Hacking, Ian: *Why Is There Philosophy of Mathematics At All?* Cambridge U. P. 2014

> [The historian]… takes us away from simple and absolute judgements and by returning to the historical context entangles everything up again... If history can do anything it is to remind us of those complications that undermine our certainties...
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> Herbert Butterfield *The Whig Interpretation of History* (1931) p. 58

This is a book of stories. Hacking has always been a historically-minded philosopher and this book continues in this vein. He develops each topic by offering vignettes, telling details and in some cases reminiscences. The effect of these is as Butterfield describes in the quotation above, to tangle things up and make it difficult to sustain simple, clean notions of proof, or of applied mathematics (to pick the two structuring concepts of this work). This is not to say that Hacking is a historian, because historians have another objective (which Butterfield discusses elsewhere), which is to tell a joined-up story with no significant gaps or omissions. This, for the most part, Hacking does not do (nor need he, given his aims). His procedure, rather, is to break up the concepts that figure in the stories and theories of philosophers. Are you a philosopher of mathematics? Be aware that mathematics is a 'motley'. Do you have a philosophical view about proof? Be aware that there are at least two notions of proof in the philosophical literature, which invite very different philosophical commentaries, and many more in the historical record. Are you developing a sophisticated view about Platonism in mathematics? Be aware that there are many Platonisms, and the Platonisms of mathematicians (those few who commit themselves to print on philosophical topics) are different species from the platonisms of philosophers.

The frontispiece has three quotations (from Wittgenstein, Lakatos, Howard Stein) and a little gem of a proof quoted from Littlewood. In the introduction, Hacking says that he remains 'infatuated' with Wittgenstein's *Remarks on the Foundations of Mathematics*, which he bought, so he tells us, on 6 April 1959 (p. 2). Wittgenstein is indeed a visible presence, and influences this book in several dimensions. At the end of the book, just before the references, is a list of 'disclaimers', in which Hacking records his contacts with some of the philosophers and mathematicians who appear in the main text. One of the earliest and longest refers to Lakatos, with whom Hacking coincided at Cambridge when they were both doctoral students. Hacking observes that, "There is much more of Imre's influence in the present book than meets the eye" (p. 258). Lakatos may not be the deepest influence, however; his entry in the index of this book is much shorter than that of Wittgenstein, and his influence is indeed of a less obvious sort. Wittgenstein has by far the longest index-entry, easily twice that of his nearest rival, Kant. I shall come to the influences of Wittgenstein and Lakatos shortly, but as a reading expereince, this book is nothing like their works. Hacking deals here with deep questions about mathematics and rationality, but his writing does not demand of readers that they share the strain and pain of digging-down-to-bedrock, as Wittgenstein's work seems to. Nor does Hacking share Lakatos's taste for combat or his Manichean division of scientists and philosophers into friends and enemies of reason. Hacking says much with which many will disagree, but he does not seem to relish such disagreements or seek to heighten them as Lakatos did. If Wittgenstein digs hard with his spade until it rings against rock and Lakatos brandishes a switchblade, Hacking's approach requires a butterfly net or perhaps a trowel.

Consequently, as a reading experience, this book is more like some of the works of Foucault. This similarity is not an accident. In *Historical Ontology* (2002), Hacking wrote, "My work has been seriously influenced by Foucault (or by successive Foucaults) for many years. Books I have written and books I am writing reek of his effect on me." (*Historical Ontology* p. 70). A little later in the same work, Hacking describes Foucault's pleasure in obscure details, "Foucault's genius is to go down to the little dramas, dress them in facts that hardly anyone else had noticed..." (*Historical Ontology* p. 74). Like Foucault, Hacking delights in the historical byways that philosophers and other grand theorists overlook. He deploys the historical stories and asides for philosophical ends, but it's clear that he would relish them even if they lacked this significance.

Hacking mentions Foucault twice in this book. The first occasion is in the course of a discussion of the ancient Greek origins of proof. Hacking quotes Bruno Latour's complaint that Plato and Aristotle used the idea of mathematical proof to introduce a particularly bullying rhetoric into philosophy, namely, the claim that some arguments are self-evident and irresistible (p. 133). Against this, he sets André Lichnerowicz, who praises the Eleatic philosophers for discovering proof, but complains that they brought in with proof a focus on mathematical objects which Lichnerowicz (in keeping with his Bourbakist sympathies) deplores. These origin-stories, says Hacking, are not "history-as-fact" (p. 135). In contrast, Foucault's 'archaeology', Hacking observes, is "resolutely history-as-fact, but intended not to illuminate the past but to light up the present". I think we can take it that all these readers of the ancient past (Latour, Lichnerowicz, Foucault and Hacking) find their motivations in the present. The issue here is the extent to which fidelity to the historical record is helpful to the contemporary philosophy of contemporary mathematics. On this, Hacking is clear, "moral history, or history-as-parable,… is often much more useful in philosophy than history-as-fact." (p. 135). Here, we might reasonably conjecture, is the influence of Lakatos, who taught that real history is often the caricature of its rational reconstructions—but it's the rational reconstruction that is philosophically illuminating.[1] Philosophy may be history teaching by examples, but the examples need quite some editorial work before they can fulfil their pedagogic function. Does this mean that Hacking's parables contain falsehoods? No, but nor do they add up to history, because that is not their function.

The second mention of Foucault occurs during a discussion of Alain Connes' Platonism. Though the mention of Foucault is brief, the discussion is typical of this book, so it's worth recounting. The focus is an interview that Connes gave to the general science magazine *La Recherche*. Connes claimed that while the tools that mathematicians use in their investigations are largely man-made, the object of mathematical enquiry is a sort of primordial mathematical reality that "precedes the elaboration of concepts".[2] Connes' French term (here translated as 'primordial') is '*archaïque'*. This sends Hacking on a brief digression on the greater richness of the French term compared to the English word 'archaic'. The French word, according to Hacking, associates with archaeology, with the hidden roots of the present, with the Greek sense of an origin or ancient spring, and specifically with Homeric Greece. Then we meet Foucault:

> In conversation about 1980, Michel Foucault assured me that the 'arch' in his word 'archive' in *The Archaeology of Knowledge* was intended to recall the meaning of a source, and origin—and not just a collection of old documents. That meaning is not heard in the English word 'archive'.[3]

This is the tone of the whole book. Philological care serves philosophy by ensuring that we do not miss nuances in translation or suppose that the persistence of a word entails the unity of a concept. This little digression passes through German as well as English, French and Greek. At the same time, there is no attempt to hide the fact that this is one scholar's view, and the thoughts in it are the result of conversations with a relatively small number of philosophers and mathematicians. This is not anecdote pretending to be social science. It is, rather, philosophy that makes no attempt to hide the conditions of its production. This is another difference from Wittgenstein and Lakatos—they would insist that philosophy is not personal, even though their personalities and biographies colour ever page of their works.

[1] Lakatos 1978 p. 138.
[2] Connes (2000) quoted on p. 200 of Hacking.
[3] P. 200

The same philological spirit has Hacking distinguish platonisms from Platonisms—the latter have some non-trivial connection to the philosophy of Plato, while the former are technical positions in current philosophy of mathematics. Alain Connes' view is a Platonism (with a capital P), and sounds rather like the (capital P) Platonism of Albert Lautman. Hacking contrasts this with Tim Gowers' avowed anti-Platonism. Though Connes and Gowers both present their metaphysical stances as doctrines, they also offer them as attitudes (p. 198), and it is this that interests Hacking. His remarks at the very end of the book suggest that for him, both Gowers and Connes attitudes (to 'Platonism') capture something of the experience of doing mathematical research. On the last page of the book, he gives the last word to Lichnerowicz, who attempts to reconcile these two aspects of mathematical experience with a distinction between discovery (where the mathematician feels constrained by an external mathematical reality) and publication (where the satisfaction of formal requirements has none of that feeling). One might wonder whether Gowers could live with this, as he seems to see his anti-Platonism as true of his creative work as well as of the working up of proofs for publication.

So much, then, for the style and approach. What of the argument?

The question that gives the book its title divides into two. First, why has there been philosophy of mathematics since the recorded beginnings of western philosophy in ancient Greece? And second, why do we have philosophy of mathematics of the sort that professional philosophers currently practice? The answer to the first question is a pair of stories about mathematics; the answer to the second is a story about philosophy.

Hacking's answer to the first question is that mathematics gives us two marvels: the startling experience of compelling mathematical proof, and the surprising application of mathematics. Hacking delights in the details of this two-part answer, which he discusses over five rather oddly organised chapters. In the first chapter, he starts with the application of mathematics and in particular with the fact that it can apply to itself. Problems in geometry can turn into algebra, and problems in number theory can turn into geometry (to give just two examples). For Hacking, this is philosophically interesting, but it comes into view only when we appreciate the fact that different parts of mathematics use different methods to explore different things. Philosophical enquiries that start from the assumption that all of mathematics can be reduced to or reconstructed from a common foundational basis will fail to register how remarkable it is that there are deep connections between diverse parts of mathematics. Hacking mentions Kenneth Manders on this point (pp. 10-11), but the thought runs further back, to

Wittgenstein's insistence that mathematics is a motley of proof techniques. Here, Hacking supplies another discussion of nuances exposed by translation, and argues that Wittgenstein's original phrase "ein BUNTES *Gemisch* von Beweistechniken" indicates an especially disorderly mixture of proof techniques, a hold-all containing things of different kinds. 'Buntes' has connotations of multi-colours, like Jacob's coat (pp. 57-59) (though perhaps the often-patched 'coat of many colours' that Dolly Parton sang about is a nearer analogy, since 'buntes' has a mildly disparaging connotation).

In the second half of the first chapter, he takes up the first marvel of mathematics, namely, the experience of proof. He contrasts two modern notions, which he labels 'cartesian' and 'leibnizian' (following his convention of using lower-case letters to indicate labels that have a merely indirect connection with the philosopher whose name they invoke). 'Cartesian' proof is the dawning 'aha', the clear insight that in some cases can be induced by a few lines of argument, exemplified by the example from Littlewood on the frontispiece. One of the sources of philosophy of mathematics, according to Hacking, is the impression such 'aha' experiences makes on some philosophers (he goes on to claim that Wittgenstein may have been one) (p. 28). Hacking is surely correct that philosophers too often take such proofs to be typical of mathematical practice, when in fact proofs have been getting longer and it is not normally the case that even an expert can keep the whole of a proof in mind as a single apprehension. In contrast, the ideal 'leibnizian' proof is a mechanically checkable finite sequence of operations on well-formed formulae. This distinction cuts through philosophy of mathematics--philosophers divide into those who work on the assumption that mathematical proofs are essentially (convertible into, indicative of or otherwise related to) ideal 'leibnizian' proofs, and those who attempt to undermine this assumption by pointing to proofs that are close to the 'cartesian' ideal. For Hacking, philosophical ideals are interesting only insofar as they manifest in practice, so he motivates his distinction by suggesting that something like the Leibniz conception animates Voevodsky's univalent foundations programme, and that on the other side there is something Cartesian about Grothendieck's vision of the aims and nature of mathematics. Hacking ends this chapter with a reminder that a motley is a disorderly collection, and so these two ideals are not the start of a tidy taxonomy of proofs.

The second chapter asks what is it that makes mathematics mathematics. The philological discussion of 'motley' occurs here, and this remains Hacking's watchword as he shows the variety of definitions in the principal dictionaries of European languages and retraces the history of failed attempts at definitions and philosophical accounts of mathematics. The chapter ends with some reflections on the importance of playfulness in mathematics, with references to chess problems and Conway's game of life. Hacking does not say as much, but his discussion suggests that this ludic aspect may be distinctive of mathematics. People do study other sciences for recreation, but other sciences do not have recreational problems.

Chapter three takes up the question of the book's title and reiterates, in greater detail, the answer sketched in the first chapter. Hacking refines the question: he wants to know why mathematics elbows its way in to philosophical discussions that are not primarily motivated by curiosity about mathematics (such as Plato's moral epistemology or Kant's analysis of the structure of empirical experience). The two features he picks out are perennial: "Philosophising about mathematics comes back to proof and use, over and over again." (p. 83). Over the first half of the chapter, he explores one aspect of the experience of proof, namely, the sense that one is engaged in discovery rather than exploration. This is half of a dialectical opposition that animates the last two chapters, the other half being the sense that mathematics is a product of human activity (and so doing it is really more like invention). Hacking indicates that his sympathy is with the latter horn of the dilemma, but insists that this opposition needs some sort of resolution, which is not currently available (pp. 93-4).

The second half of this third chapter considers the philosophical efforts through the twentieth century to understand the application of mathematics. This section illustrates the sense in which a philosophical problem may be perennial. As Collingwood argued in the case of political philosophy (1993 p. 229), it's not that there is a single question that philosophers keep failing to answer. The question of political authority was for Plato a different problem from the question that troubled Hobbes. The background assumptions and success criteria of their enquiries differ so much that they are best regarded as two different questions. So with the philosophical question of the applicability of mathematics. As philosophers changed their minds about related notions such as necessity and analyticity, the question mutated so that one generation's plausible solution did not look like any sort of contribution to the next. On the other hand, all of these philosophers (starting, in Hacking's telling, with Kant) were perplexed by the use of mathematics. It's a new question every time, but the same old perplexity.

So far, half of chapter one and half of chapter three are devoted to proof, with the corresponding halves treating the applicability of mathematics. In chapter four, proof is the sole topic, while chapter five is given over to application. Where the discussions in the previous chapters exhibit the miscellaneous variety of applications and proof techniques, chapters four and five argue that both demonstrative proof and the idea of applied mathematics are contingent developments, originating in small communities that might never had existed. "Neither demonstrative proof nor the pure/applied distinction was inevitable. Both are the contingent products of historical events... They could have been otherwise, or not come into being at all." (p. 116). This claim is quite subtle; all civilisations have mathematics, but only one ancient society proved theorems (and, as Hacking might have added, several medieval traditions that knew about Euclid did not take up theorem-proving). All civilisations use mathematics, but it is only in recent centuries that we have come to distinguish pure mathematics from its applications, and therefore it is only recently that we have had to wonder at the practical usefulness of (what now seems to be) a pure product of the mind.

Chapter five traces the distinction between pure and applied mathematics from Plato on through early modern science and in to the twentieth century. Hacking describes this distinction as robust (it has lasted in form or another since antiquity) but not sharp (because the dividing line falls differently in different times and places). Late in the chapter, he takes aim at what he calls the 'representational-deductive picture' of the application of mathematics. In this picture, the scientist invents a mathematical representation of the phenomenon to be understood. Then, the mathematical consequences of the model are deduced, as an exercise in pure mathematics. Finally, these consequences are interpreted with respect to the phenomenon under study. This has the consequence that mathematics is never really applied to phenomena, and certainly not mixed with them. Hacking argues, partly by examples and partly by invoking Kuhn, that this is just one, suspiciously tidy, kind of application.

The end of chapter five is the end of Hacking's treatment of the first version of the book's title-question: why has philosophy of mathematics been a constant thread in the history of philosophy since antiquity? The concepts invoked in the answer that he gives at the outset—the twin wonders of proof and application—have, by the end of chapter five, been so thoroughly worked over historically that one hesitates to deploy them without qualification. The effect of this is to make it all the more remarkable that these two features of mathematical experience persist, in strikingly different guises, down the centuries. It is the fact of perennial philosophical perplexity about proof and application, through all the change and contingency, that makes it plausible that Hacking has put his finger on something deep.

In the final two chapters, Hacking turns his attention to the Platonisms of mathematicians and the platonisms of philosophers. As Hacking tells it, the animating dilemma of contemporary philosophy of mathematics arises from the combination of denotational semantics and causal accounts of knowledge.

On one hand, a plausible general account of the semantics of name-words is that they denote objects. Philosophers who think this, and who further think that mathematical language should have the same semantics as all other discourses, find themselves driven towards platonism, because number-words seem to work like names. On the other hand, many philosophers think that knowledge requires causal interaction between the knower and the objects of knowledge. This, together with platonism, makes a mystery of mathematical knowledge.

Hacking traces this dilemma to Benacerraf's paper of 1973 (p. 216), and tells a story about how the presuppositions of this dilemma came to be cemented into place. He directs his critical attention to the claims that semantics should be uniform across discourses and that name-words refer. He laments that Strawson's claim that "words do not refer, but that speakers use words to refer" seems to have fallen out of favour (p. 221). Inspired by Wittgenstein's maxim that meaning is (sometimes) use, he suggests that there is no need to believe that people use number-words to refer to numbers (p. 251). This all happens rather quickly, and Hacking is careful not to claim deep expertise about semantics. His retelling of the story of this kind of platonism is intended to show, I think, that this too is contingent. There is nothing inevitable about the framework within which philosophers debate platonism, nor is it especially virtuous. Hacking complains that, "much contemporary platonism in [philosophy of] mathematics fails to explain anything about mathematical practice" (p. 253). In contrast, the Platonism of Alain Connes responds to something in the experience of doing mathematical research. Hacking ends this discussion with another suggestion, drawn from Pierce, that the art of using words to conjure new things into existence may be an important kind of action in mathematics. He does not say as much, but the sequence suggests that perhaps this is what people (sometimes?) do with number-words (perhaps in something like the way banks create money).

The book ends with a recognition that the tensions in mathematical experience towards and away from Platonism remain unresolved. This may seem inconclusive, and masks the fact that Hacking in this book argues for a clear thesis: there is nothing inevitable about either our mathematics or our philosophy. There is no immutable order of being that requires us to cultivate mathematics as we happen to do, nor are we obliged to adopt a philosophical framework that drives us towards believing in an immutable order of being. The book may seem to ramble and include a lot of irrelevant material, but that is part of the point; human endeavours, including our most rational endeavours, are like that. A book that marched directly towards this conclusion with every appearance of inevitability would be self-refuting. On this point, too, Hacking may have learned a lesson from Wittgenstein.

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