Frege, Thomae, and Formalism: Shifting Perspectives*

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Abstract

Mathematical formalism is the the view that numbers are "signs" and that arithmetic is like a game played with such signs. Frege's colleague Thomae defended formalism using an analogy with chess, and Frege's critique of this analogy has had a major influence on discussions in analytic philosophy about signs, rules, meaning, and mathematics. Here I offer a new interpretation of formalism as defended by Thomae and his predecessors, paying close attention to the mathematical details and historical context. I argue that for Thomae, the formal standpoint is an *algebraic perspective* on a domain of objects, and a "sign" is not a linguistic expression or mark, but a representation of an object within that perspective. Thomae exploits a shift into this perspective to give a purely algebraic construction of the real numbers from the rational numbers. I suggest that Thomae's chess analogy is intended to provide a model for such shifts in perspective.

1 Frege, Thomae and the chess analogy

For a mathematical formalist, *numbers are signs.* So say both proponents and opponents of formalism in their earliest discussions, starting around the

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middle of the nineteenth century. Johannes Thomae was one of those early proponents. Gottlob Frege, his colleague in Jena, was one of the early critics. Today Thomae is best known in philosophy because of the extensive criticism that Frege leveled at his formalism, which occupies almost fifty pages of Frege's *Grundgesetze* (Frege, 2013a, §§86–137). For many, Frege's attack marks both the beginning and the end of Thomae's relevance in the philosophy of mathematics. I am here to say that there's more to the story: once we see Thomae's formalism without the distorting lens of Frege's criticisms, we'll be able to see both why Frege felt it needed attacking, and why it was important for later developments.

For Frege, Thomae was the focal point of the formalist milieu. Before coming to Jena, Thomae was a student and later colleague of Eduard Heine, whose formalism Frege attacks alongside Thomae's. Thomae may also have known Hermann Hankel, another of Frege's formalist targets, from his graduate studies in Göttingen; certainly he knew Hankel's work, and cited Hankel's formalism as a predecessor of his own (Thomae, 1908). At the same time, Thomae was close to Frege. They had a common dissertation advisor in Göttingen (Ernst Schering), though several years apart. After Thomae's arrival in Jena in 1879, the two worked closely together for more than 20 years[1], handling most of the university administrative tasks in mathematics together; they also met privately for discussion at Ernst Abbe's house (Dathe, 1997; Tappenden, 2008). It was thus natural for Frege's longest and most intense discussion of formalism to focus on Thomae's view, which he probably knew best, and which he regarded as more precisely worked out than Heine's (Frege, 2013a, §86).

Thomae advocates a formalist point of view in the opening pages of his *Elementare Theorie der analytischen Functionen einer complexen Veränderlichen* (Thomae, 1898), a textbook on complex analysis. To explain his "formal conception" of numbers, he offers an analogy:

> The formal conception of numbers draws itself more modest limits than the logical. It asks not: what are the numbers and what shall

[1]Frege's attack in *Grundgesetze* unfortunately coincided with the collapse of their professional relationship, which is on display in a poisonous exchange in the *Jahresbericht der Deutschen Mathematiker-Vereinigung* that begins with Thomae's reply to Frege's criticisms (Thomae, 1906b; Frege, 1906; Thomae, 1906a; Frege, 1908a; Thomae, 1908; Frege, 1908b; for discussion see Gabriel, 1979; Dathe, 1997; Wille, 2020).

> they be?[2] but rather it asks: what does one require of the numbers in arithmetic? Now, for the formal conception, arithmetic is a game with signs which one may well call empty, thereby conveying that (in the calculating game) no other content belongs to them than the content attributed to them with respect to their behavior under certain combinatorial rules (game rules). A chess player makes similar use of his figures: he attributes certain properties to them which determine their behavior in the game, and the figures are only external signs for this behavior.[3] (Thomae, 1898, 3)

Thomae is among the first to defend formalism by drawing an analogy between arithmetic and chess, though similar ideas were in the air.[4] Thomae's version of formalism has been called *game formalism* by recent commentators to mark this aspect of his view (Resnik, 1980; Linnebo, 2017; Weir, 2010, 2020).

The analogy compares the signs of arithmetic and pieces in the game of chess.

[2]This question is a clear reference to Dedekind's book *Was sind und was sollen die Zahlen?* (Dedekind, 1893), which attempts to develop purely logical foundations for the natural numbers. Thomae is contrasting his "formal" conception of numbers with the "logical" conception he sees in both Dedekind and Frege.

[3]Die formale Auffassung der Zahlen zieht sich bescheidenere Grenzen als die logische. Sie fragt nicht, was sind und was wollen [*sic*] die Zahlen, sondern sie fragt, was braucht man von den Zahlen in der Arithmetik. Die Arithmetik ist für die formale Auffassung ein Spiel mit Zeichen, die man wohl leere nennt, womit man sagen will, dass ihnen (im Rechenspiel) kein anderer Inhalt zukommt als der, der ihnen in Bezug auf ihr Verhalten gegenüber gewissen Verknüpfungsregeln (Spielregeln) beigelegt wird. Aehnlich bedient sich der Schachspieler seiner Figuren, er legt ihnen gewisse Eigenschaften bei, die ihr Verhalten im Spiel bedingen, und die Figuren sind nur äussere Zeichen für dies Verhalten.

[4]The idea of treating numbers as "signs" equipped with rules appears already in Heine (1872, 173), and Thomae already deploys this idea in the first edition of his book (Thomae, 1880); but neither makes mention of games or a comparison with chess. Paul Du Bois-Reymond, who also knew Thomae personally, came closer: he criticizes formalism as treating numbers as "figures" which one can use as "play magnitudes" (*Spielgrößen*), rather than signs for "actual magnitudes" (*wirkliche Größen*), but also makes no direct comparison with chess (Du Bois-Reymond, 1882, 50; cf. Kienzler, 2009, 278). Frege (1984, 118) gives the earliest explicit comparison I've found between the *rules* of chess and arithmetic, though as part of a criticism of formalism, not a defense. The French mathematician Louis Couturat also compared mathematical objects with chess *pieces* shortly before Thomae (1898), writing that the mathematician "creates mathematical entities by means of arbitrary conventions, in the same way that the several chessmen are defined by the conventions which govern their moves and the relations between them" (Couturat, 1896; quoted in Stenlund, 2015, 53).

The passage suggests that signs in arithmetic are “empty”: like arbitrary pieces of wood, they don’t mean anything on their own. But in the presence of a system of rules for manipulating them, they take on a kind of meaning, just as those pieces of wood take on a meaning in a chess game.

That, anyway, is how Frege himself interpreted the analogy: it’s meant to show how a system of rules can impart content to meaningless signs. Frege assumes throughout his discussion in *Grundgesetze* that “signs” are “formations that are created by means of writing or printing on the surface of a physical object (blackboard, paper)” (Frege, 2013a, §98).[5] Signs are thus concrete, perceptible items for Frege, bare pieces of syntax which we could write down or say aloud, but which are meaningless on their own. Accordingly, he assumes that Thomae’s chess analogy is meant to show how rules can infuse such concrete items with meaning or content. Against this, he objects that a system of rules *can’t* impart content, as we can see already in the case of chess:

> I acknowledge that the chess pieces are there, and also that rules have been laid down for their manipulation; but I know nothing of any content. It cannot simply be said that the black king designates something as a consequence of these rules, like the name “Sirius” designates a certain fixed star. (Frege, 2013a, §95)

Thomae’s chess analogy, and Frege’s interpretation of it, have gone on to play an important role in analytic philosophy. On the one hand, Frege’s interpretation of the analogy continues to permeate contemporary discussions of formalism in the philosophy of mathematics. Linnebo, for example, reads Thomae as a formalist who holds that “mathematics revolves around formal systems, which are syntactical games played with meaningless expressions” (Linnebo, 2017, 40). On the other hand, Frege’s interpretation inspired later developments in the philosophy of language, especially through Wittgenstein. Many discussions of how linguistic meaning relates to rules of use can be traced back to Wittgenstein’s reflections on formalism, the chess analogy, and Frege’s criticism of Thomae. (cf. Kienzler, 1997, chap. 5; Stenlund, 2015; Dehnel, 2020; Weir, 2020). Wittgenstein thought that Frege “did not see the other, justified side of formalism”, and used the chess analogy to explain a conception of meaning that he thought Frege had missed (Waismann, 1979,

[5]Frege stuck to this understanding of signs across his discussions of formalism over many years: cf. Frege (1980, 95), Frege (1984, 115), Frege (1997, 132), Frege (2013b, p. XIII).

105).

Frege's interpretation of the "signs" in Thomae's analogy as pieces of syntax in search of a semantics is admittedly natural when the analogy is read on its own. But there are some reasons to be cautious about this reading. First of all, Thomae was not writing in a context where our modern notions of a formal language and the distinction between syntax and semantics were available, and could not have thought of signs as syntactic items in our sense. He also clearly understands signs to be different from linguistic expressions or printed marks: as I will discuss below, the main kind of "sign" Thomae is interested in is an infinite decimal representation of a real number, and thus not something we could write down or say aloud. Finally, we ought to remember that Frege is a notoriously uncharitable interpreter of his intellectual opponents, including other formalists like Hermann Hankel (Tait, 1996; Tappenden, 2019; Lawrence, 2021). As the rest of this essay will show, Frege's interpretation of the chess analogy is in fact pretty far from what Thomae intends.

The real story is much more interesting, and provides an important window onto the context in which Frege developed his logicism, and from which twentieth century discussions of signs, rules and meaning begin.

When Thomae and other formalists speak of "signs", they don't mean concrete pieces of syntax, but representations of objects from a certain mathematical perspective. This is an *algebraic* perspective which they call the "formal standpoint". When we take up this perspective on a domain of objects, we focus on their relations under the arithmetical operations, ignoring other properties they have from other perspectives. We can come to view a domain of objects *as* numbers from the formal standpoint by defining the arithmetical operations on those objects.

To understand this "formal conception" of numbers, we need to look at its historical context. The formal standpoint originates in Kant's understanding of algebra, and is closely associated with an algebraic approach to the foundations of analysis advocated by Karl Weierstrass in the late nineteenth century. This program required a purely algebraic construction of the real numbers. For Thomae, formalism provides a route to that construction, because it allows us to view certain sequences of rational numbers *as* real numbers. When we adopt the formal standpoint on these sequences, we shift into a perspective in which we ignore their internal structure and just think of them as operands for arithmetic operations.[6]

This, I will conclude, is the idea that Thomae's chess analogy is meant to support. When we play chess, we shift from a perspective where we think of the pieces using physical concepts, like being brown or made of wood, into a perspective where we think of them using chess concepts, like being a white bishop or being in check. Similarly, we can shift between thinking of mathematical objects from different perspectives, using different mathematical concepts. The chess analogy's most important function is to show us that such shifts are possible.

2 Formalism as an algebraic perspective

Thomae describes his formalism as a "standpoint" (Thomae, 1898, 3), and this language should be taken seriously. Formalism is a perspective or point of view we can adopt on numbers. Thomae's view is that this perspective is useful for certain mathematical purposes, but not that we must adopt it for all of mathematics, to the exclusion of all others.

What do I mean by 'perspective' or 'standpoint'? A full answer to this question goes well beyond the scope of this paper, but here is a sketch that will suffice for now. To have a perspective on a domain of objects is to see or look at them in a certain way. In the cases I am interested in, having a 'way of looking at' some objects means applying a certain set of concepts to them,

[6] As I develop my interpretation, it will become clear that Thomae's view has parallels to the views now discussed under the heading of *mathematical structuralism*, since structuralism also involves adopting a standpoint in which we focus on certain properties of a domain of objects (namely, the 'structural' properties) and ignore others. In terms of the taxonomy of Reck and Schiemer (2023), Thomae's formalism can be usefully compared to *set-theoretic* as well as to (Dedekindian) *abstractionist* structuralism, since it involves both a particular construction of the real numbers from sequences of rationals, together with an attitude of indifference to, or abstraction from, the non-arithmetical features of that particular construction.

I would caution against taking these parallels too far, though. Much discussion of structuralism is focused on metaphysical issues for which Thomae displays little patience. Thomae is also uncomfortable with set theory, a point I discuss below. Finally, structuralism and formalism generally take different attitudes toward the results of abstractions, which shows up for example in Dedekind's talk of "free creation" of numbers (cf. Reck, 2003) in contrast to Thomae's talk of "signs". Just how far the parallels extend is an interesting but complex issue that I must leave for future work.

and *not* applying other concepts. Thus, looking at some stones, I can adopt a geological perspective and think of them as *metamorphic* or *from different strata*, and within this geological perspective, I refrain from thinking of the stones as *paperweights* or from considering one *more beautiful* than another.

There are mathematical perspectives in this sense, too. The formal standpoint is an *algebraic* perspective. To adopt it means to conceive and reason about a domain of objects using algebraic concepts. Such reasoning considers their relations to each other under the arithmetic operations; it considers them only as operands for these operations. An algebraic perspective likewise excludes conceiving and reasoning about these objects using non-algebraic concepts, such as geometric ones. I will explain the notion of algebraic perspective here and argue that the formal standpoint is best seen as an algebraic perspective. The next section will then explain the role of this perspective in Thomae's presentation of analysis, and what it tells us about his understanding of "signs".

2.1 Form, content and algebra

To understand the idea that the formal standpoint is an algebraic perspective, it helps to look at how algebraic problem solving was understood in Thomae's context. I shall start the story with Kant, who had a major influence on that context. As Shabel (1998) explains, in Kant's time, algebra was not seen as having its own subject matter. It was instead conceived as a general tool for manipulating 'magnitudes', different kinds of which were studied in the branches of mathematics that *did* have subject matter, namely, arithmetic and geometry. One would translate a problem given about a certain number, or a certain figure, into algebraic notation in order to apply algebraic problem solving methods, and then translate the result back to a specification of the solution in the original problem domain.

Here is how Kant expresses this understanding of algebra in the pre-critical essay "Inquiry concerning the distinctness of the principles of natural theology and morality":

> In both [algebra and arithmetic], there are posited first of all not things themselves but their signs … *one operates with these signs according to easy and certain rules*, by means of substitution, combination, subtraction and many kinds of transformation, so

> that *the things signified are themselves completely forgotten in the process*, until eventually, when the conclusion is drawn, the meaning of the symbolic conclusion is deciphered. (Kant, 1992, 250, emphasis added)

According to Kant's understanding here, algebraic problem solving involves temporarily adopting a perspective in which we ignore what signs originally signify, and just manipulate them "according to easy and certain rules". The rules Kant has in mind are laws governing the arithmetic operations. For example, the law of the distribution of multiplication over addition,

$$a(b + c) = ab + ac$$

may be understood as a rule that permits transforming, say, $x(x + 1)$ into $x^2 + x$ or vice versa. When such manipulations have transformed the problem statement into a more useful form, we can then return to our original perspective, "deciphering" the result into a conclusion about the original "things signified".

For Kant, the algebraic perspective was 'formal' because it was indifferent to the kind of content signified. It operated with whatever content it inherited from the original problem domain: that is, numerical or geometric magnitudes. So long as those magnitudes could be equipped with an appropriate definition of the arithmetic operations—addition, multiplication, exponentiation, and their inverses—then problems concerning those magnitudes could be translated into the language of algebra, and the rules governing those operations could be used to find solutions to those problems via algebraic manipulation.

Although Kant calls the representations transformed in such manipulations "signs", we should not understand them in a syntactic sense, akin to words in a language. In the passage where this quote appears, Kant is drawing a distinction between mathematical and philosophical reasoning, and he does so in part by *contrasting* mathematical signs with words. The problem with philosophical reasoning is that "the signs employed. . . are never anything other than words", which do not reveal the relationships between the concepts they signify. Mathematical signs, in contrast, are particular representations like geometric diagrams, which "facilitate thought" by exhibiting such conceptual relationships. (Kant, 1992, 251) So mathematical signs here are less like arbitrary pieces of syntax, and more like what Kant will later describe as constructions of concepts in intuition: they are particular representations that support reasoning to universally valid conclusions in mathematics. The algebraic perspective is thus one in which we can represent any kind of magnitude as an arithmetic operand, and transform those representations according to the laws of arithmetic.

This association between rule-governed algebraic manipulation of signs and the 'form' side of the form-content distinction persisted into the nineteenth century, even as algebra broke new ground and significantly expanded its scope. Indeed, the association still holds today. Modern algebra studies structures such as groups or fields which pair a set of objects with some operations on those objects. The structure is defined by the properties of the operations—the rules that they follow, such as being associative or transitive. Even more so than in Kant's day, the study of the structure is agnostic to the *kind* of objects being operated on: they might be numbers, or polynomials, or matrices, or transformations of planar figures, or any other objects that can be equipped with suitable definitions of the relevant operations. Applying algebraic methods means moving to a perspective in which we temporarily ignore the features which distinguish these kinds of objects, and just focus on the common features they share as operands of these operations. Once we have obtained a result from this formal perspective, we can re-interpret it as a result about numbers, or polynomials, or whatever.

2.2 Signs, numbers, and the focus on operations

In speaking of a "formal standpoint", Thomae and other formalists of the nineteenth century are invoking the understanding of algebra just laid out. The formal standpoint is *formal* because it is content-agnostic, focused on the common features different kinds of objects have *qua* operands for a set of operations. Like Kant, they call the content-agnostic representations of these objects *signs*.

While modern algebra takes a more general view of which operations can induce an algebraic perspective, Thomae and other nineteenth century formalists remained focused on the operations of elementary arithmetic. They emphasized the role of laws governing these operations for defining systems of numbers. Their idea was that numbers, from the formal perspective, should just be conceived as 'whatever the operations operate on'. For example, in one widely read defense of formalism, Hermann Hankel says:

> I call the signs of such a system numbers, and thus set their concept in a necessary context with the operations through which they are formed and pass into one another. Every change of the operations brings a change of the numbers with it. (Hankel, 1867, 36)

Similarly, Eduard Heine, Thomae's teacher and later colleague in Halle, emphasized the arithmetic operations in his formalist construction of the real numbers:

> I adopt the purely formal standpoint for the definition, *in that I call certain graspable signs numbers... A main emphasis is to be laid on arithmetic calculation*, and the number-sign must be so chosen, or equipped with such an apparatus, that it affords an insight into the definition of the operations.[7] (Heine, 1872, 173)

Thomae also thinks of systems of numbers as being defined by the arithmetic operations, emphasizing in several places that this *exhausts* the conception of numbers we get from such a definition:

> If one puts forward the postulate that these operations should always be able to be carried out, one arrives at new number-constructs, at zero, the negative numbers, and the rational numbers. These can be understood as purely formal constructs, i.e., as concepts whose content is exhausted through their behavior with respect to the calculating rules.[8] (Thomae, 1898, 4)

Notice that these authors all speak of numbers as "signs" in connection with the idea that numbers are to be defined in relation to the arithmetic operations. Again, this language can be traced back to the view of algebra expressed by Kant: when we adopt an algebraic perspective, we temporarily ignore any content associated with signs outside this perspective, and just work with the signs themselves, which are representations that facilitate algebraic reasoning.

[7] Ich stelle mich bei der Definition auf den rein formalen Standpunkt, *indem ich gewisse greifbare Zeichen Zahlen nenne... Ein Hauptgewicht is auf die Rechenoperation zu legen*, und das Zahlzeichen muss so gewählt, oder mit einem solchen Apparate ausgerüstet werden, dass es einen Anhalt zur Definition der Operationen gewährt.

[8] Stellt man aber die Forderung, dass diese Operationen immer ausführbar sein sollen, so gelangt man zu neuen Zahlengebilden, der Null, den negativen und gebrochenen Zahlen. Diese lassen sich als rein formale Gebilde auffassen, d. h. als Begriffe, deren Inhalt durch ihr Verhalten gegen die Rechnungsregeln erschöpft ist.

For Kant, the "things signified" by such signs were paradigmatically quantities given in intuition, and adopting an algebraic perspective meant temporarily ignoring their other given features, such as geometric relations or empirical properties. But we might just as well take signs to be representations of other kinds of mathematical objects, like vectors or sequences. As we will see below, that is exactly what Thomae did. For Thomae, a "sign" is a representation of a certain kind of sequence. This representation focuses on its behavior under the arithmetic operations, and ignores its other features; it is a way of thinking of the sequence *as* an arithmetic operand—that is, as a number.

This reading helps clarify one of Thomae's more puzzling remarks. As we saw above, Thomae introduces the formal standpoint by saying that it "asks not: what are the numbers and what shall they be?" (Thomae, 1898, 3) In saying that formalism does not ask what the numbers "are", he is invoking the content-agnostic nature of algebraic methods: numbers are anything it makes sense to add, subtract, multiply, and divide. Formalism instead asks "what does one require of the numbers in arithmetic?" because algebraic methods presuppose a definition of these operations for the objects to be operated on. Such definitions are what we require of the numbers (or whatever other objects we're working with) in order to take up an algebraic perspective.

2.3 The formal standpoint is not exclusive

When we think of the formal standpoint as an algebraic perspective in this sense, it becomes clear that it is only one perspective among others. When we look at, say, geometric quantities from the formal standpoint, we focus only on adding, subtracting, multiplying and dividing them. But we could also look at those same quantities from a geometric perspective, focusing on whether they are parallel, whether they are commensurable, how to construct them, and so on. The formal standpoint makes no claim to being the *only* perspective we can adopt. As Thomae says, the formal standpoint "draws itself more modest limits than the logical".

Thomae suggests in several places that there are other perspectives on numbers we can adopt. For example, he thinks "named" numbers take us beyond the formal standpoint:

> However there are cases in which not merely a formal meaning (*formale Bedeutung*) is attributed to the numbers, for example in

> the sentence "This equation is of degree three", that is, when the numbers *occur as named*.[9] (Thomae, 1898, 3, emphasis added)

Despite the mathematical character of his example, Thomae's main reason for this view is that "named" numbers[10] occur in *applications* of arithmetic, when some quantity is counted or measured. (The terminology appears to derive from the fact that in applying arithmetic, we name the *unit* of the quantity to which a number applies: three 'apples', 14.8 'degrees', etc.) This relation to another quantity gives the numbers a non-formal meaning or content. We can think of such applications as involving a shift in perspective, in which we stop viewing the numbers purely as operands for the arithmetic operations and instead view them as properties of something else, like quantities in the empirical world.

But Thomae insists that *in analysis*, we do not need such a perspective on the numbers:

> In the formulae of arithmetic and analysis, the numbers are unnamed, and one needs nothing more from them as a medium for operations than that which the formal definition gives them.[11] (Thomae, 1898, 4)

The "unnamed" numbers Thomae has in mind here are those in the domain or range of a function—the "medium" on which that function operates. Of course, particular numbers are sometimes named in analysis, for example as coefficients. Thomae's point is not that we never refer to individual numbers

[9]Allerdings giebt es Fälle, in denen auch in der Arithmetik den Zahlen nicht blos eine formale Bedeutung zukommt, z.B. in dem Satze, "diese Gleichung ist vom Grade drei", also wenn die Zahlen als benannte auftreten.

[10]The distinction between *benannte* (named) and *unbenannte* (unnamed) numbers which Thomae invokes here was evidently an established idiom. In for example Müller (1900, 309), a French-German mathematical dictionary, *benannt* is equated with French *concret* (concrete) and *unbenannt* with *abstrait* (abstract). (Thanks to Ansten Klev for bringing this to my attention, and to Tabea Rohr for the reference.) I have not yet found a more exact explanation, and so have not attempted a non-literal translation; but it is clear that the *benannt-unbenannt* distinction goes beyond the literal meaning of "naming" numbers and expresses a distinction between numbers as used in applied vs. pure arithmetic. Interestingly, Frege also uses the terminology this way in Frege (1884, 46), though he later objected to it as confusing (Frege, 2013a, §73 n. 2).

[11]In den Formeln der Arithmetik und Analysis sind die Zahlen *unbenannte*, und *man braucht von ihnen als Operationsmedium nichts, als was die formale Definition in sie legt.*

in analysis, but that analysis is not concerned with how the elements of a function's domain are interpreted in applications of arithmetic—say, as lengths or temporal durations. From the perspective we adopt in analysis, all that matters about a domain of numbers is their behavior under the basic arithmetic operations.

Thus, when Thomae compares arithmetic to a "calculating game" in which

> no other content belongs to [signs] than the content attributed to them with respect to their behaviour under certain combinatorial rules

the negative form of this claim is important: *no other* content belongs to the signs in this perspective. Thomae is emphasizing that when we adopt the formal standpoint in analysis, we ignore or "screen off" the non-algebraic content we attach to numbers from other perspectives, such as when we make applications of arithmetic in geometry or physics.

Thomae's formal standpoint is therefore a *non-exclusive* algebraic perspective. It is a useful perspective on the numbers in analysis; but adopting it does not preclude us from adopting other perspectives. Why, though, might it be useful to adopt this perspective *in analysis*? That is the question to which I now turn.

3 Formalism and the history of analysis

Thomae presents his formalism in a textbook about complex analysis. Analysis is often thought of as a part of mathematics which *contrasts* with algebra: non-algebraic concepts and operations play an important role in analysis, especially those that come from calculus, such as continuity and differentiation. If Thomae's formal standpoint is an algebraic standpoint, what is it doing in a textbook on complex analysis? What mathematical purpose does the formal standpoint serve?

As Thomae's title ("Elementary theory of functions of a complex variable") indicates, his goal is to give an *elementary* presentation of his topic. He explains what he means by this in the foreword to the book's first edition, which appeared in 1880:

> Although the thought of grounding the theory of functions

> elementarily—I mean without application of infinitesimal calculus, but only on the representation through power series—is an old one, and although this method possesses so many advantages because of the absolute rigor that it permits, a consistent execution of such a plan has nevertheless not been undertaken to my knowledge by any author.[12] (Thomae, 1898, Foreword to first ed.)

An "elementary" presentation thus means avoiding presenting the concepts of analysis in a way that irreducibly relies on certain ideas from calculus, and instead representing functions using *power series*. In a power series, a function $f(x)$ is defined via a possibly infinite sum of terms, where the nth term involves the nth power of x, multiplied by a coefficient a_n which does not depend on x:

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 \dots$$

Such a series is "centered" at the point c, and converges for values of x in a certain neighborhood around c.[13]

Notice that in a power series representation, though there may be infinitely many non-zero terms in the sum, each term involves only basic arithmetic operations. By restricting himself to power series representations, Thomae is thus adopting an approach where algebraic methods are particularly applicable and emphasized. As Thomae says, the advantage of this approach is that it offers "absolute rigor".

In fact, this passage signals a commitment to a definite mathematical point of view. The method and conception of rigor that Thomae evokes here come from Karl Weierstrass, who Thomae names a few sentences later. In the period in

[12] So alt der Gedanke ist, die Functionentheorie elementar, ich meine ohne Anwendung der Infinitesimalrechnung, nur auf die Darstellung durch Potenzreihen zu gründen, so viele Vorzüge diese Methode besitzt, wegen der absoluten Strenge, die sie gestattet, so ist trotzdem meines Wissens eine consequente Durchführung eines solchen Planes noch von keinem Autor unternommen.

[13] Notice, for example, that when $x = c$, all terms but the first are zero, so the series converges to a_0. But once the distance between x and c is greater than 1, the powers $(x-c)^n$ start growing infinitely large, so the series will diverge unless the coefficients a_n compensate. The Cauchy-Hadamard theorem describes the values of x for which the series converges in terms of these coefficients a_n.

which Thomae was writing, there was a rivalry among mathematicians working on the foundations of complex analysis in Germany, between the broadly "conceptual" methods favored by Bernhard Riemann and his followers, and broadly "computational" methods favored by Weierstrass and his followers (Bottazzini, 1994, 2002; Tappenden, 2006; Gray, 2015; Ferreirós and Reck, 2020). The two approaches are now mostly integrated, but that process only began in the early twentieth century. For the second half of the nineteenth century, they were in competition, disagreeing even about how to define the object of study in complex analysis. Members of each camp were critical of the methods and results of the other. Thomae's pursuit, in 1880, of an *elementary* approach to analysis via power series representations, is a clear declaration of Weierstrassian commitments, and his formalism should be understood against that background.

3.1 Thomae as a Weierstrassian[14]

Weierstrass emphasized that analysis should be founded on "algebraic truths". As he wrote in an 1875 letter to his student and colleague H. A. Schwarz, for example:

> The more I think about the principles of function theory—and I do it incessantly—the more I am convinced that this must be built on the foundation of algebraic truths, and that it is consequently not correct when the 'transcendental', to express myself briefly, is taken as the basis of simple and fundamental algebraic propositions. (Weierstrass, 1894a, 235; translated in Gray, 2015, 204)

Because Weierstrass regarded algebraic truths as fundamental for analysis, it was natural for him to work with algebraic methods. That in turn made power series representations of functions a central part of his approach: because they involve only arithmetic operations, power series representations are especially convenient for algebraic manipulation and thus for algebraic proofs of facts about, for example, convergence properties of those functions. The (sometimes long) algebraic manipulations in such proofs give them a very "calculational" feeling. Although these algebraic methods are not always

[14] I am indebted in this section to much detective work by others, especially Liebmann (1921), Bottazzini (1994), Göpfert (1999), Bottazzini (2002), Epple (2003), O'Connor and Robertson (2006), Tappenden (2006), and Gray (2015, Ch. 19).

convenient, and sometimes resulted in difficulties Weierstrass could not solve, he felt they were more rigorous than the "transcendental" approach in Cauchy and Riemann's work.

Exactly which methods did Weierstrass reject as too "transcendental" for the foundations of analysis? Gray (2015, 204) characterizes them as methods "involving the integral" and notes that integral calculus hardly played any role in Weierstrass' lectures; Bottazzini (1994, 428) says more specifically that Weierstrass avoided using Cauchy's integral theorem and the theory of residues. Thomae's "elementary" presentation of complex analysis follows Weierstrass in this respect, only mentioning integrals in a few asides and footnotes. It is also worth noting that in other texts, Weierstrass often thinks of the "transcendental" (usually opposed to the "rational") as involving essential appeal or passage to the infinite. His remark to Schwarz thus suggests a certain skepticism about using the infinite in the foundations of analysis. Thomae explicitly voices such skepticism; I will return to this point below.

Thomae was not an exclusively Weierstrassian mathematician. He completed his doctorate in Riemann's Göttingen, and in fact would have been Riemann's doctoral student, if Riemann hadn't become too ill. He brought ideas from Riemann with him after leaving Göttingen, and some of his work adopts a Riemannian approach.[15] But in the early years of his academic career, Thomae was in close contact with Weierstrassian mathematics, during a period in which it scored some major victories in the rivalry with Riemann. These developments had an influence on Thomae, and a brief look at them will help us understand his formalism.

Thomae studied as an undergraduate in Halle in 1861 and 1862, where he attended Heine's lectures on analysis. In 1862 he moved to Göttingen, finishing his doctorate there in 1864. Meanwhile, in the early 1860s, Weierstrass had begun lecturing in analysis and developing his own foundational approach. In 1861 he offered a course on "Differential and Integral Calculus" in which he proved some important results about infinite series representations of functions that made them a viable foundation for analysis.[16] This course was

[15]In his obituary of Thomae, Liebmann (1921) refers to Thomae as a "pupil of Riemann" [*Riemannschüler*] and cites a number of places in which Thomae built on Riemann's work, such as his 1867 Habilitation thesis in Halle, "De Propositione Quadam Riemanniana Ex Analysi Situs" (Thomae, 1867).

[16]For example, he proved that term-by-term differentation is valid for such a series when

attended by Schwarz, who would later bring its ideas to Heine and Thomae in Halle. Weierstrass became increasingly concerned in this period with rigor in the foundations of analysis, and began to develop his work through a four-semester cycle of lectures on analysis. Since Weierstrass refrained from publishing, attending these lectures was generally the only way to access his research.

For this reason, like many young mathematicians of the day, Thomae went to Berlin for two semesters to attend Weierstrass' lectures after finishing his doctorate. Thomae then spent a couple of years developing his own work, submitting two Habilitation theses, first in 1866 in Göttingen, and then in 1867 in Halle, where he became a lecturer. Thomae arrived back in Halle in the same year as Schwarz and Georg Cantor, who had just finished his doctorate under Weierstrass in Berlin.

So in 1867, three young mathematicians who had direct contact to Weierstrass' new program of rigor in analysis converged in Halle, where Heine was already a professor. Schwarz moved on in 1869, but Heine, Thomae and Cantor remained together in Halle from 1867 until 1874, when Thomae moved to Freiburg.

The years Thomae spent in Halle brought a string of successes for the Weierstrass camp. In 1870, Weierstrass presented a challenge to Dirichlet's principle (Weierstrass, 1894c), which had been an important foundational principle for the Riemannian approach to analysis. Then in 1872, Weierstrass presented his example of a function which is everywhere continuous but nowhere differentiable (Weierstrass, 1894b), dealing a serious blow to the role of geometric intuition in analysis, which again challenged Riemann's more geometric approach. These results spurred Riemannians to search for more rigorous foundations in their own school. Meanwhile, also in 1872, Heine and Cantor both published constructions of the irrational numbers providing for the continuity of the real line, which at the time represented an important and novel advance. Dedekind, whose construction using "cuts" is often used today, notes that he was prompted to publish his own construction because he received a copy of Heine's article (Dedekind, 1963, 3).

Heine's construction (1872) is of particular interest here. Heine opens his article by invoking Weierstrass, saying that Weierstrass' ideas have been

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the series converges uniformly (Gray, 2015, 201).

spread in lectures and conversation but have not been presented together, and that as a result there are still certain doubts in the foundations of analysis. Indeed, he says that he has hesitated to publish the article for a long time because he didn't think he was making any contribution himself, but just setting out Weierstrass' ideas and eliminating gaps in the presentation. Only then does Heine introduce his "purely formal standpoint" on which numbers are "certain graspable signs" and "a main emphasis is to be laid on arithmetic calculation", noting in a footnote that he generally introduces this perspective in his lectures on "algebraic analysis". Heine thus explicitly links Weierstrass, the formal standpoint, and an algebraic approach to analysis.

The first edition of Thomae's book appeared eight years after these developments, in 1880. By the time Thomae published the second edition of his book in 1898, Weierstrass' algebraic, power series-based approach was ascendent. In the foreword to the second edition, Thomae mentions that other such elementary presentations have appeared in the meantime, especially lectures from Weierstrass published as a textbook by Biermann (Biermann, 1887).[17] Thomae is thus clearly positioning his presentation of complex function theory within the Weierstrassian approach to analysis. That was also the approach of his colleagues and teachers; and it was gaining ground during most of his academic career. Even though Thomae took a more Riemannian approach in other work, the formalism he defends in Thomae (1898) should be seen in that historical context.

3.2 The mathematical advantages of the power series approach

The Weierstrass approach is not just about rigor for rigor's sake; it offers some real mathematical advantages. These advantages stem from the restriction to working with power series representations of functions. For example, Bottazzini (2002) and Gray (2015, 197) point out that the power series approach can be readily generalized to functions of more than one variable, which seems to have been Weierstrass' own reason for adopting this approach.

The most relevant feature of power series representations for my purposes here, though, is that a power series representation is composed only of elementary

[17] As Tappenden (2006, 126) has noted, this textbook presents itself and was received as a standard presentation of the Weierstrass approach.

arithmetic operations. It is thus "content-agnostic" in the sense described above. In a power series representation of a function $f(x)$, it doesn't matter what *sort* of object x is: so long as we know how to do elementary arithmetic operations with it, it makes sense to plug it into that representation of the function.

This means that power series representations are particularly useful when we want to generalize functions to a wider *domain*. Consider for example the problem of extending the exponential function e^x to the complex plane. What does it mean to raise e to a complex power z? If you think about exponentiation in terms of repeated multiplication, it's not clear that e^z has any meaning at all. But if you think about this function in terms of its power series representation,

$$f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

then you reduce the problem of raising a number to a complex power to the problem of doing elementary arithmetic operations with complex numbers.[18] So long as we have an account of the elementary operations for complex numbers, we can use the power series representation to extend the exponential function to the complex plane.

This basic idea lies at the heart of a technique called *analytic continuation*, one of the most powerful problem-solving methods in complex analysis. Every power series representation has a limited domain of convergence around its center: the infinite sum only converges when the terms $a_n(x-c)^n$ get smaller with increasing n. The basic idea of analytic continuation is that, by "recentering" the power series at a different point inside its domain of convergence, one obtains a new power series representation which can have a different domain of convergence. The two representations agree on every point where their domains overlap, but the new representation can also converge at points where the original representation didn't, thus "extending" or "continuing" the original function to a wider domain. By iterating this procedure, one can extend a function from one part of the complex plane to another part, sometimes even from the real line to the entire complex plane (as with e^x). Power series representations are thus a powerful conceptual

[18] Notice especially that z^n *can* be thought of in terms of repeated multiplication of z by itself.

tool for extending the theory of functions of a *real* variable to results about functions of *complex* variables. Giving an elementary presentation of complex analysis is a way to obtain access to this powerful conceptual tool.

This observation adds another dimension to Thomae's remark that the formal standpoint only asks "what does one need from the numbers in arithmetic?" The formal standpoint can be seen as part of a two-part strategy for building up complex analysis from algebraic foundations. First, we restrict attention to functions that have power series representations—that is, functions which can be defined in terms of (perhaps infinitely many) basic arithmetic operations. This restriction forbids certain ways of defining functions. But in return, it means that such functions can be readily extended to a new domain, such as the complex numbers. "What one needs" from the numbers in that domain is a definition of the arithmetic operations, because that is what makes them suitable inputs for these functions and thus a suitable target for the extension. The second part of the strategy is then to supply these definitions of the basic operations for the new domain. Let us see how Thomae does this.

4 Signs and infinitary representations

In the context of analysis, we need a representation of numbers that provides for the *continuity* of the domains of real and complex numbers.

Before Weierstrass, the real and complex numbers were generally thought of as geometric quantities, and this representation automatically provides for their continuity. But for a Weierstrassian mathematician, a geometric conception of the reals is not a suitable foundation for analysis. A Weierstrassian needs an elementary, algebraic representation of the numbers, independent of geometric intuition. Thus it became important, in the last third of the nineteenth century, to provide this representation.[19]

One strategy, going back to Gauss, was to start with the natural numbers and introduce successively-wider classes of numbers as inverse elements, while

[19] This need was also felt outside the Weierstrass camp. Weierstrass' results in the 1870s, mentioned above, put pressure on *any* view of functions on the real numbers which made essential appeal to geometric intuition. Thus Dedekind, for example, writes in his introduction to "Continuity and the irrational numbers" that geometric intuition is "useful, from the didactic standpoint" but "this form of introduction into the differential calculus can make no claim to being scientific" (Dedekind, 1963, 1).

preserving the laws of arithmetic (Boniface, 2007; cf. also Detlefsen, 2005; Tappenden, 2019). Thus, starting with the natural numbers, we introduce the negative numbers to ensure that addition always has an inverse, i.e., that there is always a number x such that $a + x = b$, even when $a > b$. Similarly, we introduce the rational numbers as inverse elements for multiplication. But the strategy of inverse elements does not work to develop the real numbers out of the rationals, so the Weierstrassians looked for a different construction that could still be considered purely arithmetical. Heine and Cantor, Thomae's colleagues in Halle, published such a construction in 1872, based on treating real numbers as convergent infinite sequences of rationals (Heine, 1872; Cantor, 1872). Thomae's construction essentially follows theirs in its technical details.

4.1 Thomae's construction of the real numbers

To motivate the need for this construction, think about what π means for a Weierstrassian. If we ignore any geometric interpretation of π, it is just a certain irrational number. To get an arithmetical grip on this number, we can perhaps think of it in terms of its infinite decimal representation $3.1415\ldots$. But what does it mean to add another number to that, especially another irrational number, given by another infinite decimal? More importantly, what does it mean to pass such numbers as arguments into functions, like the exponential function?

In Weierstrassian analysis, we need a way to do arithmetic with such irrational values. But it would be circular, in Thomae's view, to assume from the outset that we can calculate with infinite decimals. An infinite decimal is fundamentally a power series representation, an infinite sum over terms based on successive powers. The infinite decimal representation of π, for example, really means

$$3 \cdot \left(\frac{1}{10}\right)^0 + 1 \cdot \left(\frac{1}{10}\right)^1 + 4 \cdot \left(\frac{1}{10}\right)^2 + 1 \cdot \left(\frac{1}{10}\right)^3 + 5 \cdot \left(\frac{1}{10}\right)^4 + \ldots$$

which is simply the value at $x = \frac{1}{10}$ of a certain function $f(x)$, represented as a power series. Accordingly, calculation with infinite decimals must first be grounded in the elementary theory of functions which Thomae is in the process of giving; it would be circular for that theory to presuppose such calculation (Thomae, 1898, 5).

Thomae therefore adopts a different strategy based on infinite *sequences* of rationals. We think of π in terms of a sequence like $(3; 3.1; 3.14; 3.141; \ldots)$ whose successive values approximate π ever more closely. Each term in the sequence is a finite decimal, so this sequence representation does not involve the notion of *summing* over infinitely many terms and thus does not raise the circularity problem. Thomae's strategy for constructing the reals is to define real numbers as a certain class of these infinite sequences, and then to define the arithmetic operations on those sequences.

He begins by defining this class of sequences using an epsilon-delta style definition. A *regular* sequence (today known as a Cauchy sequence) is an infinite sequence of rational numbers such that one can always find a point in the sequence beyond which the differences between terms remain inside a given bound.[20] Every rational number can be trivially represented as a regular sequence: just take the number itself as each term of the sequence, so that the difference between terms is always zero. But there are also non-trivial regular sequences; among these are sequences of the initial segments of an infinite decimal. Thus, "we can calculate with infinite decimals"—and thus with irrational numbers like π—"as soon as we can calculate with regular sequences" (Thomae, 1898, 7).[21]

Thomae then proceeds to define arithmetic operations on these regular sequences of rationals. Each definition essentially defines the operation on two *sequences* by applying the operation pairwise on its *terms*. For example, given two regular sequences a and b:

$$a = (a_1; a_2; a_3; \ldots) \qquad b = (b_1; b_2; b_3; \ldots)$$

Thomae defines their sum $a + b$ as the sequence:

$$a + b := (a_1 + b_1; a_2 + b_2; a_3 + b_3; \ldots)$$

Note that the '+' being defined on the left here applies to sequences, while the '+' being used on the right applies to rational numbers. These definitions

[20]More formally and in modern notation, a regular sequence $(a_1; a_2; \ldots a_n \ldots)$ is one such that

$$\forall \epsilon > 0 \exists N \forall n \geq N, m \geq 1 |a_{n+m} - a_n| < \epsilon$$

See Thomae (1898, 7).

[21]Wir können also mit unendlichen Decimalbrüchen rechnen, sobald wir mit regulären Folgen rechnen können.

'lift' the arithmetic operations on the rationals up to infinite sequences of rationals.

Finally, Thomae proves that, given these definitions for the arithmetic operations, regular sequences are closed under these operations (i.e., if a and b are regular sequences, so is $a + b$, etc.) and they obey trichotomy (i.e., if a and b are regular sequences, then $a < b$ or $b < a$ or $a = b$). He also points out that addition and multiplication are associative and commutative, and that multiplication distributes over addition. In effect, he shows that from an algebraic perspective, the regular sequences form a number system—a field, in modern terminology.[22]

A modern reader might worry here: shouldn't real numbers be identified as something like *equivalence classes* of regular sequences? Aren't their identity conditions coarser than those of the sequences themselves? Thomae sidesteps this issue, though, by focusing on *equality* rather than *identity* of regular sequences. He expresses equality with "=" and defines it as co-convergence, that is, convergence of the difference of two sequences to 0 (Thomae, 1898, §8).[23] Since he works directly with equalities, there is no need for a further abstraction over sequences to produce the elements of the field. Instead, this definition of equality, like the definitions of the arithmetic operations, should be considered as introducing one of the concepts needed to take up an algebraic perspective on the regular sequences.

At this point, Thomae says we can call the "signs" for these sequences numbers:

> Since it has been shown that the signs assigned to regular sequences satisfy the fundamental rules ... the signs for regular sequences may be incorporated among the numbers, and we call

[22]Thomae's presentation does not follow the modern axiomatic definition of a field, but the regular sequences satisfy those axioms under the operations as he defines them. Besides the laws of commutativity, associativity, and distributivity for multiplication and addition, the modern axioms also require the existence of additive and multiplicative identity elements, and the existence of inverse elements for addition and multiplication. Thomae uses the constant regular sequences $(0; 0; \ldots)$ and $(1; 1; \ldots)$ as identity elements, and the existence of inverse elements follows immediately from his definitions of subtraction and division.

[23]In fact, Thomae *objects* to understanding "=" as expressing identity on grounds that will be familiar to Frege's readers: if we did that, "we would remain stuck at the trivial knowledge... that $a = a$" (Thomae, 1898, 2).

them numbers.[24] (Thomae, 1898, 10)

The signs Thomae has in mind here are infinite decimals: "an infinite decimal is... an abbreviation, a sign for an infinite sequence of the usual finite decimals, or a sign that is assigned to such a sequence"[25] (Thomae, 1898, 5). Thus the construction justifies calculations with infinite decimals, since these represent calculations with regular sequences of rationals. With this construction in place, then, Thomae has an elementary definition of the real numbers which allows for our ordinary way of doing arithmetic with them, and is an adequate domain for functions given by power series representations.

4.2 Why "signs"?

In the passages just quoted, Thomae follows Heine (1872) in referring to irrationals represented by infinite decimals as "signs", and says that his construction justifies us in "calling" those signs numbers. This language is puzzling for several reasons. First of all, what does Thomae mean by "sign" here? As we have just seen, Thomae calls infinite decimal representations of irrational numbers "signs", so he is clearly not thinking of signs as marks we could actually write down. So how *does* he think of them? Second, why does Thomae feel the need to speak of "signs" in this context at all? This seems like an extra level of indirection: why not just talk about doing arithmetic with regular sequences themselves, instead of with signs for them? Finally, there is a historical puzzle. By 1880 and certainly by 1898, the early results in set theory had been established and a set-theoretic construction of the reals was available.[26] Why, in 1898, does Thomae stick with a formalist conception of the irrationals, instead of using these newer tools?

[24] Nachdem gezeigt worden ist, dass die regulären Folgen zugeordneten Zeichen den Fundamentalregeln... genügen, dürfen die Zeichen für reguläre Folgen unter die Zahlen aufgenommen werden, wir nennen sie Zahlen.

[25] Ein unendlicher Decimalbruch... ist eine Abkürzung, ein Zeichen für eine unendliche Folge von gewöhnlichen endlichen Decimalbrüchen, oder ein Zeichen, das einer solchen Folge zugeordnet ist.

[26] Both Dedekind's construction of the reals using cuts, and Cantor's early results in set theory (including his first proof that the cardinality of the continuum is greater than the cardinality of the natural numbers) followed soon after Heine's formalist construction in 1872, and appeared well before the first edition of Thomae's book in 1880. Further developments in set theory, including Dedekind's construction of the natural numbers, appeared before the second edition. Thomae was aware of these developments and discusses them briefly (cf. Thomae, 1898, §3 and §11), but he shies away from making use of them.

Part of the answer to all three puzzles is that Thomae expresses a deep skepticism about actual infinity. He says that "the actual infinite is… best banished from arithmetic and analysis" (Thomae, 1898, 6).[27] This bars him from using the newer set-theoretic tools, since they involve thinking of infinite sets as actually completed totalities. For Thomae, infinite sequences cannot be viewed as completed mathematical objects. He is thus unwilling to look at the definitions he has given as defining arithmetic operations *on* regular sequences. Officially, the completed sequences aren't 'there' to be operated 'on'; and the sequence output by such an operation is no more completed than its inputs. Instead, we must think of the definitions as giving a rule which *correlates* the terms of the input sequences with those of the output sequence, so that the nth term of, say, $a + b$ is completely determined by the nth terms of a and b, for any finite n. For any given sequences, we can continue such a process of calculation as far as we need to, but the calculated sequence, like the input sequences, is only potentially infinite.

From the point of view of such skepticism, it would be misleading to speak of adding or multiplying the sequences themselves, insofar as such talk involves thinking of them as completed objects which are the input or output of an infinite process. But Thomae also holds that it is sometimes unproblematic to pass in thought from a potentially infinite process to its completed result. There is no problem with such "idealism" *so long as it only relies on the same rules as formalism*:

> But it appears to be a need of the human mind (which one can also observe in other areas of thought) to attribute to every unending process of taking more and more decimals an eventual end, which is in a certain sense real and only inaccessible to our powers and therefore ideal but not imaginary[28], which one can actually reach in the case of a periodic decimal and to which we are especially driven by geometric representation in other cases, for example in the determination of $\sqrt{2}$ (diagonal of a square). Such an idealism is also entirely harmless, insofar as it only makes use of the same

[27]Das actuelle, das wirkliche Unendlich wird… am besten aus der Arithmetik und der Analysis verbannt.

[28]Compare §12, where Thomae uses "ideal or imaginary" to describe points introduced in projective geometry as an analogy to justify the introduction of complex numbers, which he also describes as "ideal" (Thomae, 1898, 13). Thomae thus appears to be gesturing here toward the broader debate about 'ideal elements' which began in the nineteenth century.

> rules for the recognition of its ideals as formalism.[29] (Thomae, 1898, 6)

That is, it is harmless to shift from thinking about an unending process of calculation to its completed result so long as that result is 'reachable' just via an infinite sequence of arithmetic operations. That is precisely the form that Thomae's construction takes: he's defined arithmetic *operations on infinite sequences*—Thomae's real numbers—via an *infinite sequence of operations* on rational numbers. Because we can do arithmetic operations with rational numbers any finite number of times, in Thomae's view we are justified in passing from such regular sequences of rationals to their limit in a real number and doing arithmetic with those real numbers.

Still, why call these limits "signs"? It is instructive at this point to look back at Heine. When Heine introduces his formalism, he *contrasts* the formal standpoint with a strategy of introducing the irrationals as limits:

> If I do not want to stop at the positive rational numbers, I do not answer the question of what a number is by conceptually defining[30] the number, for example by introducing the irrationals as limits, whose *existence* would be presupposed. I adopt the purely formal standpoint for the definition, *in that I call certain graspable signs numbers*, so that the existence of these numbers is not in question.[31] (Heine, 1872, 173)

Because Heine calls signs "graspable" or "tangible" (*greifbar*), his formalism

[29]Aber es scheint dem menschlichen Geiste ein Bedürfniss zu sein (was man auch auf andern Denkgebieten beobachten kann), jenem endlosen Processe mehr und mehr Decimalen zu nehmen, ein schliessliches, gewissermaassen reales, nur unsern unzulänglichen Kräften unerreichbares und deshalb ideales, aber nicht imaginäres Ende zuzuschreiben, zu dem man bei einem periodischen Decimalbruch wirklich gelangen kann und wozu wir in andern Fällen, z. B. bei Bestimmung von $\sqrt{2}$, durch geometrische Vorstellungen (Diagonale eines Quadrates) besonders veranlasst werden. Ein solcher Idealismus ist auch ganz ungefährlich, sofern er sich nur zur Wiedererkennung seiner Ideale derselben Regeln bedient als der Formalismus.

[30]Heine's phrasing here may be a jab at the rival Riemannian approach, which emphasizes "conceptual definitions"; see Tappenden (2006) and Ferreirós and Reck (2020).

[31]Die Frage, was eine Zahl sei, beantworte ich, wenn ich nicht bei den rationalen positiven stehen bleiben will, nicht dadurch dass ich die Zahl begrifflich definire, die irrationalen etwa gar als Grenze einführe, deren *Existenz* eine Voraussetzung wäre. Ich stelle mich bei der Definition auf den rein formalen Standpunkt, *indem ich gewisse greifbare Zeichen Zahlen nenne*, so dass die Existenz dieser Zahlen also nicht in Frage steht.

is often read as a simple-minded empiricism.[32] But the preceding sentence indicates that Heine has a more mathematical worry in mind, namely, that introducing irrationals as limits presupposes the existence of those limits. Weierstrass expressed this worry more clearly in 1886:

> If we start from the existence of rational numerical magnitudes, it makes no sense to define the irrationals as limits of them, because we cannot know at first whether there are other magnitudes besides the rational ones. (Weierstrass, 1886; quoted in Boniface, 2007, 327, translation slightly adapted)

The problem is that, if we attempt to construct the real numbers assuming only the rational numbers as given, then we have nothing to identify with the limits, since in general those limits don't exist among the rationals. We cannot identify an irrational number as *the* limit of a sequence a, since we often won't be able to show that there *is* such a limit, unless we simply assume or postulate its existence. For Heine, speaking of "signs" instead of limits is a way to avoid this problem. The existence of a sign for a sequence is not in doubt, even when the existence of a limit for that sequence is.

Although Heine does not specify exactly what he means by "sign", the results of section 2 offer us a natural interpretation: the sign is simply the sequence of rationals itself, *as viewed from an algebraic perspective*.[33] The existence of the sign is then just as obvious as the existence of the rationals in the sequence. We don't need to find a *new* object, outside the rationals, to identify with the sign. We just need to look at the objects we've already got from a different perspective, namely, from the formal standpoint. As we saw above, this means that we focus on sequences of rationals as operands for the arithmetic operations (once these operations have been defined). Within this perspective, we can ignore the other, non-algebraic properties of sequences, such as their internal structure or their infinite length. By ignoring these properties, we arrive at a different representation, a "sign" for the sequence as originally conceived.

Thus it is Heine's commitment to *constructing* the reals out of the rationals

[32] See for example the comments about Heine in Epple (2003) or Detlefsen (2005).

[33] Heine indicates that he identifies the sign with the sequence itself, though his language does not clearly separate the sign from his notation for it: "One introduces the sequence itself, set in brackets, as sign, so that e.g. the sign belonging to the sequence a, b, c, etc. is $[a, b, c, \text{etc.}]$" (Heine, 1872, 176).

which motivates his formalism and his talk of "signs". Thomae is following Heine in using "sign" this way, and I see his remarks about "harmless idealism" as intended to explain and justify this aspect of Heine's approach. Thomae seems more aware than Heine that, when we shift from thinking about a regular sequence of rationals to a sign for that sequence, we make a nontrivial abstraction, which involves representing the sequence as a completed object, rather than something unending and incomplete. But for both authors, the advantage of calling this object a *sign* for the sequence is that this does not presuppose the existence of anything but the underlying rationals of the sequence.[34] In contrast, talk of limits either already presupposes the existence of irrational numbers, or would need to be justified by exactly the same sort of construction that Thomae has given.

So for Thomae, a "sign" for a regular sequence of rationals is a representation of that sequence. It is a view of that sequence within an algebraic perspective, in which we just look at it as an operand for the arithmetic operations. In order to adopt this perspective—the formal standpoint—we first need to define the arithmetic operations on regular sequences and prove that they are closed under these operations and obey the usual laws of arithmetic. Then within this perspective, we think of signs as completed objects, even though the sequences they represent are infinite, and calculations with them involve an infinity of calculations with the underlying rationals. The abstraction of signs idealizes away these aspects of operating with sequences; but this idealization is justified, in Thomae's view. His intention is that signs are a relatively harmless abstraction from the underlying sequences, an abstraction which does not presuppose the existence of any objects besides the rationals themselves.

Thus, Thomae's conception of "sign" is not a linguistic or syntactic one, but a notion with a Kantian pedigree: a sign is a representation that has been obtained by adopting a new perspective on something, by temporarily

[34] This does not mean there are no other conditions for the sign's existence: the sequence also needs to converge, for example. But the rationals in the sequences are the only *objects* whose existence is presupposed by the existence of the sign. This may seem strange to a modern reader: in modern set theory, the rationals *in* a sequence are different from the sequence itself, so the existence of the rationals in that sequence doesn't necessarily guarantee the existence of the sequence. But for Heine and Thomae, the sequence simply *consists* of the rationals within it, given in a certain order. This was a natural assumption for them to make in their historical context, at the very beginning of set theory.

ignoring some features it has in the original perspective, and focusing on others. Thomae's chess analogy, I now want to suggest, gives us a model for thinking about such shifts in perspectives.

5 Some reflections on the chess analogy

We have seen above that the crucial step in Thomae's formalist construction of the real numbers involves a shift into an algebraic perspective: by defining the arithmetic operations on regular sequences of rationals, we come to see those regular sequences *as* numbers. In the formal standpoint, we think of regular sequences simply as things we can add, subtract, multiply, and divide, ignoring their other properties.

Such a shift in perspective involves re-conceiving a domain of objects, representing or thinking of them under a new set of concepts. The chess analogy is important for Thomae because it provides an example of this process which is familiar and easy to grasp.

How does this work? Consider a perspective from which we view the pieces on a chessboard as ordinary physical objects. Within that perspective, we might describe them as made of wood or plastic or stone, as brown or gray, or as being 3.4cm apart. But when we actually play chess, we adopt a different perspective, in which these physical properties are ignored. We consider the pieces only as bishops or knights, as black or white, as occupying certain board positions. Material properties like what they are made of or the distance between them are irrelevant. Shifting into the chess-playing perspective thus means conceiving a domain of objects in a new way, ignoring their properties which are not relevant for playing chess.

The first purpose of the analogy is simply to show that such shifts are possible. It may not seem obvious that it is possible to think of regular sequences of rationals as real numbers. But it is obvious that it is possible to think of a piece of wood as the black queen in a chess game, at least to anyone who has learned to play chess. By comparing the formal standpoint to the perspective we adopt to play chess, Thomae is seeking to show that the important step in his formalist construction is of a familiar, unmysterious kind.

Because the material properties of chess pieces are mostly irrelevant in chess, it is possible to play with many different sorts of pieces. Indeed, one can play chess without any physical pieces at all, for example using diagrams, or a computer program. This makes the chess-playing perspective "content-agnostic" in the same sense as algebra: from the chess-playing perspective, it doesn't matter whether the pieces we move are made of wood, or plastic, or ink on paper; all that matters are their relations to each other and the squares of the board. Similarly, from an algebraic perspective, it doesn't matter whether the objects we operate with are numbers, matrices, or sequences; all that matters are their relations to each other under the arithmetic operations.

So the second purpose of the analogy is to provide another example of a formal, content-agnostic perspective. The formal standpoint is 'formal' in the same way that the chess-playing perspective is 'formal': when we adopt these perspectives on different types of objects, we ignore the features which distinguish them. Again, this serves to dispel doubts about the coherence of the formal standpoint. If anyone objects that it makes no sense to perform arithmetic operations on regular sequences, because they are *sequences* and not *numbers*, Thomae has a straightforward reply: does it also make no sense to play chess with both wooden and plastic pieces, or via post?

Thus, the purpose of the analogy goes beyond the idea that chess is a game structured by rules. Indeed, we have not mentioned this aspect of chess at all yet. For Thomae's mathematical purposes, it is much more important that the formal standpoint, like playing chess, involves adopting a new perspective, thinking of a domain of objects in a new way from the one in which they are initially given. In ongoing discussions of the chess analogy and mathematical formalism, this aspect of Thomae's view should not be overlooked.

Still, it is clear that rules play an important role in the analogy for Thomae. When he formulates the analogy between the formal standpoint and chess, he calls the laws governing arithmetic operations "game rules" to underscore this as a point of comparison, and emphasizes the role of rules in other places too. So how exactly do rules fit into the analogy?

Roughly, Thomae's idea is that in both chess and the formal standpoint, rules *determine* the concepts we use to conceive of objects from the new perspective. This is why he says that from the formal standpoint, signs have "no other content" than that which belongs to them "under certain combinatorial rules", and numbers are "concepts whose content is exhausted through their behavior with respect to the calculating rules" (Thomae, 1898, 3, 4). Again, the comparison with chess helps make this idea clearer. From the chess-playing perspective, there is nothing more to being, say, a bishop, than to being a piece which starts at a certain board position, is only allowed to move diagonally, and so on. Any *other* aspect of bishops not specified in the rules of chess is irrelevant from the chess-playing perspective and can vary from one chess game to the next, and in this sense does not belong to the concept of a bishop. Thus the rules determine the concept of a bishop.

But if we leave things at that, we are led straight to Frege's objection. As we saw in the introduction, Frege understood the chess analogy as an attempt to show that signs can acquire a meaning or content in the presence of a system of rules for manipulating them, just as pieces of wood take on a meaning in chess. He argued that this attempt fails: a system of rules simply does not suffice to attach content to meaningless marks, in chess or arithmetic. Rules can't give us concepts where there were none already.

Given Thomae's relationship to Frege, it is very likely that he was aware of this objection: by the time Thomae (1898) was published, Frege had been criticizing formalism, including this aspect of the chess analogy, for almost fifteen years (cf. Frege, 1984, 2013b, XIII). Yet Thomae held onto the chess analogy as a good device for explicating and defending formalism. Why was he unconcerned about Frege's objection?

To answer this question, we need a clearer picture of the relationship between rules, signs, and the concepts they signify. The key idea we need is that for Thomae, this relationship is *non-arbitrary.* The problem with Frege's objection is that he thinks of the relationship between sign and signified in linguistic terms, as the relationship between a linguistic expression and what it means or designates, like the relationship of "Sirius" to a certain star. His objection is that rules for manipulating an expression cannot produce *that* kind of relationship. Thomae's understanding of signs, on the other hand, traces back to Kant's understanding of algebra, and as we saw above, Kant *contrasts* mathematical signs with words or linguistic expressions, which are only arbitrary marks for the concepts they signify. For Kant, mathematical signs are non-arbitrary representations, in the sense that operating with signs facilitates thinking with the concepts they signify.

This Kantian conception of a non-arbitrary relationship between sign and signified shows up in a *disanalogy* that Thomae stresses between chess and arithmetic. After introducing the chess analogy, Thomae immediately contrasts the "arbitrary" rules of chess with the rules of arithmetic, which allow the numbers to "perform essential service for us in the knowledge of nature", because they are not arbitrary in the same sense (Thomae, 1898, 3).[35] Thomae thus implies that, since the rules of arithmetic are non-arbitrary in an appropriate sense, manipulation of signs can facilitate thinking and yield knowledge.

Thomae's thought here is admittedly quite vague, but here is one way we might spell it out. The representational relationship in question is that between an infinite decimal (a sign, in Thomae's sense) and the corresponding regular sequence. For example, as we saw above, the decimal representation of π, $3.1415\ldots$, corresponds to the regular sequence of rationals

$$(3; 3.1; 3.14; 3.141; 3.1415; \ldots)$$

Notice first that this relationship is very different from the (linguistic) relationship of the letter 'π' to this sequence. 'π' is an unstructured piece of syntax, and has a merely arbitrary relationship to the sequence, in the sense that you could not produce the sequence from 'π' if you had not specifically learned the connection. By contrast, the decimal representation *encodes* the sequence: a simple procedure, applied to the decimal representation, allows you to produce the sequence. This encoding relationship is a regular, non-arbitrary one, and *can be expressed as a general rule* relating infinite decimals to regular sequences. Moreover, this rule sets up a correspondence of the sort Kant envisioned, in which "easy and certain" manipulations of the signs (akin to our school algorithms for decimal arithmetic) correspond to the arithmetic operations on the sequences as Thomae defines them.

Here then is a different and potentially important role for rules in formal arithmetic: rules set up a non-linguistic, non-arbitrary representational relation in which signs (decimals) encode what they signify (regular sequences). We manipulate the signs because it is cognitively easier; the encoding relationship ensures that these manipulations at the sign level correspond to operations at the level of what they signify, and thus that sign manipulations can give us genuine knowledge. By contrast, chess pieces do not encode anything in this sense, which is why manipulations of chess pieces are arbitrary and do not

[35] Die Schachspielregeln sind willkürliche, das System der Regeln der Arithmetik ist ein solches, dass die Zahlen mittels einfacher Axiome auf anschauliche Mannigfaltigkeiten bezogen werden können und uns in Folge dessen wesentliche Dienste in der Erkenntniss der Natur leisten.

produce knowledge. This suggestion is speculative, but it seems to capture at least part of what Thomae (and Kant) had in mind about how rules relate signs to what they signify in arithmetic. It also explains why Thomae was unconcerned about Frege's objection: the objection presupposes a linguistic relationship between sign and signified, not an encoding relationship. Rules clearly *do* determine encoding relationships, even if they can't determine the relationship Frege called "designating".

Let me close by making a suggestion about how the above discussion of Thomae's formalism and the chess analogy can help us understand the subsequent history. Frege too had a way of talking about a relationship that mediates between a sign and its *Bedeutung*: the notion of *Sinn*, which he at least sometimes seems to think of as providing a rule or non-arbitrary connection between signs and their *Bedeutungen*. But Frege's notion of *Sinn* is notoriously underdeveloped. It may be that, had Frege reflected more charitably on his colleague's view, he would have been led to something like the notion of encoding I sketched above, which could have sharpened his own semantic picture.[36] Certainly Wittgenstein thought that Frege had missed something important in formalism about the relationship of signs and their meanings, and he used the chess analogy to argue that Frege had missed the possibility that "the signs can be used the way they are in the game" without meaning anything in Frege's sense (Waismann, 1979, 105). With a clearer picture of Thomae's formalism in hand, we are now also in a better position to explicate these passages in Wittgenstein's thought.

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[36]Compare Costreie (2013), who suggests that Frege's engagement with formalism may actually have led him to draw the *Sinn-Bedeutung* distinction.

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