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Knowledge and the Philosophy of Number

Richard Lawrence

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BOOK REVIEW

Knowledge and the Philosophy of Number, K. Hossak, London, Bloomsbury Publishing, 2020, ix + 206 pp., £90.00, ISBN 978-1-3501-0290-3

Hossack's project in this book is to provide a new foundation for the philosophy of number inspired by the traditional idea that numbers are *magnitudes*. Hossack advocates understanding magnitudes as properties of *quantities*, a genus which includes pluralities, continua, and series as species. Thus a natural number is a property of a plurality, a real number is a property of a continuum, and an ordinal is a property of a series. The book works out a modern theory of quantity and magnitude, based on the ancient theory of quantity in Aristotle and Euclid, in order to argue that we can have a priori knowledge of the natural numbers, the real numbers, and the ordinals. It is a refreshing and innovative attempt at that longstanding goal.

The book divides roughly in half around its central result, the Homomorphism Theorem, which says that the mereological algebra of a system of quantities gives rise to an algebra of magnitudes equipped with addition, restricted subtraction, and a linear order (Chapter 5). The first half develops the metaphysical and logical foundations needed to demonstrate this theorem, introducing a metaphysics of properties (Chapter 1), an understanding of quantity (Chapter 3) and an axiomatization of a mereology common to all species of quantities (Chapter 4). The main thread in these chapters argues that this mereology is an a priori theory of quantity, so that the Homomorphism Theorem implies we also have a priori knowledge of the associated magnitudes. The second half of the book then builds on the Homomorphism Theorem, supplementing the general axioms of quantity with additional axioms for particular species of quantities, to supply the natural numbers (Chapters 6 and 7), real numbers (Chapters 8 and 9), and ordinals (Chapter 10) with their familiar algebraic structures. In each case, according to Hossack, the additional axioms are also knowable a priori.

Following Aristotle, Hossack takes quantities to essentially admit of equality and inequality, and thus to obey a set of equality axioms derived from Euclid's Common Notions, like 'If (disjoint) equals are added to equals, the wholes are equal'. The Homomorphism Theorem shows that these axioms induce an equivalence relation on a system of quantities, and thus that one can define a quotient algebra on their equivalence classes equipped with addition, restricted subtraction, and a linear order. Mathematically, the resulting algebra of magnitudes is what Hossack calls a *positive semigroup*, which is stronger than a semigroup (which lacks restricted subtraction and thus ordering) but weaker than a group. Metaphysically, Hossack interprets magnitudes, the elements of this algebra, as the properties which unify the underlying equivalence classes of quantities: two quantities are equal when they have identical magnitudes.

Throughout the book, Hossack deliberately steers away from much of the understanding of number that has developed in logic and philosophy of mathematics since Frege. For example, given his Aristotelian ontology, he views the Fregean thesis that numbers are *objects* as a category mistake: numbers are predicated of other things, not impredicable objects. He likewise avoids both Fregean second order logic and modern set theory. This does not mean that Hossack eschews the tools of modern logic entirely. Indeed, he argues for expanding our logical primitives. He makes essential use of plural logic, for example, and argues for an interpretation of Tarski's mereology as an a priori logic of quantity, implicit in the grammar of 'and'. The final chapter supplies a constructive notion of ordinal numbers, and in the final pages, Hossack even sketches plans for a restricted set theoretic universe.

The result is a precise and modern mathematical theory of magnitude, expressed in contemporary formalism and in crisp mathematical prose. The book itself is an attractive hardback, with an index and bibliography which the curious reader will find useful. The mathematical details are mostly given in appendices to each chapter, so that readers can easily focus on the philosophical or mathematical material that interests them most. There are occasional copy-editing errors in the mathematical notation, but none so egregious that a careful reader cannot recover the thread; hopefully the publisher will correct these in future printings.

The book synthesizes ideas drawn from many historical figures. In addition to Aristotle and Euclid, Hossack draws inspiration from Hume, Kant, Tarski, Quine, and others. But the book is primarily a work of logic and mathematics, and aims at theory construction, not scholarship; readers seeking a careful exposition of these thinkers' views should look elsewhere. Similarly, readers coming from other contemporary approaches to the philosophy of number, such as structuralism or neo-Fregean logicism, will find their point of view almost entirely unaddressed except for some brief remarks in chapter 2. I do not regard this as a shortcoming of the book – every author surely has the right to put the proof in his own pudding before he tries to entice others with it – but it does raise the question of who the intended audience is. The book will appeal most to those who, having surveyed current approaches to the philosophy of number, find themselves desirous of an alternative. Philosophers concerned about applications of number in empirical science may find Hossack's metaphysical outlook especially agreeable. It will likely be less appealing to devoted platonists and those interested in the epistemology of pure, as opposed to applied, mathematics.

Toward the end of the nineteenth century, the notion of magnitude and the aristotelian conception of mathematics as the science of quantity were abandoned. There was good reason for this: these traditional foundations were too narrow to support the great expansion of mathematical knowledge which began then and has continued ever since. Hossack's neo-aristotelianism appears to have the same restrictions as the traditional version, which raises the worry that it is likewise too narrow for our modern conception of number. Hossack does not even attempt to explain knowledge of *negative* numbers, for example, much less complex numbers or quaternions. Hossack defends these omissions, remarking that 'none of these are pure magnitudes, so none of them are numbers in our sense: all of them are *vectors*, which is to say, they are mathematical entities having both magnitude and direction' and thus to be treated elsewhere (*Hossack 2020*, 106).

Still, it is hard not to feel that this simply moves the goalposts. The central problem in the foundations of mathematics, as it has been understood since about 1860, is to explain our knowledge of the full taxonomy of numbers which appear in pure mathematics. Hossack seems to be abandoning this problem, or at least restricting focus to a subset of it, without much concern for the unsolved remainder.[1] Maybe that is because a solution is forthcoming, though. By the end of the book, Hossack has developed resources – in particular the notions of reference to, and predications of, series – that could underwrite the usual constructions of the integers as ordered pairs of natural numbers, the rationals as ordered pairs of integers, complex numbers as ordered pairs of reals, and so on.

A more serious shortcoming of the book, given its aim, is that Hossack nowhere explains his conception of the a priori. Thus it is hard to assess his claims that his axioms are knowable a priori, some of which receive little or no argument. For guidance, readers might look to Hossack's earlier *Metaphysics of Knowledge*, where he explains that 'the *a priori* facts are the primitive facts and all their logical consequences' and postulates cognitive faculties of *intellect* and *deduction*

[1] Øystein Linnebo discusses a related worry in a new collection engaging with Hossack's work (*Linnebo 2022*). Linnebo argues that there is pressure to generalize Hossack's view to account for other kinds of magnitudes, and that this generalization will raise many of the same problems facing later philosophies of number, such as neo-Fregean logicism.

which enable us to know primitive facts and their consequences (*Hossack 2007*, 129). Hossack provides deductive arguments here for some of the facts that he claims are a priori; others he leaves up to the reader's intellect, but not always persuasively. He claims, for example, that every instance of his axiom of Plural Comprehension is 'primitively evident' (*Hossack 2020*, 54).

The book is nonetheless a valuable contribution which promises to reinvigorate current debates. It is an outstanding example of the value of reinterpreting historical ideas in a contemporary setting; and it offers plenty of footholds where new philosophical discussions of number can begin.

References

Hossack, K. 2007. *The Metaphysics of Knowledge*, Oxford: Oxford University Press.
Hossack, K. 2020. *Knowledge and the Philosophy of Number*, London: Bloomsbury Academic.
Linnebo, Ø. 2022. 'Aristotelian aspirations, Fregean fears: Hossack on numbers as Magnitudes', in N. Kürbis, B. Assadian and J. Nassim (eds.), *Knowledge, Number and Reality: Encounters with the Work of Keith Hossack*, London: Bloomsbury Academic, pp. 189–201.

Richard Lawrence
Institute Vienna Circle, University of Vienna, Vienna, Austria
rwl@recursewithless.net

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