}^w_{\Box_i}v$, by $\mathbf{AT}_i$. That $v \in Z$ now follows from the argument for $\mathcal{R}^w_{\Box_i}$.

(2). For the left-to-right direction, suppose that $v\mathcal{R}^w_{\Box_n}u$. By $\mathbf{AR}_n$, $u\mathcal{R}^w_{@_n}u$, so $u \in im(\mathcal{R}^w_{@_n})$. For the converse, assume that $u \in im(\mathcal{R}^w_{@_n})$, and let z be the element in W^w such that $\mathcal{R}^w_{@_n}(w) = z$. Now, consider any point $v \in W^w$. We show that (2a) $v\mathcal{R}^w_{\Box_n}z$, and then that (2b) $z\mathcal{R}^w_{\Box_n}u$, from which $v\mathcal{R}^w_{\Box_n}u$ follows by transitivity of $\mathcal{R}^w_{\Box_n}$, thereby proving the lemma.

(2a) Since $v \in W^w$, it follows by an argument similar to the one establishing $(*)$ in (1a) that there are $z_n, ..., z_2 \in W^w$ such that

$$w\mathcal{R}^w_{\Box_n}z_n\mathcal{R}^w_{\Box_{n-1}}z_{n-1}\mathcal{R}^w_{\Box_{n-2}}z_{n-2}...\mathcal{R}^w_{\Box_3}z_3\mathcal{R}^w_{\Box_2}z_2\mathcal{R}^w_{\Box_1}v.$$

By symmetry of $\mathcal{R}^w_{\Box_1}$, it follows that $v\mathcal{R}^w_{\Box_1}z_2$. Now consider $z'_3 \in \mathcal{R}^w_{@_2}(z_3)$. By $\mathbf{AT}_2$, it follows that $z_3\mathcal{R}^w_{\Box_2}z'_3$. So $z_3\mathcal{R}^w_{\Box_2}z_2$ and $z_3\mathcal{R}^w_{\Box_2}z'_3$, from which $z_2\mathcal{R}^w_{\Box_2}z'_3$ follows from the Euclidean property of $\mathcal{R}^w_{\Box_2}$. So $v\mathcal{R}^w_{\Box_1}z_2$ and $z_2\mathcal{R}^w_{\Box_2}z'_3$, hence $v\mathcal{R}^w_{\Box_2}z'_3$, by upward transitivity. Now consider $z'_4 = \mathcal{R}^w_{@_3}(z_4)$. By $\mathbf{AT}_3$, it follows that $z_4\mathcal{R}^w_{\Box_3}z'_4$. So $z_4\mathcal{R}^w_{\Box_3}z_3$ and $z_4\mathcal{R}^w_{\Box_3}z'_4$, from which $z_3\mathcal{R}^w_{\Box_3}z'_4$ follows from the Euclidean property of $\mathcal{R}^w_{\Box_3}$. Since $z_3\mathcal{R}^w_{\Box_2}z'_3$, it follows that $z'_3\mathcal{R}^w_{\Box_3}z'_4$, by upward Euclideanity. And since $v\mathcal{R}^w_{\Box_2}z'_3$, it follows that $v\mathcal{R}^w_{\Box_3}z'_4$, by upward transitivity. By repeating this argument multiple times, it follows that $v\mathcal{R}^w_{\Box_{n-1}}z'_n$. Now consider $z = \mathcal{R}^w_{@_n}(w)$. By $\mathbf{AT}_n$, it follows that $w\mathcal{R}^w_{\Box_n}z$. So $w\mathcal{R}^w_{\Box_n}z_n$ and $w\mathcal{R}^w_{\Box_n}z$, whence $z_n\mathcal{R}^w_{\Box_n}z$, by the Euclidean property of $\mathcal{R}^w_{\Box_n}$. Then $z_n\mathcal{R}^w_{\Box_{n-1}}z'_n$ (by a previous iteration of the argument) and $z_n\mathcal{R}^w_{\Box_n}z$, so $z'_n\mathcal{R}^w_{\Box_n}z$ follows by upward Euclideanity. Therefore, $v\mathcal{R}^w_{\Box_{n-1}}z'_n$ and $z'_n\mathcal{R}^w_{\Box_n}z$, from which

$v\mathcal{R}^w_{\Box_n} z$ follows by upward transitivity.

(2b). Since $u \in W^w$, by hypothesis, it follows by an argument similar to the one establishing $(*)$ in (1a) that there are $z_n, ..., z_2 \in W^w$ such that

$$w\mathcal{R}^w_{\Box_n} z_n \mathcal{R}^w_{\Box_{n-1}} z_{n-1} \mathcal{R}^w_{\Box_{n-2}} z_{n-2} ... \mathcal{R}^w_{\Box_3} z_3 \mathcal{R}^w_{\Box_2} z_2 \mathcal{R}^w_{\Box_1} u.$$

Consider $z = \mathcal{R}^w_{@_n}(w)$. Then $w\mathcal{R}^w_{\Box_n} z$, by $\mathbf{AT}_n$. Since $\mathcal{R}^w_{\Box_n}$ is Euclidean, it follows that $z\mathcal{R}^w_{\Box_n} z_n$. Since $u \in im(\mathcal{R}^w_{@_n})$, there is a $u_n \in W^w$ such that $u_n \mathcal{R}^w_{@_n} u$. By $\mathbf{AT}_n$, $u_n \mathcal{R}^w_{\Box_n} u$, whence $u\mathcal{R}^w_{@_n} u$, by $\mathbf{AR}_n$, and so $u\mathcal{R}^w_{\Box_n} u$, by $\mathbf{AT}_n$. By $\mathbf{A}_n$, in turn, it also follows that $u_n \mathcal{R}^w_{\Box_{n-1}} u$, so $u\mathcal{R}^w_{@_{n-1}} u$ by $\mathbf{AR}_{n-1}$, whence $u\mathcal{R}^w_{\Box_{n-2}} u$ by $\mathbf{A}_{n-1}$. Then $u\mathcal{R}^w_{@_{n-2}} u$ by $\mathbf{AR}_{n-1}$, whence $u\mathcal{R}^w_{\Box_{n-3}} u$ by $\mathbf{A}_{n-2}$. By repeating this argument multiple times, it follows that $u\mathcal{R}^w_{\Box_3} u$ by $\mathbf{A}_4$ and, similarly, that $u\mathcal{R}^w_{\Box_2} u$ by $\mathbf{A}_3$. Now, since $z_2 \mathcal{R}^w_{\Box_1} u$ and $u\mathcal{R}^w_{\Box_2} u$, $z_2 \mathcal{R}^w_{\Box_2} u$ follows by upward transitivity. And because $z_3 \mathcal{R}^w_{\Box_2} z_2$ and $z_2 \mathcal{R}^w_{\Box_2} u$, $z_3 \mathcal{R}^w_{\Box_2} u$ follows by transitivity of $\mathcal{R}^w_{\Box_2}$. But since $z_3 \mathcal{R}^w_{\Box_2} u$ and $u\mathcal{R}^w_{\Box_3} u$, $z_3 \mathcal{R}^w_{\Box_3} u$ follows by upward transitivity, and so $z_4 \mathcal{R}^w_{\Box_3} u$ follows by transitivity of $\mathcal{R}^w_{\Box_3}$. By repeating this argument multiple times it follows that $z_n \mathcal{R}^w_{\Box_{n-1}} u$. But since $u\mathcal{R}^w_{\Box_n} u$, it follows that $z_n \mathcal{R}^w_{\Box_n} u$ by upward transitivity. And since $z\mathcal{R}^w_{\Box_n} z_n$, it follows that $z\mathcal{R}^w_{\Box_n} u$ by transitivity of $\mathcal{R}^w_{\Box_n}$, as desired. □

Lemma 1.3 *Let* $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n})$ *be an* $\mathbf{S}@_n$*-frame, and* $v \in W$*. Then* $\mathcal{R}_{@_i}(\mathcal{R}_{@_{i+1}}(v)) = \mathcal{R}_{@_{i+1}}(v)$*, for* $1 \leq i < n$*.*

Proof. Let $\mathcal{R}_{@_{i+1}}(v) = z$. Then $v\mathcal{R}_{\Box_{i+1}} z$, by $\mathbf{AT}_{i+1}$, and so $z\mathcal{R}_{@_{i+1}} z$ follows by $\mathbf{AR}_{i+1}$. Now $z\mathcal{R}_{\Box_i} z$ follows by $\mathbf{A}_{i+1}$, and $z\mathcal{R}_{@_i} z$ follows by $\mathbf{AR}_i$. So $v\mathcal{R}_{@_{i+1}} z\mathcal{R}_{@_i} z$, and hence

$$\mathcal{R}_{@_i}(\mathcal{R}_{@_{i+1}}(v)) = \mathcal{R}_{@_i}(z) = z = \mathcal{R}_{@_{i+1}}(v),$$

as desired. □

In order to establish the main completeness result of this section we now want to show the existence and relevant properties of certain families of surjective functions, which are essential to construct the relevant bounded morphism from a sequence frame onto an $\mathbf{S}@_n$-frame (or, more precisely, a point-generated subframe of an $\mathbf{S}@_n$-frame). In Fritz's (2014), Lemma 2.8, completeness proof for the two-dimensional case, this is done as follows. Let $\mathfrak{F} = (W, \mathcal{R}_{\Box_1}, \mathcal{R}_{\Box_2}, \mathcal{R}_{@_1}, \mathcal{R}_{@_2})$ be an $\mathbf{S}@_2$-frame, and $\mathfrak{F}^w = (W^w, \mathcal{R}^w_{\Box_1}, \mathcal{R}^w_{\Box_2}, \mathcal{R}^w_{@_1}, \mathcal{R}^w_{@_2})$ the subframe of $\mathfrak{F}$ generated by w. Validity is preserved by taking generated subframes,[14] and hence the $\mathbf{S}@_2$ axioms hold

[14]This is proved in Blackburn et al. (2001: 140), Theorem 3.14(ii).

in $\mathfrak{F}^w$. Now, for every element $w \in W^w$ there is a surjective function $g^1_w : W^w \to \mathcal{R}^w_{\Box_1}[g^0_\emptyset(w)]$ such that $g^1_w(w) = \mathcal{R}^w_{@_2}(g^0_\emptyset(w))$, where $g^0_\emptyset = Id_{W^w}$. The family g^1 of surjections exists since for every $w \in W^w$, $\mathcal{R}^w_{\Box_1}[g^0_\emptyset(w)] \subseteq W^w$, and also because $\mathcal{R}^w_{@_2}$ is a function such that $\mathcal{R}^w_{@_2} \subseteq \mathcal{R}^w_{\Box_1}$, by $\mathbf{A}_2$. We then construct a two-dimensional sequence frame, $\mathfrak{F}' = (W', \mathcal{R}'_{\Box_1}, \mathcal{R}'_{\Box_2}, \mathcal{R}'_{@_1}, \mathcal{R}'_{@_2})$, where $W' = W^w \times W^w$, and define a map f from W' onto W^w by $f((w,v)) = g^1_v(w)$, which is a bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$—this can be verified by a standard back-and-forth argument. Since we know by Sahlqvist completeness (Theorem 1.1) that $\mathbf{S@}_2$ is strongly complete with respect to $\mathbf{S@}_2$-frames, completeness of $\mathbf{S@}_2$ relative to the class of two-dimensional sequence frames now follows since modal satisfaction is invariant under taking bounded morphisms between models, and the fact that every formula in the language is satisfiable in an $\mathbf{S@}_2$-model if and only if it is satisfiable in a point-generated submodel of it.[15] This is by and large Fritz's argument except for minor notational differences, the absence of an $\mathcal{R}_{@_i}$ relation, and the fact that he uses an intermediate class of frames from Restall (2012) indirectly instead of the point-generated subframes $\mathfrak{F}^w$.

To see how this method can be generalized for n dimensions it is instructive first to briefly mention the adaptation of the argument above for three dimensions, as the definition of the bounded morphism in this case will need multiple families of surjections which are constructed in stages, just as in the n case. So, let $\mathfrak{F}$ be an $\mathbf{S@}_3$-frame, and $\mathfrak{F}^w$ the subframe of $\mathfrak{F}$ generated by w. In order to prove that $\mathbf{S@}_3$ is complete relative to three-dimensional sequence frames, we construct a three-dimensional sequence frame, $\mathfrak{F}' = (W', (\mathcal{R}'_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}'_{@_i})_{1\leq i\leq n})$, where $W' = W^w \times W^w \times W^w$, and a bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$. This is done as follows. Let $g^0_\emptyset = Id_{W^w}$. Then for every $w \in W^w$ there is a surjective function $g^1_w : W^w \to \mathcal{R}^w_{\Box_2}[g^0_\emptyset(w)]$ such that

(i) $g^1_w(w) = \mathcal{R}^w_{@_3}(g^0_\emptyset(w))$.

The family g^1 of surjections exists since for any $w \in W^w$, $\mathcal{R}^w_{\Box_2}[g^0_\emptyset(w)] \subseteq W^w$, and also because $\mathcal{R}^w_{@_3}$ is a function such that $\mathcal{R}^w_{@_3} \subseteq \mathcal{R}^w_{\Box_2}$, by $\mathbf{A}_3$. Additionally, for every $w, v \in W^w$ there is a surjective function $g^2_{w,v} : W^w \to \mathcal{R}^w_{\Box_1}[g^1_v(w)]$ such that

(ii) $g^2_{w,v}(w) = \mathcal{R}^w_{@_2}(g^1_v(w))$.

[15] See Blackburn et al. (2001), §2.1, for the definitions of bounded morphisms between models and generated submodels, as well as the relevant invariance results.

The family g^2 of surjections exists since for any $w, v \in W^w$, $\mathcal{R}^w_{\Box_1}[g^1_v(w)] \subseteq W^w$, and also because $\mathcal{R}^w_{@_2}$ is a function such that $\mathcal{R}^w_{@_2} \subseteq \mathcal{R}^w_{\Box_1}$, by $\mathbf{A}_2$. Then for every $w \in W^w$, since $g^2_{w,w}(w) = \mathcal{R}^w_{@_2}(g^1_w(w))$, by (ii), $g^1_w(w) = \mathcal{R}^w_{@_3}(g^0_\emptyset(w))$, by (i), and $\mathcal{R}^w_{@_2}(\mathcal{R}^w_{@_3}(w)) = \mathcal{R}^w_{@_3}(w)$, by Lemma 1.3, it follows that

(iii) $g^2_{w,w}(w) = \mathcal{R}^w_{@_3}(w)$.

The map f from W' onto W^w defined by $f((w, v, z)) = g^2_{v,z}(w)$ is then a bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$, which can again be verified by a standard back-and-forth argument.

The aim of the next two lemmas is to register the existence of similar surjections in the general n-dimensional case as well as to construct the desired bounded morphism for the completeness proof.

Lemma 1.4 *Let $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be an $\mathbf{S}@_n$-frame. Then:*

(1) There is a sequence $g^0, ..., g^{n-1}$ of families of surjective functions defined on W, where the family g^i is indexed by W^i, $g^0_\emptyset = Id_W$, and for each $1 \leq i \leq (n-1)$ and every $w_1, ..., w_i \in W$,

$$g^i_{w_1,...,w_i} : W \to \mathcal{R}_{\Box_{n-i}}[g^{i-1}_{w_2,...,w_i}(w_1)]$$

such that

$$g^i_{w_1,...,w_i}(w_1) = \mathcal{R}_{@_{n-(i-1)}}(g^{i-1}_{w_2,...,w_i}(w_1)).$$

(2) If an initial segment of $w_1, ..., w_{n-1}$ is such that $w_1 = ... = w_j = z$, for $1 \leq j \leq (n-1)$, and $z \in W$, then

$$g^{n-1}_{w_1,...,w_{n-1}}(z) = \mathcal{R}_{@_{j+1}}(g^{n-(j+1)}_{w_{j+1},...,w_{n-1}}(w_j)).$$

Proof. (1). The proof is by induction on $1 \leq i \leq (n-1)$. For $i = 1$, since for any $w_1 \in W$, $\mathcal{R}_{\Box_{n-1}}[g^0_\emptyset(w_1)] \subseteq W$, and $\mathcal{R}_{@_n}$ is a function such that $\mathcal{R}_{@_n} \subseteq \mathcal{R}_{\Box_{n-1}}$, by $\mathbf{A}_n$, there is a family of surjective functions

$$g^1_{w_1} : W \to \mathcal{R}_{\Box_{n-1}}[g^0_\emptyset(w_1)]$$

such that

$$g^1_{w_1}(w_1) = \mathcal{R}_{@_n}(g^0_\emptyset(w_1)).$$

Assume the induction hypothesis, for $i = k$, where $1 \leq k < (n-1)$, that there is family of surjective functions

$$g^k_{w_1,\ldots,w_k} : W \to \mathcal{R}_{\Box_{n-k}}[g^{k-1}_{w_2,\ldots,w_k}(w_1)]$$

such that

$$g^k_{w_1,\ldots,w_k}(w_1) = \mathcal{R}_{@_{n-(k-1)}}(g^{k-1}_{w_2,\ldots,w_k}(w_1)).$$

We show that the lemma holds for $i = k+1$. Since, by the induction hypothesis, there are functions $g^k_{w_1,\ldots,w_k}$, and both $\mathcal{R}_{\Box_{n-(k+1)}}[g^k_{w_2,\ldots,w_{k+1}}(w_1)] \subseteq W$ and $\mathcal{R}_{@_{n-k}} \subseteq \mathcal{R}_{\Box_{n-(k+1)}}$, by $\mathbf{A}_{n-k}$, it follows that there is a family of surjective functions

$$g^{k+1}_{w_1,\ldots,w_{k+1}} : W \to \mathcal{R}_{\Box_{n-(k+1)}}[g^k_{w_2,\ldots,w_{k+1}}(w_1)]$$

such that

$$g^{k+1}_{w_1,\ldots,w_{k+1}}(w_1) = \mathcal{R}_{@_{n-k}}(g^k_{w_2,\ldots,w_{k+1}}(w_1)),$$

as desired.

(2). The proof is by induction on j. For $j = 1$, it follows from (1) that

$$g^{n-1}_{w_1,\ldots,w_{n-1}}(z) = \mathcal{R}_{@_2}(g^{n-2}_{w_2,\ldots,w_{n-1}}(w_1)).$$

Now let $j = k$, where $1 \leq k < (n-1)$, and assume that

$$g^{n-1}_{w_1,\ldots,w_{n-1}}(z) = \mathcal{R}_{@_{k+1}}(g^{n-(k+1)}_{w_{k+1},\ldots,w_{n-1}}(w_k)).$$

Suppose that $w_1 = \ldots = w_{k+1} = z$. Then the family of surjections

$$g^{n-(k+1)}_{w_{k+1},\ldots,w_{n-1}} : W \to \mathcal{R}_{\Box_{n-(n-(k+1))}}[g^{(n-(k+1))-1}_{w_{k+2},\ldots,w_{n-1}}(w_{k+1})]$$

is such that

$$g^{n-(k+1)}_{w_{k+1},\ldots,w_{n-1}}(w_k) = \mathcal{R}_{@_{n-((n-(k+1))-1)}}(g^{(n-(k+1))-1}_{w_{k+2},\ldots,w_{n-1}}(w_{k+1})),$$

by (1), and hence

$$g^{n-1}_{w_1,\ldots,w_{n-1}}(z) = \mathcal{R}_{@_{k+1}}(\mathcal{R}_{@_{n-((n-(k+1))-1)}}(g^{(n-(k+1))-1}_{w_{k+2},\ldots,w_{n-1}}(w_{k+1}))).$$

Note, moreover, that $n - ((n-(k+1))-1) = k+2$, and hence

$$g^{n-1}_{w_1,\ldots,w_{n-1}}(z) = \mathcal{R}_{@_{n-((n-(k+1))-1)}}(g^{(n-(k+1))-1}_{w_{k+2},\ldots,w_{n-1}}(w_{k+1})),$$

by Lemma 1.3, as desired. □

Lemma 1.5 *Let $\mathfrak{F} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be an $\mathbf{S}@_n$-frame, $w \in W$, and $\mathfrak{F}^w = (W^w, (\mathcal{R}^w_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}^w_{@_i})_{1\leq i\leq n})$ the subframe of $\mathfrak{F}$ generated by w. Then $\mathfrak{F}^w$ is a bounded morphic image of a sequence frame.*

Proof. Let $\mathfrak{F}$, $w \in W$, and $\mathfrak{F}^w$ be as in the hypothesis. Because generated subframes preserve validity between frames, all of the $\mathbf{S}@_n$ axioms hold in $\mathfrak{F}^w$. So in what follows we construct a sequence frame, $\mathfrak{F}' = (W', (\mathcal{R}'_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}'_{@_i})_{1\leq i\leq n})$, such that $W' = (W^w)^n$, where $(W^w)^n$ is the n-fold Cartesian product of W^w, and a surjective bounded morphism $f : \mathfrak{F}' \twoheadrightarrow \mathfrak{F}^w$. Then, where $\sigma \in (W^w)^n$, let $f : (W^w)^n \to W^w$ be defined as

$$f(\sigma) = g^{n-1}_{\sigma_2,...,\sigma_n}(\sigma_1),$$

where the sequence $g^0, ..., g^{n-1}$ of families of surjective functions is defined on W^w. We prove that f so defined is a bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$ by checking the back and forth conditions for each accessibility relation as follows:

$[\mathcal{R}_{\Box_1}]$ Suppose that $\sigma\mathcal{R}'_{\Box_1}\sigma'$. Then σ and σ' are identical beyond 1. So $f(\sigma) = g^{n-1}_{\sigma_2,...,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)]$, and $f(\sigma') = g^{n-1}_{\sigma'_2,...,\sigma'_n}(\sigma'_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)]$, from which it follows that $f(\sigma)\mathcal{R}^w_{\Box_1}f(\sigma')$. Conversely, suppose that $f(\sigma)\mathcal{R}^w_{\Box_1}z$. Since

$$f(\sigma) = g^{n-1}_{\sigma_2,...,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)],$$

it follows that $z \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)]$. And since $g^{n-1}_{\sigma_2,...,\sigma_n}$ is a surjection, there is a $v \in W^w$ such that $g^{n-1}_{\sigma_2,...,\sigma_n}(v) = z$, whence $f(\sigma^1_v) = g^{n-1}_{\sigma_2,...,\sigma_n}(v) = z$. Furthermore, $\sigma\mathcal{R}'_{\Box_1}\sigma^1_v$.

$[\mathcal{R}_{\Box_i}, 1 < i < n]$ Suppose that $\sigma\mathcal{R}'_{\Box_i}\sigma'$. Then σ' is i-diagonal and identical with σ beyond i. So $f(\sigma') = g^{n-1}_{\sigma'_2,...,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_i}(g^{n-i}_{\sigma'_{i+1},...,\sigma'_n}(\sigma'_i))$, by Lemma 1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,...,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)]$. Since $g^{n-i}_{\sigma'_{i+1},...,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{@_i}f(\sigma')$, it follows that $g^{n-i}_{\sigma'_{i+1},...,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{\Box_i}f(\sigma')$, by $\mathbf{AT}_i$. Also, $g^{n-i}_{\sigma'_{i+1},...,\sigma'_n}(\sigma'_i) \in \mathcal{R}^w_{\Box_i}[g^{(n-i)-1}_{\sigma_{i+2},...,\sigma_n}(\sigma_{i+1})]$, since the sequences σ and σ' are identical beyond i, whence $g^{(n-i)-1}_{\sigma_{i+2},...,\sigma_n}(\sigma_{i+1})\mathcal{R}^w_{\Box_i}f(\sigma')$, by transitivity of $\mathcal{R}^w_{\Box_i}$. But

$$\begin{aligned}
&g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2) \in \mathcal{R}^w_{\Box_2}[g^{n-3}_{\sigma_4,...,\sigma_n}(\sigma_3)],\\
&g^{n-3}_{\sigma_4,...,\sigma_n}(\sigma_3) \in \mathcal{R}^w_{\Box_3}[g^{n-4}_{\sigma_5,...,\sigma_n}(\sigma_4)],\\
&...,\\
&g^{n-i}_{\sigma_{i+1},...,\sigma_n}(\sigma_i) \in \mathcal{R}^w_{\Box_i}[g^{(n-i)-1}_{\sigma_{i+2},...,\sigma_n}(\sigma_{i+1})].
\end{aligned}$$

So $g^{n-i}_{\sigma_{i+1},...,\sigma_n}(\sigma_i)\mathcal{R}^w_{\Box_i}f(\sigma')$, by the Euclidean property of $\mathcal{R}^w_{\Box_i}$. Now, $g^{n-(i-1)}_{\sigma_i,...,\sigma_n}(\sigma_{i-1}) \in \mathcal{R}^w_{\Box_{i-1}}[g^{n-i}_{\sigma_{i+1},...,\sigma_n}(\sigma_i)]$. So $g^{n-(i-1)}_{\sigma_i,...,\sigma_n}(\sigma_{i-1})\mathcal{R}^w_{\Box_i}f(\sigma')$, by upward Euclideanity. By an analogous argument, since $g^{n-(i-2)}_{\sigma_{i-1},...,\sigma_n}(\sigma_{i-2}) \in \mathcal{R}^w_{\Box_{i-2}}[g^{n-(i-1)}_{\sigma_i,...,\sigma_n}(\sigma_{i-1})]$, it follows that $g^{n-(i-2)}_{\sigma_{i-1},...,\sigma_n}(\sigma_{i-2})\mathcal{R}^w_{\Box_i}f(\sigma')$, by upward

Euclideanity. By multiple repetitions of this argument, it follows that $g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_i} f(\sigma')$. Since $g^{n-1}_{\sigma_2,...,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)]$, by upward Euclideanity, again, it follows that $g^{n-1}_{\sigma_2,...,\sigma_n}(\sigma_1)\mathcal{R}^w_{\Box_i} f(\sigma')$, and therefore $f(\sigma)\mathcal{R}^w_{\Box_i} f(\sigma')$. Conversely, suppose that $f(\sigma)\mathcal{R}^w_{\Box_i} z$. Then $f(\sigma) = g^{n-1}_{\sigma_2,...,\sigma_n}(\sigma_1)$ $\in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)]$. So

$$g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_1} f(\sigma)\mathcal{R}^w_{\Box_i} z,$$

from which $g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_i} z$ follows by upward transitivity. Now, $g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)$ $\in \mathcal{R}^w_{\Box_2}[g^{n-3}_{\sigma_4,...,\sigma_n}(\sigma_3)]$, whence

$$g^{n-3}_{\sigma_4,...,\sigma_n}(\sigma_3)\mathcal{R}^w_{\Box_2} g^{n-2}_{\sigma_3,...,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_i} z,$$

from which $g^{n-3}_{\sigma_4,...,\sigma_n}(\sigma_3)\mathcal{R}^w_{\Box_i} z$ follows by upward transitivity. Similarly, $g^{n-3}_{\sigma_4,...,\sigma_n}(\sigma_3) \in \mathcal{R}^w_{\Box_3}[g^{n-4}_{\sigma_5,...,\sigma_n}(\sigma_4)]$, whence

$$g^{n-4}_{\sigma_5,...,\sigma_n}(\sigma_4)\mathcal{R}^w_{\Box_3} g^{n-3}_{\sigma_4,...,\sigma_n}(\sigma_3)\mathcal{R}^w_{\Box_i} z,$$

from which $g^{n-4}_{\sigma_5,...,\sigma_n}(\sigma_4)\mathcal{R}^w_{\Box_i} z$ follows by upward transitivity. By repeating this argument, $g^{n-i}_{\sigma_{i+1},...,\sigma_n}(\sigma_i)\mathcal{R}^w_{\Box_i} z$, but since $g^{n-i}_{\sigma_{i+1},...,\sigma_n}(\sigma_i) \in \mathcal{R}^w_{\Box_i}[g^{(n-i)-1}_{\sigma_{i+2},...,\sigma_n}(\sigma_{i+1})]$, it follows that

$$g^{(n-i)-1}_{\sigma_{i+2},...,\sigma_n}(\sigma_{i+1})\mathcal{R}^w_{\Box_i} g^{n-i}_{\sigma_{i+1},...,\sigma_n}(\sigma_i)\mathcal{R}^w_{\Box_i} z,$$

and so $g^{(n-i)-1}_{\sigma_{i+2}...\sigma_n}(\sigma_{i+1})\mathcal{R}^w_{\Box_i} z$ follows by transitivity of $\mathcal{R}^w_{\Box_i}$. Then $z \in \mathcal{R}^w_{\Box_i}[g^{(n-i)-1}_{\sigma_{i+2},...,\sigma_n}(\sigma_{i+1})]$. Now, since $g^{n-i}_{\sigma_{i+1},...,\sigma_n}$ is a surjective function, it follows that there is a $v \in W^w$ such that

$$(*)\ g^{n-i}_{\sigma_{i+1}...\sigma_n}(v) = z.$$

So consider the i-diagonal sequence $(v, ..., v, \sigma_{i+1}, ..., \sigma_n) \in (W^w)^n$, which is identical with σ beyond i. Then $\sigma\mathcal{R}'_{\Box_i}(v, ..., v, \sigma_{i+1}, ..., \sigma_n)$, and so:

$$\begin{array}{rcll} f((v, ..., v, \sigma_{i+1}, ..., \sigma_n)) & = & g^{n-1}_{v,...,v,\sigma_{i+1},...,\sigma_n}(v) & \text{[by def. of } f\text{]} \\ & = & \mathcal{R}^w_{@_i}(g^{n-i}_{\sigma_{i+1},...,\sigma_n}(v)) & \text{[by Lemma 1.4(2)]} \\ & = & \mathcal{R}^w_{@_i}(z). & \text{[by } (*)\text{]} \end{array}$$

So $z\mathcal{R}^w_{@_i} f((v, ..., v, \sigma_{i+1}, ..., \sigma_n))$, whence $z\mathcal{R}^w_{\Box_i} f((v, ..., v, \sigma_{i+1}, ..., \sigma_n))$, by $\mathbf{AT}_i$. Then $f(\sigma)\mathcal{R}^w_{\Box_i} f((v, ..., v, \sigma_{i+1}, ..., \sigma_n))$ follows by transitivity of $\mathcal{R}^w_{\Box_i}$. Furthermore, $z\mathcal{R}^w_{@_i} z$ follows from the original hypothesis and $\mathbf{AR}_i$, in which case both $z\mathcal{R}^w_{@_i} f((v, ..., v, \sigma_{i+1}, ..., \sigma_n))$ and $z\mathcal{R}^w_{@_i} z$, from which $f((v, ..., v, \sigma_{i+1}, ..., \sigma_n)) = z$ follows by $\mathbf{F}_{@_i}$, as desired.

[$\mathcal{R}_{\Box_n}$] Suppose that $\sigma\mathcal{R}'_{\Box_n}\sigma'$. So σ' is n-diagonal. Then $f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_n}(\sigma'_n)$, by Lemma 1.4(2), whence $f(\sigma') \in im(\mathcal{R}^w_{@_n})$. It follows that $f(\sigma)\mathcal{R}^w_{\Box_n}f(\sigma')$, by Lemma 1.2(2). Conversely, suppose that $f(\sigma)\mathcal{R}^w_{\Box_n}z$. So $z\mathcal{R}^w_{@_n}z$, by $\mathbf{AR}_n$, in which case there is a $v \in W^w$, namely, u, such that

$$(\star)\ \mathcal{R}^w_{@_n}(v) = z,$$

as $\mathcal{R}^w_{@_n}$ is a function. Now consider the sequence $\sigma^n_v \in (W^w)^n$. Then:

$$\begin{array}{rcll} f(\sigma^n_v) & = & g^{n-1}_{v,\ldots,v}(v) & \text{[by def. of } f\text{]} \\ & = & \mathcal{R}^w_{@_n}(v) & \text{[by Lemma 1.4(2)]} \\ & = & z. & \text{[by } (\star)\text{]} \end{array}$$

Additionally, $\sigma\mathcal{R}'_{\Box_n}\sigma^n_v$.

[$\mathcal{R}_{@_1}$] Suppose that $\sigma\mathcal{R}'_{@_1}\sigma'$. Then $\sigma = \sigma'$. Since $\mathcal{R}^w_{@_1}$ is reflexive, it follows that $f(\sigma)\mathcal{R}^w_{@_1}f(\sigma')$. Now suppose that $f(\sigma)\mathcal{R}^w_{@_1}z$. Because $\mathcal{R}^w_{@_1}$ is reflexive, $f(\sigma)\mathcal{R}^w_{@_1}f(\sigma)$, hence $f(\sigma) = z$, as $\mathcal{R}^w_{@_1}$ is a function. Also, $\sigma\mathcal{R}'_{@_1}\sigma$.

[$\mathcal{R}_{@_i}, 1 < i < n$] Suppose that $\sigma\mathcal{R}'_{@_i}\sigma'$. Then σ' is i-diagonal and identical with σ beyond $i-1$. So

$$f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_i}(g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)),$$

by Lemma 1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. Then $f(\sigma)\mathcal{R}^w_{\Box_1}g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)$, by symmetry of $\mathcal{R}^w_{\Box_1}$. Also,

$$\begin{array}{l} g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2) \in \mathcal{R}^w_{\Box_2}[g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3)], \\ g^{n-3}_{\sigma_4,\ldots,\sigma_n}(\sigma_3) \in \mathcal{R}^w_{\Box_3}[g^{n-4}_{\sigma_5,\ldots,\sigma_n}(\sigma_4)], \\ \ldots, \\ g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1}) \in \mathcal{R}^w_{\Box_{i-1}}[g^{n-i}_{\sigma_{i+1},\ldots,\sigma_n}(\sigma_i)]. \end{array}$$

Now, since σ' is identical with σ beyond $i-1$, it also follows that $g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1}) \in [g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)]_{\mathcal{R}^w_{\Box_{i-1}}}$. And since $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{@_i}f(\sigma')$, it follows that $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{\Box_{i-1}}f(\sigma')$, by $\mathbf{A}_i$, and so

$$g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1})\mathcal{R}^w_{\Box_{i-1}}f(\sigma'),$$

as $\mathcal{R}^w_{\Box_{i-1}}$ is Euclidean. But since $g^{n-(i-2)}_{\sigma_{i-1},\ldots,\sigma_n}(\sigma_{i-2}) \in \mathcal{R}^w_{\Box_{i-2}}[g^{n-(i-1)}_{\sigma_i,\ldots,\sigma_n}(\sigma_{i-1})]$, it follows that $g^{n-(i-2)}_{\sigma_{i-1},\ldots,\sigma_n}(\sigma_{i-2})\mathcal{R}^w_{\Box_{i-1}}f(\sigma')$, by upward Euclideanity. And since $g^{n-(i-3)}_{\sigma_{i-2},\ldots,\sigma_n}(\sigma_{i-3}) \in \mathcal{R}^w_{\Box_{i-3}}[g^{n-(i-2)}_{\sigma_{i-1}\ldots\sigma_n}(\sigma_{i-2})]$, it follows that

$$g^{n-(i-3)}_{\sigma_{i-2},\ldots,\sigma_n}(\sigma_{i-3})\mathcal{R}^w_{\Box_{i-1}}f(\sigma'),$$

again by upward Euclideanity. By repeating this argument, it follows that $g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_{i-1}}f(\sigma')$, by upward Euclideanity. Then $f(\sigma)\mathcal{R}^w_{\Box_{i-1}}f(\sigma')$, follows from upward Euclideanity, too. But since $g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)\mathcal{R}^w_{@_i}f(\sigma')$, it follows that $f(\sigma')\mathcal{R}^w_{@_i}f(\sigma')$, by $\mathbf{AT}_i$ and $\mathbf{AR}_i$, respectively. So $f(\sigma)\mathcal{R}^w_{@_i}f(\sigma')$, by $\mathbf{A4}_i$. Conversely, suppose that $f(\sigma)\mathcal{R}^w_{@_i}z$. Consider the sequence $\sigma' \in (W^w)^n$ such that σ' is i-diagonal and identical with σ beyond $i-1$. Then $\sigma\mathcal{R}'_{@_i}\sigma'$. So

$$f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_i}(g^{n-i}_{\sigma'_{i+1},\ldots,\sigma'_n}(\sigma'_i)),$$

by Lemma 1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. By the same argument as in the forth direction, it follows that $f(\sigma)\mathcal{R}^w_{@_i}f(\sigma')$. But since $f(\sigma)\mathcal{R}^w_{@_i}z$, by assumption, it follows that $f(\sigma') = z$, by $\mathbf{F}_{@_i}$.

$[\mathcal{R}_{@_n}]$ Suppose that $\sigma\mathcal{R}'_{@_n}\sigma'$. Then σ' is n-diagonal and identical with σ beyond $(n-1)$. So

$$f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_n}(\sigma'_n),$$

by Lemma 1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. Since $\sigma_n = \sigma'_n$, it follows that $\sigma_n\mathcal{R}^w_{@_n}f(\sigma')$, and so both $\sigma_n\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, by $\mathbf{A}_n$, and $f(\sigma')\mathcal{R}^w_{@_n}f(\sigma')$, by $\mathbf{AR}_n$. Now, $g^1_{\sigma_n}(\sigma_{n-1}) \in \mathcal{R}^w_{\Box_{n-1}}[\{\sigma_n\}]$, so it follows that $g^1_{\sigma_n}(\sigma_{n-1})\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, as $\mathcal{R}^w_{\Box_{n-1}}$ is Euclidean. Also, $g^2_{\sigma_{n-1},\sigma_n}(\sigma_{n-2}) \in \mathcal{R}^w_{\Box_{n-2}}[g^1_{\sigma_n}(\sigma_{n-1})]$. So $g^2_{\sigma_{n-1},\sigma_n}(\sigma_{n-2})\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$ by upward Euclideanity. Analogously,

$$g^3_{\sigma_{n-2},\sigma_{n-1},\sigma_n}(\sigma_{n-3}) \in \mathcal{R}^w_{\Box_{n-3}}[g^2_{\sigma_{n-1},\sigma_n}(\sigma_{n-2})].$$

So $g^3_{\sigma_{n-2},\sigma_{n-1},\sigma_n}(\sigma_{n-3})\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, again by upward Euclideanity. By repetitions of this argument, it follows that $g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, by upward Euclideanity. Since $f(\sigma)\mathcal{R}^w_{\Box_1}g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)$, by symmetry of $\mathcal{R}^w_{\Box_1}$, it follows that $f(\sigma)\mathcal{R}^w_{\Box_{n-1}}f(\sigma')$, by upward transitivity. Therefore, $f(\sigma)\mathcal{R}^w_{@_n}f(\sigma')$, by $\mathbf{A4}_n$. Conversely, suppose that $f(\sigma)\mathcal{R}^w_{@_n}z$. Now consider the sequence $\sigma' \in (W^w)^n$ such that σ' is n-diagonal and identical with σ beyond $n-1$. Also $\sigma\mathcal{R}'_{@_n}\sigma'$. So

$$f(\sigma') = g^{n-1}_{\sigma'_2,\ldots,\sigma'_n}(\sigma'_1) = \mathcal{R}^w_{@_n}(\sigma'_n),$$

by Lemma 1.4(2), and $f(\sigma) = g^{n-1}_{\sigma_2,\ldots,\sigma_n}(\sigma_1) \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{\sigma_3,\ldots,\sigma_n}(\sigma_2)]$. By the same argument as in the forth direction, it follows that $f(\sigma)\mathcal{R}^w_{@_n}f(\sigma')$. But since $f(\sigma)\mathcal{R}^w_{@_n}z$, it follows that $f(\sigma') = z$, by $\mathbf{F}_{@_n}$.

This concludes the back and forth cases. It remains to show that f is surjective. Consider any $u \in W^w$. By Lemma 1.2(1),

$$u \in \mathcal{R}^w_{\Box_1}[\mathcal{R}^w_{\Box_2}[...[\mathcal{R}^w_{\Box_i}[...[\mathcal{R}^w_{\Box_n}[\{w\}]]]]]],$$

so there are elements $v_n, ..., v_2 \in W^w$, and a chain

$$w\mathcal{R}^w_{\Box_n} v_n \mathcal{R}^w_{\Box_{n-1}} v_{n-1} \mathcal{R}^w_{\Box_{n-2}} v_{n-2} ... \mathcal{R}^w_{\Box_3} v_3 \mathcal{R}^w_{\Box_2} v_2 \mathcal{R}^w_{\Box_1} u.$$

Consider the mapping $g^1_{v_n} : W^w \to \mathcal{R}^w_{\Box_{n-1}}[\{v_n\}]$. As this is surjective, and $v_{n-1} \in \mathcal{R}^w_{\Box_{n-1}}[\{v_n\}]$, there is a $z_{n-1} \in W^w$ such that $g^1_{v_n}(z_{n-1}) = v_{n-1}$. Now consider the mapping $g^2_{z_{n-1},v_n} : W^w \to \mathcal{R}^w_{\Box_{n-2}}[g^1_{v_n}(z_{n-1})]$. As this is surjective, and $v_{n-2} \in \mathcal{R}^w_{\Box_{n-2}}[g^1_{v_n}(z_{n-1})]$, there is a $z_{n-2} \in W^w$ such that $g^2_{z_{n-1},v_n}(z_{n-2}) = v_{n-2}$. After multiple repetitions of this argument, consider the mapping $g^{n-1}_{z_2,...,z_{n-1},v_n} : W^w \to \mathcal{R}^w_{\Box_1}[g^{n-2}_{z_3,...,z_{n-1},w}(z_2)]$. By a previous iteration of the argument, $g^{n-2}_{z_3,...,z_{n-1},v_n}(z_2) = v_2$, and so $u \in \mathcal{R}^w_{\Box_1}[g^{n-2}_{z_3,...,z_{n-1},v_n}(z_2)]$. As $g^{n-1}_{z_2,...,z_{n-1},v_n}$ is also surjective, there is a $z_1 \in W^w$ such that $g^{n-1}_{z_2,...,z_{n-1},v_n}(z_1) = u$. Therefore, there is a sequence $(z_1, z_2, ..., z_{n-1}, v_n) \in (W^w)^n$ such that

$$f((z_1, z_2, ..., z_{n-1}, v_n)) = g^{n-1}_{z_2,...,v_n}(z_1) = u,$$

as desired. □

This is sufficient to prove that $\mathbf{S}@_n$ is the multidimensional logic of sequence frames:

Theorem 1.2 (Soundness and completeness) $\mathbf{S}@_n$ *is sound and strongly complete with respect to the class* F *of sequence frames.*

Proof. Soundness is clear given the correspondence between the axioms of $\mathbf{S}@_n$ and the properties of n-dimensional sequence frames. For strong completeness, by Theorem 1.1, $\mathbf{S}@_n$ is strongly complete with respect to the class of $\mathbf{S}@_n$-frames, that is, $\mathsf{F}_{\mathbf{S}@_n}$. So it suffices to prove that a set of formulas is satisfiable in a sequence frame if and only if it is satisfiable in an $\mathbf{S}@_n$-frame. Suppose that a set of formulas Γ is satisfiable in a sequence frame $\mathfrak{F} \in \mathsf{F}$. Because every sequence frame is an $\mathbf{S}@_n$-frame, it follows that Γ is also satisfiable in an $\mathbf{S}@_n$-frame. Conversely, suppose that Γ is satisfiable in an $\mathbf{S}@_n$-frame, say, $\mathfrak{F}$. Then Γ is satisfiable in a model $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}_{@_i})_{1 \leq i \leq n}, V)$ based on $\mathfrak{F}$. So there is a point w in $\mathfrak{M}$ such that $\mathfrak{M}, w \vDash \Gamma$. Let $\mathfrak{M}^w = (W^w, (\mathcal{R}^w_{\Box_i})_{1 \leq i \leq n}, (\mathcal{R}^w_{@_i})_{1 \leq i \leq n}, V^w)$ be the submodel of $\mathfrak{M}$ generated by w. Since modal satisfaction is invariant under taking generated submodels, and $w \in W^w$, it follows that $\mathfrak{M}^w, w \vDash \Gamma$, and so Γ is satisfiable in the point-generated subframe $\mathfrak{F}^w$ of $\mathfrak{F}$. But, by Lemma 1.5, $\mathfrak{F}^w$ is a bounded

morphic image of a sequence frame, say, $\mathfrak{F}' = (W', (\mathcal{R}'_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}'_{@_i})_{1\leq i\leq n})$. Let $f : \mathfrak{F}' \twoheadrightarrow \mathfrak{F}^w$ be a surjective bounded morphism from $\mathfrak{F}'$ onto $\mathfrak{F}^w$, and $\mathfrak{M}' = (W', (\mathcal{R}'_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}'_{@_i})_{1\leq i\leq n}, V')$ a model based on $\mathfrak{F}'$ such that for each $p \in \mathsf{PROP}$, $V'(p) = \{\sigma \in W' \mid f(\sigma) \in V^w(p)\}$. Because f is surjective, and $w \in W^w$, let σ' be a sequence in W' such that $f(\sigma') = w$. Then $\mathfrak{M}^w, f(\sigma') \vDash \Gamma$. But as $\mathfrak{M}^w$ is a bounded morphic image of $\mathfrak{M}'$, this implies that $\mathfrak{M}', \sigma' \vDash \Gamma$, since modal satisfaction is invariant under bounded morphisms between models. Therefore, Γ is satisfiable in the sequence frame $\mathfrak{F}'$. □

Recall that in Fritz (2014: 386) a class of frames with distinguished elements is derived from matrix frames—or, according to our nomenclature, two-dimensional sequence frames (again, for a language without $@_1$). The logic of these frames is then derived syntactically from **2Dg**, and a completeness proof is presented in Fritz (2014: 394). Given the completeness result established just above, we show that Fritz's argument can once more be generalized for the n-dimensional case. We say that an *n-dimensional sequence frame with distinguished elements* is a quadruple,

$$\mathfrak{F}^D = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, D)$$

where $(W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ is an n-dimensional sequence frame such that $W = S^n$, and $D = \{\overline{s} \mid s \in S\}$. Let F^D be the class of n-dimensional sequence frames with distinguished elements.

Models based on such frames with distinguished elements are defined in the obvious way, and the logical notions of validity and consequence are now relativized to generalized diagonal points $\overline{s}$, as expected. Even though, as pointed out by Fritz (2014: 286), such logics characterizing frames with distinguished elements are not, in general, normal, for they do not have to be closed under the rule of generalization (or necessitation) for each $\mathcal{O}_i$, they can be defined syntactically from their normal counterparts, as it were, as follows:

Definition 1.9 $\vdash_n^D \varphi$ iff $\vdash_n @_n\varphi$.

Let $\mathbf{S}@_n^D$ be the logic characterizing n-dimensional sequence frames with distinguished elements. Given the strong completeness of $\mathbf{S}@_n$ relative to the class of n-dimensional sequence frames, the argument for the strong completeness of $\mathbf{S}@_n^D$ relative to F^D is then a simple adaptation of the argument in Fritz (2014: 394), Theorem 2.11.

Theorem 1.3 (Soundness and completeness) *$\mathbf{S}@_n^D$ is sound and strongly complete with respect to F^D.*

2 n-dimensional tableaux

It is very natural, as will be seen below, to define indexed tableau systems for $\mathbf{S}@_n$ following the style of Melvin Fitting's prefixed tableaux for modal logics.[16] In fact, generalizations of indexed tableau systems for a variety of two-dimensional modal logics appear in Lampert (2018) as well as in Gilbert (2016).

In order to define indices and index-sequences for the tableaux, we apply similar conventions as found in Definition 1.2 for sequences of possible worlds. A nice feature of these tableaux is their simplicity, for we only need, in effect, a single rule for each modal operator in the language rather than multiple rules corresponding to multiple properties of the accessibility relations, as is usually the case. Next we define the notions of *index-sequences, indexed formulas,* and *root* of an n-dimensional tableau. These will consist in slight modifications of the notions appearing in Definition 1.2 for sequences of possible worlds.

Definition 2.1 (Index-sequence, formula, root) An *index* is a natural number greater than 0. Let $\mathtt{s} = (x_1, ..., x_n)$ be an *index-sequence*, where each x_i is an index. We will often write $\mathtt{s}_i$ for x_i. An *indexed formula* is an expression, $[\varphi]\mathtt{s}$, where $\mathtt{s}$ is an index-sequence and φ is a formula of $\mathcal{L}_n^@$. All indexed formulas are enclosed within brackets. The *root* of an n-dimensional tableau always contains the negation of the formula we are attempting to prove indexed by $(1, 2, ..., n)$, that is, an index-sequence with the natural numbers ordered from 1 to n.

Definition 2.2 (Notation) Let $\mathtt{s}_z^i$ be that index-sequence $\mathtt{t}$ for which $\mathtt{t}_1 = \mathtt{t}_2 = ... = \mathtt{t}_i = z$ and for $j > i$, $\mathtt{t}_j = \mathtt{s}_j$. Additionally, for an index x, $\overline{x}$ is that sequence $\mathtt{s}$ for which $\mathtt{s}_i = x$ for all $i \in \{1, ..., n\}$.

As per usual, a branch of a tableau is *closed* if for some formula, φ, and index-sequence $\mathtt{s}$, both $[\varphi]\mathtt{s}$ and $[\neg\varphi]\mathtt{s}$ occur on the branch. A tableau is *closed* just in case all of its branches are closed. Otherwise, the tableau is *open.* Finally, a *tableau proof* of a sentence, φ, is a closed tableau for $[\neg\varphi]\mathtt{s}$, where $\mathtt{s}$ is the index-sequence $(1, 2, ..., n)$. The tableau rules for $\mathbf{S}@_n$ are displayed in Figure 8 with some provisos for the modal rules explained below.

The rules $(\neg\neg)$, $(\wedge)$, and $(\vee)$ comprise the portion from the classical propositional calculus. The necessity rules, (ν_i), are applied to any index z already occurring on the branch, and the possibility rules, (π_i), may be used provided the index z is new to the branch. These are just the usual restrictions on

[16] These can be found in several papers as well as in Fitting and Mendelsohn (1998).

$$(\neg\neg): \frac{[\neg\neg\varphi]\mathtt{s}}{[\varphi]\mathtt{s}}$$

$$(\wedge): \frac{[\varphi \wedge \psi]\mathtt{s}}{\begin{array}{c}[\varphi]\mathtt{s}\\ [\psi]\mathtt{s}\end{array}} \quad \frac{[\neg(\varphi \vee \psi)]\mathtt{s}}{\begin{array}{c}[\neg\varphi]\mathtt{s}\\ [\neg\psi]\mathtt{s}\end{array}} \quad \frac{[\neg(\varphi \rightarrow \psi)]\mathtt{s}}{\begin{array}{c}[\varphi]\mathtt{s}\\ [\neg\psi]\mathtt{s}\end{array}} \quad \frac{[\varphi \leftrightarrow \psi]\mathtt{s}}{\begin{array}{c}[\varphi \rightarrow \psi]\mathtt{s}\\ [\psi \rightarrow \varphi]\mathtt{s}\end{array}}$$

$$(\vee): \frac{[\varphi \vee \psi]\mathtt{s}}{[\varphi]\mathtt{s} \mid [\psi]\mathtt{s}} \quad \frac{[\neg(\varphi \wedge \psi)]\mathtt{s}}{[\neg\varphi]\mathtt{s} \mid [\neg\psi]\mathtt{s}} \quad \frac{[\varphi \rightarrow \psi]\mathtt{s}}{[\neg\varphi]\mathtt{s} \mid [\psi]\mathtt{s}}$$

$$\frac{[\neg(\varphi \leftrightarrow \psi)]\mathtt{s}}{[\neg(\varphi \rightarrow \psi)]\mathtt{s} \mid [\neg(\psi \rightarrow \varphi)]\mathtt{s}}$$

$$(\nu_i): \frac{[\Box_i\varphi]\mathtt{s}}{[\varphi]\mathtt{s}^i_z} \quad \frac{[\neg\Diamond_i\varphi]\mathtt{s}}{[\neg\varphi]\mathtt{s}^i_z} \qquad (\pi_i): \frac{[\Diamond_i\varphi]\mathtt{s}}{[\varphi]\mathtt{s}^i_z} \quad \frac{[\neg\Box_i\varphi]\mathtt{s}}{[\neg\varphi]\mathtt{s}^i_z}$$

$$(@_i): \frac{[@_i\varphi]\mathtt{s}}{[\varphi]\mathtt{s}^i_{\mathtt{s}_i}} \quad \frac{[\neg@_i\varphi]\mathtt{s}}{[\neg\varphi]\mathtt{s}^i_{\mathtt{s}_i}}$$

Figure 8: n-Dimensional Tableau Rules.

indexed-tableau rules but generalized for n-dimensional modal logic. In the case of $(@_i)$, the rule says to take the ith element of the sequence in question and copy it $(i-1)$ times towards the very first element in the sequence. For instance, the following is a proof of $\Diamond_3 p \rightarrow \Box_2\Box_1@_3\Diamond_3 p$ in a four-dimensional tableau:

1. $[\neg(\Diamond_3 p \rightarrow \Box_2\Box_1@_3\Diamond_3 p)](1,2,3,4)$
2. $[\Diamond_3 p](1,2,3,4)$
3. $[\neg\Box_2\Box_1@_3\Diamond_3 p](1,2,3,4)$
4. $[p](5,5,5,4)$
5. $[\neg\Box_1@_3\Diamond_3 p](6,6,3,4)$
6. $[\neg@_3\Diamond_3 p](7,6,3,4)$
7. $[\neg\Diamond_3 p](3,3,3,4)$
8. $[\neg p](5,5,5,4)$

$\times$

Items 2 and 3 result from the rule $(\wedge)$ applied to the negated conditional. Item 4 results from applying the (π_3) rule to item 2, and so the index-sequence

$(5, 5, 5, 4)$ appears on the tableau because the index 5 is new, that is, it had no previous occurrences. Items 5 and 6 result from applications of (π_2) and (π_1) to items 3 and 5, respectively. The index-sequence $(6, 6, 3, 4)$ appears in 5 because we needed a new index for $\neg\Box_2$ from item 3, and the index-sequence $(7, 6, 3, 4)$ occurs because we needed a new index for $\neg\Box_1$ from item 5. Finally, 7 results from 6 by an application of $(@_3)$, and so the index 3 is copied down in the index-sequence twice, and 8 results by applying (ν_3), and so any index could have been chosen to compose its index-sequence. The index-sequence $(5, 5, 5, 4)$, therefore, is chosen so that the tableau closes.

2.1 Soundness

Definition 2.3 (Satisfiability) Let F be a set of indexed formulas. We say F is *satisfiable* in a model $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, where $W = S^n$, if there is a function f assigning to each single index $\mathtt{s}_i$ occurring in a sequence $\mathtt{s}$ in F a possible world $f(\mathtt{s}_i) \in S$, and, where g is a function such that $g(\mathtt{s}) = (f(\mathtt{s}_1), ..., f(\mathtt{s}_n))$,

- If $[\varphi]\mathtt{s} \in F$, then φ is true at $g(\mathtt{s})$, i.e. $\mathfrak{M}, g(\mathtt{s}) \vDash \varphi$.
- If the index-sequences $\mathtt{s}$ and $\mathtt{s}^i_z$ are in F, then $g(\mathtt{s})\mathcal{R}_{\Box_i} g(\mathtt{s})^i_{f(z)}$. If, moreover, $z = \mathtt{s}_i$, then also $g(\mathtt{s})\mathcal{R}_{@_i} g(\mathtt{s})^i_{f(z)}$.

Definition 2.4 A tableau branch $\mathfrak{b}$ is *satisfiable* if the set of indexed formulas on it is satisfiable in some model, and a tableau is *satisfiable* if some branch of it is satisfiable.

It follows immediately from the definitions that a closed tableau is not satisfiable. Then the following lemma can be established by induction on formulas:

Lemma 2.1 *If one of the rules is applied to a tableau that is satisfiable in an n-dimensional sequence model $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, it results in another tableau satisfiable in $\mathfrak{M}$.*

Theorem 2.1 (Soundness) *If φ has an n-dimensional tableau proof, then φ is valid on the class of n-dimensional sequence frames F.*

Proof. Suppose φ has a tableau proof, in which case there is a closed tableau, $\mathcal{T}$, beginning with $[\neg\varphi]\mathtt{s}$, where $\mathtt{s}$ is the sequence $(1, 2, ..., n)$. For a contradiction, assume that φ is not valid. Thus, there is an n-dimensional sequence model $\mathfrak{M} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, where $W = S^n$, for some

non-empty set S, such that $\mathfrak{M}, \sigma \nvDash \varphi$ for some sequence $\sigma \in W$. Define a function f such that for each $\mathtt{s}_i \in \mathtt{s}$ and $\sigma_i \in \sigma$, $f(\mathtt{s}_i) = \sigma_i$, such that $g(\mathtt{s}) = (f(\mathtt{s}_1), ..., f(\mathtt{s}_n)) = \sigma$. Then $\{[\neg\varphi]\mathtt{s}\}$ is satisfiable in $\mathfrak{M}$. Moreover, since the one point tableau $[\neg\varphi]\mathtt{s}$ is satisfiable in $\mathfrak{M}$, $\mathcal{T}$ is also satisfiable in $\mathfrak{M}$, by Lemma 2.1. Therefore, $\mathcal{T}$ is both closed and satisfiable, which is impossible, whence $\mathfrak{M}, \sigma \vDash \varphi$, as desired. □

2.2 Completeness

We establish completeness by constructing a systematic tableau proof procedure in the style of Fitting (1983, ch. 8) producing a tableau proof in case there is one, and directing us to a countermodel otherwise. As it is shown below, this will also give us a decision procedure for the validities.

2.2.1 Systematic proof procedure

For the purposes of a systematic proof procedure, we do not want to apply the rules for a single occurrence of an indexed formula more than one time, so we need to make sure that each occurrence of an indexed formula is used only once. This is not difficult to keep track of, as we can just introduce a device to declare formulas *used*—a check mark, for instance, as in Smullyan (1968). The procedure is defined by stages, and for stage $n = 1$, introduce $[\neg\varphi]\mathtt{s}$ at the tableau's root, where $\mathtt{s}$ is the sequence $(1, 2, ..., n)$. Next, suppose n stages of the procedure have been completed. If the tableau is closed, or every occurrence of an indexed formula is used, then stop. If, on the other hand, the tableau remains open, then we proceed to stage $n + 1$ as follows: take the highest occurrence of an indexed formula in the tree, say, $[\psi]\mathtt{s}$, that is not yet used.[17] Now for each open branch $\mathfrak{b}$ through the occurrence of $[\psi]\mathtt{s}$, do the following:

I If $[\psi]\mathtt{s}$ is atomic or a negation thereof, then declare it used. This ends stage $n + 1$.

II If $[\psi]\mathtt{s}$ is $[\neg\neg\chi]\mathtt{t}$, add $[\chi]\mathtt{t}$ to the end of $\mathfrak{b}$.

III If $[\psi]\mathtt{s}$ is $[(\zeta \wedge \chi)]\mathtt{t}$, add both $[\zeta]\mathtt{t}$ and $[\chi]\mathtt{t}$ to the end of $\mathfrak{b}$ (analogously for the other conjunctive cases).

[17] If there are multiple occurrences of unused formulas at the same level, chose the unused formula occurring on the leftmost branch.

IV If $[\psi]\mathtt{s}$ is $[(\zeta \vee \chi)]\mathtt{t}$, split the end of $\mathfrak{b}$ into $\mathfrak{b}_1$ and $\mathfrak{b}_2$, adding $[\zeta]\mathtt{t}$ and $[\chi]\mathtt{t}$ to the end of $\mathfrak{b}_1$ and $\mathfrak{b}_2$, respectively (analogously for the other disjunctive cases).

V If $[\psi]\mathtt{s}$ is $[\Diamond_i \chi]\mathtt{t}$, take the smallest integer, z, which is new to $\mathfrak{b}$, and add $[\chi]\mathtt{t}^i_z$ to the end of $\mathfrak{b}$ (analogously for $\neg\Box_i$).

VI If $[\psi]\mathtt{s}$ is $[\Box_i \chi]\mathtt{t}$, for every index z occurring on the branch, add $[\chi]\mathtt{t}^i_z$ to the end of $\mathfrak{b}$ (analogously for $\neg\Diamond_i$), and then add a fresh occurrence of $[\Box_i \chi]\mathtt{t}$ to the end of $\mathfrak{b}$. If, however, all possible $[\chi]\mathtt{t}^i_z$ already occur on the branch, do nothing (not even checking off the original boxed formula).

VII If $[\psi]\mathtt{s}$ is $[@_i \chi]\mathtt{t}$, add $[\chi]\mathtt{t}^i_{\mathtt{t}_i}$ to the end of $\mathfrak{b}$ (analogously for $\neg@_i$).

Once this procedure is completed for each open branch $\mathfrak{b}$ through $[\psi]\mathtt{s}$, mark that formula as used. This completes stage $n+1$. Now, either the systematic procedure resulted in a closed tableau, producing a proof; the procedure terminated, producing an open branch; or it does not terminate, producing a possibly infinite open branch.

Let $\mathfrak{b}$ be a *complete* open branch of a tableau $\mathcal{T}$ if any application of a rule to an indexed formula occurring on the branch would only introduce indexed formulas already occurring on it. In order to prove completeness we show that if $\mathfrak{b}$ is any complete open branch of a tableau, then $\mathfrak{b}$ is satisfiable in an n-dimensional sequence model constructed out of the formulas and index-sequences occurring on $\mathfrak{b}$. Once completeness is established we come back to the case where the systematic procedure does not terminate.

Definition 2.5 We define a frame $\mathfrak{F}_\mathfrak{b} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ induced by $\mathfrak{b}$ as follows. Let the set of indices in $\mathfrak{b}$ be $S = \{x \mid \text{for some } \mathtt{s} \in \mathfrak{b} \text{ and some } i, x = \mathtt{s}_i\}$, and let $W = S^n$. For every $\mathtt{s}, \mathtt{s}^i_z \in W$, set $\mathtt{s}\mathcal{R}_{\Box_i}\mathtt{s}^i_z$, and when $z = \mathtt{s}_i$, set $\mathtt{s}\mathcal{R}_{@_i}\mathtt{s}^i_z$, too. A model $\mathfrak{M}_\mathfrak{b} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, based on $\mathfrak{F}_\mathfrak{b}$, is defined as follows: for each $p \in \mathsf{PROP}$, let $\mathtt{s} \in V(p)$ provided that $[p]\mathtt{s}$ occurs on $\mathfrak{b}$; otherwise set $\mathtt{s} \notin V(p)$.

It is simple to verify that the frames defined above are in fact n-dimensional sequence frames. The following lemma is then established by induction on formulas:

Lemma 2.2 (Truth lemma) *Let $\mathfrak{b}$ be a complete open branch of a tableau. Then, let $\mathfrak{F}_\mathfrak{b} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$ be a frame induced by $\mathfrak{b}$, and $\mathfrak{M}_\mathfrak{b} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$ a model based on $\mathfrak{F}_\mathfrak{b}$. For every indexed formula, $[\varphi]\mathtt{s}$,*

$$[\varphi]\mathtt{s} \text{ occurs on } \mathfrak{b} \Leftrightarrow \mathfrak{M}_{\mathfrak{b}}, \mathtt{s} \vDash \varphi.$$

Theorem 2.2 (Completeness) *If φ is valid on the class of n-dimensional sequence frames F, then φ has an n-dimensional tableau proof.*

Proof. We prove the contrapositive. Suppose φ does not have a tableau proof, in which case any tableau $\mathcal{T}$ beginning with $[\neg\varphi]\mathtt{s}$ remains open. Let $\mathfrak{b}$ be a complete open branch of $\mathcal{T}$ such that $[\neg\varphi]\mathtt{s}$ occurs on $\mathfrak{b}$. By Lemma 2.2, there is an n-dimensional sequence frame $\mathfrak{F}_{\mathfrak{b}} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n})$, induced by $\mathfrak{b}$, and index-sequence $\mathtt{s} \in W$, such that for an n-dimensional sequence model, $\mathfrak{M}_{\mathfrak{b}} = (W, (\mathcal{R}_{\Box_i})_{1\leq i\leq n}, (\mathcal{R}_{@_i})_{1\leq i\leq n}, V)$, based on $\mathfrak{F}_{\mathfrak{b}}$, we have $\mathfrak{M}_{\mathfrak{b}}, \mathtt{s} \nvDash \varphi$. Consequently, φ is not valid on the class of n-dimensional sequence frames F. □

n-dimensional tableaux can be easily adapted for n-dimensional sequence frames with distinguished elements. For an index x, let $\overline{x}$ be that index-sequence $\mathtt{s} = (x_1, ..., x_n)$ for which $x_i = x$ for all $1 \leq i \leq n$. Then a tableau proof in this case for a sentence φ is a closed tableau for $[\neg\varphi]\overline{x}$. The soundness and completeness results above (as well as the decidability argument below) are also straightforward to adapt.

2.3 Decidability

In this section we prove decidability for $\mathbf{S}@_n$ by showing that the systematic tableau procedure used to prove completeness is in fact a decision procedure. For this purpose we can adapt the argument in Fitting (1983: 410-413). Say we are attempting a proof by way of the systematic procedure described above. However, we now modify the procedure so that no branch contains multiple occurrences of the same indexed formulas: if the procedure says we should add an indexed formula to the end of a branch that already contains an occurrence of it, refrain from adding the repeated occurrence of that indexed formula. This involves a simple modification to our step VI above—which is completely analogous to Fitting's (1983: 411). Now, we prove that the (modified) systematic procedure we have described guarantees termination for n-dimensional tableaux, whereby they either deliver a proof or a countermodel.

Proposition 2.1 (Termination) *Let φ be an $\mathcal{L}_n^{@}$-formula. A systematic attempt to prove φ terminates.*

Proof. For a contradiction, suppose a systematic attempt at proving φ never terminates. Then there is a sequence $\mathcal{T}_0, \mathcal{T}_1, ...$ of tableaux which successively properly extend each other. Let $\mathcal{T}$ be their limit, i.e.

$$\mathcal{T} = \bigcup_{n \geq 0} \mathcal{T}_n.$$

Then $\mathcal{T}$ is a finitely branching infinite labeled tree. Hence, it follows by König's lemma that $\mathcal{T}$ has an infinite branch, say, $\mathfrak{b}$. Consider the set $\Sigma = \{\mathtt{s} \mid [\psi]\mathtt{s} \in \mathfrak{b}$ for some formula $\psi\}$. Because $\mathfrak{b}$ is infinite, the set Σ must be infinite, too. Otherwise, the set of indexed formulas, say, Φ, occurring on $\mathfrak{b}$ would be finite, since $\Phi \subseteq sb(\varphi) \times \Sigma$, where $sb(\varphi)$ is the set of subformulas of φ, which is itself finite. Moreover, because no multiple occurrences of the same indexed formulas are allowed, it is not possible for some index-sequence in Σ to occur infinitely many times postfixing a formula on $\mathfrak{b}$.

Let the *height of an index-sequence* $\mathtt{s}$, written $height(\mathtt{s})$, be the greatest natural number occurring in $\mathtt{s}$, and for every $k \in \mathbb{N}$, define $\Sigma_k = \{\mathtt{s} \mid \mathtt{s} \in \Sigma$ and $height(\mathtt{s}) = k\}$. Then we let the set Σ of all index-sequences on $\mathfrak{b}$ be the union of all sets Σ_k. There are two cases that could allow Σ to be infinite:

(1) For some height j, $card(\Sigma_j) = \aleph_0$. But this is impossible. Consider the set $S_j = \{\mathtt{s} \mid height(\mathtt{s}) = j\}$ of all possible sequences (with length n) of height j. Note that $card(S_j) = j^n - (j-1)^n$. Since $card(S_j)$ is finite, and $\Sigma_j \subseteq S_j$, there is no height j such that $card(\Sigma_j) = \aleph_0$.[18]

(2) For each height j, $card(\Sigma_j) < \aleph_0$. Because Σ is infinite, and Σ is the union of all Σ_k, each of which is of finite cardinality, there must be infinitely many heights j such that $\Sigma_j \neq \emptyset$. Let the *modal degree* of a formula occurring on $\mathfrak{b}$ be the number of modal operators occurring in that formula. Because the modal degree of formulas gets smaller as the index-sequences increase in height, there is a height k such that any formula on the branch $\mathfrak{b}$ postfixed by an index-sequence of that height has modal degree 0. And since the modal rules do not apply in this case, there cannot be further index-sequences on $\mathfrak{b}$ with height $> k$, in which case there are not infinitely many heights j such that $\Sigma_j \neq \emptyset$, a contradiction. □

Corollary 2.1 (Decidability) $\mathbf{S@}_n$ *is decidable.*

Proof. Immediately from the fact that systematic tableau constructions always terminate in a finite number of steps. □

[18]The original argument for (1) was much more complicated and, in fact, unnecessary, in light of $card(S_j)$ being finite, as a referee very helpfully pointed out.

Conclusion

We have developed and proved several results about n-dimensional sequence modal logics with actuality operators. These are natural generalizations of two-dimensional modal logics known in the philosophical literature as logics for a priori knowledge, necessity, and actuality. In particular, the structures investigated here were shown to be extensions to any finite dimension of the structures studied in Fritz (2014) for the system **2Dg**. The completeness argument from Fritz was also generalized to any finite dimension, and it can now be seen as a special case of the completeness proof presented in §1.3 by just setting $n = 2$ (given that $@_1$ is directly eliminable from the language). Additionally, we have developed sound and complete tableau calculi for the logics herein considered, and used these to show decidability by means of a systematic tableau construction. There are, of course, many questions left open by the present paper which can be settled by future research, namely, questions in proof-theory, model-theory, and even in complexity theory. Some examples of these include the following:

- Fritz's axiom system **2Dg** was naturally generalized for any arbitrary dimension, and the tableau calculi presented here also involved a natural generalization of tableaux for basic modal logic. Is there a natural generalization for n dimensions of the hypersequents developed in Restall (2012) for two-dimensional modal logic? What about different proof systems for modal logic such as natural deduction systems?

- What are the n-dimensional modal logics generated with multiple actuality operators such as A and D? There are many possibilities here, including classes of frames defined with a single distinguished point $z \in S$, and so formulas in the scope of $\mathsf{D}@_n$ would be evaluated relative to the generalized diagonal point $(z, ..., z)$; or even frames with a distinguished point $(z, ..., z)$ for a language containing A, and so any formula in the scope of A would be evaluated relative to that point. It is expected that the main theorems proved here, such as completeness and decidability, would transfer to these cases, although it would be interesting to compare the expressive power of such languages relative to distinct notions of validity. One relevant question would be under which circumstances these actuality operators are eliminable from their respective languages, that is, whether the eliminability of the actuality operator, for instance, from the basic (one-dimensional) modal language when real-world valid-

ity is assumed (see, for example, Hazen et al. (2013)) carries over to n-dimensions.

- Antonelli and Thomason (2002) proved that adding propositional quantifiers to a modal logic with two **S5** modalities results in a system that is (recursively) intertranslatable with full second-order logic. What about in this framework? Even though $\Box_1$ is an equivalence relation, any $\Box_i$, for $i > 2$, is not. So, in case there is a translation, it has to be modified accordingly, for the procedure described by Antonelli and Thomason does not seem to generalize for $\mathbf{S}@_2$ (or $\mathbf{S}@_n$) with propositional quantification. Still, is there any recursive translation with full second-order logic, such as in Antonelli and Thomason's case, that shows second-order $\mathbf{S}@_2$ and hence $\mathbf{S}@_n$ $(1 < n)$ to be undecidable?

- Even though the validity problem for n-dimensional sequence modal logics was shown to be decidable, what is its complexity?

- Is the sequence modal logic of ω-dimensions complete or decidable?

- Standefer (forthcoming) investigates relevant logics with the actuality operator. It would be interesting to see the relevant counterparts of multidimensional modal logics with, possibly, many actuality operators.

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