LOGIC IN THE DEEP END

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Abstract. Weak enough relevant logics are often closed under depth substitutions. To determine the breadth of logics with this feature, we show there is a largest sublogic of R closed under depth substitutions and that this logic can be recursively axiomatized.

1. Introduction

In relevant logic, a conditional is provable only if its antecedent is relevant to its consequent—the provability of conditionals must *respect relevance*. What exactly respecting relevance amounts to has, however, been up for debate essentially ever since the notion was introduced. One of the more promising approaches to formally explicating respect for relevance is to articulate it in terms of *variable sharing results*.[1] There are four such results in the extant literature, viz. ordinary variable sharing, strong variable sharing, depth relevance, and strong depth relevance. We'll have more to say about these below. What's important to note is that (a) each of these codifies a different way logics can be (or perhaps, can be said to be) relevant and (b) most of the results concerning variable-sharing have roughly the form 'thus and so logics enjoy such and such a variable-sharing property'. What has been lacking until recently was a serious exploration of the mechanisms by which logics come to have these properties. And—apart from the tantalizing-but-tentative results in the work of Gemma Robles and José Méndez, (see e.g. Robles and Méndez 2011; Méndez and Robles 2012)—there's been little to no exploration of what the broadest possible class of logics enjoying any of these properties might be.

[1]Variable sharing isn't the *only* way to formalize relevance. An alternative pursued in e.g. Brady 1988, 1989a involves using a formal theory of contents in one's semantics. The two approaches aren't entirely disconnected, but we'll focus our entire attention in this paper on the former, variable-sharing approach to capturing the relevance of relevant logics. We also note that while we confine our attention here to variable sharing results for propositional relevance logics here are also versions of such results for various extensions of propositional relevant logics; see e.g. Standefer 2022, Savic and Studer 2019, and Tedder and Bilková 2022.

We said above that exploration of the mechanisms by which logics come to have variable sharing properties had been lacking 'until recently'. This is because the results in Logan 2022b, which suggest that for at least the depth relevance properties, there is a plausible mechanism: weak enough relevant logics are, it turns out, *hyperformal* in the vocabulary introduced there. And in the presence of hyperformalism, the ordinary variable sharing results immediately give rise to depth relevance results.

Having mentioned hyperformalism, we'll now define it. To do so, we first introduce *depth substitutions*. Like ordinary substitutions, depth substitutions are ways of replacing atomic formulas with formulas. But for depth substitutions the formula replacing a given atomic formula is allowed to vary with the number of conditionals the atomic formula occurs in the scope of. A logic is then said to be *hyperformal* when it contains a formula as a theorem only if it contains all depth-substitution instances of that same formula as theorems. Philosophically, a hyperformal logic can be understood to respect a criterion of validity that is in the spirit of—but even more stringent than—Bolzano's notion of analyticity which requires that truth must be preserved by arbitrary uniform replacements of specified variables (see Hale and Wright 2015 for a lucid presentation and contextualization). Hyperformal logics strengthen this by restricting to theorems whose correctness is preserved even by *nonuniform* replacements of sentence letters, so long as those replacements are uniform within a given depth.

We can at last state the purpose of the paper at hand: in Logan 2022b, it was shown that weak-enough relevant logics are closed under depth substitutions. To prove this, Logan isolated a particular logic (DR^{-}), all of whose sublogics are closed under depth substitutions. This left open both of the following questions:

- Is there a *strongest* relevant logic closed under depth substitutions?
- If so, what is it?

We also note that the closure property we are looking for is somewhat analogous to the Post-completeness property of classical logic: classical propositional logic is closed under substitution and detachment, and has no non-trivial extensions with this property. The logic we're after should be closed under depth-substitution and detachment, and have no *relevant* extensions with this property.

Our goal in this paper is to begin to answer the above questions. Unfortunately, while we can answer (positively!) the first of our two questions, we can only give a partial answer to the second. Put more plainly, what we show in this paper is (a) that there is a largest sublogic of R closed under depth substitutions and (b) that it can be recursively axiomatized. We leave finding a nice presentation of it—perhaps by a small finite number of axiom schemes—to later work.

2. Setup

We work in a language $\mathcal{L}$ with a countably infinite set $\mathsf{At} = \{p_i\}_{i=0}^{\infty}$ of propositional variables and $\neg$, $\wedge$, and $\rightarrow$ as connectives. We define $\vee$ in the usual way and we write $\mathbb{N}$ for the natural numbers.

A depth substitution is a function $d : \mathsf{At} \times \mathbb{N} \rightarrow \mathcal{L}$. Corresponding to each depth substitution d there is an associated function $d^+ : \mathcal{L} \times \mathbb{N} \rightarrow \mathcal{L}$ defined as follows:

- $d^+(p, n) = d(p, n)$ for $p \in \mathsf{At}$
- $d^+(\neg A, n) = \neg d^+(A, n)$.
- $d^+(A \wedge B, n) = d^+(A, n) \wedge d^+(B, n)$.
- $d^+(A \rightarrow B, n) = d^+(A, n + 1) \rightarrow d^+(B, n + 1)$.

A normal substitution corresponds to a degenerate depth substitution s for which $s(A, n) = s(A, 0)$ for any $n > 0$. A particularly important notion in this paper is the 0-level depth substitution $d^+(A, 0)$, which we abbreviate as $A[d]$.

Intuitively, a depth substitution is a "locally uniform" replacement of atomic sentences. A normal substitution is "globally" uniform: as we go through the formula and make our substitution, we must *always* replace two occurrences of a given sentence letter A with occurrences of the same fixed formula $s^+(A)$. When making a depth substitution, we have a little more flexibility. We must replace two occurrences of a given sentence letter A with the same fixed formula $d^+(A, n)$ *if* those two occurrences are at the same depth n, (where the depth of a sentence-letter occurrence is measured by the number of conditionals that appear above that occurrence in the parsing tree of the formula). So two occurrences of a given sentence letter can be replaced by occurrences of two *different* formulas, so long as the two sentence letters are at different depths.

We call a depth substitution an injective atomic depth substitution (or just an injective atomic) when (a) it is injective and (b) its range is a subset of At. A set of formulas S is closed under depth substitutions when for all $A \in S$ and all depth substitutions d, $A[d] \in S$.

We're interested in the logic R, whose axiomatization is well-known. We include the axiom and rules that will under discussion in this paper.

$$\text{A1}\ \ (A \wedge B) \to A \qquad \text{R1}\ \frac{A, A \to B}{B} \qquad \text{R2}\ \frac{A, B}{A \wedge B}$$

When we use the term *logic* more generally, we speak of a set of formulas closed under R1, R2, and normal substitution.

We define a proof of A to be a sequence $B_1, \dots, B_n = A$ such that, for all $1 \leq i \leq n$, either B_i is an axiom of R or there are B_j and B_k with $j < i$ and $k < i$ so that $\frac{B_j, B_k}{B_i}$ is an instance of either R1 or R2. R itself is then the set of provable formulas.

Our goal is to show that the largest subset of R closed under depth substitutions is recursively axiomatizable. But we first need to show that there is such a set.

Lemma 1. *Let d_1 and d_2 be depth substitutions, and define $d_2 \bullet d_1 : \mathsf{At} \times \mathbb{N} \to \mathcal{L}$ by $\langle p, n\rangle \mapsto d_2^+(d_1(p, n), n)$. Then for all n and all A, $(d_2 \bullet d_1)^+(A, n) = d_2^+(d_1^+(A, n), n)$.*

Proof. By induction on the complexity of A. □

Corollary 2. $A[d_2 \bullet d_1] = A[d_1][d_2]$.

Theorem 3. *The set* DQ $= \{A : A[d] \in R$ *for all depth substitutions* $d\}$ *is the largest subset of R closed under depth substitutions.*

Proof. To see DQ is closed under depth substitutions, suppose that $A \in$ DQ and let d be a depth substitution. We need to show that for an arbitrary depth substitution d', $A[d][d'] \in$ R. But since $A \in$ DQ, it follows that for all depth substitutions d'', $A[d''] \in$ R. Thus in particular $A[d' \bullet d]$—which by Corollary 2 is identical to $A[d][d']$—is in R, as required.

To see that DQ is a subset of R, notice that the function $\mathbf{1} : (p, n) \mapsto p$ is a depth substitution and that $\mathbf{1}^+$ is the identity function on $\mathcal{L}$. Thus if $A[d] \in$ R for all depth substitutions, then $A[\mathbf{1}] = A \in$ R.

Finally, suppose that S is a subset of R closed under depth substitutions. Then for all $A \in S$ and all depth substitutions d, $A[d] \in S \subseteq$ R. So $S \subseteq$ DQ. □

Given this, our goal is now to show that DQ is recursively axiomatizable. We do this in the following way. First, we note that by Craig's Trick (which we show in §3 is applicable in our setting), it suffices to show that DQ is recursively enumerable. Next, in §4, we show that injective atomics are *decisive* in the sense that $A \in$ DQ iff for some injective atomic i, $A[i] \in$ R. Finally, in §5, we apply the decisiveness result from §4 to give a recursive enumeration of DQ. By Craig's Trick this gives what we want. A final section concludes and gestures in the direction of future work.

3. Craig's Trick

Craig's Trick, introduced in Craig 1953, shows that any recursively enumerable set of sentences closed under some notion of consequence is in fact recursively axiomatizable. That is to say, if there's a computable surjection $\mathbb{N} \to T$, then there's a set of axioms $T' \subseteq T$ generating T whose characteristic function is computable.[2] Craig's Trick is fairly robust, and it would probably be a publishable result if it *didn't* work here (see for example Kuznetsov et al. 2019, which presents a failure of Craig's Trick for the Lambek Calculus). But in this section we briefly go through the argument in order to verify that the trick applies in our setting.

Here's the fundamental idea behind Craig's trick, abstractly:

Lemma 4 (Abstract Craig's Trick)**.** *If $f : \mathcal{L} \times \mathbb{N} \to \mathcal{L}$ and $g : \mathcal{L} \to \mathbb{N}$ are computable functions, and $\sim$ is a relation such that*

(1) $g(f(A, i)) = i$

(2) $f(A, i) \sim A$ *and* $A \sim f(A, i)$

[2]We grant ourselves sensible encodings of syntax and pairs so that it makes sense to talk about computable sets of sentences, and computable functions of more than one argument.

then for any recursively enumerable $T \subseteq \mathcal{L}$ closed under $\sim$, there's a recursive $T' \subseteq T$ such that T is the closure of T' under $\sim$.

Proof. Let $\{A_i\}_{i\in\mathbb{N}}$ be a recursive enumeration of T. Let T' be $\{f(A_i, i)\}_{i\in\mathbb{N}}$. Because T is closed under $\sim$ and $A_i \sim f(A_i, i)$, it's clear that $T' \subseteq T$. Conversely, because $f(A_i, i) \sim A_i$, it's clear that T is the closure of T' under $\sim$. It remains to show that T' is recursive. To check whether $A \in T'$, compute $g(A)$. Suppose this turns out to be n. Then, compute $f(A_n, n)$. If this is A, then $A = f(A_n, n)$ is in T' by construction. If not, then A can't be in T', since $g(A) = n$, but the only sentence in T' that g maps to n is $f(A_n, n)$. □

We want to apply Craig's trick to find a recursive $T' \subseteq$ DQ that will serve as the basis for an axiomatization of DQ. For our $\sim$, we propose the symmetric closure of

$$A \sim B \Leftrightarrow \text{ there is } j \in \mathbb{N} \text{ so that } B = \bigwedge_{i=0}^{j} A$$

so that a formula's only equivalents are repeated left-associated conjunctions with itself. For f and g, we propose:

$$f(A, n) = \bigwedge_{i=0}^{n+1} A$$

$$g(A) = \begin{cases} n & \text{if } A = \bigwedge_{i=0}^{n+1} B \\ 0 & \text{otherwise} \end{cases}$$

letting the number of repeated top-level conjuncts do the work of encoding n.[3] It's then trivial that $f(A, i) \sim A$ and conversely. So to verify that we can apply Craig's trick to DQ with this f, g, and $\sim$ we only need to show that DQ is closed under $\sim$.

Lemma 5. *If $\Gamma \subseteq \mathcal{L}$ contains all instances of A1, is closed under R1, and is closed under R2, then for all $n \geq 1$ and all A, Γ contains $\bigwedge_{i=1}^{n} A$ iff Γ contains A.*

Proof. Notice first that $(\bigwedge_{i=1}^{n} A) \to A$ is an instance of A1. Thus it's in Γ. So since Γ is closed under R1, if it contains $\bigwedge_{i=1}^{n} A$, it contains A. For the other direction, it suffices to note that since Γ is closed under R2, if it contains A, then it contains $\bigwedge_{i=1}^{n} A$ as well. □

[3]The +1 means that $f(A, 0)$ is $A \wedge A$, so that g doesn't accidentally give us a result based on a conjunction internal to A; it also ensures that formulas have at most one representation as $\bigwedge_{i=0}^{n+1} B$, so that g is well-defined. If we used $\bigwedge_{i=0}^{n}$, then we could write e.g. $A \wedge A$ as both $\bigwedge_{i=0}^{1} A$, and as $\bigwedge_{i=0}^{0}(A \wedge A)$.

Lemma 6. *For any depth substitution d, there is another depth substitution d^- such that $(A \to B)[d^-] = A[d] \to B[d]$.*

Proof. One example is defined as follows:

$$d^-(p,n) = \begin{cases} p & \text{if } n = 0 \\ d(p, n-1) & \text{otherwise} \end{cases}$$

It is straightforward to show by induction on A that for $m \geq 1$, $d^-(A,m) = d(A, m-1)$. Then $(A \to B)[d^-] = d^-(A \to B, 0) = d^-(A,1) \to d^-(B,1) = d(A,0) \to d(B,0) = A[d] \to B[d]$. □

Lemma 7. DQ *contains all instances of A1 and is closed under R1 and R2.*

Proof. If d is an arbitrary depth substitution, then $((A \wedge B) \to A)[d] = (d(A,1) \wedge d(B,1)) \to d(A,1)$. So $((A \wedge B) \to A)[d]$ is still an instance of $A1$, and so an element of R. Thus every instance of A1 is in DQ.

If d is an arbitrary depth substitution and $A, B \in$ DQ then $A[d]$ and $B[d]$ are in R, so, applying $R2$ in R, $A[d] \wedge B[d] \in$ R. But $A[d] \wedge B[d] = (A \wedge B)[d]$. Hence, every depth substitution instance of $A \wedge B \in$ R. So $A \wedge B \in$ DQ.

If d is an arbitrary depth substitution and $A, A \to B \in$ DQ, then $A[d]$ and $(A \to B)[d^-]$ are in R. But $(A \to B)[d^-] = A[d] \to B[d]$. So applying $R1$ in R, we find that $B[d] \in$ R. Hence every depth substitution instance of $B \in$ R. So $B \in$ DQ. □

Corollary 8. *If DQ is recursively enumerable, then it is recursively axiomatizable*

Proof. By Lemma 5 together with Lemma 7, DQ is closed under the $\sim$ defined above. So given an enumeration $\{A_i\}_{i \in \mathbb{N}}$ of DQ, using the $f, g, \sim$ above, we have that $\{\bigwedge_{i=0}^{j+1} A_j\}_{i \in \mathbb{N}}$ is a recursive subset of DQ. Each sentence $(\bigwedge_{i=0}^{j+1} A_j) \to A_j$ is an instance of A1. So the set

$$Ax = \left\{\bigwedge_{i=0}^{n+1} A_n\right\}_{n \in \mathbb{N}} \cup \left\{(\bigwedge_{i=0}^{n+1} A_n) \to A_n\right\}_{n \in \mathbb{N}}$$

is also a recursive subset of DQ. Every sentence of A_n of DQ can be derived from two sentences in Ax by one application of R1. So Ax, taken with the rule R1, is our recursive axiomatization of DQ. □

4. Injective Atomics are Decisive

We now turn to showing that DQ is recursively enumerable. We begin by noting that we can extend ordinary substitutions s so that they not only act on arbitrary formulas, but on arbitrary *proofs*. Explicitly, where $\Pi = B_1, \ldots, B_n$ is a proof, we define $s^+(\Pi)$ to be the sequence $s^+(B_1), \ldots, s^+(B_n)$.

Lemma 9. *If* Π *is a proof of* A *and* s *is an ordinary substitution, then* $s^+(\Pi)$ *is a proof of* $s^+(A)$.

Proof. By induction on the length of Π. □

Note that in general *depth* substitutions do *not* extend to give functions from proofs to proofs.

Lemma 10. *If* i *is an injective atomic and* d *is a depth substitution, then there are (ordinary, non-depth) substitutions* i_d *so that* $i_d(A[i]) = A[d]$.

Proof. One example of such a substitution is this:

$$i_d(p) = \begin{cases} p & \text{if } p \text{ isn't in the range of } i \\ d(i^{-1}(p)) & \text{otherwise} \end{cases}$$

This is well-defined by the injectivity of i. Note also that since i is injective, i^+ is also injective. It's not too hard to see from here that $i_d(A) = d(i^{-1}(A))$. Thus:

$$i_d(A[i]) = d(i^{-1}(i(A, 0))) = d(A, 0) = A[d]$$

□

Theorem 11. *If* i *is an injective atomic, then* $A[i] \in \mathrm{R}$ *iff* $A \in \mathrm{DQ}$.

Proof. If $A \in \mathrm{DQ}$, then $A[i] \in \mathrm{R}$ by the definition of DQ. For the reverse direction, let i be an injective atomic and d be an arbitrary depth substitution. Then by Lemma 10, there is an ordinary substitution i_d so that $i_d(A[i]) = A[d]$. But then by Lemma 9, if $A[i]$ is provable, so is $i_d(A[i]) = A[d]$. Thus if $A[i] \in \mathrm{R}$ then $A[d] \in \mathrm{R}$ for all depth substitutions d, which is to say that $A \in \mathrm{DQ}$. □

5. DQ is Recursively Axiomatizable

To enumerate DQ, it therefore suffices to enumerate all A such that $A[i]$ is provable in R for some fixed injective atomic depth substitution i. One might proceed as follows: let $\{A_i\}_{i\in\mathbb{N}}$ be an enumeration of the formulas of our language. Let i be an injective atomic depth substitution. On the nth day of computation, consider whether any of $A_0[i] \ldots A_{n-1}[i]$ has a proof in R whose length does not exceed n characters. If you encounter a proof of $A[i]$, output A. In the fullness of time, every A such that $A[i]$ is provable will be enumerated.

We take this to suffice as a proof of the following:

Theorem 12. *DQ is recursively enumerable.*

Corollary 13. *DQ is recursively axiomatizable.*

Proof. By Corollary 8, if DQ is recursively enumerable, then it is recursively axiomatizable. Thus, by Theorem 12, it is recursively axiomatizable. □

6. Concluding Remarks

The strongest (essentially by means of being the only) logic shown to be closed under depth substitutions is the logic that in Logan 2021 is called DR^-.[4] However, it's not hard to find members of DQ that aren't in DR^-. In fact, we can give a general way of constructing such formulas. To do so, we first provide a lemma whose proof can be found in the literature (see e.g. Restall 2002 or Logan 2022a).

Lemma 14. *Let B be a formula and $B[A]$ be B with a particular occurrence of A highlighted, and let $B[C]$ be the formula that we get by replacing that occurrence of A with C. Then if $A \to C \in \mathrm{R}$, then*

- *If the occurrence of A is positive, then $B[A] \to B[C] \in \mathrm{R}$; and*
- *If the occurrence of A is negative, then $B[C] \to B[A] \in \mathrm{R}$.*

We note that $(\neg A \to A) \to A \in \mathrm{R}$ and $A \to \neg(A \to \neg A) \in \mathrm{R}$. It follows that if $B[A]$ is a theorem of R with a highlighted positive (resp. negative) occurrence of A as a depth

[4] Logan (2022b) claims, incorrectly, to show that DR is closed under depth substitutions, but it was shown by Tore Fjetland Øgaard that this is in fact not the case.

n subformula, then $B[\neg(A \to \neg A)]$ (resp. $B[\neg A \to A]$) is a theorem of R in which the corresponding occurrences of A are at depth $n + 1$.

Having noticed this, it's quick work to produce members of DQ that aren't in DR$^-$. One way to do so is to begin with a theorem A of R that isn't in DQ, and then perform replacements of the above form on atomic subformulas of A until no atom occurs at more than one depth. For example, we might do the following:

$$\begin{aligned} p \to ((p \to q) \to q) &\rightsquigarrow (\neg p \to p) \to ((p \to q) \to q) \\ &\rightsquigarrow ((p \to \neg p) \to \neg(p \to \neg p)) \to ((p \to q) \to q) \\ &\rightsquigarrow ((p \to \neg p) \to \neg(p \to \neg p)) \to ((p \to q) \to \neg(q \to \neg q)) \end{aligned}$$

The end result is a member of DQ that (as can be checked by e.g. using MaGIC; see Slaney 1995) isn't a member of DR$^-$.

We note that one needn't restrict to the case where the replacements are performed on atomic subformulas. In fact, the observation that the above trick works was motivated by another observation of Robles and Méndez (2014) that, for the sake of producing a depth relevant logic, Brady needn't have eliminated either of the following axioms from the logic DT that he investigated in Brady 1989b:

- $(\neg(A \to B) \to (A \to B)) \to ((B \to C) \to (A \to C))$
- $(\neg(A \to B) \to (A \to B)) \to ((C \to A) \to (C \to B))$

Each of these formulas is the result of replacing a negative, depth 1 occurrence of the complex formula $A \to B$ with an occurrence of $\neg(A \to B) \to (A \to B)$. The result in both cases is a member of DQ. Importantly, each remains a member of DQ even if A is itself complex and even if A contains the same atom at two different depths—for example, even if A has the form $p \to (p \to q)$. And, as with the example given above, so also here one can verify MaGICally that what we're looking at are formulas that, while in DQ, aren't in DR$^-$.

All that to say that while the main result of this paper leaves us optimistic about the search for a friendly axiomatization of DQ, this optimism remains quite cautious.

Finally, a note on the philosophical significance of all this. Relevant logics have been objects of serious philosophical investigation for a long time. In recent decades, logics at the weaker end of the spectrum in particular have seen a great deal of attention. Many logics at that end of the spectrum are closed under depth substitutions. Regardless of whether this closure is taken to be a good thing or a bad thing, it's certainly both unexpected and significant. From one perspective, it suggests that the logics in question are—in some sense that remains nebulous and in need of fleshing out—"ignorant" of the fact that occurrences of the same atom at different depths are in fact occurrences of the very same atom. From another perspective, it suggests that those logics, because hyperformal, track a novel and well-behaved species of truth in virtue of form. What we've shown in this paper is that the class of logics that (depending on your perspective) either suffer from or enjoy closure under depth substitutions is natural enough to be amalgamated into a single strongest logic, and that this logic is reasonably well-behaved. Fans of depth substitution can thus read in our results the possibility of a systematic way to exploit it while opponents can instead see a systematic way to avoid it.

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