DISCUSSION

POPPER ON INDUCTION AND INDEPENDENCE*

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K. R. Popper, in his *Logic of Scientific Discovery,* Appendix *vii, considers and rejects an important claim, that "once we have found some [object] k_i to possess the property A, the probability increases that another k_j possesses the same property; and even more so if we have found the property in a number of cases."[1] Popper asserts the contrary, that the propositions that k_i is A and that k_j is A are mutually 'independent'. That is, he asserts the following two (equivalent) statements of logical probability:

$$p(Ak_j|Ak_i) = p(Ak_j)$$

$$p(Ak_n|Ak_1 \cdot Ak_2 \cdots Ak_{n-1}) = p(Ak_n)$$

Popper holds that, since these statements are true, there can be no effective probabilistic principle of induction.

However, by adapting materials supplied by Popper himself, it is possible to prove the negations of Popper's two independence statements. In the first part of the argument, I show the inconsistency of a certain set of premises. In the second, I argue that, of these premises, it is the independence statements which should be abandoned. Most of the discussion will hold relative to the assumption, implicitly made by Popper in Appendix *vii, of a domain of individuals which is denumerably infinite.[2]

J. Humburg and H. Gaifman (in [4] and [3] respectively) have used different premises to derive somewhat stronger results: for example, Humburg obtains the statement of positive instantial relevance, that $p(Ak_j|Ak_i) > p(Ak_j)$. The main advantage of the proof offered below lies in its elementary nature.

Consider a contingent universal statement '$(x)Ax$'. Let Γ be the

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[1] [5], p. 369.

[2] Cf. also Postulate 1 of Popper's formal theory of probability in Appendix *iv of [5], p. 332.

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language in which this statement is expressed. The quantifiers of Γ range over a denumerably infinite universe of discourse. Γ contains a denumerably infinite supply of names, each of which consists of a letter of the English alphabet with an integral subscript. There are a small number of assignments each of which sets up a 1-1 correspondence between names containing a given English letter and the individuals in the universe of discourse. Each name is the name of the corresponding individual. Accordingly, each assignment provides an ordering of the individuals as the first, the second, etc. Thus each individual has several names, given it in accordance with various different assignments; each individual has as many names as any other; and each name applies to exactly one individual.

Let 'a' denote the statement that $(x)Ax$. Let there be an assignment K which gives each individual one name containing the English letter 'k' and an integral subscript. Then let 'a^n' denote the conjunction $a_1 \cdot a_2 \cdots a_n$, where a_i is the statement that Ak_i, and where, according to assignment K, k_i is the i-th individual. Popper holds that a is logically equivalent to the infinite conjunction $a_1 \cdot a_2 \cdot a_3 \cdots$

(1) $p(a) = \text{Lim}_{n\to\infty}\, p(a^n)$ premiss

(2) For all i, j, $p(a_i) = p(a_j)$ symmetry premiss

(3) $p(a_n|a^{n-1}) = p(a_n)$ independence premiss

(4) $p(a^n) = p(a_n \cdot a^{n-1})$ probability calculus

(5) $p(a^n) = p(a_n|a^{n-1})\, p(a^{n-1})$

$= p(a_n|a^{n-1})\, p(a_{n-1}|a^{n-2}) \cdots p(a_2|a^1)\, p(a_1)$

(4), prob. calc.

(6) $p(a^n) = p(a_n)\, p(a_{n-1}) \cdots p(a_1)$ (5), (3)

(7) $p(a^n) = [p(a_1)]^n$ (6), (2)

(8) $[p(a_1)]^n < 1$ or $[p(a_1)]^n = 1$ probability calculus

(9) $\text{Lim}_{n\to\infty}\, [p(a_1)]^n = 0$ or $\text{Lim}_{n\to\infty}\, [p(a_1)]^n = 1$ from (8)

(10) $\text{Lim}_{n\to\infty}\, p(a^n) = 0$ or $\text{Lim}_{n\to\infty}\, p(a^n) = 1$ (9), (7)

(11) $p(a) = 0$ or $p(a) = 1$ (10), (1)

(12) $p(a) < 1$ premiss

(13) $p(a) = 0$ (12), (11)

So far everything has been in the spirit of Popper ([5], pp. 364-368);

the conclusion (13) is exactly what Popper wants to prove. One might have thought that Popper would have tried to avoid (13) and so have been led to question (3). If every contingent universal statement has zero absolute logical probability, then absolute logical probability is not after all related to content in the way suggested throughout the bulk of *The Logic of Scientific Discovery*. Popper goes on in Appendix *vii to speak of the "fine structure" of probability and content. This seems to me quite bankrupt. For we now have no idea in Popper of how to understand the notion of absolute or relative logical probability. But let us leave this criticism and press on.

Consider now g, the statement that $(\exists x)Ax$, and $\sim a$, the statement that $(\exists x) \sim Ax$. Let 'a^N' denote the disjunction $a_1 \vee a_2 \vee \ldots \vee a_n$, and let '$b^N$' denote the disjunction $b_1 \vee b_2 \vee \ldots \vee b_n$, where b_i is the statement that $\sim Ak_i$, and where, according to assignment K, k_i is the i-th individual in the domain.

(14) $p(g) = \text{Lim}_{n \to \infty} p(a^N)$ — premiss

(15) $p(\sim a) = \text{Lim}_{n \to \infty} p(b^N)$ — premiss

(16) $p(a^N) = p(a_n \vee a^{N-1})$ — probability calculus

(17)
$$\begin{aligned} p(a^N) &= p(a_n) + p(a^{N-1}) - p(a_n \cdot a^{N-1}) \\ &= p(a_n) + p(a^{N-1}) - p(a^{N-1}|a_n)\, p(a_n) \\ &= p(a_n) + p(a_{n-1}) + \ldots + p(a_1) - p(a^{N-1}|a_n)\, p(a_n) \\ &\quad - p(a^{N-2}|a_{n-1})\, p(a_{n-1}) - \ldots - p(a^1|a_2)\, p(a_2) \end{aligned}$$
(16), probability calculus

Similarly,

(18)
$$\begin{aligned} p(b^N) &= p(b_n) + p(b_{n-1}) + \ldots + p(b_1) - p(b^{N-1}|b_n)\, p(b_n) \\ &\quad - p(b^{N-2}|b_{n-1})\, p(b_{n-1}) - \ldots - p(b^1|b_2)\, p(b_2) \end{aligned}$$

(19) If $p(a_1) = 0$ then $p(a_n) = p(a_{n-1}) = \ldots = p(a_2) = 0$ — from (2)

(20) If $p(a_1) = 0$ then $p(a^N) = 0$ — (19), (17)

(21) If for each n, $p(a^N) = 0$ then $\text{Lim}_{n \to \infty} p(a^N) = 0$ — premiss

(22) If $p(a_1) = 0$ then $\text{Lim}_{n \to \infty} p(a^N) = 0$ — (20), (21)

(23) If $p(a_1) = 0$ then $p(g) = 0$ — (22), (14)

(24) $p(g) > 0$ — premiss

(25) $p(a_1) > 0$ (24), (23), prob. calc.

(26) $p(b_1) = 1 - p(a_1)$ probability calculus

(27) $p(b_1) < 1$ (26), (25)

(28) $\sim Ak_1$ is logically equivalent to $(x)(x = k_1 \supset \sim Ax)$ premiss

Let 'c' denote the statement that $(x)(x = k_1 \supset \sim Ax)$, and let '$c^n$' denote the conjunction $c_1 \cdot c_2 \cdot \cdot \cdot c_n$, where c_j is the statement that $(l_j = k_1) \supset \sim Al_j$, and where, according to some assignment L, l_j is the j-th individual in the domain.

(29) $p(c) = p(b_1)$ (28), probability calculus

(30) $p(c) < 1$ (29), (27)

(31) $p(c) = \text{Lim}_{n\to\infty}\, p(c^n)$ premiss

(32) For all s, t, $p(c_s) = p(c_t)$ symmetry premiss

(33) For all n, $p(c_n | c^{n-1}) = p(c_n)$ independence premiss

(34) $p(c^n) = [p(c_1)]^n$ proof follows (4)-(7)

(35) $\text{Lim}_{n\to\infty}\, p(c^n) = 0$ or $\text{Lim}_{n\to\infty}\, p(c^n) = 1$ proof follows (8)-(10)

(36) $p(c) = 0$ or $p(c) = 1$ (35), (31)

(37) $p(c) = 0$ (36), (30)

(38) $p(b_1) = 0$ (37, (29)

(39) For all i, j, $p(b_i) = p(b_j)$ symmetry premiss

(40) $p(b_n) = p(b_{n-1}) = \ldots = p(b_2) = 0$ (39), (38)

(41) $p(b^N) = 0$ (40), (18)

(42) If for each n, $p(b^N) = 0$ then $\text{Lim}_{n\to\infty}\, p(b^N) = 0$ premiss

(43) $\text{Lim}_{n\to\infty}\, p(b^N) = 0$ (42), (41)

(44) $p(\sim a) = 0$ (43), (15)

(45) $p(a) = 0$ and $p(\sim a) = 0$ (44), (13)

But the probability calculus assures us that this is a contradiction.

From various uncontroversial principles of the probability calculus and elementary logic and mathematics, together with certain other premises, a contradiction has been deduced. One concludes that at least one of the other premises must be false.

Consider firstly (1), (14), (15) and (31), which assert the equality of the absolute logical probabilities of various universal statements with the limits of the probabilities of increasingly long conjunctions or disjunctions of singular statements. Popper explicitly asserts (1), hinting that it can be justified by Kolgomorov's "axiom of continuity", an axiom added to the axioms of the probability calculus to enable treatment of infinite formulae.[3] Clearly, Popper's assertion of (1) commits him to the analogous (14), (15), (31).

(2), (32), (39) have been described as "symmetry premises," following Carnap's rather than Popper's terminology.[4] Leaving aside the difficulty noted earlier concerning the fine structure of probability and content, one might argue that, since the expressions 'k_i', 'k_j', etc. are pure individual constants, the sentences 'Ak_i', 'Ak_j', etc. do not differ in degree of content, and so not in absolute logical probability. Popper himself adopts such a premiss: in [5], p. 366.[5]

The premises (12) and (24), that $p[(x)Ax] < 1$ and that $p[(\exists x)Ax] > 0$ are impregnable in the present context. Popper is firmly committed to them: it is his constant theme that the logical probability of a statement decreases with increasing degree of falsifiability, and that universal statements are, other things being equal, more highly falsifiable than existential ones; cf. also [5], p. 366.

The truth of the limit theorems (21) and (42) is intuitively obvious: if each member of a series has a certain value, then that series is not converging on some other value. Adapting a standard definition of 'limit', we can say that

> $\mathrm{Lim}_{n\to\infty}\, p(a^N) = q$ if and only if for every positive ϵ, however small, there exists a (finite) number m such that, if $n > m$, $|p(a^N) - q| < \epsilon$.

Consider any number $q > 0$. Then one can choose positive ϵ such that $q > \epsilon$. Now if $p(a^N) = 0$ then $|p(a^N) - q| = q$ and so $|p(a^N) - q| > \epsilon$. But, ex hypothesi, there is no n for which $p(a^N) \neq 0$. So $q \neq \mathrm{Lim}_{n\to\infty}\, p(a^N)$.

(28), the premiss that $\sim Ak_1$ is logically equivalent to $(x)(x = k_1 \supset \sim Ax)$, is bound up with such principles as the Indiscernibility of Identicals. It involves a concept of strict identity, rather than various relative identity concepts. To this extent it is controversial. But it is also highly plausible.

[3] [5], pp. 366, 365, 346.

[4] [1], pp. 483ff; [5], pp. 326f, 331.

[5] Also suggestive of a symmetry assumption are Popper's remarks on relatively atomic statements, [5], p. 128: text and footnote *2.

Symmetry premiss (32) has been formulated in the light of the fact that $p[(k_1 = k_1) \supset \sim Ak_1) \neq p[(k_2 = k_1) \supset \sim Ak_2]$. Even if, say, $l_3 = k_1$ and so $l_4 \neq k_1$, $p[(l_3 = k_1) \supset \sim Al_3] = p[(l_4 = k_1) \supset \sim Al_4]$.

If we accept all the above premises, and also acknowledge that their conjunction with the independence premises (3) and (33) is inconsistent, then we have no option but to reject these independence premises.

REFERENCES

[1] Carnap, R. *Logical Foundations of Probability.* (2nd ed.). Chicago: University of Chicago Press, 1965.

[2] Carnap, R. and Jeffrey, R. (eds.). *Studies in Inductive Logic and Probability.* Vol. I. Berkeley and Los Angeles: University of California Press, 1971.

[3] Gaifman, H. "Applications of DeFinetti's Theorem to Inductive Logic." In [2], pages 235-251.

[4] Humburg, J. "The Principle of Instantial Relevance." In [2], pages 225-233.

[5] Popper, K. R. *The Logic of Scientific Discovery.* (2nd ed.). London: Hutchinson, 1968.