A Metasemantic Analysis of Gödel's Slingshot-Argument

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Abstract. Gödel's slingshot-argument proceeds from a referential theory of definite descriptions and from the principle of compositionality for reference. It outlines a metasemantic proof of Frege's thesis that all true sentences refer to the same object—as well as all false ones. Whereas Frege drew from this the conclusion that sentences refer to truth-values, Gödel rejected a referential theory of definite descriptions. By formalising Gödel's argument, it is possible to reconstruct *all* premises that are needed for the derivation of Frege's thesis. For this purpose, a reference-theoretical semantics for a language of first-order predicate logic with identity and referentially treated definite descriptions will be defined. Some of the premises of Gödel's argument will be proven by such a reference-theoretical semantics, whereas others can only be postulated. For example, the principle that logically equivalent sentences refer to the same object cannot be proven but must be assumed in order to derive Frege's thesis. However, different true (or false) sentences can refer to different states of affairs if the latter principle is rejected and the other two premises are maintained. This is shown using an identity criterion for states of affairs according to which two states of affairs are identical if and only if they involve the same objects and have the same necessary and sufficient condition for obtaining.

Keywords.
slingshot-argument; reference; definite descriptions; substitutivity; state of affairs; metasemantics

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1 Introduction

Gödel puts forth his *slingshot-argument* as a *metasemantic* argument in the following passage:

> "What do the so-called descriptive phrases (i.e., phrases as, e.g. 'the author of *Waverley*' or 'the king of England') denote or signify and what is the meaning [i.e. denotation] of sentences in which they occur? The apparently obvious answer that, e.g., 'the author of *Waverley*' signifies [i.e. denotes] Walter Scott, leads to unexpected difficulties. For, if we admit the further apparently obvious axiom, that the signification of a composite expression, containing constituents which have themselves a signification, depends only on the signification of these constituents [...], then it follows that the sentence 'Scott is the author of *Waverley*' signifies the same thing as 'Scott is Scott'; and this again leads almost inevitably to the conclusion [(FCG)] that all true sentences have the same signification (as well as all false ones). Frege actually drew this conclusion; [...] 'the True' being the name he uses for the common signification [i.e. denotation] of all true propositions [i.e. sentences]."[1]

He reacts to this argument as follows:

> "But different true sentences may indicate [i.e. denote] different things. Therefore this view concerning sentences makes it necessary either to drop the above-mentioned principle [(R)] about the signification [...] of composite expressions or to deny that [(A)] a descriptive phrase denotes the object described."[2]

In order to standardise the terminology of these various semantic notions, the notion of reference will be used in this paper to replace Gödel's notions of denotation, signification, and description, as well as Frege's notion of meaning[3] and Russell's notion of indication. Thus, Gödel *explicitly* puts forth the following theses as premises of his argument:

(A) Definite descriptions without free individual variables refer to individuals as do individual constants (i.e. the fundamental assumption of a *referential theory of definite descriptions*).

[1] Cf. (Gödel 1944, p.450). This argument is discussed, e.g. in (Neale 1995 and 2001).
[2] Cf. (Gödel 1944, p.451).
[3] Cf. (Frege 1892).

(R) The referent of a composite expression containing constituents which have themselves a referent depends only on the referent of these constituents (i.e. the *principle of compositionality for referents*). In other words, it is always possible to substitute coreferential expressions for each other within a composite expression without changing the referent of this composite expression.

Gödel claims that from (A) and (R), it follows that

(FCG) all true sentences have the same referent—as well as all false ones (i.e. the Frege-Church-Gödel thesis).

Gödel mentions that from (FCG), Frege drew the conclusion that

(T) truth-values are the referents of sentences (i.e. Frege's answer as to what the referents of sentences are).

By contrast, Gödel considers (FCG) to be absurd because different true sentences may refer to different objects.

In this paper, Gödel's slingshot-argument[4] will be formalised in the logical framework of first-order predicate logic with identity (hereinafter abbreviated to $PL1^{=}$)[5]. The aim of formalising Gödel's argument is to obtain from this formalisation *all* premises that are necessary for deriving (FCG) from these very premises. Observe that Gödel's argument is based on a language for first-order predicate logic with identity and definite descriptions (hereinafter abbreviated to $\mathscr{L}^{PL1^{=}\iota}$) where definite descriptions are treated referentially, that is, as referring to individuals.

How could these properties of truth and falsity of sentences (that is, their being true and being false) be defined by means of the concepts and methods of a semantics for such a language in order to formalise Gödel's argument? To investigate this question, consider the relation "x has y as semantic value". This relation can either be understood as the relation "x refers to y" or as the relation "x has y as extension". The truth or falsity of a sentence can be defined by assigning a unique truth-value (True or False) to the sentence as its semantic value. This semantic value is either the referent or the extension of the sentence. On the other hand, the truth

[4] Cf. (Gödel 1944).

[5] Cf. (Kalish/Montague/Mar ^{2}1980).

or falsity of a sentence can also be defined without assigning a semantic value to the sentence.

Consider, for instance, the language of a propositional logic with atomic sentences (hereinafter abbreviated to $\mathscr{L}^{\text{ALA}}$). The atomic sentences of such a language are of the form $\ulcorner Fa_1 \ldots a_n \urcorner$ where F is an n-ary predicate constant and $a_1, \ldots, a_n$ are individual constants. Suppose individual constants have individuals and n-ary predicate constants have sets of ordered n-tuples of individuals as their semantic values. Suppose further the relation "x has y as semantic value" is taken to be left-total and right-unique, i.e. it is construed as the function "the semantic value of x is y". Then the question is: How could the properties of truth and falsity of all sentences of $\mathscr{L}^{\text{ALA}}$ be defined in order to formalise Gödel's argument? In logical semantics, this can be done in two different ways:

(V1) No semantic values are assigned to sentences.

(V2) Semantic values are assigned to sentences.

In (V2), one can distinguish between two cases, depending on the type of the assigned values:

(V2a) The semantic values of sentences are referents.

(V2b) The semantic values of sentences are extensions.

ad (V1) According to this answer, an atomic sentence of $\mathscr{L}^{\text{ALA}}$ of the form $\ulcorner Fa_1 \ldots a_n \urcorner$ is true iff the ordered n-tuple of the semantic values of $a_1, \ldots, a_n$ is an element of the set that is the semantic value of F. Furthermore, a sentence of the form $\ulcorner \neg S \urcorner$ is true iff S is not true (i.e. false), and a sentence of the form $\ulcorner (S_1 \wedge S_2) \urcorner$ is true iff S_1 and S_2 are true, and so on for the remaining connectives.

By this recursive procedure, the truth and falsity of all atomic sentences of $\mathscr{L}^{\text{ALA}}$ can be defined depending only on their form and on the semantic values of the descriptive constants of which these atomic sentences are composed. In (V1), it is only the descriptive constants that have unique semantic values. The x-position of the function "the semantic value of x is y" is therefore restricted to individual constants and n-ary predicate constants. Using common semantic clauses for the connectives, the truth and falsity of all sentences composed of such atomic sentences can be defined depending only on the form of these sentences and on the truth and falsity of the atomic sentences which are contained in those sentences.

ad (V2) According to this answer, an atomic sentence of $\mathscr{L}^{\mathrm{ALA}}$ of the form $\ulcorner Fa_1 \ldots a_n \urcorner$ is true iff the semantic value of $\ulcorner Fa_1 \ldots a_n \urcorner$ is the truth-value 1. The semantic value of $\ulcorner Fa_1 \ldots a_n \urcorner$ is the truth-value 1 iff the ordered n-tuple of the semantic values of $a_1, \ldots, a_n$ is an element of the set that is the semantic value of F. Furthermore, the semantic value of a sentence of the form $\ulcorner \neg S \urcorner$ is the truth-value 1 iff the semantic value of S is not the truth-value 1 (i.e. the semantic value of S is the truth-value 0), and the semantic value of a sentence of the form $\ulcorner (S_1 \wedge S_2) \urcorner$ is the truth-value 1 iff the semantic values of S_1 and S_2 are the truth-value 1, and so on for the remaining connectives.

By this recursive procedure, the function "the semantic value of x is y", which in (V1) was yet limited to individual constants and n-ary predicate constants, is now *extended* to all sentences of $\mathscr{L}^{\mathrm{ALA}}$. Thus, the function "the semantic value of x is the truth-value y" (where $y \in \{1, 0\}$) is defined for all atomic sentences, depending only on their form and on the semantic values of the descriptive constants of which these atomic sentences are composed. Using common semantic clauses for the connectives, this function can be uniquely extended from atomic sentences to all sentences composed of atomic sentences. Once this function has been defined for all sentences, the truth or falsity of a sentence S of $\mathscr{L}^{\mathrm{ALA}}$ can be defined in the following way: S is true iff the semantic value of S is the truth-value 1 (or: S is false iff the semantic value of S is the truth-value 0).

In the case of true singular sentences, Gödel's argument proceeds from an assumption for Conditional Proof which states that two singular sentences σ_1 and σ_2 are true. These singular sentences contain at least one individual constant. On the basis of this assumption for CP (and his other assumptions, such as the one that reference is a function), he aims to prove that σ_1 and σ_2 refer to the same object. However, he provides no answer as to how the truth or falsity of a sentence—let alone of a singular sentence—ought to be defined. Were he to define it as stated in (V2a), any further assumptions than his assumption for Conditional Proof and the assumption that reference is a function would be wholly superfluous. For in (V2a), from the mere fact that σ_1 and σ_2 are true, it follows by symmetry and transitivity of identity that σ_1 and σ_2 refer to the same object.

Proof:

1. σ_1 is true	Ass. f. CP
2. σ_1 is true $\Leftrightarrow$ the truth-value that σ_1 refers to = 1	(V2a)
3. the truth-value that σ_1 refers to = 1	BMP 1–2
4. σ_2 is true	Ass. f. CP
5. σ_2 is true $\Leftrightarrow$ the truth-value that σ_2 refers to = 1	(V2a)
6. the truth-value that σ_2 refers to = 1	BMP 4–5
7. the truth-value that σ_1 refers to = the truth-value that σ_2 refers to	Identity 3, 6
8. σ_1 and σ_2 refer to the same truth-value	7 [6]

Hence, if one proceeds according to (V2a), no further premises than the assumption for Conditional Proof and the provision of a reference function are needed.

Following Gödel's argument, the question of how the truth or falsity of a sentence ought to be defined should not be answered in the same way as in (V2a) (or (V2b)), lest his argument become trivial. In what follows, we shall take particular account of this point.

[6] If two sentences refer to identical truth-values, then they refer to the same truth-value.

2 Reference-Theoretical Models

In order to formalise Gödel's slingshot-argument, a language of first-order predicate logic with identity and with a definite description operator (i.e. $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$) is required as the object language. Furthermore, a semantics for such a language is required. In what follows, the language $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ will be defined, and the reference-theoretical models for it and the reference-theoretical clauses associated with it will be specified.

Let the alphabet of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ contain a countably infinite amount of individual constants and n-ary predicate constants ($n \geq 1$) as its descriptive (that is, non-logical) signs. Let it further contain as its logical signs a countably infinite amount of individual variables as well as the all quantifier '$\bigwedge$', the existential quantifier '$\bigvee$', the definite description operator 'ι', the usual connectives '$\neg$', '$\wedge$', '$\vee$', '$\rightarrow$', '$\leftrightarrow$', and the sign for identity '='. Finally, let it contain as its auxiliary signs the left and right parentheses '(', ')'.

In what follows, a simultaneous-recursive definition of the formulas and singular terms of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ is given[7]:

(D1) The formulas and singular terms of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$
(simultaneous-recursive definition):

a. c is an individual constant $\Rightarrow$
 c is a singular term

b. v is an individual variable $\Rightarrow$
 v is a singular term

c. v is an individual variable & ϕ is a formula $\Rightarrow$
 $\ulcorner \iota v \phi \urcorner$ is a singular term

d. t_1, t_2 are singular terms $\Rightarrow$
 $\ulcorner t_1 = t_2 \urcorner$ is a formula

e. F is an n-ary predicate constant &
 $t_1, \ldots, t_n$ are singular terms $\Rightarrow$
 $\ulcorner F t_1 \ldots t_n \urcorner$ is a formula

f. ϕ, ψ are formulas $\Rightarrow$
 $\ulcorner \neg\phi \urcorner, \ulcorner(\phi \wedge \psi)\urcorner, \ulcorner(\phi \vee \psi)\urcorner, \ulcorner(\phi \rightarrow \psi)\urcorner, \ulcorner(\phi \leftrightarrow \psi)\urcorner$ are formulas

[7] Cf. (Gamut 1991, pp.159 f.) and (Kalish/Montague/Mar ^{2}1980, pp.308 f.).

g. v is an individual variable & ϕ is a formula $\Rightarrow$ $\ulcorner(\bigwedge v)\phi\urcorner$, $\ulcorner(\bigvee v)\phi\urcorner$ are formulas
h. Nothing else is a formula or a singular term of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$

The sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ are those formulas in which no individual variable occurs free (these are the closed formulas). Furthermore, we write for the set of individual constants '$\mathrm{IC}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$', for the set of individual variables '$\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$', for the set of definite descriptions '$\mathrm{DD}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$', for the set of singular terms '$\mathrm{ST}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$', for the set of n-ary predicate constants '$\mathrm{PC}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$', and finally for the set of all formulas '$\mathrm{FM}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$'. Thus, it holds, by (D1), for the set of singular terms that

(1) $\mathrm{ST}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}} = \mathrm{IC}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}} \cup \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}} \cup \mathrm{DD}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$.

Subsequently, we define the expressions of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ that are capable of referring in the proper and improper sense:

(D2) x is an expression of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ capable of referring $:\Leftrightarrow$ $x \in \mathrm{ST}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}} \vee x \in \mathrm{PC}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}} \vee x \in \mathrm{FM}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$

(D3) x is an expression of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ capable of referring in the proper sense $:\Leftrightarrow$ x is an expression of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ capable of referring & $\sim(\exists v)\,(v \in \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ & v occurs free in $x)$

(D4) x is an expression of $\mathscr{L}^{\mathrm{PL1}=\iota}$ capable of referring in the improper sense $:\Leftrightarrow$ x is an expression of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ capable of referring & $(\exists v)\,(v \in \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ & v occurs free in $x)$

In the following definition, the expression 'x is a primitive expression capable of referring in the proper sense of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$' is formalised as '$PRfEp\,(x, \mathscr{L}^{\mathrm{PL1}^{=}\iota})$':

(D5) $PRfEp\,(x, \mathscr{L}^{\mathrm{PL1}^{=}\iota}) :\Leftrightarrow x \in \mathrm{IC}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}} \vee x \in \mathrm{PC}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$.

In a *model-theoretical semantics*, a domain D can be understood as a set of individuals. Subsequently, it is possible to define what an entity relative to such a domain is. In so doing, the expression 'y is an entity relative to D' is formalised as '$Ent\,(y, D)$', and the sign '$\emptyset$' stands for the empty set:

(D6) Let D be a set of individuals.
$Ent(y, D) :\Leftrightarrow D \neq \emptyset \ \& \ y \in D \cup \mathbb{P}(D^n)$

In (D5) and (D6), the two relata-sets for a reference function for the descriptive constants—the primitive expressions capable of referring in the proper sense—were defined. The elements of the power set $\mathbb{P}$ of D^n are considered to be the referents of n-ary predicate constants.

In what follows, a reference function *ref* for $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ which is only defined for descriptive constants is introduced:

(D7) Let X and Y be sets, and let D be a non-empty set of individuals.
ref is a reference function from X to Y for $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ relative to $D :\Leftrightarrow$
ref is a reference relation with the relata-sets X and Y for $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ relative to D—that is, $X \subseteq \{x \mid PRfEp(x, \mathscr{L}^{\mathrm{PL1}^{=}\iota})\}$ &
$Y \subseteq \{y \mid Ent(y, D)\}$ & $ref \subseteq X \times Y$—&
$(\forall x)(x \in X \Rightarrow (\exists! y)(y \in Y \ \& \ \langle x, y\rangle \in ref))$ &
$(\forall x)(x \in X \ \& \ x \in \mathrm{IC}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}} \Rightarrow ref(x) \in D)$ &
$(\forall x)(x \in X \ \& \ x \in \mathrm{PC}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}} \Rightarrow ref(x) \in \mathbb{P}(D^n))$

Hence, the referents of an individual constant and of an n-ary predicate constant relative to *ref* are an individual from the domain and a set of ordered n-tuples of such individuals, respectively.

It is not reference functions but variable assignments that are responsible for interpreting individual variables of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$. In what follows, a variable assignment and its $[v|d]$-variants are defined:

(D8) Let D be a non-empty set of individuals.
g is a variable assignment from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ to $D :\Leftrightarrow$
g is a function from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ to D

(D9) Let g be a variable assignment from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ to D, and let $v \in \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ and $d \in D$.
g' is a $[v|d]$-variant of $g :\Leftrightarrow$
$g' = g[v|d] \ \& \ g[v|d] = (g \setminus \{\langle v, g(v)\rangle\}) \cup \{\langle v, d\rangle\}$

A $[v|d]$-variant of g is a function g' which differs from g at most in the value d it assigns to v (and therefore, it assigns to any other argument than v the same value as g). Thus, the number of $[v|d]$-variants of a variable assignment is equal to the number of individuals of the domain. Any $[v|d]$-variant of a variable assignment is in itself a variable assignment from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$ to D. Furthermore, for any $v \in \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$ and any $d \in D$: $g[v\,|\,d\,](v) = d$. These variable assignments and their $[v\,|\,d\,]$-variants are necessary for interpreting open formulas, quantified formulas, and definite descriptions of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$.

The definition of a reference-theoretical model for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ is as follows:

(D10) A reference-theoretical model M for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ is a triple $\langle D, ref, d^{\circ}\rangle$, such that
D is a set of individuals &
$(\exists X, Y)$ (*ref* is a reference function from X to Y for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ relative to D) &
$d^{\circ} \in D$ is the designated nil-individual[8]

A reference-theoretical model of this form has to be supplemented with its associated recursive clauses. In what follows, such clauses will be framed according to (V2a) and (V1). From this, two alternative semantics for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ arise, based on the method of using the *designated nil-individual.*[9] The formalisation of Gödel's slingshot-argument, however, will not be based on the first but on the second semantics (though the first one will later be used for the purpose of comparison).

2.1 Reference-Theoretical Clauses According to (V2a)

The reference function *ref* of a reference-theoretical model M for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ is only defined for descriptive constants. By means of a simultaneous-recursive definition, this reference function *ref* is *extended* to a reference function $ref_{M,g}$ for all expressions of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ capable of referring. In the course of such a simultaneous-recursive definition of an extension $ref_{M,g}$ of *ref*, the same entities that the reference function *ref* had previously assigned to the descriptive constants of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ are now assigned to them as their referents relative to $ref_{M,g}$.

[8] Owing to the third condition, D is non-empty, which is why this addition can be omitted from the first condition.

[9] Cf. method IIIb in (Carnap ^{2}1956, §8, pp.35 ff.).

Let $t_1, \ldots, t_n \in \mathrm{ST}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$, let further $F \in \mathrm{PC}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$ and $\phi, \psi \in \mathrm{FM}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$, and let $v \in \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$.

(D11) Let $M = \langle D, ref, d^{\circ} \rangle$ be a reference-theoretical model for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$, let g be a variable assignment from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$ to D, and let $g[v|d]$ be a $[v|d]$-variant of g.

a. $t \in \mathrm{IC}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}} \Rightarrow ref_{M,g}(t) := ref(t)$

b. $t \in \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}} \Rightarrow ref_{M,g}(t) := g(t)$

c. $ref_{M,g}(F) := ref(F)$

Thus, the referent of a descriptive constant relative to $ref_{M,g}$ is exactly that entity relative to D that the reference function *ref* of M had previously assigned to it. Similarly, the referent of an individual variable relative to $ref_{M,g}$ is exactly that entity relative to D that the variable assignment g had previously assigned to it.

Subsequently, the referent of a definite description of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ relative to $ref_{M,g}$ is defined as follows[10]:

(D11) (Continues (D11))

d. $(\exists! d)(d \in D \;\&\; ref_{M,g[v|d]}(\phi) = 1 \;\&\; d = e) \Rightarrow$
$ref_{M,g}(\ulcorner \iota v \phi \urcorner) = e$
otherwise $ref_{M,g}(\ulcorner \iota v \phi \urcorner) = d^{\circ}$

Therefore, the referent of a definite description relative to $ref_{M,g}$ is of the same kind as the referent of an individual constant, namely an individual d in D. If there is a unique individual d in D such that $ref_{M,g[v|d]}(\phi) = 1$ and $d = e$, then e is the referent of a definite description $\ulcorner \iota v \phi \urcorner$ relative to $ref_{M,g}$. If there is no such unique individual, then the referent relative to $ref_{M,g}$ is the designated nil-individual d°.

Subsequently, the referent of all formulas of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ relative to $ref_{M,g}$ can be defined as follows:

[10] Cf. (Scott 1967, p.32) and (Gamut 1991, p.161).

(D11) (Continues (D11))

e. $ref_{M,g}(\ulcorner Ft_1 \ldots t_n \urcorner) = 1 \Leftrightarrow$
$\langle ref_{M,g}(t_1), \ldots, ref_{M,g}(t_n) \rangle \in ref_{M,g}(F)$[11]

f. $ref_{M,g}(\ulcorner t_1 = t_2 \urcorner) = 1 \Leftrightarrow ref_{M,g}(t_1) = ref_{M,g}(t_2)$

g. $ref_{M,g}(\ulcorner \neg\phi \urcorner) = 1 \Leftrightarrow ref_{M,g}(\phi) = 0$

h. $ref_{M,g}(\ulcorner (\phi \wedge \psi) \urcorner) = 1 \Leftrightarrow ref_{M,g}(\phi) = ref_{M,g}(\psi) = 1$

i. $ref_{M,g}(\ulcorner (\phi \vee \psi) \urcorner) = 1 \Leftrightarrow ref_{M,g}(\phi) = 1 \vee ref_{M,g}(\psi) = 1$

j. $ref_{M,g}(\ulcorner (\phi \to \psi) \urcorner) = 1 \Leftrightarrow ref_{M,g}(\phi) = 0 \vee ref_{M,g}(\psi) = 1$

k. $ref_{M,g}(\ulcorner (\phi \leftrightarrow \psi) \urcorner) = 1 \Leftrightarrow ref_{M,g}(\phi) = ref_{M,g}(\psi)$

l. $ref_{M,g}(\ulcorner (\bigwedge v)\phi \urcorner) = 1 \Leftrightarrow (\forall d)(d \in D \Rightarrow ref_{M,g[v|d]}(\phi) = 1)$

m. $ref_{M,g}(\ulcorner (\bigvee v)\phi \urcorner) = 1 \Leftrightarrow (\exists d)(d \in D \; \& \; ref_{M,g[v|d]}(\phi) = 1)$

This completes the simultaneous-recursive definition of an extension $ref_{M,g}$ of *ref* to all expressions of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ capable of referring.

Furthermore, a concept of relativised truth based on a reference function $ref_{M,g}$ can be introduced as follows, where 'x is true relative to $ref_{M,g}$' is formalised as '$True(x, ref_{M,g})$':

(D12) Let x be a formula of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$, let further $M = \langle D, ref, d^{\circ} \rangle$ be a reference-theoretical model for $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$, and let g be a variable assignment from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ to D.
$True(x, ref_{M,g}) :\Leftrightarrow ref_{M,g}(x) = 1$

This first semantics for $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ based on the method of using the designated nil-individual was developed according to (V2a). The next step would be to define additional semantic notions such as the notions of logical truth and of logical equivalence. However, we will not pursue this course any further, as this would lead to the trivialisation of Gödel's argument. For from the mere assumption that two sentences are true relative to $ref_{M,g}$, it follows by (D11) and (D12) that they refer to the same object. Instead, these semantic notions will be defined according to (V1).

[11] For any $d \in D$, the standard convention applies: $\langle d \rangle = d$.

2.2 Reference-Theoretical Clauses According to (V1)

In what follows, a semantics for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ will be presented which, though based on the method of using the designated nil-individual, will yet be developed according to (V1). For this purpose, a reference function *ref* will be introduced that will only be defined for the descriptive constants. Thus, the semantic values of descriptive constants will be considered as referents. Furthermore, the truth and falsity of all formulas (and thus of all sentences) of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ will be defined in the form of a simultaneous-recursive definition—without assigning any referents relative to $ref_{M,g}$ to these formulas. In so doing, the aforementioned trivialisation of Gödel's argument can be avoided because the question of what the referents of sentences are is left open for the time being.

Let $t_1, \ldots, t_n \in \mathrm{ST}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$, let further $F \in \mathrm{PC}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$ and $\phi, \psi \in \mathrm{FM}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$, and let $v \in \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$.

(D13) Let $M = \langle D, ref, d^{\circ} \rangle$ be a reference-theoretical model for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$, let further g be a variable assignment from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}}$ to D, and let $g[v|d]$ be a $[v|d]$-variant of g.

a. $t \in \mathrm{IC}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}} \Rightarrow ref_{M,g}(t) := ref(t)$

b. $t \in \mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=\iota}}} \Rightarrow ref_{M,g}(t) := g(t)$

c. $ref_{M,g}(F) := ref(F)$

d. $(\exists! d)(d \in D \;\&\; True(\phi, ref_{M,g[v|d]}) \;\&\; d = e) \Rightarrow$
$ref_{M,g}(\ulcorner \iota v \phi \urcorner) = e$
otherwise $ref_{M,g}(\ulcorner \iota v \phi \urcorner) = d^{\circ}$

e. $True(\ulcorner F t_1 \ldots t_n \urcorner, ref_{M,g}) \Leftrightarrow$
$\langle ref_{M,g}(t_1), \ldots, ref_{M,g}(t_n) \rangle \in ref_{M,g}(F)$

f. $True(\ulcorner t_1 = t_2 \urcorner, ref_{M,g}) \Leftrightarrow ref_{M,g}(t_1) = ref_{M,g}(t_2)$

g. $True(\ulcorner \neg\phi \urcorner, ref_{M,g}) \Leftrightarrow {\sim} True(\phi, ref_{M,g})$
(i.e. $False(\phi, ref_{M,g})$)

h. $True(\ulcorner (\phi \wedge \psi) \urcorner, ref_{M,g}) \Leftrightarrow True(\phi, ref_{M,g}) \;\&\; True(\psi, ref_{M,g})$

i. $True(\ulcorner (\phi \vee \psi) \urcorner, ref_{M,g}) \Leftrightarrow True(\phi, ref_{M,g}) \curlyvee True(\psi, ref_{M,g})$

j. $True(\ulcorner (\phi \rightarrow \psi) \urcorner, ref_{M,g}) \Leftrightarrow False(\phi, ref_{M,g}) \curlyvee True(\psi, ref_{M,g})$

k. $True(\ulcorner (\phi \leftrightarrow \psi) \urcorner, ref_{M,g}) \Leftrightarrow$
$(True(\phi, ref_{M,g}) \;\&\; True(\psi, ref_{M,g})) \curlyvee$
$(False(\phi, ref_{M,g}) \;\&\; False(\psi, ref_{M,g}))$

l. $True\,(\ulcorner(\bigwedge v)\,\phi\urcorner, ref_{M,g}) \Leftrightarrow (\forall d)\,(d \in D \Rightarrow True\,(\phi, ref_{M,g[v|d]}))$
m. $True\,(\ulcorner(\bigvee v)\,\phi\urcorner, ref_{M,g}) \Leftrightarrow (\exists d)\,(d \in D\,\&\, True\,(\phi, ref_{M,g[v|d]}))$

The fundamental assumption of a *referential theory of definite descriptions* for a formal language, i.e. the assumption that

(A) definite descriptions without free individual variables refer to individuals as do individual constants,

has promptly been adopted into clause (D13d). Since definite descriptions are—according to (D1c)—singular terms, (A) is not only included in clause (D13d) but also in clauses (D13e) and (D13f). This is owing to the fact that these three clauses directly concern singular terms and, thus, definite descriptions. For if in (D13e) and (D13f), the metalinguistic variable t_1 stands for a definite description, then $ref_{M,g}\,(t_1)$ is defined precisely because of clause (D13d). In his argument, Gödel aims at disproving (A); in other words, he seeks to show that (A)—together with further assumptions—entails absurd consequences, i.e. (FCG). He considers (FCG) to be absurd because different true sentences can refer to different objects (and different false sentences can refer to different objects as well). Thus, his argument is a case of reductio-ad-absurdum. To formalise Gödel's argument, I have therefore opted for a semantics for $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ that is grounded in (A).

Based on such a concept of relativised truth, the semantic notions of logical truth (formal: '$LTrue\,(x)$'), of logical equivalence (formal: '$LE\,(x_1, x_2)$'), and of satisfiability (formal: '$Satisfiable\,(x)$') can be defined as follows:

(D14) Let x be a formula of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$.
$LTrue\,(x) :\Leftrightarrow (\forall M, g)\, True\,(x, ref_{M,g})$

(D15) Let x_1 and x_2 be formulas of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$.
$LE\,(x_1, x_2) :\Leftrightarrow LTrue\,(\ulcorner x_1 \leftrightarrow x_2\urcorner)$[12]

(D16) Let x be a formula of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$.
$Satisfiable\,(x) :\Leftrightarrow (\exists M, g)\, True\,(x, ref_{M,g})$

[12] This paper follows the usage in which the outermost pair of parentheses in a formula can be omitted.

This second semantics for $\mathscr{L}^{\text{PL1}^{=}\iota}$ based on the method of using the designated nil-individual is developed according to (V1) and shall serve as the foundation for the formalisation of Gödel's slingshot-argument.

Let A_1 and A_2 be expressions of $\mathscr{L}^{\text{PL1}^{=}\iota}$ capable of referring. Then, the relation of coreferentiality can be defined as follows:

(D17) $A_1 \equiv_{ref_{M,g}} A_2 :\Leftrightarrow ref_{M,g}(A_1) = ref_{M,g}(A_2)$

Owing to the fact that identity is an equivalence relation, the relation $\equiv_{ref_{M,g}}$ is—according to the definition above—an equivalence relation as well, i.e. a relation that is reflexive, symmetrical, and transitive. Thus, it holds that

(Rf) $(\forall A_1)\, A_1 \equiv_{ref_{M,g}} A_1$

(Sy) $(\forall A_1, A_2)\, (A_1 \equiv_{ref_{M,g}} A_2 \Rightarrow A_2 \equiv_{ref_{M,g}} A_1)$

(Tr) $(\forall A_1, A_2, A_3)\, (A_1 \equiv_{ref_{M,g}} A_2 \;\&\; A_2 \equiv_{ref_{M,g}} A_3 \Rightarrow A_1 \equiv_{ref_{M,g}} A_3)$

3 Premises of Gödel's Argument

The premises of Gödel's argument can be divided into three categories. There are

(i) premises that are provable using only the assumptions of the second semantics, i.e. (D13) (these premises are the first two metatheorems and the first lemma—see §3.1),

(ii) premises that are not provable using only the assumptions of the second semantics and which therefore have to be postulated as valid (these are the postulates and principles—see §3.2),

and

(iii) premises that are provable using the assumptions of the second semantics and some of the postulates and principles from (ii) (these are the second lemma and the fourth metatheorem—see §3.3).

3.1 Three Metatheorems and the First Lemma

Singular sentences contain at least one individual constant. Let σ be a singular sentence, and let c be an individual constant that occurs in σ. Then, this singular sentence is written as '$\sigma[c]$'. Now let $\sigma[c]$ be a singular sentence, and let v be an individual variable that does not occur in $\sigma[c]$. Replacing all occurrences of c in $\sigma[c]$ with v results in a formula that is written as '$\sigma[c/v]$'.

The following premises are provable using only the second semantics; in their proofs, there is no need for any principles and postulates of (ii) above. In what follows, two metatheorems concerning singular sentences will be proven. These will come in useful for the formalisation of Gödel's slingshot-argument.

3.1.1 The first metatheorem: (T1)

Let $M = \langle D, ref, d^\circ \rangle$ be a reference-theoretical model for $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$, let further g be a variable assignment from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ to D. Let σ be a singular sentence of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ that contains an individual constant c of $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$, and let v be an individual variable of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ that does not occur in $\sigma[c]$. Then, the first metatheorem for such sentences states that

(T1) {(D13), (D9)} $\vdash_{\mathrm{PL1}^=}$
$(\forall \sigma, c, v, M, g)$
$(\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g}) \Rightarrow \mathit{True}\,(\ulcorner c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner, \mathit{ref}_{M,g}))$.

The proof of (T1) can be found in the appendix. By way of illustration, consider an example in everyday language: if the sentence 'Salzburg is a city' is true, then the sentence 'Salzburg is the thing such that it is a city and identical to Salzburg' must be true as well.

3.1.2 Formal equivalent of the fundamental assumption (A) of a referential theory of definite descriptions: lemma (L1)

The first lemma is a formal equivalent of (A). It is therefore exactly that metatheoretical sentence that Gödel wants to refute. Using (T1) together with the reference-theoretical clause for identity (see (D13f)), we obtain

(L1) {(D13), (D17), (T1)} $\vdash_{\mathrm{PL1}^=}$
$(\forall \sigma, c, v, M, g)$
$(\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g}) \Rightarrow c \equiv_{\mathit{ref}_{M,g}} \ulcorner \iota v\,(\sigma[c/v] \wedge v = c)\urcorner)$.

Proof: Lemma (L1) follows from (D13f), (D17), and (T1). □

By way of illustration, consider yet again an example in everyday language: if the sentence 'Salzburg is a city' holds, then the proper noun 'Salzburg' and the definite description 'the thing such that it is a city and identical to Salzburg' refer to the same object.

This lemma will facilitate the formalisation of Gödel's argument. It states that if the singular sentence $\sigma[c]$ is true, then the individual constant c refers to the same object as the definite description $\ulcorner \iota v\,(\sigma[c/v] \wedge v = c)\urcorner$, viz. an individual. Since v is an individual variable that does not occur in $\sigma[c]$, $\ulcorner \iota v\ (\sigma[c/v] \wedge v = c)\urcorner$ contains no free individual variables. To put it simply, (L1) states that if a certain singular sentence holds, then a certain individual constant refers to the same object as a certain definite description without free individual variables. Lemma (L1) is therefore a formal equivalent of (A). Consequently, Gödel aims to prove in his argument that not only (A) but also its formal equivalent, (L1), entail absurd consequences. According to him, (L1) and thus (A) are to be abandoned. Since (L1) is a metatheoretical sentence provable only on the basis of the second semantics—a sentence that entails absurd consequences—Gödel's argument is directed against any semantics, on the basis of which (L1) can be proven.

3.1.3 The second metatheorem: (T2)

Theorem (T1) leads to the following metatheorem for singular sentences:

(T2) $\{(\mathrm{D13}), (\mathrm{T1})\} \vdash_{\mathrm{PL1}^=}$
$(\forall \sigma, c, v, M, g)$
$(\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g}) \Rightarrow \mathit{True}\,(\ulcorner \sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner, \mathit{ref}_{M,g}))$

The proof of (T2) can be found in the appendix. Once again, consider an example in everyday language: if the sentence 'Salzburg is a city' holds, then so does the sentence 'Salzburg is a city iff Salzburg is the thing such that it is a city and identical to Salzburg'.

What follows is a corollary of (T2):

(Cr1) $\{(\mathrm{T2}), (\mathrm{D14}), \text{PL1-theorem}\} \vdash_{\mathrm{PL1}^=}$
$(\forall \sigma, c, v)\,(\mathit{LTrue}\,(\sigma[c]) \Rightarrow \mathit{LTrue}\,(\ulcorner \sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner))$

The proof can be found in the appendix, lines 3–9 of the proof of (L2).

3.1.4 The third metatheorem: (T3)

By (T2), a further metatheorem concerning singular sentences can be proven, despite the fact that it is not needed for the formalisation of Gödel's argument. However, it can—together with corollary (Cr1) of metatheorem (T2)—be used to discuss an interesting point that is of no little importance for this argument.

This further metatheorem is as follows:

(T3) $\{(\mathrm{D16}), (\mathrm{T2}), \text{PL1-theorem}\} \vdash_{\mathrm{PL1}^=}$
$(\forall \sigma, c, v)$
$(\mathit{Satisfiable}\,(\sigma[c]) \Rightarrow \mathit{Satisfiable}\,(\ulcorner \sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner))$

Proof: (T3) holds by (T2), (D16), and by the PL1-theorem '$(\bigwedge \mathtt{x})\,(\mathtt{Fx} \rightarrow \mathtt{Gx}) \rightarrow ((\bigvee \mathtt{x})\,(\mathtt{Fx}) \rightarrow (\bigvee \mathtt{x})\,(\mathtt{Gx}))$'. □

Corollary (Cr1) of metatheorem (T2) shows that

(1) $LTrue(\sigma[c])$

is a sufficient condition for

(2) $LTrue(\ulcorner\sigma[c] \leftrightarrow c = \iota v(\sigma[c/v] \wedge v = c)\urcorner)$

and thus, by (D15), it is also a sufficient condition for

(3) $LE(\sigma[c], \ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner)$.

By ($\mathrm{L_R}$), i.e. the principle that logically equivalent sentences refer to the same object, (3) gives

(4) $\sigma[c] \equiv_{ref_{M,g}} \ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner$

if (1). Now, (T3) shows that

(5) $Satisfiable(\sigma[c])$

is a sufficient condition for

(6) $Satisfiable(\ulcorner\sigma[c] \leftrightarrow c = \iota v(\sigma[c/v] \wedge v = c)\urcorner)$.

The question that will now be addressed is as to whether the assumption that

(7) $True(\sigma[c], ref_{M,g})$,

from which it follows that (5), is sufficient for (2) and thus by (D15) also sufficient for (3). If so, then it is possible to deduce from (3) by ($\mathrm{L_R}$) that (4) if (7). In this case, Gödel could—if he assumes ($\mathrm{L_R}$)—abandon his assumption (4) that $\sigma[c] \equiv_{ref_{M,g}} \ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner$ (for all σ, c, v, M, g) holds *unconditionally*. This assumption of his cannot be proven in the context of the second semantics (and neither can ($\mathrm{L_R}$)); instead, it has to be postulated as valid.

However, it is possible to prove that (5) is not a sufficient condition for (2). At a particular step in the process of formalising Gödel's argument, it is therefore not possible to forego Gödel's additional assumption (4) (see (G) in §3.2.4). This is the case because the following metatheoretical sentence is false in the context of the second semantics and thus refutable (see the proof of the refutability of (NT) in §6.3):

(NT) $(\forall \sigma, c, v)$
$(\textit{Satisfiable}\, \sigma[c] \Rightarrow \textit{LTrue}\, (\ulcorner \sigma[c] \leftrightarrow c = \iota v\, (\sigma[c/v] \wedge v = c)\urcorner)$

Using a singular sentence that does not contain the individual variable v, such as the atomic (and thus singular) sentence $\ulcorner Fc \urcorner$, we prove that (NT) is false by showing that

(INT) $\textit{Satisfiable}\, (\ulcorner Fc \urcorner) \Rightarrow \textit{LTrue}\, (\ulcorner Fc \leftrightarrow c = \iota v\, (Fv \wedge v = c)\urcorner)$

does not hold. For (INT) is an instantiation and thus a logical consequence of (NT). However, if a logical consequence of (NT) is false, then so is (NT). □

Therefore, assuming the satisfiability of $\ulcorner Fc \urcorner$ is not sufficient for the logical truth of $\ulcorner Fc \leftrightarrow c = \iota v\, (Fv \wedge v = c)\urcorner$ and thus not sufficient for the logical equivalence of $\ulcorner Fc \urcorner$ and $\ulcorner c = \iota v\, (Fv \wedge v = c)\urcorner$ either. In other words, the logical equivalence of two sentences is understood to mean that the two sentences are both true (or false) relative to the same models and variable assignments. It may be the case that the two sentences $\ulcorner Fc \urcorner$ and $\ulcorner c = \iota v\, (Fv \wedge v = c)\urcorner$ are true relative to the same models and variable assignments. But there remain models and variable assignments relative to which $\ulcorner Fc \urcorner$ is false, $\ulcorner c = \iota v (Fv \wedge v = c)\urcorner$, however, is true (e.g. the model used in (5) in the proof of the refutability of (NT)). Thus, the condition of satisfiability of $\ulcorner Fc \urcorner$ is to weak to allow the application of (L_R) to both of the sentences. Therefore, it is not possible to justify by (INT), (D15), and by principle (L_R) that $\ulcorner Fc \urcorner$ and $\ulcorner c = \iota v (Fv \wedge v = c)\urcorner$ refer to the same object if $\ulcorner Fc \urcorner$ is satisfiable. Nor is it possible to justify by (NT), (D15), and by (L_R) that it holds for all σ, c, v that $\sigma[c]$ and $\ulcorner c = \iota v (\sigma[c/v] \wedge v = c)\urcorner$ refer to the same object if $\sigma[c]$ is satisfiable.

Furthermore, assuming the logical truth of $\sigma[c]$ is of no use to the formalisation of Gödel's argument, for Gödel does not aim to derive the thesis that only all *logically* true singular sentences refer to the same object, but rather the thesis that *all true*—and thus also all *contingently* true—singular sentences do so.

3.1.5 A consequence of the reference-theoretical clause for negation: (N)

(N) $\{(D13)\} \vdash_{PL1^{=}}$
$(\forall \sigma, c, M, g)\, (\textit{False}\, (\sigma[c], \textit{ref}_{M,g}) \Rightarrow \textit{True}\, (\ulcorner \neg \sigma[c] \urcorner, \textit{ref}_{M,g}))$

3.2 Principles and Postulates

In order to avoid any trivialisation of Gödel's argument, it is left open in (D13) what the referents of sentences (and formulas) are. For this reason, the definition of coreferentiality in (D17) is not applicable to sentences (and formulas). Consequently, the following premises of Gödel's argument are not provable using only the second semantics. Therefore, the validity of the principles contained in these premises has to be postulated on a *metasemantic level*, thus enabling them to guide the search for such referents without any supposition as to what the referents of sentences are.

3.2.1 Specification of (R) for closed singular terms: (RT)

(RT) $(\forall S_1, S_2, t_1, t_2, M, g)$
$(Res(S_2, S_1(t_1//t_2))\ \&\ t_1 \equiv_{ref_{M,g}} t_2 \Rightarrow S_1 \equiv_{ref_{M,g}} S_2)$[13]

3.2.2 Specification of (R) for sentences: (RS)

(RS) $(\forall S_1, S_2, T_1, T_2, M, g)$
$(Res(S_2, S_1(T_1//T_2))\ \&\ T_1 \equiv_{ref_{M,g}} T_2 \Rightarrow S_1 \equiv_{ref_{M,g}} S_2)$

3.2.3 Logically equivalent sentences refer to the same object: (L_R)

(L_R) $(\forall S_1, S_2, M, g)\,(LE(S_1, S_2) \Rightarrow S_1 \equiv_{ref_{M,g}} S_2)$

Observe that in the second semantics, the principle (L_R) is not provable by means of the definitions of truth in (D13k) and of logical equivalence in (D15): based on the logical truth of $\ulcorner S_1 \leftrightarrow S_2 \urcorner$ alone, it is not possible to draw any conclusions (by means of these definitions) as to the referents of S_1 and S_2.

From the point of view of a state-of-affairs semantics, two logically equivalent sentences (that are not identical) describe states of affairs which involve different objects and have different necessary and sufficient conditions for obtaining. Hence, those states of affairs are not identical, although they are described by logically equivalent sentences.

[13] '$Res(S_2, S_1(A_1//A_2))$' abbreviates 'S_2 is a result of substituting one ore more occurrences of A_1 in S_1 with A_2'.

Carnap assumed the principle (L_R) in his early semantics[14], and Church adopted it as a premise of his slingshot-argument[15]. The principle (L_R) and theorem (T2) enable us to take a certain step in the proof of Gödel's argument. And yet, this principle was often criticised. Barwise and Perry, for instance, rejected this principle as untenable in their critique of premises of slingshot-arguments.[16] The *standard version of the slingshot-argument*—according to Barwise and Perry—states that the (FCG)-thesis (the assumption that all true sentences refer to the same object, as do all false sentences) can be derived from (L_R) and (RT).

3.2.4 Gödel's assertion: (G)

(G) $(\forall \sigma, c, v, M, g)$
$\sigma[c] \equiv_{ref_{M,g}} \ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner$

Observe that $\sigma[c]$ and $\ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner$ are not logically equivalent in the second semantics, as was demonstrated in §3.1.4.

The postulate (G) does not hold in a state-of-affairs semantics, for instance because the state of affairs that the sentence 'Scott is an author of *Waverley*' refers to is different from the state of affairs that the sentence 'Scott is the thing such that it is an author of Waverley and identical to Scott' refers to. The necessary and sufficient condition for obtaining of the first state of affairs is the condition that Scott has the property of being an author of *Waverley*. By contrast, the necessary and sufficient condition for obtaining of the second state of affairs is the condition that Scott is identical to the thing that is an author of *Waverley* and identical to Scott. Since these two conditions are different, the two states of affairs that those two sentences describe must be different as well.

[14] Cf. (Carnap 1942).
[15] Cf. (Church 1943).
[16] Cf. (Barwise/Perry 1981, p.378).

3.3 The Second Lemma and Fourth Metatheorem

The following premises are provable using the assumptions of the second semantics and some of the postulates and principles from §3.2.

3.3.1 Logically true singular sentences refer to the same object as certain identity sentences: lemma (L2)

The postulate (G) leads to a consequence that can be proven independently of (G) itself. This consequence is lemma (L2).[17] Using this lemma, the first step in the proof of theorem (T4)—that is, the case of true singular sentences—can be justified without any need for the postulate (G) itself. This lemma, which can be proven by (L_R), (T2), (D14), (D15), and by a PL1-theorem is as follows:

(L2) $\{$(D14), (D15), (T2), PL1-theorem, (L_R)$\} \vdash_{PL1^=}$
$(\forall \sigma, c, v, M, g)\,(LTrue\,(\sigma[c]) \Rightarrow \sigma[c] \equiv_{ref_{M,g}} \ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner)$

The proof of (L2) can be found in the appendix. Consider the following example in everyday language: if the sentence 'Salzburg is or is not a city' is logically true, then the two sentences 'Salzburg is or is not a city' and 'Salzburg is the thing such that it is (a city or not a city) and identical to Salzburg' refer to the same object.

3.3.2 The theorem for true singular sentences: (T4)

(T4) $(\forall \sigma_1, \sigma_2, a, b, M, g)$
$(True\,(\sigma_1[a], ref_{M,g})\ \&\ True\,(\sigma_2[b], ref_{M,g}) \Rightarrow \sigma_1[a] \equiv_{ref_{M,g}} \sigma_2[b])$

This theorem will be proven in §4.1 by (L1), (L2), (RT), (G), and three further premises. It serves as a premise for the case of false singular sentences.

[17] By the rules of first-order predicate logic, (L2) can be derived from (G), but not vice versa; therefore, (G) is logically stronger than (L2). Observe that '$(\bigwedge x)(Gx) \rightarrow (\bigwedge x)(Fx \rightarrow Gx)$' is a PL1-theorem, while '$(\bigwedge x)(Fx \rightarrow Gx) \rightarrow (\bigwedge x)(Gx)$' is not.

4 Formalisation of Gödel's Argument

As stated previously, Gödel presupposes a referential theory of definite descriptions, according to which definite descriptions without free individual variables refer to individuals as do individual constants. We have shown that lemma (L1) is a more formal equivalent of (A), for it states, to put it simply, that

(L1′) if a certain singular sentence is true, then a certain individual constant and a certain definite description refer to the same object (i.e. an individual of the domain).

In (V1), which serves as the basis of our formalisation, it remains undefined of which type the referents of sentences are. However, if every sentence is supposed to refer to one and only one object, then the question arises as to how it is possible to *extend*—according to (V1)—the reference function *ref*, which is only defined for descriptive constants, to a reference function $ref_{M,g}$ that is defined for all sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$.

But what does Gödel's argument tell us about the possible ways in which such an extension could be defined? On the basis of the premises of Gödel's argument, Frege could argue that all true sentences refer to the same object (and that all false sentences do so as well). Yet, what is it—in addition to their truth-value—that all true and false sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ have in common in terms of semantics? It is merely the two truth-values, and not a plethora of different states of affairs, that all true and false sentences have in common and, thus, refer to. Hence, so long as one adheres to the premises of Gödel's argument, it is mandatory to extend any reference function *ref* from descriptive constants to sentences (and to formulas) in such a way that their referents relative to $ref_{M,g}$ are the two truth-values, and not a variety of different states of affairs. Thus, a semantics such as that in (D11) would arise.

Gödel, however, considers his argument to be a reductio-ad-absurdum of (A); in other words, he believes that this assumption (together with the principle of compositionality) entails absurd consequences, viz. (FCG). According to him, (A) should therefore be abandoned. Furthermore, since (L1) is nothing but a more formal equivalent of (A), (L1) would also have to be abandoned and, by extension, so too would any semantics, on the basis of which this lemma can be proven.

In what follows, Gödel's preliminary draft of this argument will be brought to a conclusion and thereby formalised using a calculus of natural deduction for first-order predicate logic with identity. First, the argument will be formalised for the case of true and, subsequently, for the case of false singular sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$. Next, the result will be generalised to apply to all sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$.

The question that we will examine in this context is as to whether the premises of Gödel's argument suffice to precisely determine the kind of the entities referred to by the sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$.

4.1 The Case of True Singular Sentences: (T4)

In addition to the premises (L1), (L2), (RT), (G), and (D14) the proof of theorem (T4) requires the two following premises regarding the composition of certain formulas of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$:

(Pr1) $Res(\ulcorner a = \iota v((v = b \lor v \neq b) \land v = a)\urcorner, \ulcorner a = \iota v(\sigma_1[a/v] \land v = a)\urcorner$
$(\ulcorner \iota v(\sigma_1[a/v] \land v = a)\urcorner // \ulcorner \iota v((v = b \lor v \neq b) \land v = a)\urcorner))$

(Pr2) $Res(\ulcorner b = \iota v((a = v \lor a \neq v) \land v = b)\urcorner, \ulcorner b = \iota v(\sigma_2[b/v] \land v = b)\urcorner$
$(\ulcorner \iota v(\sigma_2[b/v] \land v = b)\urcorner // \ulcorner \iota v((a = v \lor a \neq v) \land v = b)\urcorner))$

Let σ_1, σ_2 be arbitrary but fixed singular sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$, let further a be an individual constant of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ that occurs in σ_1, and let b be an individual constant of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ that occurs in σ_2. Then, Gödel's slingshot-argument for the case of true singular sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ is as follows:

(T4) $\{(\mathrm{L1}), (\mathrm{L2}), (\mathrm{RT}), (\mathrm{G}), (\mathrm{D14}), (\mathrm{Pr1}), (\mathrm{Pr2})\} \vdash_{\mathrm{PL1}^{=}}$
$(\forall \sigma_1, \sigma_2, a, b, M, g)$
$(True\,(\sigma_1[a], ref_{M,g})\ \&\ True\,(\sigma_2[b], ref_{M,g}) \Rightarrow \sigma_1[a] \equiv_{ref_{M,g}} \sigma_2[b])$

The proof of theorem (T4) requires an equivalence relation $\equiv_{ref_{M,g}}$ (see (D17)). Observe that the definition of coreferentiality (D17) is, by (D13) and without the premises of Gödel's argument, only applicable to singular terms and n-ary predicate constants. It becomes applicable to sentences once we assume the premises of Gödel's argument, for these premises specify a number of conditions as to when two sentences are coreferential.

Let us first outline the main idea of the proof of this theorem, which will be proven in the following three steps by Conditional Proof and, subsequently, by Universal Generalisation:

Step 1

The assumption for Conditional Proof states that

(1) $\mathit{True}\,(\sigma_1[a], \mathit{ref}_{M,g})$ & $\mathit{True}\,(\sigma_2[b], \mathit{ref}_{M,g})$.

Since

(2) $\mathit{LTrue}\,(\ulcorner a = b \vee a \neq b \urcorner)$,

we have, by (D14),

(3) $(\forall M, g)\,\mathit{True}\,(\ulcorner a = b \vee a \neq b \urcorner, \mathit{ref}_{M,g})$.

By (L2), (2) gives

(4) $\ulcorner a = b \vee a \neq b \urcorner \equiv_{\mathit{ref}_{M,g}} \ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a) \urcorner$

and

(5) $\ulcorner a = b \vee a \neq b \urcorner \equiv_{\mathit{ref}_{M,g}} \ulcorner b = \iota v((a = v \vee a \neq v) \wedge v = b) \urcorner$.

The singular sentence $\ulcorner a = b \vee a \neq b \urcorner$ contains two individual constants, a and b. Using our notation for singular sentences, this sentence can therefore be represented in two different ways, '$\sigma[a]$' or '$\sigma[b]$'. In the first consequence (4), $\sigma[a]$ is the singular sentence $\ulcorner a = b \vee a \neq b \urcorner$; thus, $\sigma[a/v]$ is the formula $\ulcorner v = b \vee v \neq b \urcorner$. In the second consequence (5), $\sigma[b]$ is the singular sentence $\ulcorner a = b \vee a \neq b \urcorner$; thus, $\sigma[b/v]$ is the formula $\ulcorner a = v \vee a \neq v \urcorner$. The relationship between these two consequences can be illustrated as follows:

$$\ulcorner a = b \vee a \neq b \urcorner$$

$$\equiv_{\mathit{ref}_{M,g}} \qquad\qquad \equiv_{\mathit{ref}_{M,g}}$$

$$\ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a) \urcorner \qquad \ulcorner b = \iota v((a = v \vee a \neq v) \wedge v = b) \urcorner$$

This first step can be justified by (L2), into the proof of which (L_R) has been incorporated. Although (L2) is a consequence of (G), it is not necessary to make use of the logically stronger postulate (G) itself. Instead, we can justify this step by the logically weaker lemma (L2), which cannot be proven using only the second semantics.

Step 2

By (1) and (3), we have

(6) $\mathit{True}(\sigma_1[a], \mathit{ref}_{M,g})$

and

(7) $\mathit{True}(\ulcorner a = b \vee a \neq b \urcorner, \mathit{ref}_{M,g})$.

Hence, it follows from (6) and (7) by (L1) that

(8) $a \equiv_{\mathit{ref}_{M,g}} \ulcorner \iota v(\sigma_1[a/v] \wedge v = a) \urcorner$

and

(9) $a \equiv_{\mathit{ref}_{M,g}} \ulcorner \iota v((v = b \vee v \neq b) \wedge v = a) \urcorner$.

From (8) and (9), we obtain by symmetry and transitivity of $\equiv_{\mathit{ref}_{M,g}}$

(10) $\ulcorner \iota v(\sigma_1[a/v] \wedge v = a) \urcorner \equiv_{\mathit{ref}_{M,g}} \ulcorner \iota v((v = b \vee v \neq b) \wedge v = a) \urcorner$.

Furthermore, the identity sentence

(11) $\ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a) \urcorner$

of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ is a result of partially substituting

(12) $\ulcorner \iota v(\sigma_1[a/v] \wedge v = a) \urcorner$

in

(13) $\ulcorner a = \iota v(\sigma_1[a/v] \wedge v = a) \urcorner$

with

(14) $\ulcorner \iota v((v = b \vee v \neq b) \wedge v = a) \urcorner$.

As (10) gives

(15) $\ulcorner \iota v(\sigma_1[a/v] \wedge v = a) \urcorner \equiv_{\mathit{ref}_{M,g}} \ulcorner \iota v((v = b \vee v \neq b) \wedge v = a) \urcorner$,

it follows from (11)–(15) by (RT) that

(16) $\ulcorner a = \iota v(\sigma_1[a/v] \wedge v = a)\urcorner \equiv_{ref_{M,g}} \ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner$.

On the other hand, it also follows from (1) and (3) that

(17) $\mathit{True}\,(\sigma_2[b], ref_{M,g})$

and

(18) $\mathit{True}\,(\ulcorner a = b \vee a \neq b\urcorner, ref_{M,g})$.

Similarly to (6)–(16), from (17) and (18), we also obtain by (L1) and (RT)

(19) $\ulcorner b = \iota v(\sigma_2[b/v] \wedge v = b)\urcorner \equiv_{ref_{M,g}} \ulcorner b = \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner$.

The illustration above can thus be continued in a similar vein:

$$\ulcorner a = b \vee a \neq b\urcorner$$

$$\equiv_{ref_{M,g}} \qquad\qquad \equiv_{ref_{M,g}}$$

$$\ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner \qquad \ulcorner b = \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner$$

$$\equiv_{ref_{M,g}} \qquad\qquad \equiv_{ref_{M,g}}$$

$$\ulcorner a = \iota v(\sigma_1[a/v] \wedge v = a)\urcorner \qquad \ulcorner b = \iota v(\sigma_2[b/v] \wedge v = b)\urcorner$$

This second step can be justified by (L1) and (RT), the former being provable by (T1) and by the reference-theoretical clause for identity. At this point, we incorporate the critical assumption (A) since (L1) is a more formal equivalent of (A). Furthermore, we also have to make use of the principle of compositionality (R)—or rather its specification (RT) for closed singular terms.

Step 3
By (G), we have

(20) $\sigma_1[a] \equiv_{ref_{M,g}} \ulcorner a = \iota v(\sigma_1[a/v] \wedge v = a)\urcorner$

and

(21) $\sigma_2[b] \equiv_{ref_{M,g}} \ulcorner b = \iota v(\sigma_2[b/v] \wedge v = b)\urcorner$.

Thus, it follows from (4), (16), (20); (5), (19), and (21) by the transitivity of $\equiv_{ref_{M,g}}$ that

(22) $\sigma_1[a] \equiv_{ref_{M,g}} \sigma_2[b]$.

This completes the Conditional Proof. UG. □

Even if we combine the assumptions that $\sigma_1[a]$ and $\sigma_2[b]$ are true relative to $ref_{M,g}$—which would imply the satisfiability of $\sigma_1[a]$ and $\sigma_2[b]$—with (NT), (D15), and (L_R), we cannot justify this third step. The reasons for this have already been discussed in §3.1.4; accordingly, (NT) is refutable in the second semantics, which is why postulate (G) is of paramount importance here.

Synopsis

The three steps can be illustrated in the following way:

$$\ulcorner a = b \lor a \neq b \urcorner$$

$$\equiv_{ref_{M,g}} \qquad\qquad \equiv_{ref_{M,g}}$$

$$\ulcorner a = \iota v((v = b \lor v \neq b) \land v = a)\urcorner \qquad \ulcorner b = \iota v((a = v \lor a \neq v) \land v = b)\urcorner$$

$$\equiv_{ref_{M,g}} \qquad\qquad \equiv_{ref_{M,g}}$$

$$\ulcorner a = \iota v(\sigma_1[a/v] \land v = a)\urcorner \qquad \ulcorner b = \iota v(\sigma_2[b/v] \land v = b)\urcorner$$

$$\equiv_{ref_{M,g}} \qquad\qquad \equiv_{ref_{M,g}}$$

$$\sigma_1[a] \equiv_{ref_{M,g}} \sigma_2[b]$$

Proof of (T4):

1.	$(\forall \sigma, c, v, M, g)\,(True\,(\sigma[c], ref_{M,g}) \Rightarrow$ $c \equiv_{ref_{M,g}} \ulcorner \iota v\,(\sigma[c/v] \land v = c)\urcorner$	(L1)
2.	$(\forall \sigma, c, v, M, g)\,(LTrue\,(\sigma[c]) \Rightarrow$ $\sigma[c] \equiv_{ref_{M,g}} \ulcorner c = \iota v(\sigma[c/v] \land v = c)\urcorner)$	(L2)
3.	$(\forall S_1, S_2, t_1, t_2, M, g)$ $(Res\,(S_2, S_1(t_1//t_2))\ \&\ t_1 \equiv_{ref_{M,g}} t_2 \Rightarrow$ $S_1 \equiv_{ref_{M,g}} S_2)$	(RT)
4.	$(\forall \sigma, c, v, M, g)$ $\sigma[c] \equiv_{ref_{M,g}} \ulcorner c = \iota v(\sigma[c/v] \land v = c)\urcorner$	(G)
5.	$LTrue\,(\ulcorner a = b \lor a \neq b\urcorner)$	Metatheorem
6.	$(\forall ref, M, g)\,True\,(\ulcorner a = b \lor a \neq b\urcorner, ref_{M,g})$	(D14) 5

7. $Res(\ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner,$ $\ulcorner a = \iota v(\sigma_1[a/v] \wedge v = a)\urcorner(\ulcorner \iota v(\sigma_1[a/v] \wedge v = a)\urcorner // \ulcorner \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner))$ — (Pr1)

8. $Res(\ulcorner b = \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner,$ $\ulcorner b = \iota v(\sigma_2[b/v] \wedge v = b)\urcorner(\ulcorner \iota v(\sigma_2[b/v] \wedge v = b)\urcorner // \ulcorner \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner))$ — (Pr2)

9. $True(\sigma_1[a], ref_{M,g})$ & $True(\sigma_2[b], ref_{M,g})$ — Ass. f. CP

10. $True(\sigma_1[a], ref_{M,g})$ — SIM 9

11. $True(\sigma_2[b], ref_{M,g})$ — SIM 9

12. $\sigma_1[a] \equiv_{ref_{M,g}} \ulcorner a = \iota v(\sigma_1[a/v] \wedge v = a)\urcorner$ — UI 4

13. $\sigma_2[b] \equiv_{ref_{M,g}} \ulcorner b = \iota v(\sigma_2[b/v] \wedge v = b)\urcorner$ — UI 4

14. $a \equiv_{ref_{M,g}} \ulcorner \iota v(\sigma_1[a/v] \wedge v = a)\urcorner$ — MP 10, UI 1

15. $b \equiv_{ref_{M,g}} \ulcorner \iota v(\sigma_2[b/v] \wedge v = b)\urcorner$ — MP 11, UI 1

16. $True(\ulcorner a = b \vee a \neq b\urcorner, ref_{M,g})$ — UI 6

17. $\ulcorner a = b \vee a \neq b\urcorner \equiv_{ref_{M,g}}$ $\ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner$ — MP 5, UI 2

18. $a \equiv_{ref_{M,g}} \ulcorner \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner$ — MP 16, UI 1

19. $\ulcorner \iota v(\sigma_1[a/v] \wedge v = a)\urcorner \equiv_{ref_{M,g}}$ $\ulcorner \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner$ — (Tr) (Sy) 14, 18

20. $\ulcorner a = \iota v(\sigma_1[a/v] \wedge v = a)\urcorner \equiv_{ref_{M,g}}$ $\ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner$ — MP ADJ 7, 19, UI 3

21. $\sigma_1[a] \equiv_{ref_{M,g}} \ulcorner a = \iota v((v = b \vee v \neq b) \wedge v = a)\urcorner$ — (Tr) 12, 20

22. $\sigma_1[a] \equiv_{ref_{M,g}} \ulcorner a = b \vee a \neq b\urcorner$ — (Tr) 21, (Sy) 17

23. $\ulcorner a = b \vee a \neq b\urcorner \equiv_{ref_{M,g}}$ $\ulcorner b = \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner$ — MP 5, UI 2

24. $b \equiv_{ref_{M,g}} \ulcorner \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner$ — MP 16, UI 1

25. $\ulcorner \iota v(\sigma_2[b/v] \wedge v = b)\urcorner \equiv_{ref_{M,g}}$ $\ulcorner \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner$ — (Tr) (Sy) 15, 24

26. $\ulcorner b = \iota v(\sigma_2[b/v] \wedge v = b)\urcorner \equiv_{ref_{M,g}}$ $\ulcorner b = \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner$ — MP ADJ 8, 25, UI 3

27. $\sigma_2[b] \equiv_{ref_{M,g}} \ulcorner b = \iota v((a = v \vee a \neq v) \wedge v = b)\urcorner$ — (Tr) 13, 26

28. $\sigma_2[b] \equiv_{ref_{M,g}} \ulcorner a = b \vee a \neq b\urcorner$ — (Tr) 27, (Sy) 23

29. $\sigma_1[a] \equiv_{ref_{M,g}} \sigma_2[b]$ (Tr) 22, (Sy) 28

30. $True\,(\sigma_1[a], ref_{M,g})$ & $True\,(\sigma_2[b], ref_{M,g}) \Rightarrow$ CP 9–29
$\sigma_1[a] \equiv_{ref_{M,g}} \sigma_2[b]$

31. $(\forall \sigma_1, \sigma_2, a, b, M, g)$ UG 30
$(True\,(\sigma_1[a], ref_{M,g})$ & $True\,(\sigma_2[b], ref_{M,g}) \Rightarrow$
$\sigma_1[a] \equiv_{ref_{M,g}} \sigma_2[b])$

Thus, it follows from the premises (L1), (L2), (RT), (G), (D14), (Pr1), and (Pr2) that all true singular sentences of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ refer to the same object.

□

4.2 The Case of False Singular Sentences: (T5)

To prove the following theorem for the case of false singular sentences, we need—in addition to the premises ($\mathrm{L_R}$), (T4), (N), and (RS)—a premise which states that

(Pr3) $Res(\ulcorner\neg\neg\sigma_4[b]\urcorner, \ulcorner\neg\neg\sigma_3[a]\urcorner\,(\ulcorner\neg\sigma_3[a]\urcorner//\ulcorner\neg\sigma_4[b]\urcorner))$.

Gödel's slingshot-argument for the case of false singular sentences of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ is as follows:

(T5) $\{(\mathrm{L_R}), (\mathrm{T4}), (\mathrm{N}), (\mathrm{RS}), (\mathrm{Pr3})\} \vdash_{\mathrm{PL1}^{=}}$
$(\forall \sigma_3, \sigma_4, a, b, M, g)$
$(False\,(\sigma_3[a], ref_{M,g})$ & $False\,(\sigma_4[b], ref_{M,g}) \Rightarrow \sigma_3[a] \equiv_{ref_{M,g}} \sigma_4[b])$

The proof of theorem (T5) requires once again the equivalence relation $\equiv_{ref_{M,g}}$.

Proof of (T5):

1. $(\forall S_1, S_2, M, g)\,(LE\,(S_1, S_2) \Rightarrow S_1 \equiv_{ref_{M,g}} S_2)$ ($\mathrm{L_R}$)

2. $(\forall \sigma_1, \sigma_2, a, b, M, g)$ (T4)
$(True\,(\sigma_1[a], ref_{M,g})$ & $True\,(\sigma_2[b], ref_{M,g}) \Rightarrow$
$\sigma_1[a] \equiv_{ref_{M,g}} \sigma_2[b])$

3. $(\forall \sigma, c, M, g)$ (N)
$(False\,(\sigma[c], ref_{M,g}) \Rightarrow True\,(\ulcorner\neg\sigma[c]\urcorner, ref_{M,g})$

4. $(\forall S_1, S_2, T_1, T_2, M, g)$ (RS)
$(Res(S_2, S_1(T_1//T_2))\ \&\ T_1 \equiv_{ref_{M,g}} T_2 \Rightarrow$
$S_1 \equiv_{ref_{M,g}} S_2)$

5. $Res(\ulcorner\neg\neg\sigma_4[b]\urcorner, \ulcorner\neg\neg\sigma_3[a]\urcorner(\ulcorner\neg\sigma_3[a]\urcorner//\ulcorner\neg\sigma_4[b]\urcorner))$ (Pr3)

6. $False(\sigma_3[a], ref_{M,g})\ \&\ False(\sigma_4[b], ref_{M,g})$ Ass. f. CP

7. $False(\sigma_3[a], ref_{M,g})$ SIM 6

8. $False(\sigma_4[b], ref_{M,g})$ SIM 6

9. $True(\ulcorner\neg\sigma_3[a]\urcorner, ref_{M,g})$ MP 7, UI 3

10. $True(\ulcorner\neg\sigma_4[b]\urcorner, ref_{M,g})$ MP 8, UI 3

11. $\ulcorner\neg\sigma_3[a]\urcorner \equiv_{ref_{M,g}} \ulcorner\neg\sigma_4[b]\urcorner$ MP ADJ 9, 10, UI 2

12. $\ulcorner\neg\neg\sigma_3[a]\urcorner \equiv_{ref_{M,g}} \ulcorner\neg\neg\sigma_4[b]\urcorner$ MP ADJ 5, 11, UI 4

13. $LE(\ulcorner\neg\neg\sigma_3[a]\urcorner, \sigma_3[a])$ Metatheorem

14. $\ulcorner\neg\neg\sigma_3[a]\urcorner \equiv_{ref_{M,g}} \sigma_3[a]$ MP 13, UI 1

15. $LE(\ulcorner\neg\neg\sigma_4[b]\urcorner, \sigma_4[b])$ Metatheorem

16. $\ulcorner\neg\neg\sigma_4[b]\urcorner \equiv_{ref_{M,g}} \sigma_4[b]$ MP 15, UI 1

17. $\sigma_3[a] \equiv_{ref_{M,g}} \sigma_4[b]$ (Tr) (Sy) 14, 12, 16

18. $False(\sigma_3[a], ref_{M,g})\ \&\ False(\sigma_4[b], ref_{M,g}) \Rightarrow$ CP 6–17
$\sigma_3[a] \equiv_{ref_{M,g}} \sigma_4[b]$

19. $(\forall \sigma_3, \sigma_4, a, b, M, g)$ UG 18
$(False(\sigma_3[a], ref_{M,g})\ \&\ False(\sigma_4[b], ref_{M,g}) \Rightarrow$
$\sigma_3[a] \equiv_{ref_{M,g}} \sigma_4[b])$

Thus, it follows from the premises (L_R), (T4), (N), (RS), and (Pr3) that all false singular sentences of $\mathscr{L}^{PL1^{=\iota}}$ refer to the same object. By the theorems (T4) and (T5), we arrive at the conclusion that all true singular sentences of $\mathscr{L}^{PL1^{=\iota}}$ refer to the same object—and so do all false sentences of $\mathscr{L}^{PL1^{=\iota}}$.

□

4.3 Generalisation

The conclusion that we reached in the proofs of theorems (T4) and (T5) will now be generalised to apply not only to singular sentences, but rather to all sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$. The fact of the matter is that any arbitrary non-singular sentence S is logically equivalent to a singular sentence such as $\ulcorner S \wedge c = c \urcorner$. From ($\mathrm{L_R}$), which states that logically equivalent sentences refer to the same object, it follows that any arbitrary non-singular sentence S refers to the same object as a singular sentence σ that is logically equivalent to S. Thus, two true non-singular sentences S_1 and S_2 refer to the same objects as two singular sentences σ_1 and σ_2 that are logically equivalent to S_1 and S_2, respectively. But these two singular sentences, σ_1 and σ_2, are both true since they are logically equivalent to the two true non-singular sentences S_1 and S_2, respectively. From (T4), it therefore follows that σ_1 and σ_2 refer to the same object. Furthermore, since the two true non-singular sentences S_1 and S_2 refer to the same objects as σ_1 and σ_2, respectively, and since the latter two refer to the same object, the sentences S_1 and S_2 must also refer to the same object. Therefore, all true non-singular sentences refer to the same object; the same holds true of all false non-singular sentences. But by theorems (T4) and (T5), we have already established that all true singular sentences refer to the same object and that all false singular sentences do so as well. Furthermore, all true singular sentences and all true non-singular sentences together amount to all true sentences, and the same applies to false sentences. Thus, we arrive at the conclusion that all true sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ refer to the same object, and so do all false sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$. This conclusion is precisely the (FCG)-thesis. □

In his outline of this proof, Gödel fails to include a number of premises that are necessary for deriving the (FCG)-thesis from the set of premises of his argument, e.g. ($\mathrm{L_R}$) and (G). If we now add the missing premises, all the premises of Gödel's slingshot-argument together become sufficient to restrict the set of all entities referred to by all true and all false sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ to a set containing at most two different elements (e.g. at most two different truth-values or at most two different states of affairs). However, it remains uncertain as to how these premises could be used to derive Frege's assumption (which states that truth-values are the referents of sentences) from the (FCG)-thesis. Therefore, Gödel's argument provides no information as to the kind of these entities. The members of any set that contains at most two different elements are possible candidates for the referents of the sentences of $\mathscr{L}^{\mathrm{PL1}^{=}\iota}$ relative to $ref_{M,g}$.

In sum, we have shown that the premises of Gödel's slingshot-argument only permit conclusions to be drawn as to the number of referents, but they do not suffice to precisely determine the kind of these referents.

5 Summary

If we add the missing premises of Gödel's argument, then the set of premises of this argument contains—amongst others—the following premises:

(A) Definite descriptions without free individual variables refer to individuals as do individual constants.

(L_R) Logically equivalent sentences refer to the same object.

(RT) It is always possible to substitute coreferential closed singular terms for each other within any sentence without changing *its referent.*

(RS) It is always possible to substitute coreferential sentences for each other within any sentence without changing *its referent.*

The premises (L_R), (RT), and (RS) are *neutral* reference-theoretical principles since it cannot be inferred from them of which kind the referents of sentences are.

The set of premises of Gödel's argument, however, does not contain the following *non-neutral* reference-theoretical principles:

(T) Truth-values are the referents of sentences.

(TRT) It is always possible to substitute coreferential closed singular terms for each other within any sentence without changing *its truth-value* as referent.

(TRS) It is always possible to substitute coreferential sentences for each other within any sentence without changing *its truth-value* as referent.

This has to be the case, lest his argument become trivial. For instance, it would easily follow from (TRT) that truth-values are the referents of sentences. Such a trivialisation, however, could not have been a part of Gödel's intention.

Instead, it is his aim to justify—by (A), (L_R), (RT), (RS), and by the remaining premises of his argument—the thesis that

(FCG) all true sentences refer to the same object, and so do all false sentences (i.e. the Frege-Church-Gödel-thesis).

Furthermore, the (FCG)-thesis can support Frege's thesis (T), which states that truth-values are the referents of sentences. Nevertheless, (FCG) is not sufficient for proving (T) in a strictly logical way. For, as we have shown, the premises of Gödel's argument are not sufficient for determining the kind of the referents of sentences. They merely show that the elements of any set with at most two members are possible candidates for the referents of sentences. These members can (but need not) be the two truth-values—True and False.

Gödel considers (FCG) to be absurd as it rules out the possibility that different sentences may refer to different objects. According to him, it is necessary to abandon one of the premises on which this thesis is based. Thus, he proposes that the fundamental assumption (A) of a referential theory of definite descriptions be abandoned. This assumption states that definite descriptions without free individual variables refer to individuals as do individual constants. Therefore, he proposes a non-referential theory of definite descriptions—such as Russell's theory—while maintaining the principle of compositionality (R).

From the point of view of a state-of-affairs semantics, however, there is a third possibility: we maintain (A) and (R), and instead we choose to abandon a different premise of Gödel's argument; specifically, the premise that logically equivalent sentences refer to the same object. In the context of a state-of-affairs semantics, it can be shown that logically equivalent sentences that are not identical refer to different states of affairs because they involve different objects, and because they have different necessary and sufficient conditions for obtaining. For instance, the state-of-affairs that Mars is a thing such that it rotates and exists is different from the state of affairs that Mars rotates and that Mars exists. While the first state of affairs does not involve any sub-state-of-affairs, the second state of affairs involves two sub-states-of-affairs. While the condition for obtaining of the first state of affairs is the condition that Mars is an element of the set of all things that rotate and exist, the condition for obtaining of the second state of affairs is the condition that the two sub-states-of-affairs obtain. Thus, the two states of affairs are different, even though the two sentences describing them are logically equivalent to each other.

By this approach, it becomes necessary to define in (D13) an extension $ref_{M,g}$ of *ref* to the sentences (and formulas) of $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ such that logically equivalent sentences (as well as sentences that are of the same form as the two sentences in (G)) refer to and describe different states of affairs. Any state-of-affairs semantics that is based on (A) and (R) such that different sentences may refer to different states of affairs has to satisfy this condition, lest it fall victim to the slingshot-argument.

6 Appendix: Proofs of the Metatheorems and Lemmata

6.1 Proof of (T1)

Let $\sigma, c, M = \langle D, ref, d^\circ \rangle$, and g be chosen freely, and let v be an individual variable that does not occur in $\sigma[c]$. Suppose, by CP,

(1) $True\,(\sigma[c], ref_{M,g})$.

From this we have

(2) $(\exists g)\,True\,(\sigma[c], ref_{M,g})$.

Since $\sigma[c]$ is a singular sentence, it does not contain any free occurrences of individual variables. The truth of $\sigma[c]$ relative to $ref_{M,g}$ is therefore independent of the variable assignment g. This is owing to

(3) $(\exists g)\,True\,(\sigma[c], ref_{M,g}) \Leftrightarrow (\forall g)\,True\,(\sigma[c], ref_{M,g})$,

which holds for all (singular) sentences[18]. Thus from (2), it follows by (3) that

(4) $(\forall g)\,True\,(\sigma[c], ref_{M,g})$.

Now let $g[v|ref(c)]$ be a variable assignment from $\mathrm{IV}_{\mathscr{L}^{\mathrm{PL1}^{=}\iota}}$ to D as specified by (D9). Then, (4) gives

(5) $True\,(\sigma[c], ref_{M,g[v|ref(c)]})$.

Let us next show that there exists (6) at least one and (7) at most one $d \in D$ such that $True\,(\ulcorner\sigma[c/v] \wedge v = c\urcorner, ref_{M,g[v|d]})$ where $d = ref(c)$.

ad (6) It holds that

(6.1) $ref_{M,g[v|ref(c)]}(v) =_{(\mathrm{D13b})} g[v|ref(c)](v) =_{(\mathrm{D9})} ref(c) =_{(\mathrm{D13a})}$
$ref_{M,g}(c) = ref_{M,g[v|ref(c)]}(c)$.

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[18] Cf. (Gamut 1991, p.97 f.).

The functions $ref_{M,g}(c)$ and $ref_{M,g[v|ref(c)]}(c)$ are identical because c does not contain any individual constants since for all g, g^*: $ref_{M,g}(c) = ref_{M,g^*}(c)$. From (6.1), we have

(6.2) $ref_{M,g[v|ref(c)]}(v) = ref_{M,g[v|ref(c)]}(c)$.

By (D13f), it follows from (6.2) that

(6.3) $True(\ulcorner v = c \urcorner, ref_{M,g[v|ref(c)]})$.

The formula $\sigma[c/v]$ is the result of replacing all occurrences of c in $\sigma[c]$ with v (the variable v being an individual variable that does not occur in $\sigma[c]$). The only difference between $\sigma[c/v]$ and $\sigma[c]$ is that $\sigma[c/v]$ contains free occurrences of v in every place where c occurs in $\sigma[c]$. From (6.3), $True(\ulcorner v = c \urcorner, ref_{M,g[v|ref(c)]})$, and from (5), $True(\sigma[c], ref_{M,g[v|ref(c)]})$, by Leibniz' Law and semantic ascent, we obtain

(6.4) $True(\sigma[c/v], ref_{M,g[v|ref(c)]})$.

From (6.4) and (6.3), we conclude by (D13h) that

(6.5) $True(\ulcorner \sigma[c/v] \wedge v = c \urcorner, ref_{M,g[v|ref(c)]})$ & $ref(c) = ref(c)$.

Hence, by EG we have

(6.6) $(\exists d)(d \in D$ & $True(\ulcorner \sigma[c/v] \wedge v = c \urcorner, ref_{M,g[v|d]})$ & $d = ref(c))$,

i.e. there is at least one $d \in D$ such that $True(\ulcorner \sigma[c/v] \wedge v = c \urcorner, ref_{M,g[v|d]})$ where $d = ref(c)$.

ad (7) We next show that for all $d', d'' \in D$ it holds that if $True(\ulcorner \sigma[c/v] \wedge v = c \urcorner, ref_{M,g[v|d']})$ and $True(\ulcorner \sigma[c/v] \wedge v = c \urcorner, ref_{M,g[v|d'']})$, then $d' = d''$. Suppose, by CP,

(7.1) $True(\ulcorner \sigma[c/v] \wedge v = c \urcorner, ref_{M,g[v|d']})$ &
$True(\ulcorner \sigma[c/v] \wedge v = c \urcorner, ref_{M,g[v|d'']})$,

which gives

(7.2) $True(\ulcorner\sigma[c/v] \wedge v = c\urcorner, ref_{M,g[v|d']})$

(7.3) $True(\ulcorner\sigma[c/v] \wedge v = c\urcorner, ref_{M,g[v|d'']})$.

By (D13h) and (D13f), it follows from (7.2) and (7.3) that

(7.4) $ref_{M,g[v|d']}(v) = ref_{M,g[v|d']}(c)$

(7.5) $ref_{M,g[v|d'']}(v) = ref_{M,g[v|d'']}(c)$,

which leads to

(7.6) $d' =_{(D9)} g[v|d'](v) =_{(D13b)} ref_{M,g[v|d']}(v) =_{(7.4)} ref_{M,g[v|d']}(c) = ref_{M,g}(c) =_{(D13a)} ref(c) =_{(D13a)} ref_{M,g}(c) = ref_{M,g[v|d'']}(c) =_{(7.5)} ref_{M,g[v|d'']}(v) =_{(D13b)} g[v|d''](v) =_{(D9)} d''$.

We suppose again that for all g, g^*: $ref_{M,g}(c) = ref_{M,g^*}(c)$. Hence,

(7.7) $d' = d''$.

By CP and UG, (7.1) to (7.7) give

(7.8) $(\forall d', d'')(d', d'' \in D \;\&\; True(\ulcorner\sigma[c/v] \wedge v = c\urcorner, ref_{M,g[v|d']}) \;\&\; True(\ulcorner\sigma[c/v] \wedge v = c\urcorner, ref_{M,g[v|d'']}) \Rightarrow d' = d'')$,

i.e. there is at most one $d \in D$ such that $True(\ulcorner\sigma[c/v] \wedge v = c\urcorner, ref_{M,g[v|d]})$. Thus, (6.6) and (7.8) lead to

(8) $(\exists! d)(d \in D \;\&\; True(\ulcorner\sigma[c/v] \wedge v = c\urcorner, ref_{M,g[v|d]}) \;\&\; d = ref(c))$.

Hence, by (D13d),

(9) $ref_{M,g}(\ulcorner\iota v(\sigma[c/v] \wedge v = c)\urcorner) = ref(c) =_{(D13a)} ref_{M,g}(c)$.

Owing to the symmetry of identity, (9) shows by (D13f) that

(10) $True(\ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner, ref_{M,g})$.

By CP, (1) to (10) yield

(11) $True(\sigma[c], ref_{M,g}) \Rightarrow True(\ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner, ref_{M,g})$.

Finally, we apply UG to (11), which gives theorem (T1). □

6.2 Proof of (T2)

Choose $\sigma, c, M = \langle D, ref, d^\circ \rangle, g$ arbitrarily. Let v be an individual variable which does not occur in $\sigma[c]$. Suppose, by CP,

(1) $True\,(\sigma[c], ref_{M,g})$

By (T1), we have

(2) $True\,(\sigma[c], ref_{M,g}) \Rightarrow True\,(\ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner, ref_{M,g})$.

From (1) and (2), it follows by propositional logic that

(3) $(True\,(\sigma[c], ref_{M,g})$ & $True\,(\ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner, ref_{M,g})) \vee$
$(False\,(\sigma[c], ref_{M,g})$ & $False\,(\ulcorner c = \iota v(\sigma[c/v] \wedge v = c)\urcorner, ref_{M,g}))$.

Hence, by (D13k),

(4) $True\,(\ulcorner \sigma[c] \leftrightarrow c = \iota v(\sigma[c/v] \wedge v = c)\urcorner, ref_{M,g})$.

From (1)–(4) we obtain, by CP,

(5) $True\,(\sigma[c], ref_{M,g}) \Rightarrow$
$True\,(\ulcorner \sigma[c] \leftrightarrow c = \iota v(\sigma[c/v] \wedge v = c)\urcorner, ref_{M,g})$.

By UG, (5) gives theorem (T2). □

6.3 Proof of the refutability of (NT)

Suppose

(1) $Satisfiable\,(\ulcorner Fc \urcorner)$,

that is, there are at least one $M' = \langle D', ref', d^{\circ\prime} \rangle$ and one g' such that

(2) $True\,(\ulcorner Fc \urcorner, ref'_{M',g'})$.

The next step is to show that

(3) $\sim LTrue\,(\ulcorner Fc \leftrightarrow c = \iota v(Fv \wedge v = c)\urcorner)$.

The atomic sentence $\ulcorner Fc \urcorner$ is not logically true; therefore, there are at least one more M and one more g such that

(4) $\mathit{False}\,(\ulcorner Fc \urcorner, \mathit{ref}_{M,g})$.

As an example, consider the following reference-theoretical model for $\mathscr{L}^{\mathrm{PL1}^{=\iota}}$ (where the metalinguistic symbol '$\emptyset$' denotes the empty set):

(5) $M'' = \langle \{0\}, \mathit{ref}'', 0 \rangle$,
with $d^{\circ} = 0$ & $\mathit{ref}''(c) = 0$ & $\mathit{ref}''(F) = \emptyset$

The claim (3) can be justified as follows: from

(6) $\mathit{ref}''_{M'',g''}(F) = \mathit{ref}''(F) = \emptyset$

and

(7) $\mathit{ref}''_{M'',g''}(c) = \mathit{ref}''(c) = 0$

we have, by (D13e),

(8) $\mathit{False}\,(\ulcorner Fc \urcorner, \mathit{ref}''_{M'',g''})$

since $\langle \mathit{ref}''_{M'',g''}(c) \rangle \notin \mathit{ref}''_{M'',g''}(F)$. On the other hand, it follows from (6) that there is not at least one, nor, consequently, one and only one $d \in D$ such that $\mathit{True}\,(\ulcorner Fv \wedge v = c \urcorner, \mathit{ref}''_{M'',g''[v|d]})$. Thus, by (D13d),

(9) $\mathit{ref}''_{M'',g''}(\ulcorner \iota v (Fv \wedge v = c) \urcorner) = 0$.

By the laws of identity, (7) and (9) yield

(10) $\mathit{ref}''_{M'',g''}(c) = \mathit{ref}''_{M'',g''}(\ulcorner \iota v (Fv \wedge v = c) \urcorner)$.

Hence, by (D13f),

(11) $\mathit{True}\,(\ulcorner c = \iota v (Fv \wedge v = c) \urcorner, \mathit{ref}''_{M'',g''})$.

From (8) and (11), it follows by (D13k) that

(12) $\mathit{False}\,(\ulcorner Fc \leftrightarrow c = \iota v (Fv \wedge v = c) \urcorner\ \mathit{ref}''_{M'',g''})$.

Hence, $\ulcorner Fc \leftrightarrow c = \iota v\,(Fv \wedge v = c) \urcorner$ is not logically true. Accordingly, the two parts of the biconditional are not logically equivalent to each other. Consequently, the metatheoretical sentence (INT) is false; therefore, (NT) is false as well and thus refutable. □

6.4 Proof of Lemma (L2)

1.	$(\forall\sigma, c, v, M, g)\,(\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g}) \Rightarrow$ $\mathit{True}\,(\ulcorner\sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner, \mathit{ref}_{M,g}))$	(T2)
2.	$(\forall S_1, S_2, M, g)\,(\mathit{LE}\,(S_1, S_2) \Rightarrow S_1 \equiv_{\mathit{ref}_{M,g}} S_2)$	($\mathrm{L_R}$)
3.	$\mathit{LTrue}\,(\sigma[c])$	Ass. f. CP
4.	$(\forall M, g)\,\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g})$	(D14) 3
5.	$(\forall M, g)\,(\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g}) \Rightarrow$ $\mathit{True}\,(\ulcorner\sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner, \mathit{ref}_{M,g}))$	UI 1
6.	$(\forall M, g)\,(\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g}) \Rightarrow$ $\mathit{True}\,(\ulcorner\sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner, \mathit{ref}_{M,g})) \Rightarrow$ $((\forall M, g)\,\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g}) \Rightarrow (\forall M, g)$ $\mathit{True}\,(\ulcorner\sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner, \mathit{ref}_{M,g}))$	PL1-theorem
7.	$(\forall M, g)\,\mathit{True}\,(\sigma[c], \mathit{ref}_{M,g}) \Rightarrow (\forall M, g)$ $\mathit{True}\,(\ulcorner\sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner, \mathit{ref}_{M,g})$	MP 5, 6
8.	$(\forall M, g)$ $\mathit{True}\,(\ulcorner\sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner, \mathit{ref}_{M,g})$	MP 4, 7
9.	$\mathit{LTrue}\,(\ulcorner\sigma[c] \leftrightarrow c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner)$	(D14) 8
10.	$\mathit{LE}\,(\sigma[c], \ulcorner c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner)$	(D15) 9
11.	$\sigma[c] \equiv_{\mathit{ref}_{M,g}} \ulcorner c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner$	MP 10, UI 2
12.	$\mathit{LTrue}\,(\sigma[c]) \Rightarrow \sigma[c] \equiv_{\mathit{ref}_{M,g}} \ulcorner c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner$	CP 3–11
13.	$(\forall\sigma, c, v, M, g)\,(\mathit{LTrue}\,(\sigma[c]) \Rightarrow$ $\sigma[c] \equiv_{\mathit{ref}_{M,g}} \ulcorner c = \iota v\,(\sigma[c/v] \wedge v = c)\urcorner$	UG 12

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