

Logical Inquiries into a New Formal System with Plural Reference

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1 Introduction: The Semantics of Natural Language

In this paper we develop a new formal system of logic, which consists of syntactic rules, derivation rules and a model-theoretic semantics. We then make some meta-logical inquiries into the nature of this system, comparing it with the first-order predicate calculus or logic (FOL).

Our formal system is based on an analysis of the semantics of natural language sentences, an analysis which departs in several basic respects from the semantic analyses one finds in the literature. All these use some version of FOL to analyze the semantic structure of natural language sentences; the semantic structure of these sentences, so it is assumed, can be transparently represented by their translation into some version of FOL. By contrast, we think that some semantic features of some natural language sentences *cannot* be captured by means of FOL, and that one distorts the semantic structure of these sentences if one tries to represent it by such translations.

This alternative analysis of natural language semantics, together with a criticism of the analysis suggested by FOL, are developed in detail in Ben-Yami's *Logic & Natural Language* (Ashgate, 2004). We shall mention some

of this book's central claims below, but – given the purpose and space of this paper – we shall not attempt to justify them here. We have to refer the reader to this book for the full justification and development of these claims.

This semantic analysis, which is the basis of our formal system, should also serve to clarify what we tried to achieve by this system's development, and what we *did not*. Usually, when one develops a new formal system of logic, one does that in order to capture some inferences that hitherto one could not capture, and frequently could not even express, in existing formal systems. *This was not our purpose*. Rather, we tried to show that the alternative analysis of the semantics of natural language can serve as the basis for a formal system which is as powerful as some version of FOL, in a sense to be made precise below. We wanted to show that one need not abandon the semantic structure of natural language if one wants to apply a deductive system of FOL's power.

With this in mind we can proceed to a concise presentation of some of the semantic claims made in *Logic & Natural Language*.

FOL distinguishes two kinds of expression which are not variables or logical constants: predicates on the one hand and individual constants (and possibly other closed terms as well) on the other. Individual constants translate the proper-names and other *singular* referring phrases or expressions of natural language, and can thus be said to refer to or designate particulars. Accordingly, FOL can be said to recognize only one kind of referring expressions: *singular* ones (but see the discussion of many-sorted logic below).

Natural language, by contrast, contains *plural* referring expressions as well. These include plural pronouns (in English, 'we', 'you', 'they' and their declined forms), plural demonstratives ('these', 'those'), plural definite descriptions (e.g., 'my children', 'the students'), some phrases that resemble both definite descriptions and proper-names ('the Knights of the Round Table', 'the Simpsons'), and conjunctions and disjunctions of singular and plural referring expressions (e.g., 'Peter and/or Jane', 'Mary and the children'). Such expressions may have other, non-referential uses as well; but they can all be used to refer to several particulars.

The italicized words and phrases below are examples of the referential use of expressions of these kinds:

We saw the Simpsons in the supermarket. *These* are my books.
My children are asleep. *Peter and Jane* should soon be here.

What is involved in plural reference, vis-à-vis singular reference, is straightforward. Whatever is achieved in referring to a single person or thing can be achieved with respect to several persons or things, and we then have plural reference.

When we talk about plural reference we mean referring to more than a single person or thing. We *do not* mean referring to a set with many members, to a complex individual, or to any other variation on these ideas. We mean achieving with relation to more than a single thing what is achieved by reference to a single one.

The great majority of existing attempts to translate sentences that contain plural referring expressions into FOL are reductive, in the sense of trying to analyze such expressions either as singular referring ones, or as involving an implicit structure that contains referring expressions of the singular kind only. But these analyses can be shown to be either mistaken or at least implausible. Moreover, they are not motivated by any linguistic phenomenon, but by the unjustified conviction that FOL must be capable of translating the relevant sentences. Yet FOL cannot adequately represent the semantics of natural language sentences containing plural referring expressions precisely because it lacks such expressions. (Again, for the full development and justification of these claims, and of some of the following, see *Logic & Natural Language*.)

Now, the careful analysis of the functioning of common nouns in natural language shows, that in many cases, *common nouns in quantified noun phrases are plural referring expressions*. For instance, in ‘Some children are asleep’, ‘children’ is used to refer to children. Similarly, in ‘John met several members of my college’, ‘members of my college’ is used to refer to persons, several of which John met. (*N.B.*: It refers not to those met by John, but to *all* members of my college.)

This is in marked contrast with the way FOL translates these expressions. Common nouns are taken to be predicative not only when they function as grammatical predicates, but when they appear in the grammatical subject position as well. Already Frege, and as early as in his *Begriffsschrift* (§12), has translated the subjects in the four Aristotelian quantified sentences by predicates, and several times in his later writings he argued for this analysis.

Let us demonstrate the difference between the two approaches by one standard example. The sentence

- (1) All philosophers are wise

is translated into FOL by the sentence

$$\forall x(\text{Philosopher}(x) \longrightarrow \text{Wise}(x))$$

That is, the expression ‘philosophers’ is seen as contributing to the meaning of the natural language sentence in the same way that ‘wise’ does: they are both *predicative*; they are both used to say something about particulars referred to in some other way. By contrast, on our analysis, ‘philosophers’ in (1) is not predicative but *referential*; it is used to specify which are the particulars about which something is said (in this case, the philosophers). The same applies to the use of ‘philosophers’ in the sentences ‘Some/Seven/Many/Most philosophers are wise’.

The fact that his calculus did not contain plural referring expressions forced Frege to introduce quantification into it in a way that is far different from the way it functions in natural language. For Frege, and in FOL generally, quantifiers are operators that operate on sentential functions; they are second-order concepts. This is not the way quantifiers function in natural language, as we shall now explain.

When we quantify, we refer to a plurality of particulars, and say that specific quantities of them are such-and-such; quantification involves reference to a plurality. Natural language accomplishes this kind of reference by means of plural referring expressions, which designate the plurality, or pluralities, about which something is being said. And by using different expressions, natural language can refer to different pluralities. By contrast, since FOL uses concepts only as predicates, it has no plural referring expressions. The plurality about which something is said by its sentences has to be presupposed, and different sentences cannot specify different pluralities (but see again the notes on many-sorted logic below). In natural language, pluralities are introduced and specified by means of plural referring expressions; in FOL, a plurality, which is unspecified by the sentence, is introduced by presupposing a domain of discourse.

In order to speak of pluralities natural language sentences presuppose no domain of discourse, in the technical sense in which this concept is used in predicate logic semantics. A domain of discourse is a necessary component of the semantics of FOL, which has no parallel in the semantics of natural language. The idea of a domain of discourse may have important applications for formal systems, and we shall use it ourselves in that context below. But one distorts the semantics of natural language if one insists on finding a domain there.

This semantic difference results in a syntactic one as well. If the plurality is referred to by some plural referring expression, the quantifier has to be related in some syntactic way to the plural referring expression that indicates the plurality of which a quantified claim is made. Consequently, in natural language the quantifier is attached to a noun that is used to refer to a plurality, and together they form a noun phrase. However, if no expression is used to refer to a plurality, but the plurality is presupposed by the quantified construction, then the quantifier does not have to be attached to any specific component of the quantified sentence. Consequently, in FOL the quantifier operates on a sentential function.

This alternative semantic analysis of natural language can explain many features of language that create difficulties for attempts to analyze it by means of some version of FOL (including versions that use generalized quantifiers). Among other things, it explains away several alleged ambiguities of the copula; it explains some semantic features of natural kind terms and of empty concepts; it yields a natural classification of quantifiers (classifying ‘many’ and ‘most’, but not ‘more’, as belonging to the same family as ‘every’ and ‘some’); it explains the semantic need for some linguistic devices like an affirmative and negative copulas, active versus passive voice, etc.; and more (cf. [1]).

Although in *Logic & Natural Language* a consistent deductive system for natural language sentences was developed on the basis of this semantic analysis, no attempt was there made to develop a rich artificial language, with rigorous rules for wffs, derivation rules and a model-theoretic semantics. This, as was said above, is our main purpose in this paper, to which we shall now proceed. In doing this we shall also demonstrate that the new analysis can be used as a basis for a formal system which resembles FOL in its power.

A note is in order here on the use of universal and existential sentences below. When we use such sentences in our proofs, we adopt the conventions customary in mathematics. In particular, we use ‘Every A is B ’ as short for ‘If anything is A , then it is B ’. This is meant to enable a more fluent reading. Since we use these conventions consistently, the differences between this way of using sentences and the way they are commonly used in natural language should not bother us.

2 The Definitions of Our System

This part includes our definitions of a formal language and of a formula. It also includes our definitions of truth in a model and our deductive system.

2.1 Some basic definitions

Definition 1 (Formal Language). *A formal language L is a disjoint union of nine sets: \mathcal{P} – a set of one-place predicates, one of which is the predicate *Thing*; \mathcal{R} – a set of relation-signs or many-place predicates (to every one of which we assign a natural number $n > 1$, called its number of places); \mathcal{S} – a denumerable set, whose members are called singular referring expressions (or: SREs); $\mathcal{A} = \{a, a_1, a_2, \dots\}$ – the set of anaphors; $\{1, 2, 3, \dots\}$ – the set of indices; $\{\wedge, \vee, \neg, \longrightarrow\}$ – the set of sentence-connectives; $\{every, some\}$ – the set of quantifiers; $\{is, isn't\}$ – affirmative and negative copulas; $\{\}, \{\}, \langle, \rangle$ – parenthesis and comma. The members of L are its signs.*

Note: In order to fully determine a language L , it is enough to determine \mathcal{P} , \mathcal{R} and \mathcal{S} ; the rest of the constituents are the same for all languages.

As we shall explain below, one-place predicates function in our system also as plural referring expressions, as common nouns do in natural language. One might claim that the name ‘predicates’ is not appropriate for such expressions; ‘concept-letters’ might have been more suitable. However, since these expressions function also as predicates, and since the term ‘ n -place predicate’ will be convenient to use as a collective name for both one-place and many-place predicates, we shall continue using this terminology in what follows.

We shall also see that the extension of *Thing* in every model will be the whole universe. We have added such a predicate to our system in order to obtain formulas that refer to the whole domain. As we shall see, this will help us translate formulas from FOL to our system. It should be noted, however, that there is no internal need for such a predicate in our formal system, and that the system can be developed without it, as indeed is the case with the related system developed in [1].

Definition 2 (Quantified Noun-Phrase, Noun-Phrase). *If P is a one-place predicate, then every P and some P are quantified noun-phrases (QNP). If α is a QNP or an SRE, then α is a noun-phrase (NP).*

In natural language there are quantified noun-phrases that contain a defining clause of some sort; for instance, ‘every man who owns a Jaguar’, as used in ‘Every man who owns a Jaguar is rich’. Quantified noun-phrases composed in this manner are not dealt with in the present paper, and are not represented in the formal system developed below. We limit the system developed here to QNPs in which the referring expression is a simple (non-composed) one-place predicate.

The use of anaphors in our system resembles their use in natural language. As we shall see below, anaphors in our system will always relate to (an occurrence of) a noun-phrase, and their meaning will be determined with relation to that noun-phrase. The relation ‘being anaphoric on’ is syntactically defined as follows:

Definition 3 (Anaphors of a Noun-Phrase). *Let φ be a string of signs. An occurrence α of an anaphor in φ is anaphoric on an occurrence t of an NP δ in φ if the following conditions hold: t is to the left of α ; the same index k appears in parenthesis both immediately to the left of t and immediately to the left of α ; the string (k) does not occur immediately to the left of any sign that is not an anaphor between t and α . In this case, we may also say that α is an anaphor of t , and that t is the source of α .*

Example: In the string $((1)s_1, (2) \text{ every } P) \text{ is } R \longrightarrow ((2)a, (1)a) \text{ is } L$, the first (i.e. the leftmost) occurrence of a is anaphoric on the occurrence of *every P*; the second – on the occurrence of s_1 .

In natural language, a given relation can be represented in various forms: the sentences ‘John kissed Mary’ and ‘Mary was kissed by John’, for instance, represent the same relation, as do the sentences ‘John gave this book to Mary’, ‘This book was given by John to Mary’, ‘To Mary was this book given by John’, etc. We call such variations *transpositions*. To represent these in our system, we use the following definition:

Definition 4 (Transpositions). *Let R be an n -place predicate, $n > 1$, and let τ be a non-trivial permutation (i.e., not the identity permutation) of $\{1, \dots, n\}$. Then the string $R\langle\tau(1), \dots, \tau(n)\rangle$ is a transposition of R . (The symbol τ here does not belong to our formal language; it belongs to the meta-language.) Thus, if R is a 3-place predicate, its transpositions are $R\langle 1, 3, 2\rangle$, $R\langle 2, 1, 3\rangle$, etc.*

Note: For the sake of convenience, we shall sometimes refer to R as $R\langle\tau(1), \dots, \tau(n)\rangle$, where τ is the identity permutation.

Note: If t is an occurrence of a certain sign, or string of signs, in a string φ , and α is a sign, or a string of signs, then we write $\varphi[t/\alpha]$ to denote the string that is the product of replacing t with α in φ . In case several occurrences t_1, \dots, t_n are replaced by $\alpha_1, \dots, \alpha_n$, we write: $\varphi[t_1/\alpha_1, \dots, t_n/\alpha_n]$. Sometimes we would like to replace *all* occurrences of a certain sign (or string of signs) α in a string φ by another sign or string β . To refer to the product of such a replacement we write: $\varphi[\alpha/\beta]$.

Note: If α is a sign, or a string of signs, that occurs in a string φ , then, in order to emphasize the fact that φ contains α , we shall sometimes refer to φ as $\varphi(\alpha)$.

2.2 Formulas

Our formation rules are somewhat more complex than those of FOL. We shall first give a brief sketch of these rules, and several examples of formulas together with the English sentences they translate. Only then shall we proceed to give the exact definition of a formula.

Our atomic formulas include strings of the form: (s_1, \dots, s_n) *is* R , which are meant to express a relation between n individuals, and: (s_1, \dots, s_n) *isn't* R , which are meant to deny such a relation. We allow the forming of new formulas from given ones by means of sentence connective in the usual manner.

Another thing we allow is the replacement of some occurrences of an SRE by anaphors of another occurrence of the same SRE. Thus, for instance, since (s, s) *is* L is a formula, $((1)s, (1)a)$ *is* L is also a formula. The first of these two can translate 'John loves John'; the second – 'John loves himself'. Anaphors are written with indices to their left, to indicate their being anaphoric on a certain occurrence of an NP.

Under certain conditions, we also allow the replacement of an SRE by a QNP. We thus have formulas such as $(every\ M, s)$ *is* L (which can translate 'Every man loves John'), $((1)\ every\ M, (1)a)$ *is* L ('Every man loves himself') and also $(every\ M, some\ W)$ *is* L ('Every man loves some women').

Let us now turn to the exact definitions. We shall start with a definition of atomic formula, and proceed by induction.

Definition 5 (Atomic Formula). *A string of signs φ in a language L is an atomic formula if it is of one of the following forms: s_1 is s_2 ; s_1 isn't s_2 ; (s_1, \dots, s_n) is R ; (s_1, \dots, s_n) isn't R , where s_1, \dots, s_n are SREs and R is an n -place predicate ($n \geq 1$) or a transposition of such a predicate.*

Definition 6 ($\#\varphi$). *Let φ be a (finite) sequence. $\#\varphi$ is the length of φ . If φ is a string of signs in a language L , then $\#\varphi$ is the number of sign-occurrences in φ .*

Definition 7 (Formula, Sub-Formula, Main QNP). *Let φ be a string of signs in a language L .*

1. *If $\#\varphi \leq 3$, then φ is a formula iff it is an atomic formula.*
2. *Assume that $\#\varphi = n$, and that for any string ψ for which $\#\psi < n$, it is determined whether ψ is a formula.*

Define: Let δ and ψ be strings of signs in L such that $\#\delta, \#\psi < n$. Then:

- (i) *δ is a sub-formula of ψ if the following conditions hold: ψ is a formula; $\#\delta < \#\psi$; δ is contained in ψ as a string; δ itself is a formula, or the product of one or more of the following operations on a formula: substitution of anaphors (with indices to their left) for SREs, addition of indices in parenthesis to the left of some NP occurrences, substitution of NPs for other NP occurrences.*
- (ii) *An NP occurrence t in ψ is distributed in ψ if there is no sub-formula of ψ that contains both t and all its anaphors.*

Now, φ is a formula iff one of the following conditions holds:

- (a) *φ is an atomic formula.*
- (b) *There are formulas α and β such that: $\#\alpha, \#\beta < n$; α, β do not contain anaphors of SRE occurrences; $\varphi \in \{\neg(\alpha), (\alpha) \vee (\beta), (\alpha) \wedge (\beta), (\alpha) \longrightarrow (\beta)\}$.*
- (c) *There is a formula ψ and an index k such that: $\#\psi < n$; c_1, \dots, c_n are occurrences of an SRE s in ψ , ordered from left to right; none of c_1, \dots, c_n has an index in parenthesis to the left of it; the string (k) does not occur between c_1 and c_n ; if (k) occurs to the right of*

c_n and is immediately followed by an anaphor, then this anaphor has a source that lies to the right of c_n ; a_2, \dots, a_n are anaphors; φ is $\psi[c_1/(k)s, c_2/(k)a_2, \dots, c_n/(k)a_n]$.

- (d) There is a formula ψ , an SRE s and a QNP qP such that: $\# \psi < n$; c is a distributed occurrence of s in ψ ; ψ does not contain distributed occurrences of QNPs to the left of c ; other than c , no SRE occurrence in ψ has anaphors; φ is $\psi[c/qP]$. In this case, the occurrence of qP that replaced c is called the main QNP in φ .

Note: We shall sometimes omit parenthesis, where this is unlikely to cause confusion. For instance, we shall refer to $((\alpha) \wedge (\beta)) \wedge (\gamma)$ as $\alpha \wedge \beta \wedge \gamma$; to (s) is P as s is P ; and to $(\text{every } Q)$ is P as $\text{every } Q$ is P .

Theorem 1 (Induction on Formulas). *Let A be a set of formulas in a language L and assume that A satisfies the following conditions:*

1. *All the atomic formulas of L are members of A .*
2. *If $\alpha, \beta \in A$ do not contain anaphors of SRE occurrences, then $\neg(\alpha)$, $(\alpha) \wedge (\beta)$, $(\alpha) \vee (\beta)$, $(\alpha) \longrightarrow (\beta) \in A$.*
3. *If $\psi \in A$ and φ is the product of substituting anaphors for SRE occurrences in ψ as described in section 2c of the formula definition, then $\varphi \in A$.*
4. *If $\varphi(qP)$ is a formula in which an occurrence t of qP is the main QNP, then: if A contains every formula of the form $\varphi[t/s]$, where s is an SRE, then $\varphi \in A$.*

Then, A contains all the formulas in L .

To prove this theorem, one can prove, by induction on $\#\varphi$, that for any finite string φ , if φ is a formula, then $\varphi \in A$. We shall not give such a proof here.

2.3 Models, truth in a model

As we have already mentioned, our system is based upon the analysis of common nouns, in some of their uses, as referring expressions. The noun ‘whale’, for instance, is used referentially in sentences like ‘Every whale is a

mammal’. ‘Whale’ refers here to whales; it does not refer to a *set* of whales, but to the whales themselves.

Now, for a referring expression to fulfill its task, there have to be some thing or things to which it can refer. For instance, if there are no whales, then ‘whales’ in the previous example cannot fulfill its semantic task. To exclude such failures of reference, we require that the extension of any one-place predicate be non-empty. This requirement guarantees that every component of our system fulfills its semantic task. It can be compared with the requirement in FOL, that any referring expression (i.e. any closed term; e.g., individual constants) be interpreted as designating some individual. This last demand, like the one stated above, excludes failures of reference. And while it yields the result that $\exists x(x = s)$ is true in every model in FOL, our demand concerning the extensions of predicates gives the same status to formulas of the form *some P is P*.

Definition 8 (Model). *A model for a language L is an ordered pair $m = \langle M, \sigma \rangle$ such that:*

1. *M, the universe of m, is a non-empty set.*
2. *σ , the interpretation function, is a function such that:*
 - (a) *The domain of σ is the set of all SREs, predicates and predicate-transpositions of L.*
 - (b) *If s is a singular referring expression, then $\sigma(s) \in M$.*
 - (c) *$\sigma(\text{Thing}) = M$.*
 - (d) *If P is a one-place predicate, then $\sigma(P)$ is a non-empty subset of M.*
 - (e) *If R is an n-place predicate, $n > 1$, then $\sigma(R) \subseteq M^n$.*
 - (f) *If R is an n-place predicate, $n > 1$, and τ is a non trivial permutation of $\{1, \dots, n\}$, then*

$$\sigma(R\langle\tau(1), \dots, \tau(n)\rangle) = \{\langle x_{\tau(1)}, \dots, x_{\tau(n)} \rangle \mid \langle x_1, \dots, x_n \rangle \in \sigma(R)\}.$$

Note: in order to fully determine a model, it is enough to determine *M* and $\sigma(\alpha)$ for all SREs and predicates α .

It may be claimed that our requirement concerning the extensions of one-place predicates is more than is really needed: the extension of a predicate should be non-empty only if this predicate is used referentially, but in ‘*Every S is P*’, for instance, *P* is not used in this way.

Our system would have been closer to natural language had we taken the following road: instead of excluding models that assign the empty set to some one-place predicates, we could have allowed them, and say that a formula containing a QNP of the form qP expresses a (true or false) proposition only in models in which $\sigma(P)$ is non-empty. This alternative approach may also be necessary if we would like to deal with quantified noun-phrases containing a defining clause, such as ‘every man who owns a Jaguar’. It seems that the extension of ‘man who owns a Jaguar’ should be the set of all things that are both men and own a Jaguar. And requiring every two extensions to have a member in common seems to seriously limit our notion of model. The alternative approach would, however, result in a much more complicated system, and since we do not treat composed quantified noun-phrases to begin with, we shall stick to our original requirement: the extension of one-place predicates should never be empty.

Definition 9 (The Characteristic SRE). *For every Language L , let c_L be a new sign, not in L . L^* is defined as the language $L \cup c_L$, in which c_L is an SRE. c_L is the characteristic SRE of L .*

The above notion will be used in the definition of truth in a model. The idea is the following. Given a model m for L and a predicate P , we shall look at all the enrichments m' of m to the language L^* that interpret c_L as a member of the extension of P . These enrichments, which we shall call $\sigma(P)$ -enrichments, will enable us to define the truth-conditions of quantified formulas: a formula $\varphi(qP)$, in which an occurrence t of qP is the main QNP, will be true in m iff $\varphi[t/c_L]$ is true in q of the $\sigma(P)$ -enrichments of m . Let us now give the exact definitions.

Definition 10 (Enrichment, Restriction). *Let L_1, L_2 be formal languages, and assume $L_1 \subseteq L_2$.¹ Let $m_1 = \langle M_1, \sigma_1 \rangle$ and $m_2 = \langle M_2, \sigma_2 \rangle$ be models for L_1, L_2 respectively. m_2 is an enrichment of m_1 to L_2 if the following conditions hold: $M_1 = M_2$; $\sigma_1 \subseteq \sigma_2$ (i.e., for every predicate or SRE α in L_1 , $\sigma_2(\alpha) = \sigma_1(\alpha)$). m_1 , in this case, is a restriction of m_2 to L_1 .*

Definition 11 (A-enrichment). *Let $m = \langle M, \sigma \rangle$ be a model for a language L , and let $A \subseteq M$. An enrichment $m' = \langle M', \sigma' \rangle$ of m to L^* is an A-enrichment of m if $\sigma'(c_L) \in A$.*

¹We assume here that for every n , each n -place predicate of L_1 is an n -place predicate of L_2 , each SRE of L_1 is an SRE of L_2 , etc.

Note: In order to determine a specific A -enrichment m' of m , it is enough to choose a member $\alpha \in A$ and define $\sigma'(c_L) = \alpha$.

Definition 12 (Truth-conditions of Atomic Formulas). *Let φ be an atomic formula in a language L , and let $m = \langle M, \sigma \rangle$ be a model for L . The relation $m \models \varphi$ (φ is true in m) is defined as follows:*

1. If s_1, s_2 are SREs, then: $m \models [s_1 \text{ is } s_2]$ iff $\sigma(s_1) = \sigma(s_2)$;
 $m \models [s_1 \text{ isn't } s_2]$ iff $\sigma(s_1) \neq \sigma(s_2)$.
2. If R is an n -place predicate ($n \geq 1$) or a transposition of such a predicate, and s_1, \dots, s_n are SREs, then:
 $m \models [(s_1, \dots, s_n) \text{ is } R]$ iff $\langle \sigma(s_1), \dots, \sigma(s_n) \rangle \in \sigma(R)$;
 $m \models [(s_1, \dots, s_n) \text{ isn't } R]$ iff $\langle \sigma(s_1), \dots, \sigma(s_n) \rangle \notin \sigma(R)$.

Definition 13 (Truth-conditions of Formulas). *Let φ be a finite sequence, Let L be any language in which φ is a formula, and let $m = \langle M, \sigma \rangle$ be a model for L . The relation $m \models \varphi$ is defined by induction on $\#\varphi$:*

1. If $\#\varphi \leq 3$, then φ is an atomic formula in L , and its truth-conditions in m are defined as in definition 12.
2. Let $n = \#\varphi$, and assume that for any $k < n$, if ψ is a string of length k , L' is a language in which ψ is a formula, and m' is a model for L' , then it is already determined whether $m' \models \psi$. Let L be a language in which φ is a formula.
 - (a) If φ is an atomic formula, then its truth-conditions in any model for L are as in definition 12.
 - (b) If α and β are formulas in L that do not contain anaphors of SRE occurrences, then: $m \models \neg(\alpha)$ iff $m \not\models \alpha$; $m \models [(\alpha) \wedge (\beta)]$ iff $m \models \alpha$ and $m \models \beta$; $m \models [(\alpha) \vee (\beta)]$ iff $m \models \alpha$ or $m \models \beta$; $m \models [(\alpha) \longrightarrow (\beta)]$ iff it is not the case that $m \models \alpha$ and $m \not\models \beta$.
 - (c) If φ is the product of substituting anaphors for SRE occurrences in a formula ψ as in section 2c of the formula definition, then $m \models \varphi$ iff $m \models \psi$.
 - (d) If $\varphi(qP)$ is a formula that contains no anaphors of SRE occurrences, and in which an occurrence t of qP is the main QNP, then:

- i. If q is every, then: $m \models \varphi(\text{every } P)$ iff:
for every $\sigma(P)$ -enrichment m' of m , $m' \models \varphi[t/c_L]$.
- ii. If q is some, then: $m \models \varphi(\text{some } P)$ iff:
for some $\sigma(P)$ -enrichment m' of m , $m' \models \varphi[t/c_L]$.

Note: If $m \models \varphi$, we also say that m satisfies φ , and that φ holds in m .

The only two quantifiers treated in our system are ‘every’ and ‘some’. It should be noted, however, that our definition of the truth-conditions of quantified formulas can easily be extended to treat other quantifiers as well. Our basic idea was, that $\varphi(qP)$ is true in m iff $\varphi[t/c_L]$ is true in q $\sigma(P)$ -enrichments of m . And this remains true for quantifiers such as ‘seven’, ‘at least three’ and ‘most’. Our analysis of quantification gives a uniform account of all these quantifiers, as can be expected in view of the syntactic similarities between them in natural language. Such a uniform analysis is not available if we use standard versions of FOL as a tool for the analysis of natural language. As is well known, these versions cannot incorporate quantifiers such as ‘most’, which require restricted or binary quantification (cf. [1, section 6.4]; [4]).

Definition 14 (Theory). A theory T in a language L is a set of formulas in L .

Definition 15 (Model of a Theory). Let T be a theory in a language L . m is a model of T if it is a model for L and $m \models \varphi$ for all $\varphi \in T$. In that case, we may also say that m satisfies T , etc.

Definition 16 (Entailment). A theory T entails a formula φ if φ is true in every model of T . In this case, we write: $T \models \varphi$.

2.4 Deduction

We shall use a natural deduction system. Our way of writing proofs resembles the one found in Lemmon [5] and in Newton-Smith [6].

Definition 17 (Proof). Let L be a formal language. A proof in L is a finite sequence of 4-tuples of the form $\langle \alpha, (k), \varphi, J \rangle$, called the lines of the proof, where:

- (a) α is a finite (possibly empty) set of natural numbers, all of which are smaller than or equal to k . Lines $\langle \alpha', (k'), \varphi', J' \rangle$ in the proof for which $k' \in \alpha$ will be called the lines on which the k -th line relies. The formulas φ' in such lines will be called the formulas on which the k -th line relies.
- (b) k , the line's number, is a natural number. The first line in a proof has $k = 1$, the second $k = 2$, etc.
- (c) φ is a formula in L .
- (d) J , the justification of the k -th line, is written in accordance with one of the following rules.

The following derivation rules allow the beginning of a proof and the addition of lines to a given proof. In fact, these rules complete definition 17 to a precise definition of proof, by induction on the number of lines.

For the sake of convenience, we occasionally drop the parenthesis ' \langle, \rangle ' or commas when referring to lines in a proof. Also, instead of writing the set of lines on which a certain line relies, we sometimes write the members of this set. In case this set is empty, we may not write anything. Another convenient convention is the following: a proof containing a single line is identified with that line.

17.1 (Premise). *If φ is a formula in L , then $\langle 1(1)\varphi \text{ Premise} \rangle$ is a proof. Also, if D is a proof of length $k - 1$ (i.e., it has exactly $k - 1$ lines), then we may add to D the line: $\langle k(k)\varphi \text{ Premise} \rangle$ (that is: the addition of such a line to D gives a proof).*

17.2 (Thing Introduction). *If s is an SRE, then $\langle (1)s \text{ is Thing Th } I \rangle$ is a proof. Also, if D is a proof of length $k - 1$, then we may add to D the line: $\langle (k)s \text{ is Thing Th } I \rangle$.*

17.3 (Identity Introduction). *If s is an SRE, then $\langle (1)s \text{ is } s \text{ Id } I \rangle$ is a proof. Also, if D is a proof of length $k - 1$, then we may add to D the line $\langle (k)s \text{ is } s \text{ Id } I \rangle$.*

17.4 (Identity Elimination). *Let D be a proof of length $k - 1$. Assume that s and s' are SREs, and that D includes the line: $\langle \alpha(i)s \text{ is } s'J_i \rangle$. Assume also that D includes a line of the form $\langle \beta(j)\varphi J_j \rangle$, where φ contains the occurrences c_1, \dots, c_n of s (φ may contain other occurrences*

of s as well).

Then we may add to D the line $\langle \alpha \cup \beta(k)\varphi[c_1/s', \dots, c_n/s'] \text{ Id } E, i, j \rangle$.²

17.5 (Propositional Calculus Rules). We allow the usual propositional calculus derivation rules for formulas that do not contain anaphors of SRE occurrences. We shall give only two examples here:

→ **Introduction.** Let D be a proof of length $k - 1$. If D contains the lines $\langle i(i)\varphi \text{ Premise } \rangle$; $\langle \beta(j)\psi J \rangle$, where φ and ψ do not contain anaphors of SRE occurrences, then we may add to D the line $\langle \beta \setminus \{i\} (k)\varphi \rightarrow \psi \rightarrow I, i, j \rangle$.

∨ **Elimination.** Let D be a proof of length $k - 1$. Assume that D contains the lines $\langle \alpha(i)\varphi \vee \psi J_i \rangle$; $\langle j(j)\varphi \text{ Premise } \rangle$; $\langle \beta(l)\delta J_l \rangle$; $\langle m(m)\psi \text{ Premise } \rangle$; $\langle \gamma(n)\delta J_n \rangle$, where φ and ψ do not contain anaphors of SRE occurrences, $j \notin \gamma$ and $m \notin \beta$. Then we may add to D the line $\langle ((\beta \cup \gamma) \setminus \{j, m\}) \cup \alpha(k)\delta \vee E, i, j, l, m, n \rangle$.

17.6 (Transposition). Let D be a proof of length $k - 1$, and let τ and ξ be any permutations of $\{1, \dots, n\}$. Assume that D contains the line $\langle \alpha(i)(s_{\tau(1)}, \dots, s_{\tau(n)}) \text{ is } R \langle \tau(1), \dots, \tau(n) \rangle J \rangle$, where s_1, \dots, s_n are SREs. Then we may add to D the line $\langle \alpha(k)(s_{\xi(1)}, \dots, s_{\xi(n)}) \text{ is } R \langle \xi(1), \dots, \xi(n) \rangle \text{ Tr } , i \rangle$.

17.7 (Negative-Copula Introduction). Let D be a proof of length $k - 1$. Let R be an n -place predicate ($n \geq 1$) or a transposition of such a predicate, and let s_1, \dots, s_n be SREs. If D contains the line $\langle \alpha(i) \neg((s_1, \dots, s_n) \text{ is } R) J \rangle$, then we may add to D the line $\langle \alpha(k)(s_1, \dots, s_n) \text{ isn't } R \text{ NC } I, i \rangle$.

17.8 (Negative-Copula Elimination). Let D be a proof of length $k - 1$. Let R be an n -place predicate ($n \geq 1$) or a transposition of such a predicate, and let s_1, \dots, s_n be SREs. If D contains the line $\langle \alpha(i)(s_1, \dots, s_n) \text{ isn't } R J \rangle$, then we may add to D the line $\langle \alpha(k) \neg((s_1, \dots, s_n) \text{ is } R) \text{ NC } E, i \rangle$.

17.9 (Anaphors Introduction). Let D be a proof of length $k - 1$. Assume that D contains the line $\langle \alpha(i)\psi J \rangle$. If φ is the product of substituting

²It is not hard to show that substituting SRE occurrences for SREs in a formula gives a formula. Therefore, $\varphi[c_1/s', \dots, c_n/s']$ is a formula.

anaphors for SRE occurrences in ψ as in section 2c of the formula definition, then we may add to D the line $\langle \alpha(k)\varphi A I, i \rangle$.

17.10 (Anaphors Elimination). Let D be a proof of length $k-1$. Assume that D contains the line $\langle \alpha(i)\varphi J \rangle$. If φ is the product of substituting anaphors for SRE occurrences in a formula ψ as in section 2c of the formula definition, then we may add to D the line $\langle \alpha(k)\psi A E, i \rangle$.

17.11 (every Introduction). Let $\varphi(\text{every } P)$ be a formula in which an occurrence t of every P is the main QNP, and assume that φ does not contain s . Let D be a proof of length $k-1$, and assume that D includes the lines $\langle i(i)s \text{ is } P \text{ Premise} \rangle$; $\langle \beta(j)\varphi[t/s]J \rangle$. Also assume that β does not contain any number different than i of a line in which s occurs. Then, we may add to D the line $\langle \beta \setminus \{i\}(k)\varphi(\text{every } P) \text{ every } I, i, j \rangle$.

17.12 (every Elimination). Let $\varphi(\text{every } P)$ be a formula in which an occurrence t of every P is the main QNP, and let s be any SRE. Let D be a proof of length $k-1$, and assume that D includes the lines $\langle \alpha(i)\varphi(\text{every } P)J_i \rangle$; $\langle \beta(j)s \text{ is } PJ_j \rangle$. Then, we may add to D the line $\langle \alpha \cup \beta(k)\varphi[t/s] \text{ every } E, i, j \rangle$.

17.13 (some Introduction). Let $\varphi(\text{some } P)$ be a formula in which an occurrence t of some P is the main QNP. Let D be a proof of length $k-1$, and assume that D includes the lines $\langle \alpha(i)\varphi[t/s]J_i \rangle$; $\langle \beta(j)s \text{ is } PJ_j \rangle$, where s is an SRE. Then we may add to D the line $\langle \alpha \cup \beta(k)\varphi(\text{some } P) \text{ some } I, i, j \rangle$.

17.14 (some Elimination). Let $\varphi(\text{some } P)$ be a formula in which an occurrence t of some P is the main QNP. Assume that φ does not contain the SRE s , and that ψ is a formula that does not contain s . Let D be a proof of length $k-1$, and assume that D includes the lines $\langle \alpha(i)\varphi(\text{some } P)J_i \rangle$; $\langle j(j)s \text{ is } P \text{ Premise} \rangle$; $\langle k(k)\varphi[t/s] \text{ Premise} \rangle$; $\langle \beta(l)\psi J_l \rangle$. Also assume that $j, k \notin \alpha$, and that β does not contain any number, other than j and k , of a line in which s occurs. Then we may add to D the line $\langle (\alpha \cup \beta) \setminus \{j, k\}(m)\psi \text{ some } E, i, j, k, l \rangle$.

17.15 (Referential Import). Let $\varphi(\text{every } P)$ be a formula in which an occurrence t of every P is the main QNP. Let D be a proof of length $k-1$, and assume that D includes the line $\langle \alpha(i)\varphi(\text{every } P)J \rangle$. Then we may add to D the line $\langle \alpha(k)\varphi[t/\text{some } P] RI, i \rangle$.

Referential Import captures the referential use of one-place predicates in our system. With it the list of derivation rules of our system was concluded.

Definition 18 ($T_D(\alpha)$). *If α is a set of numbers of lines in a proof D , then $T_D(\alpha)$ is the set of all formulas that appear in D in lines whose numbers belong to α .*

Definition 19 (Provability of Formulas). *Let T be a theory in a language L , and let φ be a formula in L . φ is provable from T in L , if there is a proof D in L such that:*

1. *The last line in D is of the form: $\langle \alpha, (k), \varphi, J \rangle$.*
2. *$T_D(\alpha) \subseteq T$ (in other words: the last line of D relies only on members of T).*

D , in this case, is called a proof of φ from T .

Definition 20 (Provability of Theories). *Let T_1, T_2 be theories in a language L . T_2 is provable from T_1 if $T_1 \vdash \varphi$ for any $\varphi \in T_2$. In this case we write: $T_1 \vdash T_2$.*

3 Some Examples of Formalization and Proofs

We shall now give a few examples of proofs in our formal system, so that the reader gets a feel of it. These examples will also supply us with an opportunity to comment on some of the characteristics of our system, mainly in relation to other formal systems.

Consider first the following inference (Contrariety):

Every philosopher is wise; hence, it's not the case that every philosopher isn't wise.

These sentences translate into our system as:

every S is P; \neg (every S isn't P)

And the inference can be proved as follows:

1	(1)	<i>every S isn't P</i>	<i>Premise</i>
2	(2)	<i>every S is P</i>	<i>Premise</i>
2	(3)	<i>some S is P</i>	<i>RI, 2</i>
4	(4)	<i>s is S</i>	<i>Premise</i>
5	(5)	<i>s is P</i>	<i>Premise</i>
1, 4	(6)	<i>s isn't P</i>	<i>every E, 1, 4</i>
1, 4	(7)	$\neg(s \text{ is } P)$	<i>NC E, 6</i>
1, 4, 5	(8)	$(s \text{ is } P) \wedge \neg(s \text{ is } P)$	$\wedge I, 5, 7$
4, 5	(9)	$\neg(\text{every } S \text{ isn't } P)$	$\neg I, 1, 8$
2	(10)	$\neg(\text{every } S \text{ isn't } P)$	<i>some E, 3, 4, 5, 9</i>

Secondly, let us translate and prove the following inference (Darii):

Some philosophers are Athenians; every Athenian is Greek; hence, some philosophers are Greek.

Its translation:

some S is M; every M is P; some S is P

And its proof:

1	(1)	<i>some S is M</i>	<i>Premise</i>
2	(2)	<i>every M is P</i>	<i>Premise</i>
3	(3)	<i>s is S</i>	<i>Premise</i>
4	(4)	<i>s is M</i>	<i>Premise</i>
2, 4	(5)	<i>s is P</i>	<i>every E, 2, 4</i>
2, 3, 4	(6)	<i>some S is P</i>	<i>some I, 5, 3</i>
1, 2	(7)	<i>some S is P</i>	<i>some E, 1, 3, 4, 6</i>

As the reader would have noticed, these two inferences are part of the valid inferences of Aristotelian logic: the first belongs to the Square of Opposition, the second to the Syllogisms. All the other valid inferences of Aristotelian logic can also be proved in our system (cf. [1, chap. 10]). Our system thus contains Aristotelian logic. By contrast, on any acceptable translation of the four Aristotelian sentences (*every/some S is/isn't P*) into FOL, some of these inferences turn out invalid (unless some *ad hoc* axioms are added to the calculus; see below). We think this demonstrates the fact that the analysis of

the semantics of natural language on which our system is constructed is more adequate than what a similar analysis, using only the semantic categories of FOL, can supply. A formal system whose verdict on natural language inferences coincides with what logicians considered valid for more than two millennia obviously has a desirable feature.

On the other hand, unlike Aristotelian logic, our system can also prove inferences that involve multiply quantified sentences. For instance:

Some women are loved by every man; hence, every man loves some women.

Notice that these sentences use both the passive and the active form of the same verb. This is translated into our system as a relation-sign and its transposition. If we translate ‘ a loves b ’ as $(a, b) \text{ is } L$, then ‘ b is loved by a ’ should be translated as $(b, a) \text{ is } L\langle 2, 1 \rangle$. The former sentences are thus translated as:

$(\text{some } W, \text{ every } M) \text{ is } L\langle 2, 1 \rangle$; $(\text{every } M, \text{ some } W) \text{ is } L$.

Let us now prove this inference:

1	(1)	$(\text{some } W, \text{ every } M) \text{ is } L\langle 2, 1 \rangle$	<i>Premise</i>
2	(2)	$s_1 \text{ is } W$	<i>Premise</i>
3	(3)	$(s_1, \text{ every } M) \text{ is } L\langle 2, 1 \rangle$	<i>Premise</i>
4	(4)	$s_2 \text{ is } M$	<i>Premise</i>
3, 4	(5)	$(s_1, s_2) \text{ is } L\langle 2, 1 \rangle$	<i>every E, 3, 4</i>
3, 4	(6)	$(s_2, s_1) \text{ is } L$	<i>Tr, 5</i>
2, 3, 4	(7)	$(s_2, \text{ some } W) \text{ is } L$	<i>some I, 6, 2</i>
2, 3	(8)	$(\text{every } M, \text{ some } W) \text{ is } L$	<i>every I, 7, 4</i>
1	(9)	$(\text{every } M, \text{ some } W) \text{ is } L$	<i>some E, 1, 2, 3, 8</i>

Moreover, we can prove in our system inferences that involve sentences with anaphors of quantified noun phrases, a capacity which greatly increases our system’s power. We shall give one simple example:

Every man loves every man; hence, every man loves himself.

Its translation:

(every M, every M) is L; ((1) every M, (1)a) is L.

And its proof:

1	(1)	<i>(every M, every M) is L</i>	<i>Premise</i>
2	(2)	<i>s is M</i>	<i>Premise</i>
1, 2	(3)	<i>(s, every M) is L</i>	<i>every E, 1, 2</i>
1, 2	(4)	<i>(s, s) is L</i>	<i>every E, 3, 2</i>
1, 2	(5)	<i>((1)s, (1)a) is L</i>	<i>A I, 4</i>
1	(6)	<i>((1) every M, (1)a) is L</i>	<i>every I, 5, 2</i>

These examples demonstrate the nature and power of our system.

4 Many-sorted Logic

Our formal system resembles in some ways many-sorted logic. We shall therefore pause to discuss the relation between the two.

In many-sorted logic, different sorts of variables are used, each sort having its own domain. For instance, one occasionally uses x_1, x_2, x_3 etc. to range over particulars; e_1, e_2, e_3 etc. to range over events; t_1, t_2, t_3 etc. to range over times; and so on. Since in this case the variables determine the sort of pluralities about which something is said, and since different sorts of variables may determine different sorts of pluralities, it seems many-sorted logic can be considered a kind of logic with plural referring expressions, namely its variables.

Yet a significant logical distinction between many-sorted logic and our formal system (and natural language) still remains. In our system, one-place predicate letters can be used both as referring expressions and as predicates (correspondingly, in natural language some concept-words are used both as referring expressions and as one-place predicates). Consider, for instance, the two sentences:

Some Athenians are philosophers.

Every philosopher is wise.

‘Philosophers’ is used as a predicate in the first sentence and referentially in the second. These sentences translate into our system as, respectively:

some A is P
every P is W

Consequently, one can derive in our system the formula *some A is W* from these two formulas by means of syntactic derivation rules.

By contrast, many-sorted logic would either use, like ordinary one-sorted logic, the same variable when translating both sentences – in which case it would not mirror our use of different plural referring expressions in the two sentence; or it may use different variables in the two translations, e.g.:

$$\begin{aligned} &\exists x_1 P x_1 \\ &\forall y_1 W y_1. \end{aligned}$$

Here x_1 and y_1 are variables of different sorts. But in this latter case, as can be seen, the syntactic relation between the predicate P in the first formula, and the variable y_1 in the second, is lost. Consequently, one cannot derive in this case by means of syntactic derivation rules the translation of natural language's 'Some Athenians are wise' ($\exists x_1 W x_1$) from these two formulas. This is obviously an undesirable result.

To avoid this result, one may use, for instance, the same letters both as variables and as one-place predicate letters. Each one-place predicate will then be interpreted as designating its own domain of discourse. Appropriate syntactic derivation rules could then be introduced (which, although probably more complex than the usual ones, may be rather similar to those of our system).

But additional modifications of many-sorted logic should also be introduced. For instance, predicate letters in formulas of many-sorted logic usually combine with variables of specific sorts in order to form well formed formulas (see [2, pp. 295ff]). However, in order to translate both 'Every philosopher is wise' and 'Some Athenians are wise', the predicate W should combine both with the variable P and the variable A . Similarly, individual constants, which correspond in ordinary many-sorted logic to sorts, should not be classified into sorts in the modified version, in order to translate sentences like 'Socrates is an Athenian' and 'Socrates is a philosopher'. Moreover, in order to distinguish between sentences like, say, 'Every philosopher loves some philosopher' and 'Every philosopher is loved by some philosopher', several variable letters should be assigned, as usual, to each sort or predicate letter. If we use upper case letters for predicates, indexed lower case letters for variables, these sentences would be translated as, respectively:

$$\begin{aligned} &\forall p_1 \exists p_2 \text{ Loves } (p_1, p_2) \\ &\forall p_1 \exists p_2 \text{ Loves } (p_2, p_1). \end{aligned}$$

This contrasts with our system, which uses no variables (or anaphors) for such sentences:

$$\begin{aligned} &(\textit{every } P, \textit{ some } P) \textit{ is } L \\ &(\textit{every } P, \textit{ some } P) \textit{ is } L\langle 2, 1 \rangle \end{aligned}$$

As can also be seen, natural language's need for a distinction between active and passive voice, or some such linguistic device, which is preserved in our formal system, is lost in many-sorted logic, in its modified form as well. The same applies to the need for a distinction between an affirmative and a negative copula.

So many-sorted logic should be significantly modified to resemble our formal system, and even then some important distinctions would still remain.

Indeed, many-sorted logic, even in its usual form, can be shown to parallel our system in its deductive power, in the sense that this will be shown below for FOL. In fact, this follows immediately from the proofs that shall be given below, together with the fact that many-sorted logic can be reduced to one-sorted logic (see [2, pp. 296ff]). But remember that our purpose in the development of a new formal system was not to capture some new forms of inference, but to show that an alternative analysis of the semantics of natural language can serve as the basis for a formal system similar in its power to FOL. Consequently, we do not consider the fact that our system is similar in its power to FOL, on any of its versions – e.g., many-sorted logic – as a drawback, but rather as an advantage.

5 Properties of Our Formal System

In this part we shall prove some of the properties of our system. Our main goal will be to show that the system's deductive power is comparable to that of FOL. To be more precise, we shall prove that our system is equivalent to FOL, supplemented by all axioms of the form $\exists xPx$, which we shall call *axioms of existential import*. The set of all these axioms will be called *EI*. We shall correlate models in our system with models of *EI* in FOL, and define a translation of formulas in FOL into our system. We will prove this translation to be one-to-one, and to cover all the formulas in our system in the following sense: each of these formulas is both deductively and semantically equivalent to a translation of some formula of FOL. We shall also prove that the translation preserves truth in a model, entailment and provability. The

existence of such a translation, together with the completeness of FOL will entail the completeness of our system.

Let us now turn to the exact definitions and proofs.

5.1 Some basic results

This section contains some elementary lemmas and theorems that will be used in the proofs below. Some of these results will be stated without proof.

Given a formula φ and a model m , we can change some of the SREs in φ . If the new SREs are interpreted by m in exactly the same way as the old ones, the replacement should not affect the truth of φ in m . It should not matter even if the new SREs belong to a language richer than the one we started with, as long as we enrich m accordingly. This is, more or less, the content of the following lemma. Its somewhat complex formulation is meant to enable an easier proof by induction.

Lemma 1. *Let φ be a formula in a language L , and assume that φ contains the occurrences c_1, \dots, c_n of SREs s_1, \dots, s_n respectively (s_1, \dots, s_n not necessarily different). Let s'_1, \dots, s'_n and s''_1, \dots, s''_n be SREs, some (or all) of which may not belong to L ($s'_1, \dots, s'_n, s''_1, \dots, s''_n$ not necessarily different).*

Let $m' = \langle M', \sigma' \rangle$ be a model for a language L' that contains $L \cup \{s'_1, \dots, s'_n\}$, and let $m'' = \langle M'', \sigma'' \rangle$ be a model for a language L'' that contains $L \cup \{s''_1, \dots, s''_n\}$. Assume that m' and m'' coincide with m in L (that is: $M' = M'' = M$, and $\sigma'(\alpha) = \sigma''(\alpha) = \sigma(\alpha)$ for any $\alpha \in \text{dom}(\sigma)$).

If m' interprets s'_1, \dots, s'_n as m interprets s_1, \dots, s_n (i.e. $\sigma'(s'_i) = \sigma(s_i)$ for every i), and m'' interprets s''_1, \dots, s''_n as m interprets s_1, \dots, s_n , then: $m' \models \varphi[c_1/s'_1, \dots, c_n/s'_n] \iff m'' \models \varphi[c_1/s''_1, \dots, c_n/s''_n]$.

It is obvious that lemma 1 is true: from the definition of truth in a model it can be seen that SREs contribute to the truth of φ in a given model only through the way in which they are interpreted in that model; and if two SREs are assigned the same object, then they ought to have the same contribution to the truth of φ .

Theorem 2 (Agreement of Models and their Enrichments). *Let m_1 be a model for L_1 , and let m_2 be an enrichment of m_1 to L_2 . If φ is a formula in L_1 (and therefore, also in L_2), then: $m_1 \models \varphi \iff m_2 \models \varphi$.*

The following theorem shows that the reliance on c_L in the definition of truth in a model is not necessary; in order to define the truth-conditions of

a quantified formula φ , we could use any SRE of L that does not occur in φ . The notion of $\sigma(P)$ -enrichment can be replaced by that of $\sigma(P)$ -change:

Definition 21 (A-s-Change). *Let $m = \langle M, \sigma \rangle$ be a model for a language L , let s be an SRE in L , and let $A \subseteq M$. A model $m' = \langle M', \sigma' \rangle$ for L is an A-s-change of m if the following conditions hold:*

1. $M' = M$.
2. $\sigma'(s) \in A$.
3. For any $\alpha \in \text{dom}(\sigma) \setminus \{s\}$, $\sigma'(\alpha) = \sigma(\alpha)$.

Note: In order to fully determine an A-s-change m' of m , it is sufficient to choose any $\alpha \in A$, and define $\sigma'(s)$ to be α .

Theorem 3 (Truth-conditions of Quantified Formulas). *Let $\varphi(qP)$ be a formula in a language L , and assume that an occurrence t of qP is the main QNP in φ . Let $m = \langle M, \sigma \rangle$ be a model for L , and let s be an SRE not occurring in φ .*

1. *If q is every, then: $m \models \varphi(\text{every } P)$ iff: for every $\sigma(P)$ -s-change m' of m , $m' \models \varphi[t/s]$.*
2. *If q is some, then: $m \models \varphi(\text{some } P)$ iff: for some $\sigma(P)$ -s-change m' of m , $m' \models \varphi[t/s]$.*

The following two theorems follow immediately from lemma 1 and theorem 2, respectively.

Theorem 4 (Interchangeability of Identicals). *Let φ be a formula in a language L , and let $m = \langle M, \sigma \rangle$ be a model for L . Assume that φ contains the occurrences c_1, \dots, c_n of an SRE s (and maybe some other occurrences of that SRE as well). If $m \models \varphi$ and $m \models [s \text{ is } s']$, then $m \models \varphi[c_1/s', \dots, c_n/s']$.*

Theorem 5 (Agreement of Models and Their A-s-changes). *Let m be a model for a language L , let $A \subseteq M$, and let m' be an A-s-change of m . If φ is a formula (in L) that does not contain s , then: $m' \models \varphi \iff m \models \varphi$.*

The following lemma says, roughly, that any proof can be replaced by a proof in which no premise appears twice.

Lemma 2. *Let D be a proof in our system, and assume that the last line in D is $\langle \alpha(n)\varphi J \rangle$. Then there exists a proof D' of φ from $T_D(\alpha)$ in which no premise appears more than once (i.e. if $\langle i(i)\psi \text{ Premise} \rangle$ and $\langle j(j)\psi \text{ Premise} \rangle$ are both lines of D' , then $i = j$). Moreover, we can assume that for any line $\langle \beta(i)\psi J_i \rangle$ in D , there exists a line $\langle \beta'(i')\psi J'_i \rangle$ in D' such that $T_D(\beta) = T_{D'}(\beta')$.*

This lemma follows from the following fact: whenever one of our inference rules allows us to rely on a given premise, it places no restriction on the place (the line-number) of that premise in the proof. Therefore, the need to write a premise that already appears in the proof never arises.

Lemma 3 (Concatenation of Proofs). *Let D_1, D_2 be proofs in a language L , and assume that D_2 does not contain the same premise twice. Assume that the last line of D_1 is $\langle \beta(k)\psi J \rangle$. Also assume that the last line of D_2 is $\langle \alpha(n)\varphi J' \rangle$, where $T_{D_2}(\alpha) \subseteq \{\psi\}$ (that is: D_2 is a proof of φ from ψ , which is the formula in the last line of D_1). Then, there exists a proof D in L such that:*

1. *The first $\#D_1$ lines of D are exactly those of D_1 , in the same order.*
2. *The last line of D is $\langle \beta'(m)\varphi J'' \rangle$, where $\beta' = \beta$ if $T_{D_2}(\alpha) = \{\psi\}$ (i.e. if the last line of D_2 indeed relied on ψ) and $\beta' = \emptyset$ if $T_{D_2}(\alpha) = \emptyset$ (i.e. if D_2 is a proof of φ from \emptyset).*

D is called the concatenation of D_1 and D_2 , and we write: $D = (D_1, D_2)$.

We shall not give a precise proof of this lemma. The idea is the following: we start with D_1 , and apply to it the rules applied in D_2 , one by one. If, in doing that, if we have to rely on ψ , we rely on the last line of D_2 .

Lemma 4. *Let ψ, ψ' be formulas in a language L . If $\psi' \vdash \psi$, then there exists a proof D that has a last line of the form $\langle \alpha(n)\psi J \rangle$, where $T_D(\alpha) = \psi'$.*

The following lemma asserts that we can replace a premise in a proof with a stronger premise, without significantly changing the rest of the proof.

Lemma 5. *Let D be a proof in our system, and assume that no premise appears in D twice. Also assume that D includes the line: $\langle i(i)\psi \text{ Premise} \rangle$. Let ψ' be a formula such that $\psi' \vdash \psi$. Then there exists a proof D' such that:*

1. *For any $j < i$, the j -th lines of D and D' are identical.*

2. The i -th line of D' is $\langle i(i)\psi' \text{ Premise} \rangle$.
3. D' includes a line of the form $\langle i(k)\psi J_k \rangle$.
4. For any $j > i$, if D includes the line $\langle \alpha(j)\varphi J \rangle$, then D' includes a line $\langle \alpha'(j')\varphi J' \rangle$, where $T_{D'}(\alpha') = T_D(\alpha)$ up to the replacement of ψ with ψ' (that is: in case $\psi \in T_D(\alpha)$, we do not have the above equality, but instead: $T_{D'}(\alpha') = (T_D(\alpha) \setminus \{\psi\}) \cup \{\psi'\}$. Otherwise – we have equality).
A line that stands in the above relation to $\langle \alpha(j)\varphi J \rangle$ will be called a twin of $\langle \alpha(j)\varphi J \rangle$.

This lemma can be proved by induction on $\#D$. We shall not give such a proof here.

Theorem 6 (Provability of Theories is Transitive). *Let T_1, T_2, T_3 be theories in a language L . If $T_1 \vdash T_2$ and $T_2 \vdash T_3$, then $T_1 \vdash T_3$.*

This theorem follows from the fact that any proof uses only a finite number of premises, and from the following three lemmas, that hold for any φ_i, ψ_i and φ , and can be proved using our previous results:

1. If $\varphi_1 \vdash \varphi_2$ and $\varphi_2 \vdash \varphi_3$, then $\varphi_1 \vdash \varphi_3$.
2. $\varphi_1, \dots, \varphi_n \vdash \varphi \iff \varphi_1 \wedge \dots \wedge \varphi_n \vdash \varphi$.
3. If $\varphi_1 \vdash \psi_1, \varphi_2 \vdash \psi_2, \dots, \varphi_n \vdash \psi_n$, then: $\varphi_1 \wedge \dots \wedge \varphi_n \vdash \psi_1 \wedge \dots \wedge \psi_n$.

5.2 Soundness

Theorem 7 (Soundness). *Let T be a Theory in a language L , and let φ be a formula in L . If $T \vdash \varphi$, then $T \models \varphi$.*

Proof: To prove the theorem, it is convenient to prove the following proposition by induction on n : let D be a proof of length n . If the last line in D is $\langle \alpha(n)\varphi J \rangle$, then $T_D(\alpha) \models \varphi$.

The induction base is trivial. Assume now that the proposition holds for any $k < n$. To complete the proof, we need to check each of the possibilities for the justification of the n -th line in D , and prove that $T_D(\alpha) \models \varphi$ in each of them. We give the case of *some E* as an example.

If the last line is justified by *some E*, then D includes lines of the forms $\langle \alpha(i)\psi(\text{some } P)J \rangle$; $\langle j(j)s \text{ is } P \text{ Premise} \rangle$; $\langle k(k)\psi[t/s] \text{ Premise} \rangle$; $\langle \beta(l)\delta J_l \rangle$,

where: $\psi(\text{some } P)$ is a formula in which an occurrence t of *some* P is the main QNP; neither $\psi(\text{some } P)$ nor δ contains s ; $j, k \notin \alpha$; β does not contain any number, other than j and k , of a line in which s occurs. Also, the last line in D is $\langle (\alpha \cup \beta) \setminus \{j, k\} (n) \delta \text{ some } E, i, j, k, l \rangle$. Let m be a model of $T_D((\alpha \cup \beta) \setminus \{j, k\})$. We shall prove that $m \models \delta$. Since $j, k \notin \alpha$, we have $T_D(\alpha) \subseteq T_D((\alpha \cup \beta) \setminus \{j, k\})$. Therefore: $m \models T_D(\alpha)$, and by the induction hypothesis: $m \models \psi(\text{some } P)$. Since ψ does not contain s , it follows, by theorem 3, that there exists a $\sigma(P)$ - s -change m' of m such that $m' \models \psi[t/s]$. m' , as a $\sigma(P)$ - s -change, also satisfies *s is P*. That is: m' satisfies the formulas in lines j and k , or, in other words: $m' \models T_D(\{i, j\})$. Now, since $T_D(\beta \setminus \{j, k\}) \subseteq T_D((\alpha \cup \beta) \setminus \{j, k\})$, we have: $m \models T_D(\beta \setminus \{i, j\})$. And since none of the formulas in $T_D(\beta \setminus \{i, j\})$ contains s (β does not contain any number, other than j and k , of a line in which s occurs), we have: $m' \models T_D(\beta \setminus \{i, j\})$ (this follows from $m \models T_D(\beta \setminus \{i, j\})$, by theorem 5). Therefore: $m' \models T_D(\beta)$. From the induction hypothesis it now follows that $m' \models \delta$. And since δ does not contain s , it follows (by theorem 5) that $m \models \delta$. ■

5.3 The version of FOL that will be used below

We shall use a version of FOL with identity, without function signs, and without open formulas. The version of FOL to be defined and used below can be proved to be equivalent to standard versions found in the literature. This section contains the definitions of the relevant terms. Most of these terms were also used in defining our formal system. When we use a term below, we shall take care to specify, in cases where confusion might arise, whether it refers to FOL or to our system.

Definition 22 (Formal Language). *A formal language L is a disjoint union of eight sets: \mathcal{P} – a set of one-place predicates; \mathcal{R} – a set of relation-signs or many-place predicates (to every one of which we assign a natural number $n > 1$, called its number of places); $\{=\}$ – the identity sign; \mathcal{S} – a denumerable set of individual constants; $\{x_1, x_2, \dots\}$ – a denumerable set of variables; $\{\neg, \wedge, \vee, \longrightarrow\}$ – connectives; $\{\forall, \exists\}$ – quantifiers; $\{), (\}$ – parenthesis.*

Note: We use the notation $\varphi[\alpha/\beta]$ for FOL in the same way we use it for our system.

Definition 23 (Formula).

1. Any string of the form: $Rs_1 \dots s_n$ or: $s_1 = s_2$, where s_1, \dots, s_n are individual constants, and R – an n -place predicate, is a formula. Strings of these forms are also called atomic formulas.
2. If α, β are formulas, then so are $\neg(\alpha)$, $(\alpha) \wedge (\beta)$, $(\alpha) \vee (\beta)$, $(\alpha) \longrightarrow (\beta)$.
3. If ψ is a formula that contains an individual constant s and x is a variable that does not occur in ψ , then $\forall x(\psi[s/x])$ and $\exists x(\psi[s/x])$ are formulas.
4. Nothing else is a formula.

Note: When referring to formulas, we shall sometimes omit parenthesis, for the sake of convenience.

The definition of truth in a model that will be used below is close to the one we use in our system. We begin by introducing the notion of characteristic constant:

Definition 24 (Characteristic Constant). For every Language L , let c_L be a new sign, not in L . L^* is defined as the language $L \cup \{c_L\}$, in which c_L is an individual constant. c_L is the characteristic constant of L .

Definition 25 (Model). Let L be a formal language. A model for L is an ordered pair $m = \langle M, \sigma \rangle$ such that: M , the universe of m , is a non-empty set; σ , the interpretation function, is a function such that:

1. The domain of σ is the set of all constants and predicates of L .
2. If s is a constant, then $\sigma(s) \in M$.
3. If R is an n -place predicate, then $\sigma(R) \subseteq M^n$.

Enrichment and restriction of a model are defined as in our system (definition 10 above).

Definition 26 (Truth in a Model). 1. If s_1, \dots, s_n are individual constants and R an n -place predicate in a language L , and $m = \langle M, \sigma \rangle$ – a model for L , then: $m \models [Rs_1 \dots s_n]$ iff $\langle \sigma(s_1), \dots, \sigma(s_n) \rangle \in \sigma(R)$; $m \models [s_1 = s_2]$ iff $\sigma(s_1) = \sigma(s_2)$.

2. If α and β are formulas in a language L , and $m = \langle M, \sigma \rangle$ – a model for L , then: $m \models \neg\alpha$ iff $m \not\models \alpha$; $m \models [\alpha \wedge \beta]$ iff $m \models \alpha$ and $m \models \beta$; $m \models [\alpha \vee \beta]$ iff $m \models \alpha$ or $m \models \beta$; $m \not\models [\alpha \longrightarrow \beta]$ iff $m \models \alpha$ and $m \not\models \beta$.
3. Let ψ be a formula in a language L , and let $m = \langle M, \sigma \rangle$ be a model for L , assume that ψ contains the individual constant s and does not contain the variable x . Then: $m \models \forall x(\psi[s/x])$ iff $m' \models \psi[s/c_L]$ for every enrichment m' of m to L^* ; $m \models \exists x(\psi[s/x])$ iff $m' \models \psi[s/c_L]$ for at least one enrichment m' of m to L^* .

In section 5.1, we explained that the truth-conditions of quantified formulas in the system we defined can be determined without reference to c_L (by what we called *A-s-changes*). A similar theorem holds for FOL. First, we define:

Definition 27 (s-Change). Let $m = \langle M, \sigma \rangle$ be a model for a language L in FOL, let s be an individual constant in L . A model $m' = \langle M', \sigma' \rangle$ for L is an *s-change* of m if the following conditions hold:

1. $M' = M$.
2. For any $\alpha \in \text{dom}(\sigma) \setminus \{s\}$, $\sigma'(\alpha) = \sigma(\alpha)$.

Note: In order to fully determine an *s-change* m' of m , it is sufficient to choose any $\beta \in M$, and define $\sigma'(s)$ to be β .

Theorem 8 (Truth-conditions of Quantified Formulas). Let $qx\varphi[s/x]$ be a formula in a language L in FOL. Let $m = \langle M, \sigma \rangle$ be a model for L , and let s be an SRE not occurring in φ .

1. If q is \forall , then: $m \models \forall x\varphi(x)$ iff:
for every *s-change* m' of m , $m' \models \varphi[x/s]$.
2. If q is \exists , then: $m \models \exists x\varphi(x)$ iff:
for some *s-change* m' of m , $m' \models \varphi[x/s]$.

We shall not prove this theorem here.

Definition 28 (Theory). A *Theory* in a language L in FOL is a set of formulas in L .

Model of a theory and entailment are defined as in section 2.3 above (definitions 15, 16).

Definition 29 (Proof). *Let L be a formal language. A proof in L is a finite sequence of 4-tuples of the form $\langle \alpha, (k), \varphi, J \rangle$, called the lines of the proof, where:*

- a. α is a finite (possibly empty) set of natural numbers, all of which are smaller than or equal to k . Lines $\langle \alpha', (k'), \varphi', J' \rangle$ in the proof for which $k' \in \alpha$ will be called the lines on which the k -th line relies. The formulas φ' in such lines will be called the formulas on which the k -th line relies.
- b. k , the line's number, is a natural number. The first line in a proof has $k = 1$, the second – $k = 2$, etc.
- c. φ is a formula in L .
- d. J , the justification of the k -th line, is written in accordance with one of the following rules.

29.1 Premise. *If φ is a formula in L , then $\langle 1(1)\varphi \text{ Premise} \rangle$ is a proof. Also, if D is a proof of length $k - 1$, then we may add to D the line: $\langle k(k)\varphi \text{ Premise} \rangle$.*

29.2 Identity Introduction. *If s is an individual constant, then $\langle (1)s = s \text{ Id } I \rangle$ is a proof. Also, if D is a proof of length $k - 1$, then we may add to it the line $\langle (k)s = s \text{ Id } I \rangle$.*

29.3 Identity Elimination. *Let D be a proof of length $k - 1$. Assume that s and s' are individual constants, and that D includes the line $\langle \alpha(i)s = s' J_i \rangle$. Assume also that D includes a line $\langle \beta(j)\varphi J_j \rangle$, where φ contains the occurrences c_1, \dots, c_n of s (and maybe other occurrences of s as well). Then, we may add to D the line $\langle \alpha \cup \beta(k)\varphi[c_1/s', \dots, c_n/s'] \text{ Id } E, i, j \rangle$.*

29.4 Propositional Calculus Rules. *We allow the usual introduction and elimination rules for each connective. (The rules are similar to the ones we formulated for formulas with no SRE-anaphors in our system.)*

29.5 \forall Introduction. *Let D be a proof of length $k - 1$, and assume that D includes the line $\langle \alpha(i)\psi(s) J \rangle$, where s is an individual*

constant and ψ does not contain the variable x . Assume also that α does not contain any number of a line in which s occurs. Then, we may add to D the line $\langle \alpha(k)\forall x(\psi[s/x])\forall I, i \rangle$.

29.6 \forall Elimination. Let D be a proof of length $k - 1$, and assume that D includes the line $\langle \alpha(i)\forall x(\psi[s'/x])J \rangle$, where ψ is a formula containing an individual constant s' . If s is an individual constant, then we may add to D the line $\langle \alpha(k)\psi[s'/s]\forall E, i \rangle$.

29.7 \exists Introduction. Let D be a proof of length $k - 1$, and assume that D includes the line $\langle \alpha(i)\psi[s'/s]J \rangle$, where s and s' are individual constants, and ψ does not contain the variable x . Then, we may add to D the line $\langle \alpha(k)\exists x(\psi[s'/x])\exists I, i \rangle$.

29.8 \exists Elimination. Let D be a proof of length $k - 1$, and assume that D includes the lines $\langle \alpha(i)\exists x(\psi[s/x])J_i \rangle$; $\langle j(j)\psi(s)$ Premise \rangle ; $\langle \beta(k)\delta J_k \rangle$. Assume also that $j \notin \alpha$, that δ does not contain s , and that β does not contain any number, other than j , of a line in which s occurs. Then we may add to D the line $\langle (\alpha \cup \beta) \setminus \{j\}(m)\delta \exists E, i, j, k \rangle$.

Provability is defined as in our system (section 2.4 above); we also use the notation $T_D(\alpha)$ as defined there.

We state the following two theorems without proof:

Theorem 9 (Soundness and Completeness of FOL). Let L be a formal language in FOL. Let T be a theory in L and φ a formula in L . Then: $T \models \varphi \iff T \vdash \varphi$.

5.4 Translation from FOL to our system

Definition 30 (Correlate of a Formal Language). Let L be a formal language in our system. The correlate of L in FOL, L_π , is the formal language that satisfies the following conditions:

1. The individual constants of L_π are the SREs of L .
2. (a) The predicates of L_π are those of L , excluding Thing.
(b) If a predicate is n -place in L , then it is n -place in L_π .

Note: Given a language L , there is exactly one language L_π that satisfies the above conditions.

Definition 31 (Translation of Formulas). *Let L be a formal language in our system, and let φ be a formula in L_π (in FOL). The translation μ of formulas from L_π to L is defined by induction on formulas in L_π . First, we arrange the variables of L_π in an infinite list, in which each variable appears infinitely many times. (This list will enable us to make μ injective. It will be used in theorem 10 below.) Now:*

1. *The translation of atomic formulas is defined as follows:*
 - (a) *If φ is $s_1 = s_2$, where s_1 and s_2 are individual constants, then $\mu(\varphi)$ is s_1 is s_2 .*
 - (b) *Let R be an n -place predicate ($n \geq 1$) in L_π . If φ is $Rs_1 \dots s_n$, then $\mu(\varphi)$ is (s_1, \dots, s_n) is R .*
2. *If α, β are formulas in L_π , then: $\mu(\neg\alpha)$ is $\neg\mu(\alpha)$; $\mu(\alpha \wedge \beta)$ is $\mu(\alpha) \wedge \mu(\beta)$; $\mu(\alpha \vee \beta)$ is $\mu(\alpha) \vee \mu(\beta)$; $\mu(\alpha \longrightarrow \beta)$ is $\mu(\alpha) \longrightarrow \mu(\beta)$.*
3. *If $\varphi(s)$ is a formula that contains an individual constant s , and x a variable that does not occur in φ , then:*
 - (a) *$\mu(\forall x\varphi[s/x])$ is $((l)$ every Thing is Thing) \wedge $(\mu(\varphi)[s/(l)a])$, where l is the least index not occurring in $\mu(\varphi)$ such that x is on the l -th place in the above mentioned list.*
 - (b) *$\mu(\exists x\varphi[s/x])$ is $((l)$ some Thing is Thing) \wedge $(\mu(\varphi)[s/(l)a])$, where l is as above.*

Theorem 10 (The Translation μ is Injective). *Let L be a formal language in our system, and let φ be a formula in L_π . If ψ is a formula (in L_π) such that $\mu(\psi) = \mu(\varphi)$, then ψ is φ .*

To see that the theorem is true, we note that in each of the stages in the definition of μ , $\mu(\varphi)$ determines φ . A precise proof can be given by induction on $\#\varphi$.

Theorem 11 ($\mu(\varphi)$ Does Not Contain SRE-Anaphors). *If φ is a formula in L_π , then $\mu(\varphi)$ does not contain anaphors of SRE occurrences.*

The theorem is not hard to prove by induction on formulas in L_π .

As we already mentioned, universal quantification in FOL lacks existential import. Therefore, in order for the translations of formulas to be equivalent to the formulas they translate, we need to complement FOL with axioms of *existential import*. We define:

Definition 32 (EI). Let L be a formal language in our system. Then $\text{EI}(L_\pi)$ (in short: **EI**) is the set of all formulas in L_π that have the form: $\exists xPx$, where x is a variable and P is a one-place predicate.

Theorem 12 ($\mu(\text{EI})$ is Provable). Let L be a formal language in our system. If T is a theory in L , then $T \vdash \mu(\text{EI})$.

Proof: It is sufficient to prove that $\emptyset \vdash \mu(\text{EI})$. Let $\varphi \in \text{EI}$. Then φ is of the form $\exists xPx$, and $\mu(\varphi)$ is: $((l) \text{ some Thing is Thing }) \wedge ((l)a \text{ is } P)$. That this formula is provable from \emptyset follows from the existence of the following proof:

1	(1)	$s \text{ is } P$	<i>Premise</i>
	(2)	$\text{every } P \text{ is } P$	<i>every I, 1, 1</i>
	(3)	$\text{some } P \text{ is } P$	<i>RI, 2</i>
	(4)	$d \text{ is Thing}$	<i>Th I</i>
5	(5)	$d \text{ is } P$	<i>Premise</i>
5	(6)	$(d \text{ is Thing}) \wedge (d \text{ is } P)$	\wedge I, 4, 5
5	(7)	$((l)d \text{ is Thing}) \wedge ((l)a \text{ is } P)$	<i>A I, 6</i>
5	(8)	$((l)\text{some Thing is Thing}) \wedge ((l)a \text{ is } P)$	<i>some I, 4, 7</i>
	(9)	$((l)\text{some Thing is Thing}) \wedge ((l)a \text{ is } P)$	<i>some E, 3, 5, 5, 8</i>

■

Definition 33 (Correlate of a Model). Let L be a formal language in our system, and let $m = \langle M, \sigma \rangle$ be a model of $\text{EI}(L_\pi)$. The correlate of m in our system, $\mu(m)$, is the model for L defined by: $\mu(m) = \langle M, \mu\sigma \rangle$, where:

1. For any individual constant s in L , $\mu\sigma(s) = \sigma(s)$.
2. $\mu\sigma(\text{Thing}) = M$.
3. For any other n -place predicate R in L ($n \geq 1$), $\mu\sigma(R) = \sigma(R)$.

Note: The above definition indeed determines a model for L ; the requirement that $\sigma(P) \neq \emptyset$ for all one-place predicates is fulfilled since m is a model of **EI**.

Theorem 13 (The Restriction of μ to Models is a Bijection). Let L be a formal language in our system. Let A be the set of all models of $\text{EI}(L_\pi)$, and let B be the set of all models for L . Then, the restriction of μ to A is a bijection from A to B .

Proof: $\mu|_A$ is injective: if $m_1 = \langle M_1, \sigma_1 \rangle$, $m_2 = \langle M_2, \sigma_2 \rangle$ are models of **El** such that $\mu(m_1) = \mu(m_2)$, then $\sigma_1(\textit{Thing}) = M_1 = M_2 = \sigma_2(\textit{Thing})$, and also: $\sigma_1(\alpha) = \mu\sigma_1(\alpha) = \mu\sigma_2(\alpha) = \sigma_2(\alpha)$ for any predicate or individual constant α in $L \setminus \{\textit{Thing}\}$. Therefore: $m_1 = m_2$.

$\mu|_A$ is onto B : if $m = \langle M, \sigma \rangle$ is a model for L , then $\mu(m') = m$, where $m' = \langle M', \sigma' \rangle$ is the model for L_π determined by: $M' = M$; $\sigma'(\alpha) = \sigma(\alpha)$ for all predicates and individual constants α in L_π . ■

Theorem 14 (Truth under μ). *Let L be a formal language in our system, and let φ be a formula in L_π . If $m = \langle M, \sigma \rangle$ is a model of **El**(L_π), then: $m \models \varphi \iff \mu(m) \models \mu(\varphi)$.*

Proof: By induction on formulas in L_π .

1. Atomic formulas: $m \models [s_1 = s_2] \iff \sigma(s_1) = \sigma(s_2) \iff \mu\sigma(s_1) = \mu\sigma(s_2) \iff \mu(m) \models [s_1 \textit{ is } s_2] \iff \mu(m) \models \mu(s_1 = s_2)$.

$$\begin{aligned} m \models [Rs_1 \dots s_n] &\iff \langle \sigma(s_1), \dots, \sigma(s_n) \rangle \in \sigma(R) \iff \\ &\langle \mu\sigma(s_1), \dots, \mu\sigma(s_n) \rangle \in \mu\sigma(R) \iff \mu(m) \models [(s_1, \dots, s_n) \textit{ is } R] \iff \\ &\mu(m) \models \mu(Rs_1 \dots s_n) \end{aligned}$$

2. If the theorem holds for α and β , then:

$$\begin{aligned} m \models [\alpha \wedge \beta] &\iff m \models \alpha \text{ and } m \models \beta \iff \mu(m) \models \mu(\alpha) \text{ and } \\ &\mu(m) \models \mu(\beta) \iff (\text{since } \mu(\alpha), \mu(\beta) \text{ do not contain anaphors of SRE} \\ &\text{occurrences}) \mu(m) \models [\mu(\alpha) \wedge \mu(\beta)] \iff \mu(m) \models \mu(\alpha \wedge \beta). \end{aligned}$$

The proofs for $\neg\alpha$, $\alpha \vee \beta$ and $\alpha \longrightarrow \beta$ are similar.

3. If $\varphi(s)$ is a formula that contains an individual constant s , and x a variable that does not occur in φ , then:

$$\begin{aligned} m \models \forall x\varphi[s/x] &\iff \text{every } s\text{-change } m' \text{ of } m \text{ satisfies } \varphi(s) \iff (\text{by the} \\ &\text{induction hypothesis) for every } s\text{-change } m' \text{ of } m, \mu(m') \models \mu(\varphi(s)) \\ &\iff (\text{since the set of } \sigma(\textit{Thing})\text{-}s\text{-changes of } \mu(m) \text{ is } \{\mu(m')\} | m' \text{ is an} \\ &s\text{-change of } m) \text{ for every } \sigma(\textit{Thing})\text{-}s\text{-change } (\mu(m))' \text{ of } \mu(m), \\ &(\mu(m))' \models \mu(\varphi(s)) \iff \text{for every } \sigma(\textit{Thing})\text{-}s\text{-change } (\mu(m))' \text{ of } \mu(m), \\ &(\mu(m))' \models [(s \textit{ is } \textit{Thing}) \wedge \mu(\varphi(s))] \iff \text{for every } \sigma(\textit{Thing})\text{-}s\text{-change} \\ &(\mu(m))' \text{ of } \mu(m), (\mu(m))' \models [((l)s \textit{ is } \textit{Thing}) \wedge \mu(\varphi)[s/(l)a]] \iff \\ &\mu(m) \models [((l) \textit{ every } \textit{Thing} \textit{ is } \textit{Thing}) \wedge \mu(\varphi)[s/(l)a]] \iff \\ &\mu(m) \models \mu(\forall x\varphi[s/x]) \end{aligned}$$

4. The proof for $\exists x\varphi[s/x]$ is similar. ■

Theorem 15 (Entailment under μ). *Let L be a formal language (in our system), let T be a theory in L_π , and let φ be a formula in L_π . If $\mu(T) \models \mu(\varphi)$, then $T \cup \text{El} \models \varphi$.*

Proof: Assume that $\mu(T) \models \mu(\varphi)$. Let m be a model for $T \cup \text{El}$. m is a model of El that satisfies T . Therefore, by theorem 14: $\mu(m) \models \mu(T)$, and it follows that $\mu(m) \models \mu(\varphi)$. Now, by theorem 14, it follows that $m \models \varphi$. Therefore: $T \cup \text{El} \models \varphi$. ■

Theorem 16 (Provability under μ). *Let φ be a formula in a language L_π , and let T be a theory in L_π . If $T \vdash \varphi$, then $\mu(T) \vdash \mu(\varphi)$.*

Proof: The idea behind the proof is the following: given a proof in FOL, our inference rules enable us to reconstruct it in our system.

In order to prove the theorem precisely, we prove the following proposition by induction on n : Let D be a proof of length n in L_π . If the last line in D is $\langle \alpha(n)\varphi J_n \rangle$, then $\mu(T_D(\alpha)) \vdash \mu(\varphi)$.

1. If $n = 1$, then α is either $\{n\}$ or \emptyset , and the justification J_n is either *Premise*, or *Id I* respectively. In either case, one application of *Premise* or *Id I* proves $\mu(\varphi)$ from $\mu(T_D(\alpha))$.
2. Assume that the above proposition is true for any $k < n$.
 - (a) If the last line of D is justified by *Premise* or *Id I*, then the proof is as above.
 - (b) If the last line of D is justified by \longrightarrow *Introduction*, then D contains the lines $\langle i(i)\psi_1 \text{Premise} \rangle$; $\langle \beta(j)\psi_2 J_j \rangle$, and the last line in D is $\langle \beta \setminus \{i\}(n)\psi_1 \longrightarrow \psi_2 \longrightarrow I, i, j \rangle$. We shall show that $\mu(T_D(\beta \setminus \{i\})) \vdash \mu(\psi_1 \longrightarrow \psi_2)$. By the induction hypothesis, $\mu(T_D(\beta)) \vdash \mu(\psi_2)$. Therefore, there exists a proof D' , in our system, the last line of which is $\langle \gamma(k)\mu(\psi_2) J \rangle$, where $T_{D'}(\gamma) \subseteq \mu(T_D(\beta))$.

Now, if D' includes a line of the form $\langle i(i)\mu(\psi_1) \textit{ Premise} \rangle$, we can proceed as follows: we can assume that no formula appears in D' as a premise in two different lines (see lemma 2). We can now add to D' the line $\langle \gamma \setminus \{i\}(k+1)\mu(\psi_1) \longrightarrow \mu(\psi_2) \longrightarrow I, i, k \rangle$.³ Since $\mu(\psi_1)$ does not appear in D' as a premise in any line other than i , we have: $\mu(\psi_1) \notin \mu(T_{D'}(\gamma \setminus \{i\}))$. And since $T_{D'}(\gamma) \subseteq \mu(T_D(\beta))$ and line i of D is $\langle i(i)\psi_1 \textit{ Premise} \rangle$, it follows that $T_{D'}(\gamma \setminus \{i\}) \subseteq \mu(T_D(\beta \setminus \{i\}))$. Therefore: $\mu(T_D(\beta \setminus \{i\})) \vdash [\mu(\psi_1) \longrightarrow \mu(\psi_2)]$. In other words: $\mu(T_D(\beta \setminus \{i\})) \vdash \mu(\psi_1 \longrightarrow \psi_2)$, as we wanted to prove.

In case D' does not include a line of the form $\langle i(i)\mu(\psi_1) \textit{ Premise} \rangle$, we have $\mu(\psi_1) \notin \mu(T_{D'}(\gamma))$, and therefore $T_{D'}(\gamma) \subseteq \mu(T_D(\beta \setminus \{i\}))$. It follows that: $\mu(T_D(\beta \setminus \{i\})) \vdash \mu(\psi_2)$, and hence: $\mu(T_D(\beta \setminus \{i\})) \vdash [\mu(\psi_1) \longrightarrow \mu(\psi_2)]$. That is: $\mu(T_D(\beta \setminus \{i\})) \vdash \mu(\psi_1 \longrightarrow \psi_2)$.

The proofs for the cases in which the last line of D is justified by some other propositional calculus derivation rule are similar to the one above, and will not be detailed here.

- (c) If the last line of D is justified by \forall *Introduction*, then D includes a line of the form $\langle \alpha(i)\psi(s)J_i \rangle$, where s is an individual constant, ψ does not contain the variable x , and α does not contain any number of a line in which s occurs. Also, the last line in D is $\langle \alpha(n)\forall x(\psi[s/x])\forall I, i \rangle$. We shall prove that $\mu(T_D(\alpha)) \vdash \mu(\forall x(\psi[s/x]))$ or, in other words, that: $\mu(T_D(\alpha)) \vdash [((l) \textit{ every Thing is Thing}) \wedge \mu(\psi)[s/(l)a]]$.

We should note that since α does not contain numbers of lines in which s occurs, none of the formulas in $T_D(\alpha)$ contains s . And since for any formula δ , $\mu(\delta)$ contains exactly the same SREs/individual constants as δ does (this is easily proved by induction on formulas in L_π), none of the formulas in $\mu(T_D(\alpha))$ contains s .

By the induction hypothesis, $\mu(T_D(\alpha)) \vdash \mu(\psi(s))$. Therefore, there exists a proof D' , in our system, the last line of which is $\langle \gamma(k)\mu(\psi)J \rangle$, where $T_{D'}(\gamma) \subseteq \mu(T_D(\alpha))$. It will be sufficient to prove that $T_{D'}(\gamma) \vdash [((l) \textit{ every Thing is Thing}) \wedge \mu(\psi)[s/(l)a]]$.

To prove this, we add to D' the following lines:

³The \longrightarrow Introduction rule requires that the formulas involved will not contain anaphors of SRE occurrences. This condition is fulfilled, since no $\mu(\psi)$ contains such anaphors (see theorem 10).

$k + 1$	$(k + 1)$	$s \text{ is Thing}$	<i>Premise</i>
$\gamma \cup \{k + 1\}$	$(k + 2)$	$(s \text{ is Thing}) \wedge \mu(\psi)$	$\wedge I, k, k + 1$
$\gamma \cup \{k + 1\}$	$(k + 3)$	$((l)s \text{ is Thing}) \wedge \mu(\psi)[s/(l)a]$	$A I, k + 2$
γ	$(k + 4)$	$((l)\text{every Thing is Thing}) \wedge \mu(\psi)[s/(l)a]$	$\text{every } I, k + 1, k + 3$

(The *every Introduction rule* requires that $\gamma \cup \{k + 1\}$ will not contain any number, other than $k + 1$, of a line in which s occurs. This requirement is fulfilled here, since $T_{D'}(\gamma) \subseteq \mu(T_D(\alpha))$ and none of the formulas in $\mu(T_D(\alpha))$ contains s .)

The proof that results from the addition of the above lines to D' is a proof of $\mu(\forall x(\psi[s/x]))$ from $\mu(T_D(\alpha))$.

- (d) If the last line of proof D is justified by \forall *Elimination*, then D includes a line $\langle \alpha(i)\forall x(\psi[s'/x])J_i \rangle$, and the last line in D is $\langle \alpha(n)\psi[s'/s]\forall E, i \rangle$. We shall prove that $\mu(T_D(\alpha)) \vdash \mu(\psi[s'/s])$.
By the induction hypothesis: $\mu(T_D(\alpha)) \vdash \mu(\forall x(\psi[s'/x]))$. That is:
 $\mu(T_D(\alpha)) \vdash [((l)\text{every Thing is Thing}) \wedge \mu(\psi)[s'/(l)a]]$.

Therefore, there exists a proof D' , whose last line is:

$\langle \gamma(k)((l)\text{every Thing is Thing}) \wedge \mu(\psi)[s'/(l)a]J \rangle$, such that $T_{D'}(\gamma) \subseteq \mu(T_D(\alpha))$.

We can add to D' the following lines:

	$(k + 1)$	$s \text{ is Thing}$	<i>Th I</i>
γ	$(k + 2)$	$((l)s \text{ is Thing}) \wedge \mu(\psi)[s'/(l)a]$	$\text{every } E, k, k + 1$
γ	$(k + 3)$	$(s \text{ is Thing}) \wedge \mu(\psi)[s'/s]$	$A E, k + 2$
γ	$(k + 4)$	$\mu(\psi)[s'/s]$	$\wedge E, k + 3$

The proof we got shows that $\mu(T_D(\alpha)) \vdash \mu(\psi)[s'/s]$. Now, since $\mu(\psi)[s'/s]$ is in fact $\mu(\psi[s'/s])$ (this can be proved by induction on formulas in L_π), we have: $\mu(T_D(\alpha)) \vdash \mu(\psi[s'/s])$, as required.

- (e) If the last line of D is justified by \exists *introduction*, then D includes a line: $\langle \alpha(i)\psi[s'/s]J_i \rangle$, and its last line is $\langle \alpha(n)\exists x(\psi[s'/x])\exists I, i \rangle$. We shall prove that $\mu(T_D(\alpha)) \vdash \mu(\exists x(\psi[s'/x]))$. By the induction hypothesis: $\mu(T_D(\alpha)) \vdash \mu(\psi[s'/s])$. Therefore, there exists a proof D' , whose last line is: $\langle \gamma(k)\mu(\psi[s'/s])J \rangle$, such that $T_{D'}(\gamma) \subseteq \mu(T_D(\alpha))$.

We now add to D' the following lines:

$$\begin{array}{rcl} & (k+1) & s \text{ is Thing} & Th I \\ \gamma & (k+2) & (s \text{ is Thing}) \wedge \mu(\psi[s'/s]) & \wedge I, k, k+1 \end{array}$$

The formula in the last line above is, in fact: $(s \text{ is Thing}) \wedge (\mu(\psi)[s'/s])$. We can therefore add the following lines:

$$\begin{array}{rcl} \gamma & (k+3) & ((l)s \text{ is Thing}) \wedge (\mu(\psi)[s'/(l)a]) & A I, k+2 \\ \gamma & (k+4) & ((l) \text{ some Thing is Thing}) \wedge (\mu(\psi)[s'/(l)a]) & \\ & & & \text{some } I, k+1, k+3 \end{array}$$

The proof we thus get is a proof of $\mu(\exists x(\psi[s'/x]))$ from $\mu(T_D(\alpha))$.

- (f) If the last line of D is justified by \exists Elimination, then D includes lines of the forms: $\langle \alpha(i)\exists x(\psi[s/x])J_i \rangle$; $\langle j(j)\psi(s) \text{ Premise} \rangle$; $\langle \beta(k)\delta J_k \rangle$, where $j \notin \alpha$, δ does not contain s , and β does not contain any number, other than j , of a line in which s occurs. The last line in D is $\langle (\alpha \cup \beta) \setminus \{j\}(n)\delta \exists E, i, j, k \rangle$. We shall prove that $\mu(T_D((\alpha \cup \beta) \setminus \{j\})) \vdash \mu(\delta)$. By the induction hypothesis: $\mu(T_D(\alpha)) \vdash \mu(\exists x(\psi[s/x]))$, that is: $\mu(T_D(\alpha)) \vdash ((l) \text{ some Thing is Thing}) \wedge (\mu(\psi)[s/(l)a])$, and also: $\mu(T_D(\beta)) \vdash \mu(\delta)$. If $T_D(\alpha)$ contains $\psi(s)$, then, since $j \notin \alpha$, α contains some other number j' of a line in D , in which $\psi(s)$ appears as a premise. Therefore, if we omit j from $\alpha \cup \beta$, $T_D(\alpha \cup \beta)$ will remain unchanged. That is: $T_D((\alpha \cup \beta) \setminus \{j\}) = T_D(\alpha \cup \beta)$. And since $\mu(\delta)$ is provable from $\mu(T_D(\beta))$, which is a subset of $T_D(\alpha \cup \beta)$, we have: $T_D((\alpha \cup \beta) \setminus \{j\}) = T_D(\alpha \cup \beta) \vdash \mu(\delta)$, as required.

Assume now that $\psi(s) \notin T_D(\alpha)$. Since: $\mu(T_D(\alpha)) \vdash [((l) \text{ some Thing is Thing}) \wedge \mu(\psi)[s/(l)a]]$, and: $\mu(T_D(\beta)) \vdash \mu(\delta)$, there exists a proof D' , whose last two lines are:

$$\begin{array}{rcl} \gamma_1 & (k) & ((l) \text{ some Thing is Thing}) \wedge \mu(\psi)[s/(l)a] & J_1 \\ \gamma_2 & (k+1) & \mu(\delta) & J_2 \end{array}$$

where $T_{D'}(\gamma_1) \subseteq \mu(T_D(\alpha))$ and $T_{D'}(\gamma_2) \subseteq \mu(T_D(\beta))$.⁴

⁴To construct such a proof, we can start with a proof of $\mu(\delta)$ from $\mu(T_D(\beta))$, and 'insert', so to speak, a proof of $((l) \text{ some Thing is Thing}) \wedge (\mu(\psi)[s/(l)a])$ from $\mu(T_D(\alpha))$ between the last line of the proof we started with and the rest of that proof.

If $\mu(\psi) \notin T_{D'}(\gamma_2)$, then the above proof shows that $T_{D'}(\gamma_2) \setminus \{\mu(\psi)\} = T_{D'}(\gamma_2) \vdash \mu(\delta)$. We also have:

$$T_{D'}(\gamma_2) \setminus \{\mu(\psi)\} \subseteq \mu(T_D(\beta)) \setminus \{\mu(\psi)\} \subseteq \mu(T_D(\beta \setminus \{j\})) \subseteq \mu(T_D((\alpha \cup \beta) \setminus \{j\})).$$

Therefore: $\mu(T_D((\alpha \cup \beta) \setminus \{j\})) \vdash \mu(\delta)$, as required.

Assume that $\mu(\psi) \in T_{D'}(\gamma_2)$. Then, D' contains a line of the form $\langle m(m)\mu(\psi(s)) \text{ Premise} \rangle$ (it is not hard to show that every line in a proof, on which another line relies, is a premise). Since $T_{D'}(\gamma_2) \subseteq \mu(T_D(\beta))$, $T_{D'}(\gamma_2)$ contains no formula, other than $\mu(\psi)$, in which s occurs. Also, since $T_{D'}(\gamma_1) \subseteq \mu(T_D(\alpha))$, $\psi(s) \notin T_D(\alpha)$, and μ is injective (theorem 10) we have: $\mu(\psi(s)) \notin T_{D'}(\gamma_1)$. We proceed as follows: By lemma 2, we can assume that no premise appears in D' twice. By lemma 5, since $[(l)s \text{ is Thing}] \wedge \mu(\psi)[s/(l)a] \vdash \mu(\psi(s))$,⁵ there is a proof D'' that includes lines of the forms:

$$\begin{array}{llll} m & (m) & ((l)s \text{ is Thing}) \wedge \mu(\psi)[s/(l)a] & \text{Premise} \\ \gamma'_1 & (p) & ((l) \text{ some Thing is Thing}) \wedge (\mu(\psi)[s/(l)a]) & J'_1 \\ \gamma'_2 & (q) & \mu(\delta) & J'_2 \end{array}$$

where $T_{D''}(\gamma'_1)$ and $T_{D''}(\gamma'_2)$ are identical with $T_{D'}(\gamma_1)$ and $T_{D'}(\gamma_2)$ (respectively) up to the replacement of $\mu(\psi)$ by $((l) \text{ some Thing is Thing}) \wedge (\mu(\psi)[s/(l)a])$. By lemma 2, we can assume that D'' does not contain the same premise twice. γ'_2 does not contain any number, other than m , of a line in which s occurs.⁶ Also, γ'_1 does not contain m .⁷ We now add to D'' the following lines:

$$\begin{array}{ll} q+1 & (q+1) \text{ s is Thing Premise} \\ (\gamma'_1 \cup \gamma'_2) \setminus \{q+1, m\} & (q+2) \mu(\delta) \quad \text{some } E, p, m, q+1, q \end{array}$$

Call the resulting proof D''' . We have $T_{D'''}((\gamma'_1 \cup \gamma'_2) \setminus \{q+1, m\}) \vdash \mu(\delta)$. It is now sufficient to show that $T_{D'''}((\gamma'_1 \cup \gamma'_2) \setminus \{q+1, m\}) \subseteq$

⁵The proof requires one application of AE , and one application of $\wedge E$.

⁶ $T_{D'}(\gamma_2)$ contains no formula, other than $\mu(\psi)$, in which s occurs. And given the relation between $T_{D''}(\gamma'_2)$ and $T_{D'}(\gamma_2)$, it follows that $T_{D''}(\gamma'_2)$ does not contain any formula, other than $((l)s \text{ is Thing}) \wedge \mu(\psi)[s/(l)a]$, in which s occurs. And that formula appears as a premise in D'' only in line m .

⁷ $T_{D''}(\gamma'_1)$ is identical with $T_{D'}(\gamma_1)$ up to the replacement of $\mu(\psi)$ by $((l) \text{ some Thing is Thing}) \wedge (\mu(\psi)[s/(l)a])$. And $T_{D'}(\gamma_1)$ did not contain $\mu(\psi)$.

$\mu(T_D((\alpha \cup \beta) \setminus \{j\}))$). Assume that $\chi \in T_{D'''}((\gamma'_1 \cup \gamma'_2) \setminus \{q+1, m\})$. Then $\chi \in T_{D'''}(\gamma'_1 \cup \gamma'_2) = T_{D''}(\gamma'_1 \cup \gamma'_2) = T_{D''}(\gamma'_1) \cup T_{D''}(\gamma'_2)$. Also, since $\chi \notin T_{D'''}(\{q+1, m\})$, $\chi \neq ((l)s \text{ is Thing}) \wedge \mu(\psi)[s/(l)a]$. Therefore:

$\chi \in (T_{D''}(\gamma'_1) \cup T_{D''}(\gamma'_2)) \setminus \{((l)s \text{ is Thing}) \wedge \mu(\psi)[s/(l)a]\}$. And since $T_{D''}(\gamma'_1)$ and $T_{D''}(\gamma'_2)$ are identical to $T_{D'}(\gamma_1)$ and $T_{D'}(\gamma_2)$ up to the replacement of $\mu(\psi)$ by $((l) \text{ some Thing is Thing}) \wedge (\mu(\psi)[s/(l)a])$, it follows that $\chi \in (T_{D'}(\gamma_1) \cup T_{D'}(\gamma_2)) \setminus \{\mu(\psi)\} \subseteq [\mu(T_D(\alpha)) \cup \mu(T_D(\beta))] \setminus \{\mu(\psi)\} = \mu(T_D(\alpha) \cup T_D(\beta)) \setminus \{\mu(\psi)\} = \mu(T_D(\alpha \cup \beta)) \setminus \{\mu(\psi)\} \subseteq \mu(T_D((\alpha \cup \beta) \setminus \{j\}))$.

Therefore, we have

$T_{D'''}((\gamma'_1 \cup \gamma'_2) \setminus \{k+2, m\}) \subseteq \mu(T_D((\alpha \cup \beta) \setminus \{j\}))$, as required. ■

5.5 Paraphrases

In this section we show that every formula φ in our system is both semantically and deductively equivalent to a formula φ^* , which is the translation of some formula φ_π of FOL. We first correlate, with each formula, a set of paraphrases:

Definition 34 (Paraphrases). *Let L be a formal language in our system, and let φ be a formula in L . The paraphrases of φ are defined by induction on formulas:*

1. *Atomic formulas: if s_1, \dots, s_n are SREs and R is an n -place predicate ($n \geq 1$), then: the only paraphrase of s_1 is s_2 is itself; The only paraphrase of s_1 isn't s_2 is: $\neg(s_1 \text{ is } s_2)$; the only paraphrase of (s_1, \dots, s_n) is R is itself; the only paraphrase of (s_1, \dots, s_n) isn't R is: $\neg((s_1, \dots, s_n) \text{ is } R)$.*
2. *If α and β are formulas that do not contain anaphors of SRE occurrences, then: the paraphrases of $\neg\alpha$ are all the formulas of the form $\neg(\alpha')$, where α' is a paraphrase of α . Similarly, the paraphrases of $\alpha \wedge \beta$ are the formulas of the form $\alpha' \wedge \beta'$ (where α' and β' are paraphrases of α and β respectively); those of $\alpha \vee \beta$ are the formulas of the form $\alpha' \vee \beta'$; and those of $\alpha \longrightarrow \beta$ are the formulas $\alpha' \longrightarrow \beta'$.*

3. If φ results from the substitution of anaphors for SRE occurrences in a formula ψ , as in section 2-c of the formula definition, then the paraphrases of φ are those of ψ .
4. Let $\varphi(qP)$ be a formula in which an occurrence t of the QNP qP is the main QNP. Let s be an SRE not occurring in φ , and let $\psi = \varphi[t/s]$.
 - (i) If q is every: the paraphrases of $\varphi(\text{every } P)$ are all the formulas of the form: $((l) \text{ every Thing is Thing}) \wedge (((l)a \text{ is } P) \longrightarrow \psi'[s/(l)a])$, where ψ' is a paraphrase of ψ and l is an index that does not occur in ψ' .
 - (ii) If q is some: the paraphrases of $\varphi(\text{some } P)$ are the formulas of the form: $((l) \text{ some Thing is Thing}) \wedge (((l)a \text{ is } P) \wedge \psi'[s/(l)a])$, where ψ' and l are as above.

Theorem 17 (Every Formula Has a Paraphrase that Translates a Formula of FOL). *Let L be a formal language in our system. If α is a formula in L , then there exist a paraphrase α' of α and formula α_π in L_π such that α' is $\mu(\alpha_\pi)$.*

Proof: by induction on formulas in L .

1. Atomic formulas: the (only) paraphrase of $s_1 \text{ is } s_2$ is itself, and it is $\mu(s_1 = s_2)$; the paraphrase of $s_1 \text{ isn't } s_2$ is $\neg(s_1 \text{ is } s_2)$, and it is $\mu(\neg(s_1 = s_2))$; the paraphrase of $(s_1, \dots, s_n) \text{ is } R$ is itself, and it is $\mu(Rs_1 \dots s_n)$; the paraphrase of $(s_1, \dots, s_n) \text{ isn't } R$ is $\neg((s_1, \dots, s_n) \text{ is } R)$, and it is $\mu(\neg Rs_1 \dots s_n)$.
2. If α, β do not contain anaphors of SRE occurrences, and $\alpha' = \mu(\alpha_\pi)$ and $\beta' = \mu(\beta_\pi)$ are paraphrases of α and β respectively, then: $\mu(\neg\alpha_\pi) = \neg\mu(\alpha_\pi) = \neg(\alpha')$, and this formula is a paraphrase of $\neg\alpha$. Similarly, $\mu(\alpha_\pi \wedge \beta_\pi) = \mu(\alpha_\pi) \wedge \mu(\beta_\pi) = (\alpha') \wedge (\beta')$, and this formula is a paraphrase of $\alpha \wedge \beta$. The proofs for $\alpha \vee \beta$ and $\alpha \longrightarrow \beta$ are similar.
3. If φ is the product of substituting anaphors for SRE occurrences in ψ , as in section 2c of the formula definition, and $\psi' = \mu(\psi_\pi)$ is a paraphrase of ψ , then it is also a paraphrase of φ .
4. Let $\varphi(qP)$ be a formula in which an occurrence t of qP is the main QNP, and assume that the theorem holds for any formula of the form

$\varphi[t/s]$, where s is an SRE. Let ψ be $\varphi[t/s]$, where s is an SRE that does not occur in φ (we thus have $\varphi(qP) = \psi[s/qP]$). Assume that $\psi' = \mu(\psi_\pi)$ is a paraphrase of ψ .

- (i) If q is *every*: $\mu(\forall x((Ps \longrightarrow \psi_\pi)[s/x])) =$
 $((l)every \ Thing \ is \ Thing) \wedge (\mu(Ps \longrightarrow \psi_\pi)[s/(l)a]) =$
 $((l) \ every \ Thing \ is \ Thing) \wedge ((\mu(Ps \longrightarrow \mu(\psi_\pi))[s/(l)a]) =$
 $((l) \ every \ Thing \ is \ Thing) \wedge (((s \ is \ P) \longrightarrow \mu(\psi_\pi))[s/(l)a]) =$
 $((l)every \ Thing \ is \ Thing) \wedge (((l)a \ is \ P) \longrightarrow \mu(\psi_\pi)[s/(l)a]) =$
 $((l)every \ Thing \ is \ Thing) \wedge (((l)a \ is \ P) \longrightarrow \psi'[s/(l)a])$. And this formula is a paraphrase of $\varphi(every \ P)$.
- (ii) The proof for the case in which q is *some* is similar.

■

Lemma 6 (Paraphrases Do Not Contain SRE-Anaphors). *Let φ be a formula in a language L . If φ' is a paraphrase of φ , then φ' does not contain any anaphors of SRE occurrences.*

This lemma can be easily proved by induction on formulas in L .

It will be convenient to correlate with each formula in our system a unique paraphrase φ' which translates some formula of FOL.

Definition 35 (φ^*). *Let L be a formal language in our system. With each formula φ in L , we correlate a paraphrase φ^* of φ , which is the translation of some formula in L_π .⁸*

Theorem 18 (Equivalence of φ and φ^*). *Let φ a formula in a language L in our system. Then:*

- (1) φ and φ^* are semantically equivalent. That is: $m \models \varphi \iff m \models \varphi^*$ for any model m for L .
- (2) φ and φ^* are deductively equivalent. That is: $\varphi \vdash \varphi^*$ and $\varphi^* \vdash \varphi$.

⁸In case L is denumerable, we can arrange its formulas in lexicographic order, and define φ^* as the first paraphrase of φ that translates some formula in L_π . Otherwise, we can use the axiom of choice.

Proof: (1) follows from (2) and the soundness of our deductive system (theorem 7): If $\varphi \vdash \varphi^*$ and $\varphi^* \vdash \varphi$, then $\varphi \models \varphi^*$ and $\varphi^* \models \varphi$. That is: every model of $\{\varphi\}$ satisfies φ^* , and every model of $\{\varphi^*\}$ satisfies φ .

It remains to prove (2). We shall prove the following proposition by induction on $\#\varphi$:

(3) Let φ be a string. If φ is a formula in a language L (in our system), and φ' is any paraphrase of φ , then $\varphi \vdash \varphi'$ and $\varphi' \vdash \varphi$.

1. If $\#\varphi \leq 3$, then φ is an atomic formula in L . If φ is s_1 is s_2 , then $\varphi' = \varphi$ and (3) holds trivially. If φ is s_1 isn't s_2 , then φ' is $\neg(s_1$ is $s_2)$. To see that (3) holds in this case, one needs only to apply the rules *NC E* and *NC I*. The proofs for (s_1, \dots, s_n) is R and (s_1, \dots, s_n) isn't R are similar.

2. Let $\#\varphi = n$, and assume that (3) holds for any string ψ for which $\#\psi < n$.

(a) If φ is atomic, the proof is as above.

(b) Assume that $\varphi \in \{\neg\alpha, \alpha \wedge \beta, \alpha \vee \beta, \alpha \longrightarrow \beta\}$, where α and β contain no anaphors of SRE occurrences. We shall prove (3) for $\alpha \vee \beta$ (The proofs for $\alpha \wedge \beta$, $\neg\alpha$ and $\alpha \longrightarrow \beta$ are similar). By definition 34, $(\alpha \vee \beta)'$ is $\alpha' \vee \beta'$, where α' and β' are paraphrases of α and β respectively. By theorem 6, α' and β' do not contain anaphors of SRE occurrences. Since, by the induction hypothesis, α and β are deductively equivalent to α' and β' respectively, we have: $\alpha \vdash \alpha'$, $\beta \vdash \beta'$. It follows that $\alpha \vee \beta \vdash \alpha' \vee \beta'$ and $\alpha' \vee \beta' \vdash \alpha \vee \beta$. Therefore, it is not hard to prove that $\alpha \vee \beta \vdash (\alpha \vee \beta)'$. That is: $\alpha \vee \beta \vdash (\alpha \vee \beta)'$. The proof for $(\alpha \vee \beta)' \vdash \alpha \vee \beta$ is similar.

(c) Let φ be the product of substituting anaphors for SRE occurrences in a formula ψ as in section 2c of the formula definition. If (3) holds for ψ , then it obviously holds for φ ; for φ is deductively equivalent to ψ (to show this, one can apply the rules *A I* and *A E*), and they both have the same paraphrases.

(d) Let $\varphi(qP)$ be a formula in which an occurrence t of qP is the main QNP. Let ψ be $\varphi[t/s]$, where s is an SRE that does not occur in φ . We thus have $\varphi(qP) = \psi[s/qP]$.

i. If q is *every*, then φ' is of the form:

$$((l) \text{ every Thing is Thing}) \wedge (((l)a \text{ is } P) \longrightarrow \psi'[s/(l)a])$$

where ψ' is a paraphrase of ψ . We first show that $(\varphi(\text{every } P))' \vdash \varphi(\text{every } P)$. Since $(\varphi(\text{every } P))'$ contains no anaphors of SRE occurrences (lemma 6), we have the following proof:

- | | | | |
|------|-----|--|---------------------------|
| 1 | (1) | $((l) \text{ every } Thing \text{ is } Thing)$ | |
| | | $\wedge(((l)a \text{ is } P) \longrightarrow \psi'[s/(l)a])$ | <i>Premise</i> |
| 2 | (2) | $s \text{ is } P$ | <i>Premise</i> |
| | (3) | $s \text{ is } Thing$ | <i>Th I</i> |
| 1 | (4) | $((l)s \text{ is } Thing) \wedge (((l)a \text{ is } P) \longrightarrow \psi'[s/(l)a])$ | <i>every E, 1, 3</i> |
| 1 | (5) | $(s \text{ is } Thing) \wedge ((s \text{ is } P) \longrightarrow \psi')$ | <i>A E, 4</i> |
| 1 | (6) | $(s \text{ is } P) \longrightarrow \psi'$ | $\wedge E, 5$ |
| 1, 2 | (7) | ψ' | $\longrightarrow E, 2, 6$ |

Now from the induction hypothesis it follows that $\psi' \vdash \psi$. And according to lemmas 4, 2 and 3, we can add lines to the above proof until we get:

$$1, 2 \quad (k) \quad \overset{\dots}{\psi} \quad J_k$$

And since ψ is $\varphi[t/s]$, we can add the line:

$$1 \quad (k+1) \quad \varphi(\text{every } P) \quad \text{every } I, 2, k$$

It remains to prove that $\varphi(\text{every } P) \vdash (\varphi(\text{every } P))'$. Since ψ is $\varphi[t/s]$, we have the following proof:

- | | | | |
|------|-----|----------------------------|----------------------|
| 1 | (1) | $\varphi(\text{every } P)$ | <i>Premise</i> |
| 2 | (2) | $s \text{ is } Thing$ | <i>Premise</i> |
| 3 | (3) | $s \text{ is } P$ | <i>Premise</i> |
| 1, 3 | (4) | ψ | <i>every E, 1, 3</i> |

Now, from the induction hypothesis it follows that $\psi \vdash \psi'$. And according to lemmas 4, 2 and 3, we can add lines to the above proof until we get:

$$1, 3 \quad (k) \quad \overset{\dots}{\psi'} \quad J_k$$

We continue the proof:

1	(k + 1)	(s is P) → ψ'	→ I, 3, k
1, 2	(k + 2)	(s is Thing) ∧ ((s is P) → ψ')	∧ I, 2, k + 1
1, 2	(k + 3)	((l)s is Thing) ∧ (((l)a is P) → ψ'[s/(l)a])	A I, k + 2
1	(k + 4)	((l) every Thing is Thing) ∧ (((l)a is P) → ψ'[s/(l)a])	every I, 2, k + 3

From the existence of the above proof it follows that
 $\varphi(\text{every } P) \vdash (\varphi(\text{every } P))'$.

ii. If q is *some*, then $(\varphi(\text{some } P))'$ is of the form:

$((l) \text{ some Thing is Thing}) \wedge (((l)a \text{ is P}) \wedge \psi'[s/(l)a])$, where ψ' is a paraphrase of ψ . We first prove that $(\varphi(\text{some } P))' \vdash \varphi(\text{some } P)$. Since $(\varphi(\text{some } P))'$ contains no anaphors of SRE occurrences, we have the following proof:

1	(1)	((l) some Thing is Thing)	
		∧ (((l)a is P) ∧ ψ'[s/(l)a])	Premise
2	(2)	s is Thing	Premise
3	(3)	((l)s is Thing) ∧ (((l)a is P) ∧ ψ'[s/(l)a])	Premise
3	(4)	(s is Thing) ∧ ((s is P) ∧ ψ')	A E, 3
3	(5)	(s is P) ∧ ψ'	∧ E, 4
3	(6)	s is P	∧ E, 5
3	(7)	ψ'	∧ E, 5

From the induction hypothesis it follows, as before, that $\psi' \vdash \psi$. And according to lemmas 4, 2 and 3, we can add lines to the above proof until we get:

3	(k)	ψ	J _k
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Since ψ is $\varphi[t/s]$, we can apply *some* I, and add the following lines to the proof:

3	(k + 1)	ϕ(some P)	some I, 6, k
1	(k + 2)	ϕ(some P)	some E, 1, 2, 3, k + 1

It follows that $(\varphi(\text{some } P))' \vdash \varphi(\text{some } P)$.

It remains to prove that $\varphi(\text{some } P) \vdash (\varphi(\text{some } P))'$. We start with the following proof:

1	(1)	$\varphi(\text{some } P)$	<i>Premise</i>
2	(2)	$s \text{ is } P$	<i>Premise</i>
3	(3)	ψ	<i>Premise</i>

From the induction hypothesis it follows that $\psi \vdash \psi'$, and we can add lines to the above proof until we get:

3	(k)	\dots ψ'	J_k
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We proceed:

2, 3	(k + 1)	$(s \text{ is } P) \wedge \psi'$	$\wedge I, 2, k$
	(k + 2)	$s \text{ is Thing}$	<i>Th I</i>
2, 3	(k + 3)	$(s \text{ is Thing}) \wedge ((s \text{ is } P) \wedge \psi')$	$\wedge I, k + 1, k + 2$
2, 3	(k + 4)	$((l)s \text{ is Thing}) \wedge (((l)a \text{ is } P) \wedge \psi'[s/(l)a])$	<i>A I, k + 3</i>
2, 3	(k + 5)	$((l) \text{ some Thing is Thing})$ $\wedge (((l)a \text{ is } P) \wedge \psi'[s/(l)a])$	<i>some I, k + 2, k + 4</i>
1	(k + 6)	$((l) \text{ some Thing is Thing})$ $\wedge (((l)a \text{ is } P) \wedge \psi'[s/(l)a])$	<i>some E, 1, 2, 3, k + 5</i>

It follows that $\varphi(\text{some } P) \vdash (\varphi(\text{some } P))'$, as we wanted to prove. ■

Definition 36 (T^*). *Let L be a formal language in our system. If T is a theory in L , then T^* is $\{\varphi^* \mid \varphi \in T\}$.*

Theorem 19 (Invariance of Entailment and Provability under $\alpha \mapsto \alpha^*$). *Let T be a theory in a language L , and let φ be a formula in L . Then:*

1. $T \models \varphi \iff T^* \models \varphi^*$
2. $T \vdash \varphi \iff T^* \vdash \varphi^*$

Proof: 1. Assume that $T \models \varphi$. Let m be a model of T^* . If $\psi \in T$, then $m \models \psi^*$, and by theorem 18: $m \models \psi$. We therefore have: $m \models T$, and it follows that $m \models \varphi$. Therefore, by theorem 18: $m \models \varphi^*$.

The proof for $T^* \models \varphi^* \implies T \models \varphi$ is similar.

2. Assume that $T \vdash \varphi$. Then there exists a proof D of φ from T . By lemma 2, we can assume that no premise appears in D twice. Let $\langle \alpha(n)\varphi J \rangle$ be the last line in D , and let $T_D(\alpha) = \{\varphi_1, \dots, \varphi_n\} \subseteq T$.

Since (by theorem 18) $\varphi_1^* \vdash \varphi_1$, we have: $\{\varphi_1^*, \dots, \varphi_n^*\} \vdash \{\varphi_1, \varphi_2, \dots, \varphi_n\}$. And since $\varphi_2^* \vdash \varphi_2$, we have: $\{\varphi_1^*, \dots, \varphi_n^*\} \vdash \{\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n\}$. We continue by induction and get: $\{\varphi_1^*, \dots, \varphi_n^*\} \vdash \{\varphi_1, \dots, \varphi_n\}$. We also have: $\{\varphi_1, \dots, \varphi_n\} \vdash \varphi$, and since provability for theories is transitive (theorem 6), we get: $\{\varphi_1^*, \dots, \varphi_n^*\} \vdash \varphi$. Theorem 18 ensures that $\varphi \vdash \varphi^*$. And from the transitivity of \vdash for theories: $\{\varphi_1^*, \dots, \varphi_n^*\} \vdash \varphi^*$. Now, since $\varphi_1^*, \dots, \varphi_n^* \in T^*$, it follows that $T^* \vdash \varphi^*$. The proof for $T^* \vdash \varphi^* \implies T \vdash \varphi$ is similar. ■

5.6 The equivalence between our system and FOL+EI

Theorem 20 (Equivalence). *Let L be a formal language in our system, let F be the set of all formulas in L , and F_π the set of all formulas in L_π . μ , as a mapping from F_π to F , has the following properties:*

- (1) μ is injective.
- (2) μ covers F in the following sense: for each formula $\varphi \in F$ there exists a formula $\varphi^* \in \mu(F_\pi)$ that is both semantically and deductively equivalent to φ (that is: φ and φ^* are true in exactly the same models, and they are provable from each other). Also, if $T \subseteq F$, $\psi \in F$, and $T^* = \{\alpha^* \mid \alpha \in T\}$, then:
 - (a) $T \models \psi \iff T^* \models \psi^*$
 - (b) $T \vdash \psi \iff T^* \vdash \psi^*$.
- (3) μ preserves entailment and provability: for each theory $T_\pi \subseteq F_\pi$ and formula $\varphi_\pi \in F_\pi$:

- (a) $T_\pi \cup \text{El} \models \varphi_\pi \iff \mu(T_\pi) \models \mu(\varphi_\pi)$
 (b) $T_\pi \cup \text{El} \vdash \varphi_\pi \iff \mu(T_\pi) \vdash \mu(\varphi_\pi)$.

Proof: (1) is theorem 10. (2) immediately follows from theorems 18 and 19. It remains to prove (3). Let $T_\pi \subseteq F_\pi$ and let $\varphi_\pi \in F_\pi$. We have:
 $\mu(T_\pi) \models \mu(\varphi_\pi) \implies$ (by theorem 15) $T_\pi \cup \text{El} \models \varphi_\pi$
 \implies (by the completeness of the predicate calculus) $T_\pi \cup \text{El} \vdash \varphi_\pi$
 \implies (by theorem 16) $\mu(T_\pi \cup \text{El}) \vdash \mu(\varphi_\pi) \implies \mu(T_\pi) \cup \mu(\text{El}) \vdash \mu(\varphi_\pi)$
 \implies (by theorem 12 and the transitivity of provability) $\mu(T_\pi) \vdash \mu(\varphi_\pi)$
 \implies (since our system is sound) $\mu(T_\pi) \models \mu(\varphi_\pi)$. The required equivalences follow. ■

Theorem 21 (Completeness). *Let T be a theory in a language L , and let φ be a formula in L . If $T \models \varphi$, then $T \vdash \varphi$.*

Proof: Let F and F_π be as in theorem 20. We have: $T^* \subseteq \mu(F_\pi)$ and $\varphi^* \in \mu(F_\pi)$. Therefore, there exist $T_\pi \subseteq F_\pi$ and $\varphi_\pi \in F_\pi$ such that $\mu(T_\pi) = T^*$ and $\mu(\varphi_\pi) = \varphi^*$. Now, by theorem 20, and by the completeness of the predicate calculus, we get: $T \models \varphi \implies T^* \models \varphi^* \implies \mu(T_\pi) \models \mu(\varphi_\pi) \implies T_\pi \cup \text{El} \models \varphi_\pi \implies T_\pi \cup \text{El} \vdash \varphi_\pi \implies \mu(T_\pi) \vdash \mu(\varphi_\pi) \implies T^* \vdash \varphi^* \implies T \vdash \varphi$. ■

Compactness is a consequence of completeness:

Theorem 22 (Compactness). *Let T be a theory in a language L . Then, T has a model iff every finite $T_1 \subseteq T$ has a model.*

Proof: If T has a model m , then m is a model of any finite subset of T . Conversely, assume that every finite subset T_1 of T has a model. If T does not have a model, then $T \models [s \text{ isn't } s]$, and from the completeness of our system we get: $T \vdash [s \text{ isn't } s]$. Now, since any proof uses only a finite number of premises, there exists a finite $T_1 \subseteq T$ such that $T_1 \vdash [s \text{ isn't } s]$. Since our deductive system is sound, we have: $T_1 \models [s \text{ isn't } s]$. Therefore: T_1 is a finite subset of T that does not have a model. A contradiction. ■

6 Conclusion

An overview of the paper is due in this place. We started by describing, in brief outline, a new semantic analysis of natural language, according to which common nouns in noun phrases are often plural referring expressions, and not – *pace* Frege – logical predicates. We then described, again in outline, the implications of this analysis for the analysis of quantification in natural language.

This introductory discussion lead to the development of a new formal system, built on the basis of the mentioned semantic analysis of natural language. This system, unlike FOL but similarly to natural language, uses concept-letters both as plural referring expressions and as predicates; it combines quantifiers with concept-letters to form noun phrases, which occupy in sentences the same place as singular referring expressions do; the way anaphors are used in it is closer to the way anaphors are used in natural language than to that in which variables are used in FOL; and more. We have also compared and contrasted our system with many-sorted logic.

We defined formulas, derivation rules and models for our system, and proved it to be sound. We then turned to inquire its relation to FOL. For that purpose, we added to FOL a set of axioms, E1. On the other hand, while developing our system, we had introduced, having these future inquiries in mind, a special predicate to our system, *Thing*, to which any interpretation function assigns the whole domain. We correlated models in our system with models of E1 in FOL. Relying on all this, we showed how to translate formulas of FOL into our system, and proved the translation to have the following properties: first, it is one-to-one. Secondly, it covers all the formulas in our system in the following sense: every formula is both semantically and deductively equivalent to a translation of some formula of FOL. Thirdly, the translation preserves truth in a model, entailment, and provability. The completeness and compactness of our system followed immediately.

Our system can be proved to be sound, complete and compact even without the predicate *Thing*; this, however, was not done here.

Accordingly, we have demonstrated that the new analysis of the semantics of natural language can be used as a basis for the construction of a powerful formal system, sound and complete, which parallels FOL, in the sense specified above. We have thus accomplished what we set out to do in this

paper.⁹

7 Bibliography

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⁹A part of the proof of theorem 16 (p. 208) was erroneously omitted. This part is the one dealing with the case in which the last line in the proof is justified by *Id E*.