Wittgenstein on Pseudo-Irrationals, Diagonal Numbers and Decidability

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In his early philosophy as well as in his middle period, Wittgenstein holds a purely syntactic view of logic and mathematics. However, his syntactic foundation of logic and mathematics is opposed to the axiomatic approach of modern mathematical logic. The object of Wittgenstein's approach is not the representation of mathematical properties within a logical axiomatic system, but their representation by a symbolism that identifies the properties in question by its syntactic features. It rests on his distinction of descriptions and operations; its aim is to reduce mathematics to operations. This paper illustrates Wittgenstein's approach by examining his discussion of irrational numbers.

1 Tractarian heritage

In the *Tractatus*, TLP for short, Wittgenstein distinguishes between operations and functions. As do Russell and Whitehead in the *Principia Mathematica*, PM for short, he uses "functions" in the sense of "propositional functions", which are representable by symbols of the form φx within a logical formalism. In contrast, the concept of operation is Wittgenstein's own creation. According to Wittgenstein, the "basic mistake" of the symbolism of PM is the failure to distinguish between propositional functions and operations (WVC, p. 217 and TLP 4.126). In this respect, the syntax of PM suffers from the same deficiency as the syntax of ordinary language. Wittgenstein distinguishes between functions and operations by the criterion of the possibility of iterative application, TLP 5.25f.:

> (Operations and functions must not be confused with each other.)
> A function cannot be its own argument, whereas an operation can take one of its own results as its base.

Due to its possible iterative application, an operation generates a series of internally related elements. This series is defined by an initial member, η, and an operation, $\Omega(\xi)$, that must be applied to generate a new member from a previous one ξ. The form of such a definition is $[\eta, \xi, \Omega(\xi)]$. This series is not defined as an "infinite extension" but by the iterative application of an operation that determines forms. The natural numbers, for example, are defined by the operation $+1$. Starting with 0 as initial member, this yields the series 0, 0+1, 0+1+1 etc., which is denoted by $[0, \xi, \xi + 1]$, cf. TLP

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6.03. According to Wittgenstein's point of view numbers are forms defined by operations (cf. WVC, p. 223). They are neither objects denoted by names nor classes or classes of classes described by functions. While functions determine the extension of a property independent of its symbolic representation, operations determine the syntax of symbols. Operations do not refer to anything outside the symbols; they determine formal (internal) properties rather than material (external) properties. Operations do not state anything, but determine how to vary the form of their bases (inputs) without contributing any content. In contrast, functions, e.g., "x is human," state that their arguments have some property, which is not determined by the symbol of the arguments. A function determines an extension of objects, namely the "totality" or class of objects that satisfy the function.

Operations are internally related, they can "counteract the effect of another" and "cancel out another" (TLP 5.253); they form a system. In TLP Wittgenstein reconstructs so called "truth functions" such as negation, conjunction, disjunction and implication as "truth operations". They form the system of logical operations. Likewise, he understands addition, multiplication, subtraction and division as a system of "arithmetic operations". In both cases, this forces significant changes in the traditional symbolism of logic and arithmetic. In logic, he invents his ab-notation, in which the truth operators are not represented by $\neg, \wedge, \vee$ or $\rightarrow$ but by ab-operations, which assign a- and b-poles to a- and b-poles (cf. e.g., CL, letters 28, 32 NL, p. 94-96, 102, MN, p. 114-16 and TLP 6.1203). By this he intends to overcome within propositional logic the "basic mistake" of PM in failing to distinguish symbolically between operations and functions. In arithmetic he defines natural numbers by operations, cf. TLP 6.02-6.04, and indicates a symbolism of primitive arithmetic wholly resting on operations (cf. TLP 6.24f.). He explicitly opposes this to the Frege's and Russell's program to reduce mathematics to a "a theory of classes" (TLP 6.031), these classes being defined by propositional functions.

Wittgenstein called for a symbolism based on operations as a counter-program to Frege's and Russell's logicism. This still holds for his middle period. Instead of his peculiar term "operation," he frequently uses the common expression "law," and instead of the technical term "propositional function," he uses the less specific expression of "description". Yet, he still claims that mathematics is dealing with systems, operations or laws and not with totalities, functions or descriptions (cf. e.g., WVC, p. 216f. or MS 107, p. 116). Likewise, he claims that "the falsities in philosophy of mathematics" are based on a confusion of the "internal properties of a form", which are determined by operations, and "properties" in terms of material properties of daily life, which are identified by propositional functions, cf. PG, p. 476. He also calls the view that bases mathematics on functions the "extensional view" whereas he professes an "intensional view" that identifies mathematical properties by syntactic properties of an adequate symbolic representation (PG, p. 471-474, RFM, V, §34-40).

In the following we go on to illustrate Wittgenstein's intensional view in his intermediate (1929-1934) discussion of irrational numbers. Finally, we will apply this discussion to diagonal numbers, as well as to the notions of enumerability, decidability and provability. We hereby want to address two challenges faced by Wittgenstein's program: (i) How to apply it to other parts of mathematics besides primitive arithmetic? (ii) How to relate it to the basic notions and impossibility results of modern mathematical logic?

2 Irrationals

2.1 Cauchy sequences

Irrationals are customarily defined as equivalence classes of identical Cauchy sequences. A Cauchy sequence is an *infinite sequence* of rational numbers $a_1, a_2, \ldots$ such that the absolute difference $|a_m - a_n|$ can be made less than any given value $\epsilon > 0$ whenever the indices m, n are taken to be greater than some natural number k. Two Cauchy sequences $a_1, a_2, \ldots$ and $a'_1, a'_2, \ldots$ are identical if and only if for any given $\epsilon > 0$ there is some natural number k such that $|a_n - a'_n| < \epsilon$ for all n greater than k. The idea behind this definition is that all methods approximating the "true expansion" of an irrational number must once result in the same expansion up to a certain digit. For example, the methods illustrated in tables 1 and 2 both approximate the true decimal expansion of $\sqrt{2}$ in a plain manner.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
$x^2 < 2$	1			1.25	1.375		1.40625		1.4140525
$x^2 > 2$		2	1.5			1.4375		1.421875	

Table 1: Method 1

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
$x^2 < 2$	1		1.4		1.41		1.414		1.4141	
$x^2 > 2$		2		1.5		1.42		1.415		1.4142

Table 2: Method 2

At some point the methods come up with identical decimal expansions up to a certain digit. For example, from a_9 on both sequences begin with 1.41. Thus, going further and further one approximates more and more "the" expansion of the irrational number. However, no finite sequence will ever represent the "true expansion," as it is the limit of all sequences approximating it; the "true expansion" is beyond all finite sequences – it is infinite.

With respect to Wittgenstein's point of view, it is important to note that these methods of approximation do not generate the next digits by iteration. Instead, at any step it must be checked whether the square of the result is < 2 or > 2.

2.2 Wittgenstein's critique

Wittgenstein's main critique of the definition of irrational numbers in terms of Cauchy sequences is that this definition does not provide an identity criterion, which decides the identity of two real numbers (PR §186, 187, 191, 195). The problem is that, on the standard conception of irrational numbers as infinite sequences of rational numbers, for any infinite sequence s there are infinite many sequences that are identical with s up to a certain digit k. However, the definition does not provide a method to specify some upper bound for k in comparing two arbitrary real numbers. Thus, no finite comparison is sufficient to decide whether two arbitrary sequences are identical. The definition has it that the "true expansion" lies beyond all finite sequences. Therefore, it provides only a sufficient criterion for a negative answer but no sufficient criterion for a positive answer to the question of identifying arbitrary real numbers. In this respect, we have the same situation as in the case of determining within a traditional logical calculus whether some formula of first order logic is not a theorem.

One might reply to this critique that one cannot claim the decidability of things that simply are not decidable; the nature of the real numbers as infinite sequences implies that one cannot decide upon the identity of two real numbers. However, in fact it is from the purported definition that the problem arises, and it is not carved in stone that this indeed captures the "nature" of real numbers. According to Wittgenstein's analysis the definition is nothing but a consequence of the extensional view of modern mathematics. This spuriously takes the designations of real numbers by ordinary language as *descriptions* of everyday properties, which determine a certain extension. For example, in the case of $\sqrt{2}$ one wrongly analyses the ordinary explanation in terms of "the number that when multiplied by itself is identical with 2" as a description of a material, non-symbolic property. This property is then conceived as being satisfied by the "true infinite expansion", which is approximated by multiplying finite sequences with themselves and comparing the result to 2. In order to come to understand Wittgenstein's point of view, it is crucial to recognize that there is an alternative to this conception that refers to known mathematics. According to this point of view, real numbers are not defined by extensions, but by laws in the sense of Wittgenstein's operations.

2.3 Wittgenstein's alternative

In order to come to understand Wittgenstein's position one must recognize that he rejects methods of approximation such as the above illustrated methods 1 and 2. Although these kinds of methods of approximation might be called "laws," they are not "laws" in terms of operations. They are not operations because they do not generate a sequence by iteration. How to go on does not simply depend on the previous members but on a comparison between the last member and some condition. For example, at each stage in the development of the decimal expansion of $\sqrt{2}$, one must consider whether squaring the last member is greater or smaller than 2. This method is incompatible with Wittgenstein's purely syntactic foundation of mathematical properties. In his program, any sequence must be definable by an operation that determines nothing but the syntax of the members of the sequence. Only in this way is the property constituting the sequence reduced to an internal property of forms that can be identified by the symbolic features of the members of the series.

Wittgenstein's well known rejection of "arithmetical experiments" is based on his requirement to define sequences by syntactic means alone, PR §190:

> In this context we keep coming up against something that could be called an "arithmetical experiment". Admittedly the data determine the result, but I can't see *in what way* they determine it (cf. e.g., the occurrences of 7 in π.) The primes likewise come out from the method for looking for them, as the results of an experiment. To be sure, I can convince myself that 7 is a prime, but I can't see the connection between it and the condition it satisfies. – I have only found the number, not generated it. I look for it, but I don't generate it. I can certainly see a law in the rule which tells me how to find the primes, but not in the numbers that result. And so it is unlike the case $+\frac{1}{1!}, -\frac{1}{3!}, +\frac{1}{5!}$ etc., where I can see a law *in the numbers*.
> I must be able to write down a part of the series, in such a way that you can *recognize* the law.
> That is to say, no *description* is to occur in what is written down, everything must be represented.
> The approximations must themselves form what is *manifestly* a series.
> That is, the approximations themselves must obey a law.

The series of primes is Wittgenstein's paradigm of a series that cannot be generated by an operation. Although operations are available to generate an infinite series of primes, no operation is known to generate the primes in a certain order that ensures that *all* primes are enumerated. In his detailed discussions of primes in other places, Wittgenstein draws the consequence that we still lack of a clear concept of "the" primes. All we have is a concept of what "a" prime is, which allows us to decide whether a given number is prime or not (PR §159, 161, cf. (Lampert 2008)). For the same reason, he rejects the definition of a real number P as the dual fraction with a_n = 1 if n is prime and $a_n = 0$ otherwise, cf. PG, p. 475. This definition does satisfy the definition of real numbers by Cauchy sequences, but it does not satisfy Wittgenstein's criterion of being definable by an operation. In the quoted passage, Wittgenstein emphasizes that we do have a method to look for the next prime: we go through the series of natural numbers and decide one by one whether each member satisfies the condition to be divisible only by 1 and itself. However, this method does not satisfy his standard of a definition by operation. As long as we are not able to reduce the property of being a prime to some operation generating the series of primes by iteration, "we can't see the connection" between the members of the series and the condition they satisfy: we cannot "recognize the law" *in the series*. The problem is the same as with the above illustrated methods of approximating $\sqrt{2}$. Instead of generating the next member by iteration, we must decide whether some condition is satisfied or not in order to find the next member.

Wittgenstein's reference to the series of primes as an illustration of arithmetical experiments demonstrates that his concept of operation is not equivalent to that of primitive recursive function. Primes are definable by a primitive recursive function, but not by an operation.[1] Iteration in the case of operations means that the output of the n-th application of an operation is itself the input of the $n+1$-th application of the very same operation. In contrast, recursion in the case of primitive recursive functions means that the value of a primitive recursive function f for the successor of n, $S(n)$, is defined by referring to the value of the very same function f for n. This does not imply that the values of f are themselves their arguments. This is only true in case of the successor function, which itself is primitive recursive. However, the identity function, e.g., $I(x) = x$, and the zero function, $Z(x) = 0$, on which the definition of primitive recursive functions are based, are *functions* and do not define a series by iterative application. The same holds for primitive recursive characteristic functions. They have the form "$f(x) = 0$ if $\varphi(x)$ and $f(x) = 1$ otherwise". In Wittgenstein's terms, characteristic functions are a paradigm of "descriptions" and not of operations. In contrast, any iteration by applying operations has the form $a_n = \Omega'\vec{a_i}$, where $\vec{a_i}$ stands for members previous to a_n. For example, the series of Fibonacci numbers is defined by $a_n = a_{n-2} + a_{n-1}$. Recursion in the case of primitive recursive functions is part of a strategy of defining primitive recursive functions, whereas operations are not *defined* by iteration but *applied* iteratively. They are defined by some purely syntactic variation that generates a formal series of systematically varied members if iteratively applied. In the case of Fibonacci numbers, this operation consists of adding the last two members. Starting from 0 and 1, this generates the series $0, 1, 0+1, 1+(0+1), (0+1)+(1+(0+1)), (1+(0+1))+((0+1)+(1+(0+1)))$ etc.[2]

If not even primitive recursive functions satisfy Wittgenstein's standards of a purely syntactic foundation of mathematics, this causes doubts whether his programme is realizable at all. Likewise, his rejection of arithmetical experiments and his claim to "recognize the law" in the series has caused trouble. The decimal sequences of irrationals do not satisfy Wittgenstein's demand for sequences that manifestly obey a law. Do not irrationals contradict Wittgenstein's claim from their very nature? Thus, it seems unclear how Wittgenstein's point of view can even do justice to such basic irrational numbers as π and $\sqrt{2}$ (cf., e.g., (Redecker 2006), p. 212).

However, these problems only arise if one overlooks the fact that the possibility of definitions by operations depends on the mode of representation. In case of irrationals, the syntactic features of the decimal system are responsible for their "lawless" representation. However, this kind of representation is not essential; it obscures their lawful nature instead of revealing it. In MS 107, p. 91 Wittgenstein writes (translated by T.L.):

> The procedure of extracting $\sqrt{2}$ *in the decimal system*, e.g., is an arithmetical experiment, too. However, this only means that this procedure is not completely essential to $\sqrt{2}$ and a representation must exist that makes the law recognizable.

To see the connection between the members of a sequence representing a real num-

[1] The question in what sense Wittgenstein characterizes real numbers as "laws" is thoroughly discussed in the literature (cf. (Da Silva 1993), (Frascolla 1994), p.85-92, (Marion 1998), (Rodych 1999) and (Redecker 2006), chapter 5.2). However, the main reason why the identification of laws with Wittgenstein's notion of operations seemed to be insufficient to most commentators is that *operations* in Wittgenstein's sense were not distinguished sharply from the notion of primitive recursive *functions*.

[2] Brackets are merely introduced to identify a_{n-2} and a_{n-1}.

ber and the condition or property that these members satisfy, one must refer to an equivalence transformation that reduces this property to an internal property of forms. There is no equivalence transformation between $\sqrt{2}$ and a decimal number. This already shows that it is impossible to represent $\sqrt{2}$ by the decimal system; whatever decimal number one generates, it cannot be identical with $\sqrt{2}$ – referring to "infinite extensions" is just another expression of this deficiency. However, using the representation by continued fractions, it is possible to represent $\sqrt{2}$ by an operation, cf. MS 107, p. 126 (translated by T.L., cf. MS 107, p. 99):

> [...] in $\frac{1}{2}, \frac{1}{2+\frac{1}{2}}, \frac{1}{2+\frac{1}{2+\frac{1}{2}}}$ etc. one can recognize the law one cannot recognize in the decimal development.

The connection between the property of $\sqrt{2}$ as "the number that multiplied with itself is identical with 2" and its definition by its continued fraction is due to equivalence transformation:

$$
\begin{array}{ll}
x^2 = 2 & \mid \sqrt{\ } \\
x = \sqrt{2} & \mid a = 1 + (a-1) \\
x = 1 + (\sqrt{2} - 1) & \mid a = \frac{1}{\frac{1}{a}} \\
x = 1 + \frac{1}{\frac{1}{\sqrt{2}-1}} & \mid \frac{1}{a-b} = \frac{a+b}{a^2-b^2} \\
x = 1 + \frac{1}{\frac{\sqrt{2}+1}{\sqrt{2}^2-1^2}} & \mid a = \frac{a}{\sqrt{2}^2-1^2} \\
x = 1 + \frac{1}{\sqrt{2}+1} & \mid x = \sqrt{2}, a+b = b+a \\
x = 1 + \frac{1}{1+x} & \mid -1 \\
x - 1 = \frac{1}{1+x} & \mid 1 + x = 2 + (x-1) \\
x - 1 = \frac{1}{2+(x-1)} &
\end{array}
$$

Thus, $\sqrt{2}-1$ is representable by the operation $\frac{1}{2+(x-1)}$. Starting with $1-1$ for $x-1$, the iterative application of this operation yields the series $\frac{1}{2+(1-1)}, \frac{1}{2+\frac{1}{2+(1-1)}}, \frac{1}{2+\frac{1}{2+\frac{1}{2+(1-1)}}}$ etc. This is identical to the series Wittgenstein mentions if one eliminates $+(1-1)$ by an equivalence transformation. In the short notation of regular periodic continued fractions, $\sqrt{2}$ is definable by $[1; \overline{2}]$. A continued fraction of a real number is periodic if and only if the real number is a quadratic irrational (theorem of Lagrange). The notation of continued fraction identifies a common property of quadratic irrationals by a common syntactic feature, and thus shows that this property is an internal property. Other irrational numbers are representable by regular continued fractions that are not periodic but still definable by operations, such as the Euler number $e : [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots]$. Another type of irrational numbers are not definable by operations within regular continued fractions but within irregular continued fractions such as $\frac{4}{\pi} = 1 + \frac{1^2}{2+\frac{3^2}{2+\frac{5^2}{\ddots}}}$. Furthermore, the continued fraction representation for a number is finite if and only if the number is rational. This shows that this mode of representation reveals by its syntactic properties internal properties of numbers that are not identified by the decimal number system. We learn more about "the laws of numbers," their internal structure, by representing them in the notation of continued fractions.

Mathematical proofs reveal this internal structure by equivalence transformations. Consider, for example, the golden ratio. Its representation as a decimal number does not show its exceptional nature. However, by an equivalence transformation resulting in an operation defining a continued fraction, internal properties of the golden ratio are identified by the syntactic features of this adequate representation. This procedure reduces property that the ratio of two quantities a and b is identical to the ratio of the sum of them to the larger quantity a to an operation:

$$\phi = \frac{a}{b} = \frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\phi}. \tag{1}$$

By the operation $1 + \frac{1}{\phi}$, the periodic, regular continued fraction $[1; \overline{1}]$ is defined. By this representation it is proven that the golden ratio is "the most irrational and the most noble number," because these properties are identified by the lowest possible numbers in an infinite regular continued fraction. Furthermore, by this representation it is proven that the ratio of two neighboured Fibonacci numbers converges to the golden ratio. For the Fibonacci numbers are defined by $a_{n+1} = a_n + a_{n-1}$. Thus, with $a = a_n$ and $b = a_{n-1}$ we yield equation (1). The syntax of continued fractions provides symbolic connections that prove certain internal relations between numbers.

The continued fraction representation of any irrational number is unique. Thus, any definition of a real number by an operation (or "induction") defining a continued fraction satisfies Wittgenstein's criterion for representing a real number, MS 107, p. 89 (translation T.L.):

> I want a representation of the real number that reveals the number in an induction such that I have herewith the only proper, unique symbol.

It is by this property of uniqueness that the symbolic representation of irrationals by continued fractions serves as an identity criterion, which allows one to compare irrationals and rational numbers. The principle is the same as in the case of comparing fractions by converting them to fractions with identical denominators. The problem of deciding the identity of numbers results from a deficiency in their representation, allowing for ambiguity.

This does not mean that there must be one and only one proper notation for numbers. Nor does it mean that continued fractions are "the" proper notation of real numbers. Different internal properties of numbers, and herewith different types of numbers, may be identified by different systems of representation. And different types of numbers may be comparable within different modes of representation (cf. MS 107, p. 123). Natural numbers can be compared according to the conventions of the decimal system, fractions are comparable by converting them to fractions with identical denominator, rational numbers and quadratic irrationals are comparable by regular continued fractions etc. Furthermore, new proofs consist of making new symbolic connections. They invent new possibilities of comparing numbers and of revealing their internal relations.

Not all internal relations of a number to other numbers must be revealed within only one notational system. For example, instead of representing π by an irregular continued fraction (as quoted above), $\frac{2}{\pi}$ can also be represented by $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \ldots$ or $\frac{\pi}{2}$ by $\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdot \ldots$. The internal properties of different numbers may call for operations referring to different modes of representation. There need not be a "system of irrational numbers" in the sense as there is a "system of natural numbers" or a "system of rational numbers" (cf. PG, p. 479, RFM, app. 3,§33). As we have seen, only quadratic irrationals are definable by periodic, regular continued fractions, and another type of irrationals is not even definable by regular continued fractions. Different types of irrationals are definable by different kinds of operations within different modes of representation.

According to Wittgenstein's intensional point of view, our mathematical comprehension and knowledge depends on the syntax of mathematical representation. This is not due to psychological reasons. Instead, this is because mathematical proofs make symbolic connections between different modes of representation, and because the solvability of mathematical problems depends on imposing adequate notations. Instead of concluding from a specific, deficient mode of representation the lawless nature of irrational numbers, which makes it impossible to decide upon their identity and which invokes misconceptions such as "infinite extensions," one should look for adequate representations that reveal their lawful nature and make it possible to decide upon their identity. This is done by reducing their properties to operations instead of conceptualizing them in terms of functions. If such a reduction is not available, this means that one does not have a full understanding of the properties in question. We can then only refer to a vague understanding expressed within a deficient, descriptive symbolism. Only by imposing an adequate expression that depicts those properties by its syntactic features, can we be sure that those properties are properly defined.

This approach is in conflict with basic impossibility results of modern mathematical logic, such as the non-enumerability of the irrationals, the undecidability of first-order logic or the incompleteness of logical axiomatizations of arithmetics. This does not mean that Wittgenstein's point of view implies that these results are false in the sense that their negation is true. Instead, his intensional view implies that it does not make sense to speak of "the irrationals" unless an operation is known that allows us to generate them by iteration (and thus to enumerate "the irrationals"). This, of course, does not mean that he claims that such an operation is or must be available. Likewise, his intensional view implies that one cannot speak of decidability or provability in an absolute sense, such that one can say in advance that certain properties of formulae of a certain syntax are not decidable or provable, independent of the syntactic manipulations that might be invented to identify those properties. According to Wittgenstein "being a tautology" ("being true in all interpretations") or "being a theorem" of first order logic is not defined properly unless some sort of equivalence procedure is invented that converts first order formulae to an adequate representation that identifies their logical properties by its syntactic properties. From this point of view, it cannot be said that it is impossible to define such procedures, because the properties in question that are said to be undecidable or unprovable are not represented properly unless such procedures are available. Likewise, from Wittgenstein's point of view the incompleteness of axiomatic systems of arithmetic means in the first place that those systems do not properly represent the properties in question. It does not mean that we know that a certain property, e.g., the provability of a certain formula, holds, but its formal representation is not derivable. Instead, it means that we have a deficient understanding of that property expressed by an inadequate representation. In the following, we will show that this conflict between Wittgenstein's point of view and the impossibility results can all be traced back to his rejection of "descriptions" in terms of characteristic functions as adequate forms to represent real numbers.

3 Pseudo-Irrationals

Wittgenstein illustrates his point of view by providing several definitions of pseudo-irrationals. These are definitions of irrationals in terms of Cauchy sequences. However, contrary to $\sqrt{2}$ or π no reductions to operations of these definitions are available. Thus, according to Wittgenstein there are no irrationals corresponding to those definitions. Besides the above mentioned definition of P as the dual fraction $0.a_1a_2\ldots$ with $a_n = 1$ if n is prime and $a_n = 0$ otherwise, Wittgenstein discusses the following definitions (cf. PG, p. 475):

π': The decimal number $a_1.a_2a_3\ldots$ with $a_na_{n+1}a_{n+2} = 000$ if $a_na_{n+1}a_{n+2} = 777$ in π; otherwise $a_n = a_n$ of π.

F: The dual fraction $0.a_1a_2a_3\ldots$ with $a_n = 1$ if $x^n + y^n = z^n$ is solvable for n $(1 \geq x, y, z \geq 100)$; otherwise $a_n = 0$.

All these definitions are intended to define an irrational number by a characteristic function. In this case, the dots "..." refer to an "infinite extension". Thus, they are ill-defined according to Wittgenstein's standards. They do not identify a number but describe an arithmetical experiment. Wittgenstein emphasizes that even if the characteristic functions become reducible to operations, this does not mean that this shows that the definitions in fact define irrational numbers. Instead, it means that vague definitions that do not identify numbers are replaced with exact definitions that are able to identify numbers. He, for example, considers the situation when Fermat's theorem is proven. Due to his rejection of descriptions, he does not analyse this situation in terms of coming to know the number F that before was only described. Instead, the proof allows one to replace the pseudo-definition of F, which does not identify a number (neither a rational nor an irrational one), with $F = 0.11$, which is a rational number (PG, p. 480). Before, it was not decidable whether "F" denotes a number such that $F = 0.11$ or not; the definition by description simply did not define rules to do this. This demonstrates the lack of meaning that is given to "F" by the previous definition. The proof, if it is valid, makes connections to other parts of mathematics that were not recognized before and thus gives "F" a clear meaning.

Cantor's proof of the non-enumerability of irrational numbers is based on defining a diagonal number by a characteristic function. Given some enumeration of dual fractions between 0 and 1, the proof of the non-enumerability of "all" of them is based upon the following diagonal number $\mathcal{D}$:

$\mathcal{D}$**:** The dual fraction $0.a_1a_2\dots$ with $a_n = 0$ if the n'th digit of the n'th dual fraction is 1; otherwise $a_n = 1$.

To this definition, the same objections apply as to the definitions of P, π' or F: It is a definition by description in terms of a characteristic function. It describes an arithmetical experiment and does not identify a number, which can only be done by an operation. However, such an operation is not available. Thus, it is not meaningful to say that $\mathcal{D}$ is an "irrational number" not occurring in the assumed enumeration of irrationals. This, of course, does not mean that Wittgenstein claims that "the irrationals" are enumerable. Instead, he objects to identifying irrational numbers by non-periodic, infinite decimal or dual fractions. This criterion does not say anything about a certain type of numbers; it only says something about the deficiency of the decimal notation (PG, p. 474). This notation cannot serve as the unique notation for real numbers, as it does not make it possible to decide upon the identity of numbers. Likewise, Wittgenstein objects to the picture of a real number as a "point" on the "line" of real numbers. These items are elements of the extensional view. They arise from treating "is an irrational number" as well as "is a rational number" or "is a natural number" as concepts (propositional functions) identifying certain sets of numbers. This makes it possible to ask about the "cardinality" of those sets. This, in turn, allows one (i) to use "infinite" as a number word and speak of "the infinite number" of objects satisfying some concept, and (ii) to compare the cardinality of sets by coordinating their elements. Finally, from this and the method of diagonalization one comes to speak of sets with a cardinality greater than that of the set of natural numbers. First and foremost, Wittgenstein's criticism is that this conceptual machinery is rather an expression of the extensional view than a description of the nature of numbers (RFM, app. 3, §19). He cuts the roots of (transfinite) set theory by conceptualizing "types of numbers" in terms of "systems" instead of "sets". According to his intensional point of view, the criterion to identify a type of number is the possibility to generate them by an operation. As this implies their enumerability in terms of the iterative application of an operation, it does not make sense to speak of types of numbers that are not enumerable.

According to Church's thesis, the concept of decidability is representable by a primitive recursive characteristic function. Thus, on the basis of an enumeration of first-order logic formulae by their Gödel numbers, the property of being a theorem (or a tautology) is representable by the following number:

T**:** The dual fraction $0.a_1a_2\dots$ with $a_n = 0$ if $\vdash \varphi_n$ (or $\models \varphi_n$); and $a_n = 1$ otherwise.

On the basis of diagonalization, undecidability proofs demonstrate that characteristic functions such as the one defining T cannot be primitive recursive. From Wittgenstein's point of view, these proofs are based upon a confusion of material and formal properties. As a formal property, theoremhood (or being a tautology) is not representable by a characteristic function. Instead, these properties are only represented adequately by a shared syntactic property in an ideal notation. This is illustrated by the representation of tautologies via truth tables or disjunctive normal forms of propositional logic as well as by means of Venn diagrams in monadic first order logic. Wittgenstein's conception calls for equivalence transformations to identify the truth conditions of logical formulae by means of syntactic properties of their proper representation.

This conception differs from the traditional semantics of first-order logic. Presuming an endless enumeration of interpretations $\Im_1, \Im_2, \ldots$, each being either a model or a counter-model of a formula A, one might represent the truth conditions of A according to these interpretations by the following number:

$\theta(A)$**:** The dual fraction $0.a_1a_2\ldots$ with $a_n = 0$ if $\Im_n \models A$ and $a_n = 1$ otherwise.

On the contrary, Wittgenstein's approach calls for a representation of the truth conditions of a formula A that allows one to identify the truth conditions of A without deciding whether single interpretations are models or counter-models of A. Furthermore, the proper representation of first order formulae should reveal the internal relations of non-equivalent logical formulae by making it possible to generate the system of truth conditions by operations. To have an idea of what Wittgenstein envisages, one might think of a systematic generation of reduced disjunctive normal forms of the Quine-McCluskey algorithm[3], that represent all possible truth functions of propositional logic. Likewise, the task of first order logic is to define analogous disjunctive normal forms and procedures for their unique reduction within first order logic. To claim that this is impossible presumes the extensional view that is rejected by Wittgenstein's endeavour.

Likewise, Gödel represents "x is a proof of y" by a primitive recursive function xBy in definition 45 of his incompleteness proof (cf. (Gödel 1931), p. 358). On this basis, he expresses "x is provable" by $\exists y\, yBx$ in definition 46. This is incompatible with Wittgenstein's claim that the internal relation of being provable (derivable) should be defined by operations instead of propositional functions. This, in turn, presumes a proof procedure in term of equivalence transformations to an adequate symbolism that makes such a definition possible, instead of a proof procedure in terms of logical derivations from axioms. The lack of such a definition means a deficiency in the syntactic representation of the formulae in question. According to Wittgenstein's point of view, the conclusion that must be drawn from Gödel's incompleteness proof is to look for a formal representation of arithmetic that is not based upon the concept of propositional function, which is at the heart of any logical formalization.

Wittgenstein's intensional reconstruction of mathematics is not meant to be a "refutation" of the extensional view of modern mathematical logic. Instead, first and foremost it intends to propose a decisive alternative conceptualization of mathematics that radically differs in its foundations. According to him, the fruit of this endeavour should be a clarification of the philosophical problems of modern mathematics that will have the same influence on the increase of mathematics as sunshine on the growth of potato shoots, cf. PG, p. 381.

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[3]Note that the reduced disjunctive normal forms of the Quine-McCluskey algorithm are unique; any equivalent propositional formula is represented by the same reduced disjunctive normal form. Ambiguity only comes into play in the second step of the Quine-McCluskey algorithm that intends to minimize reduced disjunctive normal forms.

Abbreviations

CL: Wittgenstein, Ludwig: *Cambridge Letters*, Blackwell: Oxford 1997.

MN: Wittgenstein, Ludwig: Notes dictated to G.E.Moore in Norway, in: *Notebooks 1914 - 1916*, Blackwell: Oxford 1979, 108–119.

MS 107: Wittgenstein, Ludwig: manuscript 107 according to von Wright's catalogue. Published in *Wittgenstein's Nachlass: the Bergen Electronic Edition*, Oxford University Press: London 2000.

NL: Wittgenstein, Ludwig: Notes on Logic, in: *Notebooks 1914 - 1916*, Blackwell: Oxford 1979, 93–107.

PG: Wittgenstein, Ludwig: *Philosophical Grammar*, Blackwell: London 1974.

PR: Wittgenstein, Ludwig: *Philosophical Remarks*, Blackwell: Oxford, 1975.

RFM: Wittgenstein, Ludwig: *Remarks on the Foundations of Mathematics*, Blackwell: Oxford 1956.

TLP: Wittgenstein, Ludwig: *Tractatus Logico-Philosophicus*, Routledge: London 1994.

WVC: Waisman, Friedrich: *Wittgenstein and the Vienna Circle*, Blackwell: Oxford 1979.

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